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Abstract

The optimal premium control in a non-life insurance business is determined using dynamic programming techniques. The optimality is measured in terms of solvency and a sufficient smoothing of the premium and the surplus variations in time.

Keywords

non-life, control theory, premium control, steady state

1. Introduction

It was argued by Martin-Löf [1983] and Gray [1984] that the engineering control methods with their feedback mechanism could be helpful to control insurance systems.

Martin-Löf proposed a linear feedback control for P_t in year t ($t \geq 1$) of the following form :

$$P_t = v\ E[X_t] - w\ G_{t-1} \qquad (1)$$

where v and w are positive constants and G_{t-1} and X_t stand for respectively the surplus at the end of year t-1 and the claim in year t. The expected claim for year t is estimated by a linear function of the claims in previous years. The coefficients v and w are then determined heuristically by examining the stability of the resulting premiums and surplusses of the considered portfolio.

In this paper we will use the dynamic programming technique to prove that the optimal premium control is very similar to (1) in case the criterion expresses the demand for equity, solvency and a sufficient smoothing of the premium and surplus variations in time.

We will determine the coefficients v and w that belong to this optimal premium control, together with the supplemental term that has to be added to the right hand side in (1). This term depends on the expected claims in future years, as can be expected by the nature of the dynamic programming technique.

In the next section we will explain the model that is used in section 3 to derive the optimal control law. In the last section we give some numerical examples to illustrate the method.

2. Model.

We consider a model that is close to the model given by Martin-Löf [1983] to describe a branch of a non-life insurance business.

Suppose that the premium P_t is received at the beginning of year t to cover the operating expenses D_t and the expenses for the claims S_t that incur in year t. The claims incurred during a particular year are not always determined in the same year but often with a delay of several years. To define the surplus at the end of year t, one estimates the remaining costs for the claims of that year. The difference between the estimated costs and the actual costs for claims in preceding years is called the runoff-profit A_t. The investment earnings are denoted by I_t.

The total outflow in year t will be denoted by X_t and is defined by

$$X_t = S_t + D_t - I_t - A_t \tag{2}$$

We assume that expenses are made in the middle of the year so that the surplus G_t at the end of year t is given by :

$$G_t = R\, G_{t-1} + R\, P_t - R^{*}\, X_t \tag{3}$$

where R = 1 + i stands for the interest factor.

We will assume that we can obtain a reliable estimate for $E[X_t]$, t=1,..T, where T stands for the planning horizon. This may seem a strong assumption, but if one has some evidence that the assumed trend not longer holds, one can easily insert the new estimates in the model and calculate the optimal premium from that period on.

We want to determine the premiums in the successive years 1,...T in such a way that the following requirements are satisfied :

* the accumulated surplus G_t must be sufficient to make the ruin probability as small as prescribed
* the surplusses and the premiums should not deviate too much from prescribed values.

These criteria can be expressed mathematically as follows :

$$\underset{\{P_t\}}{\text{minimize}} \quad E\left\{ \sum_{t=1}^{T} (P_t - \alpha_t)^2 + \sum_{t=1}^{T} (G_t - \beta_t)^2 \right\} \tag{4}$$

where α_t is fixed by the premium level wanted in year t and the β_t are prescribed by ruin theory.

This provides us with a linear-quadratic control model that can be solved by control theory and which possesses some interesting properties.
For a summary of that part of control theory that we will use in our derivation, see the appendix. For a detailed overview we refer to Chow [1975].

Our model can be written in the form :

$$\begin{bmatrix} P_t \\ G_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} P_{t-1} \\ G_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ R \end{bmatrix} P_t + \begin{bmatrix} 0 \\ -R^{*} EX_t \end{bmatrix} + \begin{bmatrix} 0 \\ R^{*} (EX_t - X_t) \end{bmatrix} \tag{5}$$

and the objective function is the conditional expectation of

$$E \sum_{t=1}^{T} \{ \begin{bmatrix} P_t \\ G_t \end{bmatrix} - \begin{bmatrix} \alpha_t \\ \beta_t \end{bmatrix} \}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \{ \begin{bmatrix} P_t \\ G_t \end{bmatrix} - \begin{bmatrix} \alpha_t \\ \beta_t \end{bmatrix} \} \tag{6}$$

given G_o.

With the appropriate matrix definitions, (5) can be written as :

$$y_t = A\, y_{t-1} + C\, P_t + b_t + u_t \tag{7}$$

where the u_t denote random vectors with zero mean, and (6) equals

$$E \,[\, \sum_{t=1}^{T} (y_t - a_t)^T K \,(y_t - a_t) \,]. \tag{8}$$

3. Results

As is explained in the appendix, the optimal control can be expressed recursively starting from the last period T.
In some situations, these recursions reach a steady state for t smaller than a certain value.
We will derive this steady state for our problem and proof that this steady state control can be applied from now on up to a certain time.
For the years immediately preceding T, the optimal control must be derived using the equations given in the appendix.
It will appear from the numerical examples in section 4 that this steady state solution can be applied for the greatest part of the T years.

Lemma.

The solutions to the following system of matrix equations

$$M = -(C^T H C)^{-1} C^T H A \tag{9}$$

$$H = K + (A + C M)^T H (A + C M) \tag{10}$$

with C, A and K defined by (5), (6), (7) and (8),

are given by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \tag{11}$$

and $$M = - \frac{1}{1 + R^2 h} [\ 0 \quad R^2 h\ ] \tag{12}$$

where h is a solution to

$$h^3 R^4 + 2 h^2 (R^2 - R^4) + h (1 - 3 R^2) - 1 = 0. \tag{13}$$

Proof :

From (10) it follows that H is a symmetric matrix. If we set

$$H = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_3 \end{bmatrix}$$

we will find, after some straightforward calculations that

$$C^T H C = h_1 + 2 R h_2 + R^2 h_3 \equiv N \tag{14}$$

and $$M = - N^{-1} [\ 0 \quad R h_2 + R^2 h_3\ ]. \tag{15}$$

For A + C M we get

$$A + C M = \begin{bmatrix} 0 & - N^{-1} (R h_2 + R^2 h_3) \\ 0 & R - N^{-1} (R^2 h_2 + R^3 h_3) \end{bmatrix} \equiv \begin{bmatrix} 0 & r \\ 0 & s \end{bmatrix} \tag{16}$$

such that (10) becomes

$$\begin{bmatrix} h_1 & h_2 \\ h_2 & h_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + r^2 h_1 + 2 r s h_2 + s^2 h_3 \end{bmatrix} \tag{17}$$

from which h_1, h_2 and h_3 can be determined. Inserting these values in (14) and (15) provides us with (12).

The steady state solution for the premium control exists if the characteristic roots of (A + C M) are smaller than 1 in absolute value. Form (16) we know that this is satisfied if $|s| < 1$ or

$$\left| \frac{R}{1 + R^2 h} \right| < 1. \tag{18}$$

For realistic values of R their exists exactly one solution to (13) that satisfies (18). For some interest rates between 0 and 0.1 this solution, together with the value of the characteristic root in (18), is given in table 1.

<u>Table 1 :</u>

R	h	$R/(1+R^2h)$
1.000	1.618034	0.38197
1.005	1.620786	0.38111
1 010	1.623515	0.38025
1.015	1.626220	0.37939
1.020	1.628903	0.37852
1.025	1.631562	0.37765
1.030	1.634198	0.37678
1.035	1.636812	0.37590
1.040	1.639403	0.37502
1.045	1.641972	0.37414
1.050	1.644518	0.37326
1.055	1.647042	0.37237
1.060	1.649544	0.37148
1.065	1.652025	0.37059
1.070	1.654484	0.36970
1.075	1.656921	0.36881
1.080	1.659337	0.36792
1.085	1.661732	0.36702
1.090	1.664105	0.36613
1.095	1.666458	0.36523
1.100	1.668790	0.36433

Now we are able to derive the steady state premium control.

Theorem.

The steady state premium control is given by

$$P_t = \frac{1}{1 + R^2 h} \left(- R^2 h G_{t-1} + R^{3/2} h EX_t + \alpha_t + R d_t \right) \qquad (19)$$

where d_t is defined recursively by

$$d_t = \beta_t - \frac{R^2h}{1 + R^2h} \alpha_{t+1} + \frac{R}{1 + R^2h} (d_{t+1} + h R^{1/2} EX_{t+1}), \qquad (20)$$

with the initial condition

$$d_T = \beta_T$$

and h is determined as the solution to (13), satisfying (18).

Proof :

From the appendix it follows that the steady state solution is given by

$$P_t = M \begin{bmatrix} P_{t-1} \\ G_{t-1} \end{bmatrix} + g_t \tag{21}$$

where M is given by (12) and g_t by

$$g_t = - (C^T H C)^{-1} C^T (H b_t - h_t). \tag{22}$$

The h_t in (22) are determined by the following recursion :

$$h_t = K a_t + (A + C M)^T (h_{t+1} - H b_{t+1}) \tag{23}$$

with the initial condition

$$h_T = K a_T. \tag{24}$$

The matrix H in (22) and (23) was given in (11).

Straightforward calculations provide us with

$$h_t = \begin{bmatrix} \alpha_t \\ d_t \end{bmatrix} \tag{25}$$

where d_t is given by (20).

Calculating (22) yields then

$$g_t = \frac{1}{N} [\alpha_t + R^{3/2} h EX_t + R d_t]. \tag{26}$$

where N was defined in (14).

The premium control given by (19) and (20) clearly is a linear feedback control of a form similar to the one that was given in (1), namely a decreasing function of the surplus and an increasing function of the expected claim.

The recursion relation (20) can be solved explicitly to give

$$\begin{aligned} d_T &= \beta_T \\ d_t &= \beta_t + \sum_{i=1}^{T-t} s^{i-1} \{ s \beta_{t+i} + r \alpha_{t+i} + h R^{1/2} s EX_{t+i} \} \end{aligned} \tag{27}$$

where r and s were defined in (17) and are equal to

$$r = \frac{R^2 h}{1 + R^2 h} \quad \text{and} \quad s = \frac{R}{1 + R^2 h}. \tag{28}$$

The following special case can be useful to cope with inflation in the model :

If $\alpha_t \equiv k\ \alpha_{t-1}$ ($\alpha_1 \equiv \alpha$),

$\beta_t \equiv k\ \beta_{t-1}$ ($\beta_1 \equiv \beta$), (29)

and $\mathbf{EX}_t \equiv k\ \mathbf{EX}_{t-1}$ ($\mathbf{EX}_1 \equiv \mu$)

(27) becomes

$$d_T = k^{T-1}\ \beta$$

$$d_t = k^{t-1}\ \beta + k^t\ \{\ S\ \beta + r\alpha + h\ R^{1/2}\ s\ \mu\ \}\ \frac{1 - (k\ S)^{T-t}}{1 - k\ s}. \qquad (30)$$

If $|k\ \ s| < 1$ and the considered period is very large, the term $(k\ s)^{T-t}$ is negligible for t sufficiently small.

In case $k = 1$, we get a steady state premium control with constant coefficients.

It is intuitively clear that for constant claim amounts in the successive years ($X_t = \mu$, $t \geq 1$) the premium should be given by $R^{-1/2}\ \mu$ and that no reserve should be built up in this case. This assertion is formally stated and proved in the following corollary which gives a first indication that the model provides us with useful results :

Corollary.

If $G_0 = 0$, $\mathbf{E}\ X_t = X_t = \mu$, $\alpha = R^{-1/2}\ \mu$, $\beta = 0$ and the period T is sufficiently large then the steady state solution is given by :

$G_t = 0$ and $P_t = R^{-1/2}\ \mu$ $(t \geq 1)$.

Proof :

From (30) we get

$$d_t = \{\ + rR^{-1/2} + h\ R^{1/2}\ s\ \}\ \frac{\mu}{1 - s} = 0. \qquad (31)$$

The premium control (19) becomes then

$$P_t = -\ R\ s\ h\ G_{t-1} + \frac{R^{3/2}\ h\ \mu + R^{-1/2}\ \mu}{1 + R^2\ h} \qquad (32)$$

which is equal to

$$P_t = -\ R^2\ s\ h\ G_{t-1} + R^{-1/2}\ \mu.$$

The result is then obtained with the help of (3).

4. Example.

In this section we will illustrate the premium control by means of a numerical example.

We suppose further that $\alpha_t = 1100$, $\beta_t = 750$ and $EX_t = 1000$ for $t \geq 1$.

With a value 1.05 for R, we can derive from table 1 that h = 1.644518.

After some calculations we get that the steady state premium is given by

$$P_t = - 0.645\ G_{t-1} + 1419.041. \tag{33}$$

To investigate how long this steady state control can be applied if the time horizon is for instance 50 years, we computed the optimal premium control with the equations (A3), (A4)and (A5) in the appendix. In table 2 we expose the P_t for t = 50,49,..35. For the previous years the premium is also given by (33).

Table 2 :

t	P_t
50	$- 0.524376\ G_{49} + 1409.479$
49	$- 0.626953\ G_{48} + 1542.303$
48	$- 0.642054\ G_{47} + 1483.915$
47	$- 0.644174\ G_{46} + 1445.881$
46	$- 0.644470\ G_{45} + 1429.424$
45	$- 0.644511\ G_{44} + 1422.968$
44	$- 0.644517\ G_{43} + 1420.514$
43	$- 0.644518\ G_{42} + 1419.593$
42	$- 0.644518\ G_{41} + 1419.248$
41	$- 0.644518\ G_{40} + 1419.119$
40	$- 0.644518\ G_{39} + 1419.071$
39	$- 0.644518\ G_{38} + 1419.053$
38	$- 0.644518\ G_{37} + 1419.046$
37	$- 0.644518\ G_{36} + 1419.043$
36	$- 0.644518\ G_{35} + 1419.043$
35	$- 0.644518\ G_{34} + 1419.042$

So it is clear that the steady state control can be applied for t = 1,...43 and that the optimal premium control does not change drastically during the last few years.

Following Martin-Löf [1983] we consider the case of constant claims : $X_t = 1000$, $t \geq 1$. The premiums and the surplusses for this case are visualized in the figure with and initial surplus of zero. As can be seen in this picture, the premiums and the surplusses converge very rapidly to the constant premium 940.549 and a constant surplus of 742.405.

This premium is lower than the expected claim because the interest that is received on the surplus and on the premiums provide us with a premium reduction each year.

Remark that with our method the premiums and the surplusses converge much faster than with the method that was proposed by Martin-Löf [1975].

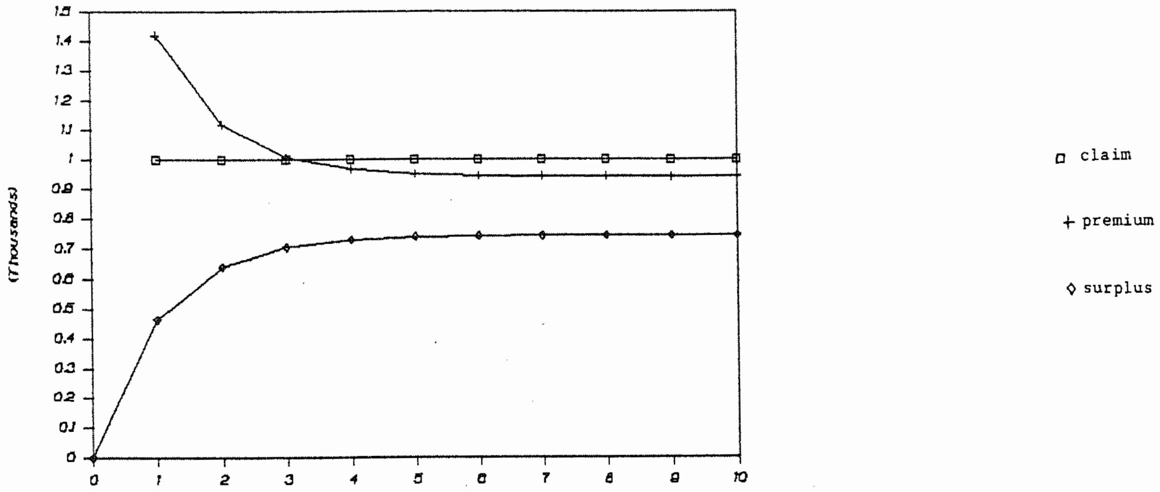

Figure : constant claims, initial surplus 0

APPENDIX

Given a linear model

$$y_t = A\, y_{t-1} + C\, P_t + b_t + u_t \qquad \text{(A1)}$$

where y_t is a vector of dependent variables and lagged control variables, P_t is the control variable, A and C are given constant matrices, b_t is a constant vector and the u_t denotes a random vector with zero mean and finite second moments that is independent of y_{t-1}.

Assume further that the following quadratic loss function is given :

$$W = \left[\sum_{t=1}^{T} (y_t - a_t)^T K\, (y_t - a_t) \right]. \qquad \text{(A2)}$$

The problem consists in choosing P_1, $P_2,\ldots P_T$ to minimize the conditional expectation E[W], given the initial condition y_0.

The solution is given by

$$P_t = M_t\, y_{t-1} + g_t \qquad \text{(A3)}$$

with $M_t = -(C^T H_t C)^{-1} (C^T H_t A)$

$$g_t = -(C^T H_t C)^{-1} C^T (H_t b_t - h_t) \qquad \text{(A4)}$$

$$H_{t-1} = K + (A + C M_t)^T H_t (A + C M_t)$$

$$h_{t-1} = K a_{t-1} + (A + C M_t)^T (h_t - H_t b_t).$$

and the initial conditions

$$H_T = K$$

$$h_T = K a_T. \qquad \text{(A5)}$$

The solution may reach a steady state for t smaller than a certain value, thus satisfying

$$M = -(C^T H C)^{-1} (C^T H A) \qquad \text{(A6)}$$

$$H = K + (A + C M)^T H (A + C M). \qquad \text{(A7)}$$

This steady state will exist if and only if all the characteristic roots of (A + C M) are smaller than 1 in absolute value.
Remark that even if a steady state is reached, g_t and h_t will change in time when a_t and b_t vary in time.

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