SOME POSITIVE DEPENDENCE NOTIONS, WITH APPLICATIONS IN ACTUARIAL SCIENCES

Michel Denuit
Institut de Statistique
Université Catholique de Louvain
Voie du Roman Pays, 20
B-1348 Louvain-la-Neuve, Belgium
denuit@stat.ucl.ac.be

Jan Dhaene
Departement Toegepaste Economische Wetenschappen
Faculteit Economische and Toegepaste Economische Wetenschappen
Katholieke Universiteit Leuven
Minderbroedersstraat, 5
B-3000 Leuven, Belgium
Jan.Dhaene@econ.kuleuven.ac.be

Carmen Ribas
Dpt. de Matemàtica Econòmica, Financiera I Actuarial
Facultat de Ciènces Econòmiques I Empresarials
Universitat De Barcelona
Avda Diagonal, 690
08034 Barcelona, Spain
cribas@eco.ub.es

December 1999

Abstract

The present paper is devoted to the study of several notions of positive dependence among risks, namely association, linear positive quadrant dependence, positive orthant dependence and conditional increasingness in sequence. Various examples illustrate the usefulness of these notions in an actuarial context.

Key words and phrases: dependence, risk theory, association, linear positive quadrant dependence, positive orthant dependence, conditional increasingness in sequence

1 Introduction

The study of the dependence among risks has become one of the main topics in actuarial sciences nowadays. It has been recognized that the assumption of mutual independence of risks is often violated in insurance practice. In many lines of business, the introduction of common shocks at the portfolio level is needed to represent the effects of catastrophes hitting several (or a large number of) policies simultaneously, like earthquakes, tornados, epidemics and so on. Consequently, the risks in the individual model are certainly not independent but merely depend on each other. The purpose of this paper is to examine some statistical models describing dependent risks, as well as to measure their consequences.

Let us briefly specify some notations. Henceforth, a non-negative random variable X with a finite expectation is called a risk. A multivariate risk $\mathbf{X}$ is a random vector $(X_1, X_2, \ldots, X_n)$ whose components $X_1, X_2, \ldots, X_n$ are univariate risks and covariances $\mathrm{Cov}[X_i, X_j]$ are finite for all $i \neq j$. All the functions used in this paper are tacitly assumed to be measurable. Finally, $\mathbb{R}$ denotes the real line $(-\infty, +\infty)$, $\mathbb{R}_+$ the half positive real line $[0, +\infty)$ and $\mathbb{N}$ the set of the non-negative integers $\{0, 1, 2, \ldots\}$.

Several notions of positive dependence were introduced in the literature to model the fact that large values of one of the components of a multivariate risk $\mathbf{X}$ tend to be associated with large values of the others. Some of these concepts appear to be relevant in actuarial sciences. We briefly review these notions in Section 2. Introducing these dependence notions will enable us to generalize several results from Dhaene and Goovaerts (1996), Dhaene, Vanneste and Wolthuis (1996) and Wang and Dhaene (1998), where only the bivariate case was considered.

To begin with, we examine the notion of association in Subsection 2.1. The risks $X_1, X_2, \ldots, X_n$ are said to be associated (or equivalently the n-dimensional risk $\mathbf{X}$ is said to possess this property) when

$$\mathrm{Cov}\left[\phi_1(X_1, X_2, \cdots, X_n), \phi_2(X_1, X_2, \cdots, X_n)\right] \geq 0 \tag{1.1}$$

for all the non-decreasing functions ϕ_1 and $\phi_2 : \mathbb{R}_+^n \to \mathbb{R}$ for which the covariances exist. A single risk X_1 is associated since the inequality

$$\mathrm{Cov}[\phi_1(X_1), \phi_2(X_1)] \geq 0 \tag{1.2}$$

holds for any non-decreasing functions ϕ_1 and ϕ_2.

Association has been first considered in actuarial sciences by Norberg (1989) in the special case $n = 2$; this author used this notion in order to investigate some alternatives to the independence assumption for multilife statuses in life insurance. Association has also recently been used by Ribas and Alegre (1999) in order to model dependency relations in the individual life model.

In Subsection 2.2, we consider a notion of dependence that is weaker than association and is defined with the aid of the positive quadrant dependence. As far as random couples are concerned ($n = 2$), positive quadrant dependence (PQD, in short) has been extensively used in actuarial sciences, e.g. by Dhaene and Goovaerts (1996) and Denuit, Lefèvre and Mesfioui (1999). Let us recall that two risks X_1 and X_2 are said to be PQD if the inequality

$$P[X_1 > x_1, X_2 > x_2] \geq P[X_1 > x_1]P[X_2 > x_2] \tag{1.3}$$

holds for any reals $x_1, x_2 \in I\!R_+$, or, equivalently, if the inequality

$$\text{Cov}[\phi_1(X_1), \phi_2(X_2)] \geq 0 \tag{1.4}$$

holds for any non-decreasing functions ϕ_1 and $\phi_2 : I\!R_+ \to I\!R$; for a proof of the equivalence of (1.3) and (1.4), proceed as for Theorem 1 in Dhaene and Goovaerts (1996). Considering (1.3), the intuitive meaning of PQD is clear: if X_1 and X_2 are PQD then the probability that they both assume "large" values is greater than if they were independent. Combining (1.1) and (1.4), it is easily seen that associated risks X_1 and X_2 are PQD.

Since the inequality

$$P[X_1 > x_1, X_1 > x_2] = P[X_1 > \max\{x_1, x_2\}] \geq P[X_1 > x_1]P[X_2 > x_2]$$

obviously holds, we get (1.2) from (1.4).

A direct multivariate ($n \geq 3$) extension of the PQD concept is known as the pairwise PQD: a n-dimensional risk $\mathbf{X}$ is said to be pairwise PQD if the components X_i and X_j of $\mathbf{X}$ are PQD for all $i \neq j$. Subsection 2.2 is devoted to a less trivial extension of PQD, the so-called linear PQD (LPQD, in short). We prove *inter alia* that the sum of the components of a LPQD risk $\mathbf{X}$ dominates the sum of the components of its independent version in the stop-loss sense.

In Subsection 2.3, in addition to pairwise PQD and LPQD, we present a third generalization of the bivariate PQD to higher dimensions, namely the positive orthant dependence (POD, in short). Whereas (1.3) compares quadrant probabilities, POD uses the corresponding orthant probabilities.

In Subsection 2.4, we review the conditional increasingness in sequence (CIS, in short). The main interest of this technical condition is that it is sufficient to imply association as well as LPQD and POD. Moreover, CIS is often easily verified in risk models.

Section 3 is devoted to various applications in actuarial sciences. We first extend to the multivariate case the main results obtained by Dhaene, Vanneste and Wolthuis (1996) for the bidimensional situation. To be specific, we compare the amounts of the net single premium relating to a joint-life or to a last-survivor annuity under the independence and the POD assumptions. Then, we examine a very particular dependence structure, namely comonotonicity. We show that a comonotonic risk possesses all the properties discussed in Section 2. We also investigate the family of counting distributions introduced by Ambagaspitiya (1998). Next, the additivity of stop-loss preserving premium calculation principles is briefly discussed. To end with, risk models similar to the one defined by Marceau, Cossette, Gaillardetz and Rioux (1999) are investigated.

In the remainder, all the vectors are tacitly assumed to be column vectors and the superscript "prime" denotes the usual transposition. The random vector $\mathbf{X}^\perp = (X_1^\perp, X_2^\perp, \ldots, X_n^\perp)$ represents an independent version of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, i.e. (i) the random variables $X_1^\perp, X_2^\perp, \ldots, X_n^\perp$ are mutually independent and (ii) for any $i = 1, 2, \ldots, n$, the random variables X_i and $X_i^\perp$ are identically distributed. Given two random variables X and Y, X is said to precede Y in the stochastic dominance (resp. stop-loss order), written as $X \preceq_{st} Y$ (resp. $X \preceq_{s\ell} Y$) if $E\phi(X) \leq E\phi(Y)$ for all the non-decreasing (resp. non-decreasing and convex) functions ϕ for which the expectations exist. The symbol "$=_d$" means "is equally distributed as".

2 Positive dependence notions

2.1 Association

It is obvious that (1.1) models a situation where the components of $\mathbf{X}$ are positively dependent, but the intuitive meaning of association is not clear. However, implicit in a conclusion that a set of risks is associated is a wealth of inequalities, often of direct use in various actuarial problems. Moreover, models recently introduced to take into account a possible dependence often generate associated risks. This point will be considered in Section 3.

The following properties of association can be found in Esary, Proschan and Walkup (1967).

Property 2.1. *Let $X_1, X_2, \ldots, X_n$ be associated risks. The following assertions hold true:*

(i) *any subset $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ of $X_1, X_2, \ldots, X_n$ is associated;*

(ii) *Let $Y_1, Y_2, \ldots, Y_k$ be associated random variables independent of the X_i's. Then X_1, X_2, $\ldots$, X_n, Y_1, Y_2, $\ldots$, Y_k are associated;*

(iii) *If the measurable functions $\psi_1, \psi_2, \ldots, \psi_k : \mathbb{R}^n \to \mathbb{R}$ are non-decreasing then the random variables $\psi_1(X_1, X_2, \cdots, X_n)$, $\psi_2(X_1, X_2, \cdots, X_n)$, $\ldots$, $\psi_k(X_1, X_2, \cdots, X_n)$ are associated.*

We will also often use the following result, which straightly follows from (1.2) together with Property 2.1 (ii).

Property 2.2. $\mathbf{X}^{\perp}$ *is associated.*

Many authors have questioned the relevance of the classical Pearson correlation coefficient as a measure of dependence. See e.g. Embrechts, McNeil and Strauman (1999). Nevertheless, under several positive dependence notions, this measure is of great interest to the practitioner, as shown in Property 2.3; for other results in that vein, see Denuit and Dhaene (1999).

Property 2.3. *Suppose that the multivariate risk $\mathbf{X}$ is associated. The X_k's are jointly independent if, and only if, $Cov[X_i, X_j] = 0$ for all $i \neq j$.*

This means that for an associated multivariate risk $\mathbf{X}$, investigating mutual independence turns out to investigate the covariances. The variance-covariance matrix of an associated risk $\mathbf{X}$ plays thus a central role in the investigation of the dependency structure of $\mathbf{X}$. A proof of Property 2.3 will be given in Section 2.2 where a more general version of the relation between positive dependence notions and correlations will be considered.

2.2 Linear positive quadrant dependence

Newman (1984) proposed the following extension of the bivariate PQD: let $\mathbf{X}$ be a multivariate risk such that for any non-negative real constants $\alpha_1, \alpha_2, \ldots$ and for any disjoint $A, B \subseteq \{1, 2, \ldots, n\}$,

$$\sum_{i \in A} \alpha_i X_i \text{ and } \sum_{i \in B} \alpha_i X_i \text{ are PQD.} \tag{2.1}$$

Then, $\mathbf{X}$ is said to be linearly PQD (LPQD, in short). Note that LPQD is essentially a symmetrical condition, in the sense that saying that $(X_1, X_2, \ldots, X_n)$ is LPQD is equivalent to say that $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$ is LPQD, for any permutation π of $\{1, 2, \ldots, n\}$. Without loss of generality, we may assume $\alpha_i \in [0, 1]$ for all i in (2.1). As a consequence, (2.1) can be interpreted as follows: for any two disjoint sets of risks A and B of the portfolio, both aggregate risks associated with a quota share reinsurance treaty are PQD, i.e. the probability that they both assume "large" values is greater than if they were independent.

It can be shown that, given a multivariate risk $\mathbf{X}$,

$$\mathbf{X} \text{ associated} \Rightarrow \mathbf{X} \text{ LPQD} \Rightarrow \mathbf{X} \text{ pairwise PQD}, \tag{2.2}$$

but the reverse is not necessarily true. The proof of the implications in (2.2) is straightforward.

The following properties are easy to prove (coming back to the definition of PQD by means of correlation order for (ii)).

Property 2.4. *Let $X_1, X_2, \ldots, X_n$ be LPQD risks. The following assertions hold true:*

(i) *any subset $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ of $X_1, X_2, \ldots, X_n$ is LPQD;*

(ii) *Let $Y_1, Y_2, \ldots, Y_k$ be LPQD random variables independent of the X_i's. Then X_1, X_2, $\ldots$, X_n, Y_1, Y_2, $\ldots$, Y_k are LPQD.*

By (2.2) together with Property 2.2, $\mathbf{X}^\perp$ is LPQD. Remark that the counterpart of Property 2.1 (iii) is not necessarily valid here.

For $\mathbf{X}$ pairwise PQD, the covariance structure reveals a lot of information about the dependence of the components $X_1, X_2, \ldots, X_n$ of $\mathbf{X}$, as it was the case for association. This is formally stated in the next result.

Property 2.5. *Suppose that the multivariate risk $\mathbf{X}$ is pairwise PQD. Then,*

(i) *$Cov[X_i, X_j] \geq 0$ for all $i \neq j$;*

(ii) *Given two disjoint subsets A and B of $\{1, 2, \ldots, n\}$, $\{X_k,\ k \in A\}$ and $\{X_k,\ k \in B\}$ are mutually independent if, and only if,*

$$Cov[X_i, X_j] = 0 \text{ for all } i \in A \text{ and } j \in B.$$

Proof. Since the couples (X_i, X_j) are PQD for all $i \neq j$, (i) directly follows from (1.4). Let us now prove (ii). Consider $i \in A$ and $j \in B$, and assume that $\text{Cov}[X_i, X_j] = 0$. Let us recall that Denuit, Lefèvre and Mesfioui (1999) have shown that

$$E[X_i X_j] = \int_{x_i=0}^{+\infty} \int_{x_j=0}^{+\infty} P[X_i > x_i, X_j > x_j] dx_i dx_j.$$

Since the covariance between X_i and X_j equals 0, the latter formula yields

$$\int_{x_i=0}^{+\infty} \int_{x_j=0}^{+\infty} \{P[X_i > x_i, X_j > x_j] - P[X_i > x_i] P[X_j > x_j]\} dx_i dx_j = 0.$$

Now, the integrand $\{.\}$ in the latter expression is non-negative for all x_i and x_j (since (X_i, X_j) is PQD); this implies that $P[X_i > x_i, X_j > x_j] = P[X_i > x_i] P[X_j > x_j]$ for all $x_i, x_j \in \mathbb{R}_+$, so that X_i and X_j are mutually independent. The opposite conclusion is straightforward. □

As a consequence of Property 2.5 (ii) we have that for a pairwise PQD risk $\mathbf{X}$, the components $X_1, X_2, \ldots, X_n$ are mutually independent if, and only if, $\mathrm{Cov}[X_i, X_j] = 0$ for all $i \neq j$. Because of (2.2) we have that the results of Property 2.5 *a fortiori* hold for LPQD or associated risks $\mathbf{X}$.

Let us now prove the following result which enhances the interest of LPQD in the study of dependent risks. More precisely, it is known from Dhaene and Goovaerts (1996, Theorem 2) that if the random couple (X_1, X_2) is PQD then the stochastic inequality

$$X_1^\perp + X_2^\perp \preceq_{s\ell} X_1 + X_2$$

holds; we provide hereafter a multivariate generalization of this result.

Theorem 2.6. *Let* $\mathbf{X}$ *be LPQD with marginal distribution functions* $F_1, F_2, \ldots, F_n$*. Then, we have*

$$X_1^\perp + X_2^\perp + \ldots + X_n^\perp \preceq_{s\ell} X_1 + X_2 + \ldots + X_n \preceq_{s\ell} F_1^{-1}(U) + F_2^{-1}(U) + \ldots + F_n^{-1}(U),$$

where U *denotes a random variable uniformly distributed on the unit interval* $[0,1]$ *and* F_i^{-1} *is the quantile function associated to* F_i*, i.e.*

$$F_i^{-1}(p) = \inf\{x \in I\!R | F_i(x) \geq p\}, \;\; 0 < p < 1.$$

Proof. The second stop-loss inequality is true in general, for risks $X_1, X_2, \ldots, X_n$ with distribution function $F_1, F_2, \ldots, F_n$; see, e.g., Dhaene, Wang, Young and Goovaerts (1997). Let us prove the first stop-loss inequality. Without loss of generality, the random vectors $\mathbf{X}^\perp$ and $\mathbf{X}$ may be considered independent. Now, proceed by induction. First, $X_1^\perp \preceq_{s\ell} X_1$ trivially holds. Now, assume that

$$X_1^\perp + X_2^\perp + \cdots + X_k^\perp \preceq_{s\ell} X_1 + X_2 + \cdots + X_k$$

holds true for $k = 1, 2, \cdots, n-1$. Then, by the closure of $\preceq_{s\ell}$ under convolution, the latter stochastic inequality yields

$$X_1^\perp + X_2^\perp + \cdots + X_{n-1}^\perp + X_n^\perp \preceq_{s\ell} X_1 + X_2 + \cdots + X_{n-1} + X_n^\perp. \tag{2.3}$$

Now, since $\mathbf{X}$ is *LPQD*, X_n and $X_1 + X_2 + \cdots + X_{n-1}$ are positively quadrant dependent, we get

$$X_1 + X_2 + \cdots + X_{n-1} + X_n^\perp \preceq_{s\ell} X_1 + X_2 + \cdots + X_{n-1} + X_n. \tag{2.4}$$

Combining (2.3) and (2.4) yields the announced result by the transitivity property of $\preceq_{s\ell}$. □

Note that Theorem 2.6 *a fortiori* holds when $\mathbf{X}$ is associated; see (2.2). It is worth mentioning that the result in Theorem 2.6 holds under much less restrictive (non-symmetrical) conditions. Indeed, assuming that the random couples (X_1, X_2), $(X_1 + X_2, X_3)$, $(X_1 + X_2 + X_3, X_4)$, $\ldots$, $(X_1 + X_2 + \ldots + X_{n-1}, X_n)$ are all PQD suffices to prove the theorem. This notion of positive dependence is used by Denuit, Dhaene, Lefèvre and Koutras (1999) to deal with dependence in the individual risk model.

From the above result, once the marginal distributions of the X_i's are fixed, the best possible bounds in the $\preceq_{s\ell}$-sense on the aggregate claims $X_1 + X_2 + \ldots + X_n$ of LPQD risks are provided by $X_1^\perp + X_2^\perp + \ldots + X_n^\perp$ and $F_1^{-1}(U) + F_2^{-1}(U) + \ldots + F_n^{-1}(U)$. Therefore, any risk-averse decision-maker will prefer $X_1^\perp + X_2^\perp + \ldots + X_n^\perp$ over $X_1 + X_2 + \ldots + X_n$ when the risks $X_1, X_2, \cdots, X_n$ are LPQD. This conclusion holds both in Von Neumann and Morgenstern expected utility theory, as well as in Yaari's dual theory of choice under risk. For more details about the interpretation of $\preceq_{s\ell}$ in decision theory, see e.g. Dhaene, Wang, Young and Goovaerts (1997).

For LPQD risks, the safest dependence structure is provided by mutual independence, for fixed marginals. When the risks are not known to be LPQD, the safest dependence structure does not always exist; see Dhaene and Denuit (1999) for more details.

We finally remark that it follows from Theorem 2.6 that making the assumption of mutual independence between the components of a LPQD risk $\mathbf{X}$ leads to an underestimation of the stop-loss premiums. In terms of utility theory this means that the insurer in fact replaces the "real" aggregate claims by a "less risky" aggregate claims, which is of course a dangerous strategy.

2.3 Positive orthant dependence

A multivariate risk $\mathbf{X}$ is said to be positively lower orthant dependent (PLOD, in short) when the inequality

$$P[X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n] \geq \prod_{i=1}^{n} P[X_i \leq x_i] \tag{2.5}$$

holds for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$; it is said to be positively upper orthant dependent (PUOD, in short) when the inequality

$$P[X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n] \geq \prod_{i=1}^{n} P[X_i > x_i] \tag{2.6}$$

holds for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$. When (2.5) and (2.6) simultaneously hold, then $\mathbf{X}$ is said to be positively orthant dependent (POD, in short). We have that

$$\mathbf{X} \text{ associated } \Rightarrow \mathbf{X} \text{ POD};$$

for a proof, see e.g. Esary *et al.* (1967, Theorem 5.1).

Note that POD is a straight extension of the bivariate PQD to dimension $n \geq 3$ (by substituting orthants for bivariate quadrants). POD has an intuitive interpretation. Indeed, from (2.5) we see that the probability that all the components of $\mathbf{X}$ are "small" is greater than in the independent case, while (2.6) means that the probability that all the components are "large" is greater than in the independent case. Note that (2.5) and (2.6) are in general not equivalent when $n \geq 3$.

It can easily be shown that for random couples ($n = 2$), the following equivalences hold true:

$$\mathbf{X} \text{ PQD} \Leftrightarrow \mathbf{X} \text{ POD} \Leftrightarrow \mathbf{X} \text{ LPQD}.$$

Of course, these equivalences no more hold in general for dimension $n \geq 3$. It is worth mentioning that the inequalities (2.5) and (2.6) are usually referred to as the Sidak inequalities, or as the first order product-type inequalities. A study of the accuracy of these inequalities can be found e.g. in Glaz and Johnson (1984). These authors also proposed a method to exploit the dependence srtucture in order to get better bounds on orthant probabilities than those furnished by (2.5) and (2.6).

From (2.5) and (2.6), it is easy to conclude that when $\mathbf{X}$ is POD, the stochastic inequalities

$$\min_i X_i^\perp \preceq_{st} \min_i X_i \text{ and } \max_i X_i \preceq_{st} \max_i X_i^\perp \tag{2.7}$$

are valid. The latter result can be found e.g. in Baccelli and Makowski (1989) and explains the usefulness of the POD notion in a variety of situations, since it suggests a natural way of generating computable bounds for the maximum or the minimum of n POD risks.

2.4 Conditional increasingness in sequence

The abstract characterization (1.1) of association is often difficult to deal with in a concrete statistical model. Therefore, a stronger notion than association but better tractable may be of interest. Conditional increasingness in sequence is such a concept of dependence.

A random vector $\mathbf{X}$ is said to be conditionally increasing in sequence (CIS, in short) if, for any $i = 2, 3, \ldots, n$, one of the equivalent following conditions hold:

(i) $P[X_i > x | X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}]$ is non-decreasing in $x_1, x_2, \ldots, x_{i-1}$ in the support of the X_i's for all x;

(ii) $[X_i | X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}] \preceq_{st} [X_i | X_1 = y_1, X_2 = y_2, \ldots, X_{i-1} = y_{i-1}]$ for any $x_1 \leq y_1$, $x_2 \leq y_2$, ..., $x_{i-1} \leq y_{i-1}$ in the support of the X_i's;

(iii) $E[\phi(X_i) | X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}]$ is a non-decreasing function of the variables $x_1, x_2, \ldots, x_{i-1}$ in the support of the X_i's for all the non-decreasing functions ϕ for which the expectations are defined.

See Cohen and Sackrowitz (1995) for further results.

As mentioned above, a CIS multivariate risk $\mathbf{X}$ is associated; for a proof, we refer the interested reader e.g. to Joe (1997, Theorem 2.4 page 16). It is easily seen that $\mathbf{X}^\perp$ is CIS.

In an actuarial context, CIS may be generalized to weaker orderings than $\preceq_{st}$ (in particular, the stop-loss order). We could require, for instance, that for any $i = 2, 3, \ldots, n$,

$$E[(X_i - x)_+ | X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}]$$

to be non-decreasing in $x_1, x_2, \ldots, x_{i-1}$ in the support of the X_i's for all x, which in turn boils down to

$$[X_i | X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}] \preceq_{s\ell} [X_i | X_1 = y_1, X_2 = y_2, \ldots, X_{i-1} = y_{i-1}]$$

for any $x_1 \leq y_1$, $x_2 \leq y_2$, ..., $x_{i-1} \leq y_{i-1}$ in the support of the X_i's. A study of this dependence concept is deferred to a subsequent work.

3 Applications

3.1 Multiple life statuses

Consider the statuses (x_1), (x_2), ..., (x_n) with remaining lifetimes $T_{(x_1)}$, $T_{(x_2)}$, ..., $T_{(x_n)}$, respectively. The joint life status $(x_1, x_2, \ldots, x_n)$ exists as long as all individual statuses exist. This status has remaining lifetime

$$T_{(x_1,x_2,\ldots,x_n)} = \min\left\{T_{(x_1)}, T_{(x_2)}, \ldots, T_{(x_n)}\right\}.$$

The last survivor status $\overline{(x_1, x_2, \ldots, x_n)}$ exists as long as at least one of the individual status is alive. Its remaining lifetime is given by

$$T_{\overline{(x_1,x_2,\ldots,x_n)}} = \max\left\{T_{(x_1)}, T_{(x_2)}, \ldots, T_{(x_n)}\right\}.$$

Let us now assume that $\mathbf{T} = (T_{(x_1)}, T_{(x_2)}, \ldots, T_{(x_n)})$ is POD. Let us also introduce the following straightforward notations:

$$T^{\perp}_{(x_1,x_2,\ldots,x_n)} = \min\left\{T^{\perp}_{(x_1)}, T^{\perp}_{(x_2)}, \ldots, T^{\perp}_{(x_n)}\right\} \text{ and } T^{\perp}_{\overline{(x_1,x_2,\ldots,x_n)}} = \max\left\{T^{\perp}_{(x_1)}, T^{\perp}_{(x_2)}, \ldots, T^{\perp}_{(x_n)}\right\}.$$

From (2.7), it follows that

$$T^{\perp}_{(x_1,x_2,\ldots,x_n)} \preceq_{st} T_{(x_1,x_2,\ldots,x_n)} \text{ and } T_{\overline{(x_1,x_2,\ldots,x_n)}} \preceq_{st} T^{\perp}_{\overline{(x_1,x_2,\ldots,x_n)}}$$

which in turn implies that

$$\ddot{a}^{\perp}_{(x_1,x_2,\ldots,x_n)} \preceq_{st} \ddot{a}_{(x_1,x_2,\ldots,x_n)} \text{ and } \ddot{a}_{\overline{(x_1,x_2,\ldots,x_n)}} \preceq_{st} \ddot{a}^{\perp}_{\overline{(x_1,x_2,\ldots,x_n)}},$$

where the superscript "$\perp$" is used to indicate that the annuity is based on $T^{\perp}_{(x_1,x_2,\ldots,x_n)}$ or $T^{\perp}_{\overline{(x_1,x_2,\ldots,x_n)}}$. This means that for POD remaining lifetimes, the independence assumption (while leaving the marginal distribution functions unchanged) leads to an underestimation of the net single premium (and reserves) of a joint life annuity. The opposite conclusion holds for the last survivor annuity. Similar conclusions can be drawn for endowment and whole life insurances. These results extend those in Dhaene, Vanneste and Wolthuis (1996), where only the bivariate case is considered.

3.2 Comonotonic risks

The risks $X_1, X_2, \cdots, X_n$ with marginal distribution functions F_1, F_2, ..., F_n are said to be mutually comonotonic when

$$(X_1, X_2, \ldots, X_n) =_d (F_1^{-1}(U), F_2^{-1}(U), \ldots, F_n^{-1}(U)),$$

with U uniformly distributed over the unit interval $[0, 1]$. Actuarial applications of the notion of comonotonicity can be found e.g. in Goovaerts, Dhaene and De Schepper (1999).

Such an extreme multivariate risk $\mathbf{X}$ fulfills all the positive dependence notions examined above. Firstly, a mutually comonotonic risk is necessarily associated, from Property 2.1(iii) together with the fact that U is associated by (1.2). A direct check of this assertion is as follows: it suffices to note that, given any non-decreasing functions ϕ_1 and $\phi_2 : \mathbb{R}^n \to \mathbb{R}$, we have that

$$\mathrm{Cov}[\phi(\mathbf{X}), \phi_2(\mathbf{X})] = \mathrm{Cov}[\psi_1(U), \psi_2(U)],$$

where

$$\psi_i(u) = \phi_i(F_1^{-1}(u), F_2^{-1}(u), \ldots, F_n^{-1}(u)), \;\; i = 1, 2.$$

It is straightforward that ψ_1 and ψ_2 are both non-decreasing. The latter covariance is then non-negative in virtue of (1.2).

A comonotonic risk $\mathbf{X}$ is even CIS. Indeed, for any non-decreasing function ϕ, we find

$$\begin{aligned} &E[\phi(X_i)|X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}] \\ &\quad = E[\phi(F_i^{-1}(U))|F_1^{-1}(U) = x_1, F_2^{-1}(U) = x_2, \ldots, F_{i-1}^{-1}(U) = x_{i-1}] \\ &\quad = E[\phi(F_i^{-1}(U))|F_j(x_j - 0) \leq U \leq F_j(x_j) \text{ for } j = 1, 2, \ldots, i-1] \end{aligned}$$

which is clearly non-decreasing in x_1, x_2, $\ldots$, x_{i-1}.

Finally, we prove (directly) that a comonotonic risk is also POD. Indeed, we have that

$$P[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n] = \min\{F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\} \geq \prod_{i=1}^{n} F_i(x_i)$$

and

$$\begin{aligned} P[X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n] &= \min\{P[X_1 > x_1], P[X_2 > x_2], \ldots, P[X_n > x_n]\} \\ &\geq \prod_{i=1}^{n} P[X_i > x_i]. \end{aligned}$$

3.3 Ambagaspityia's class of counting distributions

Ambagaspitiya (1998) proposed a new family of discrete multivariate distributions representing the number of claims in different classes of business. To be specific, the n-dimensional random vector $\mathbf{N}$ is given as

$$\begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{pmatrix},$$

where $a_{ij} \in \mathbb{N}$ for all i and j, and $\mathbf{M}$ is a random vector valued in $\mathbb{N}^k$ with independent components. Such a random vector $\mathbf{N}$ is associated since given any non-decreasing functions ϕ_1 and $\phi_2 : \mathbb{R}^n \to \mathbb{R}$, there exist two non-decreasing functions ψ_1 and $\psi_2 : \mathbb{R}^k \to \mathbb{R}$ such that

$$\mathrm{Cov}[\phi_1(\mathbf{N}), \phi_2(\mathbf{N})] = \mathrm{Cov}[\psi_1(\mathbf{M}), \psi_2(\mathbf{M})].$$

The latter covariance is non-negative since $\mathbf{M}$ has independent components, which implies that $\mathbf{M}$ is associated.

A possible generalization of Ambagaspitiya's class of multivariate distributions is as follows: let $\mathbf{M}$ be defined as above, and consider $\mathbf{N}$ with ith component N_i of the form $N_i = \varphi_i(M_1, M_2, \ldots, M_k)$, $i = 1, 2, \ldots, n$, where the functions $\varphi_i : I\!N^k \to I\!N$, $i = 1, 2, \ldots, n$, are non-decreasing. It is straightforward that $\mathbf{N}$ is also associated.

3.4 Premium calculation principle

Let us consider a premium calculation principle $H[.]$, that assigns a premium amount $H[X]$ to any risk X. We assume that the distribution function of X completely determines the premium for X. Assume further that $H[.]$ preserves the stop-loss order, i.e. given two risks X and Y,

$$X \preceq_{s\ell} Y \Rightarrow H[X] \leq H[Y].$$

Consider LPQD risks $X_1, X_2, \cdots, X_n$. The stop-loss preserving property together with Theorem 2.6 yields

$$H\left[\sum_{i=1}^{n} X_i^{\perp}\right] \leq H\left[\sum_{i=1}^{n} X_i\right] \leq H\left[\sum_{i=1}^{n} F_i^{-1}(U)\right]. \tag{3.1}$$

The inequality above states that for a stop-loss preserving premium principle, the premium of a sum of LPQD risks is maximal if the risks are comonotonic and minimal if the risks are mutually independent. We remark that the second inequality holds in general for all $\mathbf{X}$ (not necessarily LPQD); see e.g. Wang and Dhaene (1998). From (3.1), we find that if a premium principle preserves stop-loss order and is additive for independent risks, then it is super-additive for LPQD risks. This result is a generalization of the bivariate case considered in Wang and Dhaene (1998).

3.5 Marceau's model

Let us consider the model recently defined by Marceau *et al.* (1999). This model allows dependence between the risks of an insurance portfolio in the individual risk model. Consider a portfolio consisting of n policies with claim amounts $X_1, X_2, \ldots, X_n$. Let S be the aggregate claim amount for the insurance portfolio, i.e. $S = \sum_{i=1}^{n} X_i$. Let X_i, $i = 1, 2, \ldots, n$, be of the form $X_i = I_i B_i$ where the I_i's are Bernoulli random variables such that $P[I_i = 1] = p_i$ and $P[I_i = 0] = q_i$, $p_i + q_i = 1$, and where the B_i's are independent positive random variables. Assume further that the random vectors $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ are mutually independent. Now, suppose that the I_i's satisfy

$$I_i = \min\{J_i + J_0, 1\}, \;\; i = 1, 2, \ldots, n,$$

where the J_i's, $i = 0, 1, \ldots, n$ are independent Bernoulli random variables with $P[J_i = 1] = p_{ii}$ and $P[J_i = 0] = q_{ii}$, $p_{ii} + q_{ii} = 1$. The random vector $\mathbf{I}$ has clearly dependent components, so that $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ also has. Nevertheless, the I_i's remain Bernoulli distributed with $q_i = q_{00} q_{ii}$. The I_i's are associated since given any non-decreasing functions ϕ_1 and $\phi_2 : I\!R^n \to I\!R$, there exist two non-decreasing functions ψ_1 and $\psi_2 : I\!R^{n+1} \to I\!R$ such that

$$\mathrm{Cov}[\phi_1(I_1, I_2, \ldots, I_n), \phi_2(I_1, I_2, \ldots, I_n)] = \mathrm{Cov}[\psi_1(J_0, J_1, \ldots, J_n), \psi_2(J_0, J_1, \ldots, J_n)],$$

which is non-negative since the J_i's are independent, and thus associated. The X_i's are then also associated. Indeed, from Property 2.1(ii) and the independence assumption, it follows that $I_1, I_2, \ldots, I_n, B_1, B_2, \ldots, B_n$ are associated, and by Property 2.1(iii), we finally find that $\mathbf{X}$ is associated.

We remark that Marceau's model can be generalized and still remain associated. Indeed, if the I_i's are defined by

$$I_i = \varphi_i(J_0, J_1, \ldots, J_n), \;\; i = 1, 2, \ldots, n,$$

where the J_i's are arbitrary associated random variables and where the functions $\varphi_i : \mathbb{R}^n \to \{0, 1\}$ are non-decreasing, then it is easy to verify that the vector $\mathbf{X}$ remains associated. Even more general: it suffices that $\mathbf{I}$ and $\mathbf{B}$ are associated and mutually independent to imply association of $\mathbf{X}$.

4 Conclusions

In this paper, we considered several notions of positive dependence. Except for the CIS, all these notions are symmetric, in the sense that their definition is independent of the order of the components of the random vector. All these notions are qualitative (in the sense that a multivariate risk possesses or not a given dependence structure). It turned out that the independent random vectors and the comonotonic random vectors both are in accordance with all positive dependence notions we examined. Moreover, independence and comonotonicity are extremal notions in the class of positively dependent random vectors. Indeed, assuming that $\mathbf{X}$ is pairwise PQD (which is the weakest dependence notion we considered), we have that

$$0 \leq \text{Cov}[X_i, X_j] \leq \text{Cov}[F_i^{-1}(U), F_j^{-1}(U)]$$

for all $1 \leq i \neq j \leq n$. We proved that independence of the X_i's boils down to $\text{Cov}[X_i, X_j] = 0$ for all $1 \leq i \neq j \leq n$, while Denuit and Dhaene (1999) showed that comonotonicity of all components is equivalent to $\text{Cov}[X_i, X_j] = \text{Cov}[F_i^{-1}(U), F_j^{-1}(U)]$ for all $1 \leq i \neq j \leq n$. We also proved that if $\mathbf{X}$ is LPQD then the stop-loss premium relating to the sum of the components of $\mathbf{X}$ is bounded from below by the sum of the components of the independent version $\mathbf{X}^{\perp}$. In this case, the independence assumption will lead to an underestimation of the true stop-loss premium. Items for future research are the study of quantitative measures for dependence (like correlation in the bivariate case) and their appropriateness in actuarial sciences.

References

Albers, W. (1999). Stop-loss premiums under dependence. *Insurance: Mathematics and Economics* **24**, 173-185.

Ambagaspitiya, R.S. (1998). On the distribution of a sum of correlated aggregate claims. *Insurance: Mathematics and Economics* **23**, 15-19.

Baccelli, F., and Makowski, A.M. (1989). Multidimensional stochastic ordering and associated random variables. *Operations Research* **37**, 478-487.

Cohen, A., & Sackrowitz, H.B. (1995). On stochastic ordering of random vectors. *Journal of Applied Probability* **32**, 960-965.

Denuit, M., and Dhaene, J. (1999). Simple characterizations of Fréchet bounds by extremal correlations. Mimeo.

Denuit, M., Dhaene, J., Lefèvre, Cl., and Koutras, M. (1999). Collective approximations in the individual model with dependent risks. Mimeo.

Denuit, M., Lefèvre, Cl., and Mesfioui, M. (1999). A class of bivariate stochastic orderings with applications in actuarial sciences. *Insurance : Mathematics and Economics* **24**, 31-50.

Dhaene, J., & Denuit, M. (1999). The safest dependence structure among risks. *Insurance: Mathematics & Economics*, to appear.

Dhaene, J., & Goovaerts, M.J. (1996). Dependency of risks and stop-loss order. *ASTIN Bulletin* **26**, 201-212.

Dhaene, J., Vanneste, M., and Wolthuis, H. (1996). Dependencies in Multiple life statuses. Research Report, Departement Toegepaste Economische Wetenschappen, K.U. Leuven.

Dhaene, J., Wang, S., Young, V., and Goovaerts, M. (1997). Comonotonicity and maximal stop-loss premiums. Research Report, Departement Toegepaste Economische Wetenschappen. K.U. Leuven.

Embrechts, P., McNeil, P., and Strauman, D. (1999). Correlation and dependency in risk management. *Proceedings of the XXXth International ASTIN Colloquium, Tokyo, Japan*, 227-250.

Esary, J.D., Proschan, F., & Walkup, D.W. (1967). Association of random variables, with applications. *The Annals of Mathematical Statistics* **38**, 1466-1474.

Glaz, J., & Johnson, B. McK. (1984). Probability inequalities for multivariate distributions with dependence structures. *Journal of the American Statistical Association* **79**, 436-440.

Goovaerts, M., Dhaene, J., and De Schepper, A. (1999). Stochastic upper bounds for present value functions. *Journal of Risk and Insurance*, to appear.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman & Hall, London.

Marceau, E., Cossette, H., Gaillardetz, P., and Rioux, J. (1999). Dependence in the individual risk model. Mimeo.

Newman, C.M. (1984). Asymptotic independence and limit theorems for positively and negatively dependent random variables. In "Inequalities in Statistics and Probability", IMS Lecture Notes **5**, 127-140.

Norberg, R. (1989). Actuarial analysis of dependent lives. *Bulletin de l'Association Suisse des Actuaires*, 243-254.

Ribas, C., and Alegre, A. (1999). Modeling positive dependency relations between the risks of the individual life model. Mimeo.

Wang, S., and Dhaene, J. (1998). Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics* **22**, 235-242.