Supermodular Ordering and Stochastic Annuities *

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Abstract

In this paper, we consider several types of stochastic annuities, for which an explicit expression of the distribution function is not available. We will construct a random variable with the same mean and which is larger in stop-loss order, for which the distribution function can easily be obtained.

1 Introduction

In several financial-actuarial problems one is faced with the determination of the distribution function of non-negative random variables of the form

$$V = \int_0^t \gamma(\tau)\ e^{-X(\tau)} d\tau$$

where $\gamma(\tau)$ is a non-negative deterministic function and $X(\tau)$ denotes some stochastic process, such as a Wiener process e.g. The distribution of such a random variable can be used in finance in order to determine the price of an Asian option, see e.g. Geman and Yor (1993). It is also of interest in pension mathematics where the random variable V can be interpreted as the net

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present value of the cash flow of a pension scheme, see e.g. Dufresne (1990). The knowledge of the distribution function of such a random variable is also necessary for the pricing of modern life insurance products in a stochastic interest rate environment. Another example is financial reinsurance where one tends to deal with loss reserves as stochastic quantities depending on random discount factors. In this case the IBNR reserves can be seen as random variables of this form, see Goovaerts and Redant (1998).

The problem of finding the distribution function of V can be transformed into a problem consisting in determining the solution of a partial differential equation, see e.g. De Schepper et al. (1994), Vanneste et al. (1994) and Goovaerts and Dhaene (1997). Another approach consists in the use of stochastic differential equations, see e.g. Yor (1992). However, none of these approaches provides a solution which enables an explicit calculation of the distribution function valid for all values of t and for all realistic choices of the function $\gamma(\tau)$. In case of a Wiener process e.g., the only special cases for which an analytic solution for the distribution function of V is known are when $t = \infty$ and $\gamma(\tau) = e^{-\delta\tau}$ (see e.g. Dufresne (1990) and Milevsky (1997), and also when $\gamma(\tau) = \delta(\tau - \tau_0)$. In the latter case δ denotes the Dirac-delta function.

In the actuarial literature it is a common feature to replace a risk, i.e. a nonnegative random variable, by a less favorable risk, which has a simpler structure, making it easier to determine relevant quantities such as premiums. In order to clarify what we mean with a less favorable risk, we will make use of the stop-loss order, which is defined as follows:

Definition 1 *A risk V is said to precede a risk W in stop-loss order, written $V \leq_{sl} W$, if the respective stop-loss premiums are ordered uniformly:*

$$E\left[(V-d)_+\right] \leq E\left[(W-d)_+\right]$$

for all retentions $d \geq 0$.

A risk W will be said to be less favorable or more risky than a risk V, if $V \leq_{sl} W$.

In this paper, we will consider some types of random variables V as defined above, for which the distribution function cannot be determined explicitely. We will construct a new risk W with the same expectation, but which is less favorable in stop-loss order sense, meaning that for each d, the stop-loss premium with retention d of the risk V is smaller than or equal to the corresponding stop-loss premium of W. The risk W will be constructed in such a way that an expression for its distribution function can easily be obtained.

2 Supermodular Order

Supermodularity has originally been studied in the applied mathematics and operations research literature. In recent years, it has received considerable attention in the economics literature. A self-contained and up-to-date overview of the related economic theory is Topkis (1998).

Let $\mathbf{e}_i$ denote the i-th n-dimensional unit vector. For $\mathbf{x} = (x_1, \cdots, x_n)$ and an arbitrary function $f : R^n \to R$, we define $\Delta_i^\varepsilon f(\mathbf{x}) = f(\mathbf{x}+\epsilon\, \mathbf{e}_i) - f(\mathbf{x})$.

Definition 2 *A function $f : R^n \to R$ is said to be supermodular if*

$$\Delta_i^\varepsilon \Delta_j^\delta \ f(\mathbf{x}) \geq 0$$

holds for all $\mathbf{x} \epsilon R^n, 1 \leq i < j \leq n$ and all $\epsilon, \delta > 0$.

In order to derive our results, use will be made of the supermodular order, which is a partial order between multivariate distribution functions. This order has proved to be a usefull order in the applied probability literature. For actuarial applications of this order, see Müller (1997) and Bäuerle and Müller (1998).

Definition 3 *A random vector $\mathbf{X} = (X_1, \cdots, X_n)$ is said to be smaller than a random vector $\mathbf{Y} = (Y_1, \cdots, Y_n)$ in the supermodular ordering, written $\mathbf{X} \leq_{sm} \mathbf{Y}$, if $E[f(\mathbf{X})] \leq E[f(\mathbf{Y})]$ for all supermodular functions f such that the expectations exist.*

Remark that supermodular ordering can only hold if $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals, see Müller (1997)or Bäuerle and Müller (1998).

In the following theorem we present a relation between supermodular order and stop-loss order.

Theorem 1 *Let $\mathbf{X} = (X_1, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$ be n-dimensional random vectors with $\mathbf{X} \leq_{sm} \mathbf{Y}$. Further, let $\phi_1, \cdots \phi_n$ be non-increasing non-negative functions. Then*

$$\phi_1(X_1) + \cdots + \phi_n(X_n) \leq_{sl} \phi_1(Y_1) + \cdots + \phi_n(Y_n).$$

Proof. Let $g: R \to R$ be an arbitrary non-decreasing convex function and define the function $f: R^n \to R$ by $f(x_1, \cdots, x_n) = g\left(\phi_1(x_1) + \cdots + \phi_n(x_n)\right)$.

From the convexity of g we have that

$$g(x+\alpha+\beta) + g(x) \geq g(x+\alpha) + g(x+\beta).$$

for α and β both negative. Now let $1 \leq i < j \leq n$ and ε and $\delta > 0$. By choosing $x = \phi_1(x_1) + \cdots + \phi_n(x_n)$, $\alpha = \phi_i(x_i + \varepsilon_i) - \phi_i(x_i)$ and $\beta = \phi_j(x_j + \delta_j) - \phi_j(x_j)$ and inserting these expressions in the inequality above, we find that the function f is supermodular. Because $\mathbf{X} \leq_{sm} \mathbf{Y}$, this implies

$$E\left[g\left(\phi_1(X_1) + \cdots + \phi_n(X_n)\right)\right] \leq E\left[g\left(\phi_1(Y_1) + \cdots + \phi_n(Y_n)\right)\right]$$

for any non-decreasing convex function g for which the expectations exist. This last implication is equivalent with the stop-loss inequality to be proved, see e.g. Goovaerts et al. (1986). ■

Remark that the case that all the functions ϕ_i are non-decreasing is considered in Theorem 3.2.in Müller (1997).

Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random vector with marginal distribution functions $F_1, F_2, \cdots, F_n$ and let U be a random variable which is uniformly distributed on the interval $[0, 1]$. It is well-known that the random vector $\left(F_1^{-1}(U), F_2^{-1}(U), \cdots, F_n^{-1}(U)\right)$ has the same marginals as the random vector $\mathbf{X}$.

Definition 4 *Let $\mathbf{Y} = (Y_1, \cdots, Y_n)$ be a random vector with marginals F_i, $i = 1, \cdots, n$. Then $\mathbf{Y}$ is said to be comonotonic if $\mathbf{Y}$ has the same distribution function as $\left(F_1^{-1}(U), F_2^{-1}(U), \cdots, F_n^{-1}(U)\right)$, where U is a random variable which is uniformly distributed on the interval $[0, 1]$.*

The concept of comonotonicity was introduced by Schmeidler (1986) and Yaari (1987) and has since then played an important role in economic theories of decision under risk and uncertainty. For actuarial applications of the concept of comonoticity, see e.g. Dhaene and Goovaerts (1996), Wang and Dhaene (1998) and Dhaene et al. (1998).

From the following theorem, we see that a comonotonic random vector possesses a very strong form of dependence between its components.

Theorem 2 *Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random vector with marginals F_i, $i = 1, \cdots, n$, and let U be a random variable which is uniformly distributed on the interval $[0, 1]$, then*

$$(X_1, \cdots, X_n) \leq_{sm} \left(F_1^{-1}(U), F_2^{-1}(U), \cdots, F_n^{-1}(U)\right).$$

This result is due to Tchen (1980). It states that, within the class of random vectors with given marginals, the comonotonic random vectors are greater in supermodular order than any other element of this class.

3 Main Result

Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a n-dimensional random vector with marginals $F_1, \cdots, F_n$. Further, let U be a random variable which is uniformly distributed on the interval [0, 1]. Finally, let $\phi_1, \cdots \phi_n$ be non-negative and non-increasing functions. Assume that we are faced with a situation where it is impossible to find an explicit expression for the distribution function of the risk $\phi_1(X_1) + \cdots + \phi_n(X_n)$. In this situation, we could replace the unknown distribution function by the distribution function of $\phi_1\left(F_1^{-1}(U)\right) + \cdots + \phi_n\left(F_n^{-1}(U)\right)$. Combining Theorems 4 and 6, we find

$$\phi_1(X_1) + \cdots + \phi_n(X_n) \leq_{sl} \phi_1\left(F_1^{-1}(U)\right) + \cdots + \phi_n\left(F_n^{-1}(U)\right).$$

The main advantages of working with $\phi_1\left(F_1^{-1}(U)\right) + \cdots + \phi_n\left(F_n^{-1}(U)\right)$ instead of $\phi_1(X_1) + \cdots + \phi_n(X_n)$ are threefold :

- In order to compute stop-loss premiums of the random variable in the right hand side of the inequality, only one integration has to be carried out (the integration over U), while computing the stop-loss premiums of the left hand side involves n integrations. This is certainly attractive for the stochastic processes we will consider later, where n tends to infinity.

- From the expression of the stop-loss transform of $\phi_1\left(F_1^{-1}(U)\right) + \cdots + \phi_n\left(F_n^{-1}(U)\right)$, the corresponding distribution function is rather easily obtained. So we can easily find the distribution function of a random variable which is more dangerous in stop-loss order than the original random variable $\phi_1(X_1) + \cdots + \phi_n(X_n)$.

- Because $(X_1, \cdots, X_n)$ and $\left(F_1^{-1}(U),\ F_2^{-1}(U),\ \cdots,\ F_n^{-1}(U)\right)$ have the same marginals, one gets that $\phi_1(X_1) + \cdots + \phi_n(X_n)$ and $\phi_1\left(F_1^{-1}(U)\right) + \cdots + \phi_n\left(F_n^{-1}(U)\right)$ have the same mean. As these random variables are stop-loss ordered, we have that all moments of $\phi_1(X_1) + \cdots + \phi_n(X_n)$ are smaller than or equal to the moments of $\phi_1\left(F_1^{-1}(U)\right) + \cdots + \phi_n\left(F_n^{-1}(U)\right)$, see e.g. Goovaerts et al. (1986).

In the following theorem, we present an algorithm which allows to determine the distribution function of $\phi_1\left(F_1^{-1}(U)\right)+\cdots+\phi_n\left(F_n^{-1}(U)\right).$

Theorem 3 *Let* $\mathbf{X}=(X_1,\cdots,X_n)$ *be a n-dimensional random vector with marginals* $F_1,\cdots,F_n$*. Further,* U *is a random variable which is uniformly distributed on the interval* $[0,\ 1]$ *and* $\phi_1,\ \cdots\ \phi_n$ *are non-negative and non-increasing functions. Then*

$$\Pr\left[\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(U)\right)>x\right]=\sup\left\{u\epsilon\,[0,1]\mid\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(u)\right)>x\right\}$$

Proof. The expression for the tail function follows from

$$\Pr\left[\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(U)\right)>x\right]=\int_0^1 I\left[\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(u)\right)>x\right]du.$$

where $\mathrm{I}(y>x)$ denotes the indicator function which equals 1 if $y>x$ and 0 otherwise. ■

Remark that if $\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(u)\right)$ is a strictly decreasing and continuous function of u, then we have

$$\Pr\left[\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(U)\right)>x\right]=u_x$$

with u_x determined by

$$\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(u_x)\right)=x.$$

4 The Distribution Function of Annuities

4.1 Discrete Annuities

In the previous sections, we have seen that replacing the distribution function of the random variable $V=\sum_{i=1}^{n}\phi_i(X_i)$ by the distribution function of the random variable $W=\sum_{i=1}^{n}\phi_i\left(F_i^{-1}(U)\right)$ is safe, in the sense that the mean is unchanged, but the stop-loss premiums (and as a consequence also all higher order moments) are larger in the latter case.

Let us now look at the special case that the X_i are all normally distributed. Hence, assume that

$$X_i \backsim N(0, \sigma_i^2) \qquad i = 1, \cdots, n.$$

In this case, we have that

$$F_i^{-1}(u) = \sigma_i \ \Phi^{-1}(u)$$

where Φ is the distribution function of a standard normal distributed random variable. Let us now assume that the functions ϕ_i are continuous and strictly decreasing. Then we find

$$\Pr\left[W > x\right] = \Phi(v_x)$$

with v_x defined by

$$\sum_{i=1}^{n} \phi_i \left(\sigma_i \ v_x\right) = x.$$

Combining the previous results, we find the following theorem for discrete temporary annuities.

Theorem 4 *Consider the annuity*

$$V = \sum_{i=1}^{n} \alpha_i \ \exp\left(-\delta \ i - X_i\right),$$

where δ *is the risk free interest intensity,* $X_i \backsim N(0, \sigma_i^2)$*,* $i = 1, \cdots, n$*, and the* α_i*,* $i = 1, \cdots, n$*, are non-negative real numbers.*
Further, let U *be a random variable which is uniformly distributed on the interval* $[0, \ 1]$ *and let* W *be defined by*

$$W = \sum_{i=1}^{n} \alpha_i \ \exp\left(-\delta \ i - \sigma_i \ \Phi^{-1}(U)\right).$$

Then we have that

$$V \leq_{sl} W.$$

The distribution function of W *is given by*

$$\Pr\left[W > x\right] = \Phi(v_x)$$

where v_x is determined by

$$\sum_{i=1}^{n} \alpha_i \ \exp\left[-\delta \ i - \sigma_i \ v_x\right] = x.$$

From the Theorem above, we immediately find that the density function of W is given by

$$f_W(x) = \frac{\Phi'(v_x)}{\sum_{i=1}^{n} \alpha_i \ \sigma_i \ \exp\left[-\delta \ i - \sigma_i \ v_x\right]}.$$

The results above can be used to determine stop-loss more dangerous risks (with the same expectation) for the usual discrete annuities, such as constant (where $V = \sum_{i=1}^{n} \ e^{-\delta i - X_i}$) and increasing annuities (where $V = \sum_{i=1}^{n} i \ e^{-\delta i - X_i}$).

4.2 Continuous Annuities

Let us now consider the continuous temporary annuity V defined by

$$V = \int_0^t \alpha(\tau) \exp\left[-\delta \ \tau - \sigma \ X(\tau)\right] \ d\tau$$

where $X(\tau)$ represents a standard Brownian motion, δ is the risk free interest intensity and $\alpha(\tau)$ is a non-negative function of τ.

In order to be able to use the previous theory, we first approximate the annuity V by

$$V_n = \sum_{i=1}^{n} \frac{t}{n} \ \alpha(i\frac{t}{n}) \exp\left[-\delta \ i\frac{t}{n} - \sigma \ X(i\frac{t}{n})\right]$$

From our previous results, we find that $V_n \leq_{sl} W_n$ with W_n defined by

$$W_n = \sum_{i=1}^{n} \frac{t}{n} \ \alpha(i\frac{t}{n}) \exp\left[-\delta \ i\frac{t}{n} - \sigma \ \sqrt{i\frac{t}{n}} \Phi^{-1}(U)\right]$$

with U uniformly distributed on the interval $[0, 1]$.
The distribution function of W_n follows from

$$\Pr\left[W_n > x\right] = \Phi(v_x)$$

with v_x defined by

$$\sum_{i=1}^{n} \frac{t}{n} \, \alpha(i\frac{t}{n}) \exp\left[-\delta \; i\frac{t}{n} - \sigma \; \sqrt{i\frac{t}{n}} v_x\right] = x.$$

Taking limits ($n \to \infty$), we find the following result.

Theorem 5 *Consider the annuity*

$$V = \int_0^t \alpha(\tau) \exp\left[-\delta \; \tau - \sigma \; X(\tau)\right] \; d\tau$$

where $X(\tau)$ represents a standard Brownian motion, δ is the risk free interest intensity and $\alpha(\tau)$ is a non-negative function of τ.
Further, let U be a random variable which is uniformly distributed on the interval $[0, \; 1]$, and let Y be defined by

$$W = \int_0^t \alpha(\tau) \exp\left[-\delta \; \tau - \sigma \; \sqrt{\tau} \; \Phi^{-1}(U)\right] \; d\tau.$$

Then we have that

$$V \leq_{sl} W.$$

The distribution function of W is given by

$$\Pr\left[W > x\right] = \Phi(v_x)$$

where v_x is determined by

$$\int_0^t \alpha(\tau) \; \exp\left[-\delta \; \tau - \sigma \; \sqrt{\tau} \; v_x\right] \; d\tau = x.$$

From the expression for the tail function in the theorem above, we can deduce the following expression for the density function of W:

$$f_W(x) = \frac{\Phi^{'}(v_x)}{\int_0^t \alpha(\tau)\sigma \; \sqrt{\tau} \; \exp\left[-\delta \; \tau - \sigma \; \sqrt{\tau} \; v_x\right] \; d\tau}.$$

By choosing $\alpha(\tau) = 1$ or $\alpha(\tau) = \tau$, we find expressions for the case of a constant or an increasing annuity.

5 Annuities in the C.I.R.-Model.

In this paragraph, we consider the C.I.R.-model, see Cox et al. (1985), where the instantenous riskless interest rate is assumed to satisfy the differential equation

$$dR_t = a\,(b - R_t)\,dt + \sigma\sqrt{R_t}dX_t$$

where X_t denotes a standard Wiener process. Performing the substitution $F_t = \frac{2}{\sigma}\sqrt{R_t}$, one obtains the following stochastic differential equation:

$$dF_t = \left(\frac{2ab}{\sigma^2 F_t} - \frac{aF_t}{2} - \frac{1}{2F_t}\right)dt + dX_t$$

where X_t again denotes a standard Wiener process.

In order to apply this process for describing the discounting factors, we have to consider $\exp\left(-\int_0^t R_\tau d\tau\right) = \exp\left(-\frac{\sigma^2}{4}\int_0^t F_\tau^2 d\tau\right)$.
The random variable of interest is V_t which is defined as

$$V_t = \int_0^t \alpha(\tau)\ \exp\left(-\frac{\sigma^2}{4}\int_0^\tau F_s^2 ds\right)d\tau.$$

Random variables of this type (for $\alpha(\tau) = 1$ and $t \to \infty$) were also considered by Delbaen (1993). He obtained bounds for the moments $E(V_\infty^n)$, based on a classical Kac identity.

In order to apply supermodular order and stop-loss order to the present situation, we have to evaluate the distributions of F_t at different time points t.

The transition density can be cast into the form of a Feynman-Kac integral which is a special case of a more general Feynman-Kac integral presented in Vanneste et al. (1994). This integral is related to a non-stationary Calogero model, see Goovaerts (1975).

The following analytical expression in terms of the modified Bessel function can be obtained for the transition probabilities:

$$\begin{aligned} p(0, F_0; t, F_t) \quad &= \quad e^{-\sqrt{2\gamma}t\left(\sqrt{1+8g}+1\right)+\frac{\sqrt{2\gamma}}{2}(F_t^2-F_0^2)}\left(\frac{F_t}{F_0}\right)^{\sqrt{1+8g}+\frac{1}{2}}\frac{\sqrt{2\gamma}}{sh\sqrt{2\gamma}t}\sqrt{F_0F_t}. \\ &\quad I_{\sqrt{1+8g}}\left(\frac{\sqrt{2\gamma}}{sh\sqrt{2\gamma}t}F_tF_0\right)e^{-\frac{1}{2}\sqrt{2\gamma}\coth\sqrt{2\gamma}t\left(F_t^2+F_0^2\right)} \end{aligned}$$

where we introduced the constants g and γ, which are defined as

$$a = 2\sqrt{2\gamma}, \qquad \frac{4ab}{\sigma^2} = \sqrt{1+8g}.$$

The interested reader is referred to Vanneste et al. (1994).

We will derive the distribution function of a random variable W_t which in stop-loss order is larger than the original random variable under consideration V_t. Our results hold for all values of t. As a byproduct, we obtain bounds for al the moments of the finite or infinite time stochastic annuities moments of V_t.

We consider the function f defined by

$$f(F_\varepsilon, F_{2\varepsilon}, \cdots, F_{n\varepsilon}) = \left(\sum_{j=1}^{n} \varepsilon \ \alpha(j\varepsilon) e^{-\frac{\sigma^2}{4}\sum_{i=1}^{j} \varepsilon \ F_{i\varepsilon}^2} - s \right)_+ .$$

It is readily verified that f is a supermodular function. Hence, our general approach of deriving a distribution function which is more dangerous in stop-loss order applies and an upperbound for the stop-loss premium of the random variable V_t is obtained by considering the comonotonic vector instead of $(F_\varepsilon, F_{2\varepsilon}, \cdots, F_{n\varepsilon})$ and by taking the limit for $n \to \infty$ such that $n\varepsilon = t$.

Let us start at $F_0 = 0$, then the transition probabilities can be cast into the following form:

$$\begin{aligned} p(0,0;t,F_t) &= e^{-\sqrt{2\gamma}t\left(\sqrt{1+8g}+1\right)+\frac{\sqrt{2\gamma}}{2}F_t^2} \ F_t^{\sqrt{1+8g}+1} \left(\frac{\sqrt{2\gamma}}{2sh\sqrt{2\gamma}t} \right)^{\sqrt{1+8g}+1} . \\ & \quad e^{-\frac{1}{2}\sqrt{2\gamma}\coth\sqrt{2\gamma}tF_t^2} \frac{2}{\Gamma\left(\sqrt{1+8g}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{+1} \left(1-t^2\right)^{\sqrt{1+8g}-\frac{1}{2}} dt \end{aligned}$$

where use has been made of the definition of the modified Bessel function:

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{+1} \left(1-t^2\right)^{\nu-\frac{1}{2}} e^{-zt} dt.$$

In order to determine the distribution function of F_t at time t, we consider the c.d.f.

$$\int_0^F p(0,0;t,F_t) dF_t = \frac{1}{\Gamma\left(\sqrt{1+8g}+1\right)} \int_0^{\frac{e^{-\sqrt{2\gamma}t}\sqrt{2\gamma}F^2}{2sh\sqrt{2\gamma}t}} x^{\sqrt{1+8g}} e^{-x} dx.$$

Define $\overset{\frown}{F}_u$ by means of

$$\frac{1}{\Gamma\left(\sqrt{1+8g}+1\right)}\int_0^{\overset{\frown}{F}_u} x^{\sqrt{1+8g}}e^{-x}dx = u, \qquad 0 \le u \le 1.$$

One gets that the relevant set of comonotonic risks has components determined by

$$F_\tau^2(u) = \frac{2sh\sqrt{2\gamma}\tau}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}\tau}\ \overset{\frown}{F}_u\ .$$

Hence,

$$E\left((V_t-s)_+\right) \le E_U\left(\left(\int_0^t \alpha(\tau)\ e^{-\widetilde{F}_U\frac{\sigma^2}{4}\int_0^\tau \frac{2sh\sqrt{2\gamma}\tau_0}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}\tau_0}d\tau_0}d\tau - s\right)_+\right).$$

The right hand side can be cast into the following form:

$$\int_0^\infty\left(\int_0^t \alpha(\tau)\ e^{-k\frac{\sigma^2}{4}\int_0^\tau \frac{e^{2\sqrt{2\gamma}\tau_0}-1}{\sqrt{2\gamma}}d\tau_0}d\tau - s\right)_+\frac{k^{\sqrt{1+8g}}e^{-k}}{\Gamma\left(\sqrt{1+8g}+1\right)}dk.$$

Then, of course, $V_t \le_{sl} W_t$ where the density of W_t is easily determined as follows:

$$f_{W_t}(s) = \frac{k_s^{\sqrt{1+8g}}e^{-k_s}}{\Gamma\left(\sqrt{1+8g}+1\right)\int_0^t \alpha(\tau)\ e^{-k_s/\sqrt{2\gamma}\frac{\sigma^2}{4}\left(\frac{e^{2\sqrt{2\gamma}\tau}-1}{2\sqrt{2\gamma}}-\tau\right)}\frac{\sigma^2}{4}\frac{1}{\sqrt{2\gamma}}\left(\frac{e^{2\sqrt{2\gamma}\tau}-1}{2\sqrt{2\gamma}}-\tau\right)d\tau}$$

where k_s is determined by

$$\int_0^t \alpha(\tau)\ e^{-k_s/\sqrt{2\gamma}\frac{\sigma^2}{4}\left(\frac{e^{2\sqrt{2\gamma}\tau}-1}{2\sqrt{2\gamma}}-\tau\right)}d\tau = s.$$

6 Upper bounds for the moments of a perpetuity in the C.I.R.-Model.

In this section, we derive upper bounds for the moments of a perpetuity in the C.I.R.-Model: $V_\infty = \int_0^\infty \exp\left(-\frac{\sigma^2}{4}\int_0^\tau F_s^2 ds\right)d\tau$. This can easily be done by deriving upper bounds for the moments of W_∞.

Having performed the substitution $k = k(s)$ $(= k_s)$, one gets

$$E\left(W_\infty^n\right) = \int_0^\infty \left(\int_0^\infty e^{-k/\sqrt{2\gamma}\frac{\sigma^2}{4}\left(\frac{e^{2\sqrt{2\gamma}\tau}-1}{2\sqrt{2\gamma}}-\tau\right)} d\tau \right)^n \frac{k^{\sqrt{1+8g}}}{\Gamma\left(\sqrt{1+8g}+1\right)} e^{-k} dk.$$

Writing $\exp(2\sqrt{2\gamma}\tau)$ as a Taylor series, the following inequality is obtained:

$$\frac{\exp(2\sqrt{2\gamma}\tau)-1}{2\sqrt{2\gamma}} - \tau \geq \frac{1}{2\sqrt{2\gamma}} \left(2\sqrt{2\gamma}\right)^p \frac{\tau^p}{p!}$$

where we have selected only one term in the series, p can be choosen to be any integer larger than 1.
Then the following inequality results:

$$\int_0^\infty \left(\int_0^\infty e^{-k/\sqrt{2\gamma}\frac{\sigma^2}{4}\left(\frac{e^{2\sqrt{2\gamma}\tau}-1}{2\sqrt{2\gamma}}-\tau\right)} d\tau \right)^n \leq \left(\frac{(p!)^{\frac{1}{p}}}{k^{\frac{1}{p}}} \frac{\left(2\sqrt{2\gamma}\right)^{\frac{1}{p}}}{2\sqrt{2\gamma}} \left(\frac{2}{\sigma^2}\right)^{\frac{1}{p}} \Gamma(1+\frac{1}{p}) \right)^n.$$

Making use of Jensens' inequality, one obtains

$$\Gamma(1+\frac{1}{p})^n \leq \left(\int_0^1 \left(\ln\frac{1}{u}\right)^{\frac{1}{p}} du \right)^n \leq \int_0^1 \left(\ln\frac{1}{u}\right)^{\frac{n}{p}} du = \Gamma(1+\frac{n}{p}).$$

Hence, as soon as $\sqrt{1+8g} - \frac{n}{p} > -1$, the following inequality holds:

$$E\left(W_\infty^n\right) \leq (p!)^{\frac{n}{p}} \frac{\left(2\sqrt{2\gamma}\right)^{\frac{n}{p}}}{\left(2\sqrt{2\gamma}\right)^n} \left(\frac{2}{\sigma^2}\right)^{\frac{n}{p}} \Gamma(1+\frac{n}{p}) \frac{\Gamma(\sqrt{1+8g}+1-\frac{n}{p}+1)}{\Gamma(\sqrt{1+8g}+1)}$$

Because the bounds given in Delbaen (1993) have to be seen to hold for large values of n (otherwise his parameter α_n becomes negative), we consider here the case of n sufficiently large. Making a convenient choice of p, namely $p = n$ (which is certainly not the best possible one), one finally obtains

$$E\left(W_\infty^n\right) \leq \frac{2\ n! \left(2\sqrt{2\gamma}\right)^{1-n}}{\sigma^2\sqrt{1+8g}} \approx \frac{4\sqrt{2\gamma}\sqrt{2\pi n}}{\sigma^2\sqrt{1+8g}} \left(\frac{n}{2e\sqrt{2\gamma}}\right)^n.$$

Remark that Delbaen (1993) proved that there are constants K_1 and K_2 such that

$$E\left(W_\infty^n\right) \leq K_1 \left(K_2\sqrt{n}\right)^n.$$

In order to compare our moments with those derived by Delbaen (1993), we made some rough majorization of the integrals appearing in our expression for $E\left(W_{\infty}^{n}\right)$. Remark that our approach is directed towards the evaluation of the corresponding density function directly, and from this result the moments can be obtained. On the other hand, it is not evident to transform the series of moments into a density function.

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