A note on dependencies in multiple life statuses

Jan Dhaene [*] Marleen Vanneste [†] Henk Wolthuis [‡]

October 11, 2003

Abstract

We introduce the correlation order as a tool for describing and understanding dependencies in multiple life statuses. This order is well-known in the economical literature. It is a partial order in the class of all bivariate lifetime distributions with given marginals. It is shown that this order is preserved (or reversed) when pricing multiple life and last survivor insurance and annuity contracts. In particular, we establish conditions that provide information on phenomenon of over/underpricing when the usual assumption of mutual independency of the life times involved is made. The results can also be used to establish lower and upper bounds for the single premiums of insurances and annuities on joint-life and last-survivor statuses with given marginals.

Keywords: correlation order, joint-life statuses, last-survivor statuses, reversionary annuities, non-independence, single premiums.

[*]Katholieke Universiteit Leuven, Universiteit Gent, Universitaire Instelling Antwerpen, Universiteit van Amsterdam.

[†]Katholieke Universiteit Leuven.

[‡]Universiteit van Amsterdam.

1 Introduction.

A traditional assumption in the theory of multiple life contingencies is that the remaining life times of the lives involved are mutually independent. Computational feasability rather than realism seems to be the major reason for making this assumption. Indeed, a husband and his wife are more or less exposed to the same risks. The "broken heart syndrome" causes an increase of the mortality rate after the mortality of one's spouse. Such effects may have a significant influence on present values related to multiple life actuarial functions.

Carrière and Chan (1986) investigated Fréchet-bounds for pricing joint-life and last-survivor annuities. Frees, Carrière and Valdez (1996) observed a portfolio of annuities on coupled lives and concluded that the time-of-death of the paired lives were highly correlated. Carrière (1997) presents alternative ways for modeling the dependence of the time-of-deaths of coupled lives and applies these to a data set from a life annuity portfolio. These papers use copula functions to build bivariate survivorship functions. For a general introduction and a historical overview of the development of the theory of copula functions, we refer to Dall'Aglio, Kotz and Salinetti (1991). The article of Frees and Valdez (1998) introduces actuaries to the concept of copulas.

The situation where the dependence of lives arises from an exogenous event that is common to each life, can be described by a "common shock" model. A reference to this kind of models is Marshal and Olkin (1988).

To the best of our knowledge, the first actuarial textbook explicitly introducing multiple life models in which the future life time random variables are dependent is Bowers, Gerber, Hickmann, Jones and Nesbitt (1997). In Chapter 9 of this book, copula and common shock models are introduced to describe dependencies in joint-life and last-survivor statuses.

Also other models can be used to incorporate dependencies between life times, e.g. the frailty models described by Oakes (1989) or Markov models as described by Norberg (1989). For a more extensive overview of dependency models, we further refer to Frees, Carrière and Valdez (1996) and the references in that paper.

Given a certain copula or frailty model, determining the direction and the extent of over/underpricing by taking the independence assumption is straightforward. This paper allows the reader to determine this direction for a general class of models, where the only restriction is that the marginals are given. The assumption of given marginals is crucial here since our focus will lie on comparing dependencies only.

We want to compare the riskiness of several dependency relations. More precisely, we want to compare bivariate distributions with given marginals, but with different dependency structures. Although there is an extensive actuarial literature on the theory of ordering of univariate risks, see e.g. Goovaerts, Kaas, van Heerwaerden and Bauwelinckx (1990), the theory of ordering of multivariate risks has only recently been considered more extensively in the actuarial literature, see e.g. Denuit, Lefèvre and Mesfioui (1997), Müller (1997), Dhaene and Goovaerts (1996), Dhaene, Wang, Young and Goovaerts (1997), Wang and Dhaene (1997). A general reference to the theory of ordering of multivariate risks is Shaked and Shanthikumar (1994).

In this paper, we will use some results from Dhaene & Goovaerts (1996) which were obtained for portfolios where the risks involved are not necessarily mutually independent. We will see that we can use some of these general results for evaluating the effect of dependencies in case of multiple life functions. We introduce a partial ordering in the class of all bivariate lifetime distributions with given marginals. It is shown that this ordering is preserved (or reversed) when pricing certain multiple life insurance and annuity contracts. In particular, we establish conditions that provide information on the phenomenon of over/underpricing when the usual assumption of mutual independency of the life times involved is made. Combining our results with the Fréchet bounds, we establish lower and upper bounds for the single premiums of insurances and annuities on multiple life statuses. We will restrict our discussion to situations involving two lives. Generalisations to situations with more than two lives involved are possible.

In section 2, we will give some basic definitions and results. In section 3 these results will be used for deriving ordering relations between single premiums of multiple life insurances and annuities on two lives. In the sections 4 and 5 we derive lower and upper bounds for multiple life single premiums. In section 6 , we will discuss dependency between stochastically ordered remaining life times. Finally, in section 7 we will give some numerical illustrations of the results obtained in the previous sections.

2 Correlation order and positive quadrant dependency.

Let $R(F,G)$ be the set of all bivariate distributed random variables $(T(x),T(y))$ with given marginal distribution functions F and G for $T(x)$ and $T(y)$ respectively. We will interpret $T(x)$ and $T(y)$ as the remaining life times of persons of age x and y respectively. Consequently, they are assumed to be nonnegative random variables.

Definition 1 *Let $(T(x),T(y))$ and $(S(x),S(y))$ be elements of $R(F,G)$. Then $(T(x),T(y))$ is said to be less correlated than $(S(x),S(y))$, written as $(T(x),T(y)) \le_c (S(x),S(y))$, if*

$$cov\left[f(T(x)),g(T(y))\right] \le cov\left[f(S(x)),g(S(y))\right]$$

for all non-decreasing functions f and g for which the covariances exist.

The correlation order is a partial order between the joint distributions of the remaining life times in $R(F,G)$. It expresses the notion that some elements of $R(F,G)$ are more positively correlated than others.

The following theorem gives an alternative definition for correlation order in terms of bivariate distribution functions.

Theorem 1 *Let $(T(x),T(y))$ and $(S(x),S(y))$ be elements of $R(F,G)$. Then the following statements are equivalent:*

(a) $(T(x), T(y)) \leq_c (S(x), S(y))$.
(b) $Pr[T(x) \leq t, T(y) \leq s] \leq \Pr[S(x) \leq t, S(y) \leq s]$ *for all* $t, s \geq 0$.

A proof of this theorem can be found in Dhaene & Goovaerts (1996). References to the correlation order defined above are Barlow and Proschan (1975) and Tchen (1980). For economic applications, see also Epstein and Tanny (1980) and Aboudi and Thon (1993, 1995).

Often certain insured risks tend to act similarly, they possess some "positive" dependency. In order to describe such situations we introduce the well-known notion of "positive quadrant dependency", see e.g. Barlow and Proschan (1975).

Definition 2 *Two remaining life times* $T(x)$ *and* $T(y)$ *are said to be positively quadrant dependent, written as* $PQD\,(T(x), T(y))$*, if*

$$\Pr[T(x) \leq t, T(y) \leq s] \geq \Pr[T(x) \leq t] \Pr[T(y) \leq s]$$

for all $t, s \geq 0$.

Hence, saying that $T(x)$ and $T(y)$ are positive quadrant dependent means that the probability that $T(x)$ and $T(y)$ both realize small values is larger than the corresponding probability in the case of independent remaining life times. In terms of correlation order (Definition 1), one can say that $T(x)$ and $T(y)$ are actually more correlated than the corresponding couple consisting of mutually independent remaining life times. Reversing the inequality in Definition 2 leads to the notion of negative quadrant dependency (NQD). We will not use this concept in the remainder of this paper. However, for all results that we will prove for PQD, it is possible to prove a NQD version.

The notions of correlation order and positive quadrant dependency can easily be expressed in terms of Archimedean copula functions. Indeed, let $(T(x), T(y))$ and $(S(x), S(y))$ be elements of R(F,G) with respective copula functions $C_T(u,v)$ and $C_S(u,v)$. Then $(T(x), T(y)) \leq_c (S(x), S(y))$ is equivalent with $C_T(u,v) \leq C_S(u,v)$ for all $u, v \geq 0$. We also have that $PQD\,(T(x), T(y))$ is equivalent with $C_T(u,v) \geq u\ v$ for all $u, v \geq 0$.

3 Actuarial functions on two dependent lives.

Let the remaining life times of the statuses (x) and (y) as earlier be denoted by $T(x)$ and $T(y)$ respectively.
The joint-life status (xy) exists as long as (x) and (y) are both alive. Hence, the remaining life time of (xy) is given by $T(xy) = \min[T(x), T(y)]$.
The survival probabilities ${}_tp_{xy}$ of the joint-life status are given by

$$ {}_tp_{xy} = \Pr[T(xy) > t] = \Pr[T(x) > t, T(y) > t]. $$

The last-survivor status $(\overline{xy})$ exists as long as at least one of (x) and (y) is alive. Hence, the remaining life time of $(\overline{xy})$ is given by $T(\overline{xy}) = \max[T(x), T(y)]$.
The survival probabilities ${}_tp_{\overline{xy}}$ of the last-survivor status are given by

$$ {}_tp_{\overline{xy}} = \Pr[T(\overline{xy}) > t] = 1 - \Pr[T(x) \leq t, T(y) \leq t]. $$

Let T be the remaining life time of a joint-life or last-survivor status. We will consider life insurances and annuities for which the present value of future benefits (PVFB) is given by $f(T)$ with f a non-decreasing or non-increasing non-negative function. The expectation of $f(T)$ is the (pure) single premium for the insurance or annuity under consideration.
Remark that the PVFB of most of the usual joint-life and last-survivor insurances and annuities can be written as non-decreasing or non-increasing functions of the remaining life time of the joint-life or last-survivor status involved:
The PVFB of pure endowments $({}_nE_{xy},\ {}_nE_{\overline{xy}})$ and whole life annuities $(\ddot{a}_{xy},\ \ddot{a}_{\overline{xy}},\ \overline{a}_{xy},\ \overline{a}_{\overline{xy}}, \cdots)$ are non-decreasing functions of the remaining life time of the multiple life status involved. The PVFB of whole life insurances $\left(A_{xy},\ A_{\overline{xy}},\ \overline{A}_{xy},\ \overline{A}_{\overline{xy}}, \cdots\right)$ are non-increasing functions of the remaining life time of the multiple life status involved.

Before stating our main result, we need to introduce the notion of stochastic dominance.

Definition 3 *Let T and S be two remaining life times. We say that S stochastically dominates T, written $T \leq_{st} S$, if one of the following equivalent*

conditions hold:
(a) $E[f(T)] \leq E[f(S)]$ *for all non-decreasing functions* f.
(b) $\Pr[T \leq t] \geq \Pr[S \leq t]$ *for all* t.

A proof of the equivalence between the two conditions can be found e.g. in Goovaerts, Kaas, van Heerwaarden and Bauwelinckx (1990).

Lemma 1 *Let* $(T(x), T(y))$ *and* $(S(x), S(y))$ *be two bivariate remaining life times, both elements of* $R(F, G)$. *If* $(T(x), T(y)) \leq_c (S(x), S(y))$, *then the following stochastic order relations hold:*

$$T(xy) \leq_{st} S(xy),$$

$$S(\overline{xy}) \leq_{st} T(\overline{xy}).$$

Proof. From Theorem 1 we have that

$$\Pr[T(x) \leq t, T(y) \leq s] \leq \Pr[S(x) \leq t, S(y) \leq s].$$

This inequality can be transformed into

$$\Pr[T(x) > t, T(y) > s] \leq \Pr[S(x) > t, S(y) > s].$$

Hence, we find

$$\Pr[T(xy) > t] = \Pr[T(x) > t, T(y) > t] \leq \Pr[S(x) > t, S(y) > t] = \Pr[S(xy) > t],$$

which proves the first stochastic order relation.
The other relation is proven similarly. ■

In the following theorem, which states our main result, we will consider two bivariate remaining life times in $R(F, G)$ which are ordered by the correlation order. We will show that a correlation order between these bivariate remaining life times implies an ordering of the corresponding multiple life single premiums.

Theorem 2 *Let $(T(x),T(y))$ and $(S(x),S(y))$ be two bivariate remaining life times, both elements of $R(F,G)$. If $(T(x),T(y)) \leq_c (S(x),S(y))$, then the following inequalities hold for any non-decreasing function f:*

$$E[f(T(xy))] \leq E[f(S(xy))],$$

$$E[f(T(\overline{xy}))] \geq E[f(S(\overline{xy}))].$$

If f is non-increasing then the opposite inequalities hold.

Proof. From Lemma 1 we have that $S(xy)$ stochastically dominates $T(xy)$ so that the first inequality is proven.
The proof for the other inequality is similar.

The inequalities for a non-increasing function f follow immediately by remarking that $-f$ is non-decreasing in this case. ■

Theorem 2 can be interpreted as follows: Assume that the marginal distributions of the remaining life times of (x) and (y) are given. If the bivariate remaining life time of the couple increases in correlation order, then the single premiums of endowment insurances and annuities on the joint life status increase, while the single premiums of endowment insurances and annuities on the last survivor status decrease. For whole life insurances, the opposite conclusions hold.

Remark that Theorem 2 can also be used for ordering single premiums of more complex multiple life functions. Consider e.g. an annuity which pays one per year while both (x) and (y) are alive, and α per year while (y) is alive and (x) has died. The discounted value of the benefits involved is given by

$$\int_0^{T(xy)} v^t \, dt + \alpha \int_{T(xy)}^{T(y)} v^t \, dt = (1-\alpha) \int_0^{T(xy)} v^t \, dt + \alpha \int_0^{T(y)} v^t \, dt.$$

Under the conditions of Theorem 2, we find from the equality above that $(T(x),T(y)) \leq_c (S(x),S(y))$implies $a_{xy}^{(T)} + \alpha \, a_{x|y}^{(T)} \leq a_{xy}^{(S)} + \alpha \, a_{x|y}^{(S)}$, where the superscript (T) is used to indicate that the annuity is computed using the bivariate remaining lifetime $(T(x),T(y))$.

A natural measure of dependency between two random variables is the covariance. So, one could wonder whether $cov\,[T(x), T(y)] \leq cov\,[S(x), S(y)]$ is a sufficient condition for the ordering relations in Theorem 2 to hold. In the following example, we will show that the ordering of the covariances is not a sufficient condition.

Let F be the cumulative distribution function of a remaining life time that can be equal to 1/2, 3/2 or 5/2, each with probability 1/3. Now, we consider the couples $(T(x), T(y))$ and $(S(x), S(y))$, both elements of $R(F, F)$. Further, we assume that $T(x)$ and $T(y)$ are mutually independent, while the dependency structure of $(S(x), S(y))$ follows from the following relations:

$$\begin{aligned}\Pr[S(y) = 1/2 \mid S(x) = 1/2] &= 1,\\ \Pr[S(y) = 3/2 \mid S(x) = 5/2] &= 1,\\ \Pr[S(y) = 5/2 \mid S(x) = 3/2] &= 1.\end{aligned}$$

We have that $cov\,[T(x), T(y)] = 0$ and $cov\,[S(x), S(y)] = 1/3$. On the other hand, we find

$$\Pr[T(\overline{xy}) \leq t] = \begin{cases} 1/9 & : t < 3/2,\\ 4/9 & : 3/2 \leq t < 5/2,\\ 1 & : t \geq 5/2.\end{cases}$$

and

$$\Pr[S(\overline{xy}) \leq t] = \begin{cases} 1/3 & : t < 5/2,\\ 1 & : t \geq 5/2.\end{cases}$$

From the distribution functions of $T(\overline{xy})$ and $S(\overline{xy})$ we find that ${}_1E^{(T)}_{\overline{xy}} > {}_1E^{(S)}_{\overline{xy}}$, but ${}_2E^{(T)}_{\overline{xy}} < {}_2E^{(S)}_{\overline{xy}}$.

Although it is customary to compute covariances in relation with dependency considerations, one number alone cannot reveal the nature of dependency adequately. From the example above, we see that the order induced by comparing only the covariances of $(T(x), T(y))$ and $(S(x), S(y))$ will not imply a consistent ordering between the single premiums of endowment insurances on the last-survivor status. Hence, the results of Theorem 2 cannot be generalized in this way.

Instead of comparing $cov[T(x), T(y)]$ and $cov[S(x), S(y)]$ one could compare $cov[f(X_1), g(X_2)]$ with $cov[f(Y_1), g(Y_2)]$ for all non-decreasing functions f and g. The order induced in this way is the correlation order. As we see from Theorem 2, this generalisztion of an order based on comparing covariances implies a consistent ordering between single premiums of joint life and last-survivor annuities and insurances.

More generally, we could wonder if there are other bivariate orderings which lead to similar results as the one obtained in Theorem 2. Remark that the condition $E[f(T(xy))] \leq E[f(S(xy))]$ for all non-decreasing functions f is equivalent with $\Pr[T(x) \leq t, T(y) \leq t] \leq \Pr[S(x) \leq t, S(y) \leq t]$ for all $t \geq 0$. In view of Theorem 1, we can conclude that the correlation order seems to be an appropriate choice as bivariate ordering.

4 Fréchet lower and upper bounds for the single premiums.

In this section we will look at the extremal elements in $R(F, G)$, namely the one which are smaller or larger in correlation order than any other element in $R(F, G)$.

Lemma 2 *For any element $(T(x), T(y))$ in $R(F, G)$, we have that*

$$\max(F(t) + G(s) - 1, 0) \leq \Pr[T(x) \leq t, T(y) \leq s] \leq \min(F(t), G(s))$$

with the lower and upper bound being bivariate distribution functions of elements contained in $R(F, G)$.

This result can be found in Fréchet (1951).

Now let $(T_L(x), T_L(y))$ and $(T_U(x), T_U(y))$ be elements of $R(F, G)$ which correspond to the lower and the upper Fréchet bound respectively, i.e.

$$\Pr[T_L(x) \leq t, T_L(y) \leq s] = \max(F(t) + G(s) - 1, 0),$$

$$\Pr\left[T_U(x) \le t, T_U(y) \le s\right] = \min\left(F(t), G(s)\right).$$

We also introduce $T_L(xy)$ which is the remaining life time of the joint-life status associated with $(T_L(x), T_L(y))$. After some straightforward computations, we find

$$\Pr\left[T_L(xy) \le t\right] = \min\left(F(t) + G(t), 1\right).$$

Similarly, we define $T_U(xy), T_L(\overline{xy})$ and $T_U(\overline{xy})$. The distribution functions of these statuses are given by

$$\begin{aligned}
\Pr\left[T_L(\overline{xy}) \le t\right] &= \max\left(F(t) + G(t), 1\right) - 1, \\
\Pr\left[T_U(xy) \le t\right] &= \max\left(F(t), G(t)\right), \\
\Pr\left[T_U(\overline{xy}) \le t\right] &= \min\left(F(t), G(t)\right).
\end{aligned}$$

From Lemma 1 and Lemma 2, we immediately find the following inequalities for $(T_L(x), T_L(y))$, $(T_U(x), T_U(y))$ and $(T(x), T(y))$ in $R(F, G)$:

$$T_L(xy) \le_{st} T(xy) \le_{st} T_U(xy),$$

$$T_U(\overline{xy}) \le_{st} T(\overline{xy}) \le_{st} T_L(\overline{xy}).$$

Hence, from Theorem 2, we immediately find the following result.

Theorem 3 *Let $(T_L(x), T_L(y))$ and $(T_U(x), T_U(y))$ be elements of $R(F, G)$ corresponding to the Fréchet bounds. Then the following inequalities hold for any $(T(x), T(y)) \in R(F, G)$ and for any non-decreasing function f:*

$$E\left[f\left(T_L(xy)\right)\right] \le E\left[f\left(T(xy)\right)\right] \le E\left[f\left(T_U(xy)\right)\right],$$

$$E\left[f\left(T_U(\overline{xy})\right)\right] \le E\left[f\left(T(\overline{xy})\right)\right] \le E\left[f\left(T_L(\overline{xy})\right)\right].$$

If f is non-increasing then the opposite inequalities hold.

Assume that we are in a situation where we know the marginal distribution functions of $T(x)$ and $T(y)$, but where we have no information concerning the dependency relation between the two remaining life times. From the theorem above, we can compute lower and upper bounds for the single premiums of insurances or annuities on the joint-life or last-survivor statuses involved.

Let us now assume that all elements in $R(F, G)$ are stochastically ordered. Without loss of generality we can assume that

$$F(t) \geq G(t)$$

for all $t \geq 0$. A sufficient condition for this to be true is that the force of mortality related to F is always greater than or equal to the force of mortality related to G, see e.g. Goovaerts, Kaas, van Heerwaarden and Bauwelinckx (1990).
In practice, the condition above could be fullfilled e.g. if we consider a couple where $F(t)$ is the distribution function of the remaining life time of a husband which is older than his wife, who has a remaining life time with distribution function $G(t)$.

Now, let $(T_U(x), T_U(y))$ be the element in $R(F, G)$ which corresponds to the Fréchet upper bound, i.e.

$$\Pr\left[T_U(x) \leq t, T_U(y) \leq s\right] = \min\left(F(t), G(s)\right).$$

After some straightforward derivations, we find that in this case

$$\Pr\left(T_U(xy) \leq t\right) = F(t)$$

and

$$\Pr\left(T_U(\overline{xy}) \leq t\right) = G(t).$$

This means that $T_U(xy)$ has the same distribution function as $T_U(x)$, and $T_U(\overline{xy})$ has the same distribution function as $T_U(y)$. Hence, we find that for any function f the following relations hold:

$$E\left[f\left(T_U(xy)\right)\right] = E\left[f\left(T_U(x)\right)\right]$$

and similarly,

$$E\left[f\left(T_U(\overline{xy})\right)\right] = E\left[f\left(T_U(y)\right)\right]$$

We can conclude that if $F(t) \geq G(t)$ for all $t \geq 0$, or equivalently, $T(x) \leq_{st} T(y)$ for all $(T(x), T(y))$ in $R(F, G)$, then the single premiums in Theorem 3 which correspond to the Fréchet upper bound all reduce to single premiums of single life insurances or annuities.

5 Independent lives versus PQD.

In this section, we will again assume that the marginal distributions of the remaining life times $T(x)$ and $T(y)$ are given. We will compare the case where the remaining life times are mutually independent with the case where they are PQD.

Theorem 4 *Assume that the bivariate remaining life times* $(T(x), T(y))$ *and* $(T^{ind}(x), T^{ind}(y))$ *have the same marginal distributions. If* $PQD(T(x), T(y))$, *and* $T^{ind}(x)$ *and* $T^{ind}(y)$ *are mutually independent, then the following inequalities hold for any non-decreasing function* f*:*

$$E[f(T^{ind}(xy))] \leq E[f(T(xy))]$$

$$E[f(T(\overline{xy}))] \leq E[f(T^{ind}(\overline{xy}))]$$

If f *is non-increasing then the opposite inequalities hold.*

Proof. The proof follows immediately from Definition 2 and Theorem 2. ■

These inequalities have been derived for some specific types of multiple life insurances and annuities in Norberg (1989).

As an application of Theorem 4 we find that for $PQD\left[T(x), T(y)\right]$, a whole life insurance on the joint-life status (xy) has a lower single premium than in the independent case. Similarly, we can conclude that in the case of positive quadrant dependency, the independence assumption will lead to an underestimation of the single premium of an annuity on the joint-life status (xy).

From the Theorem above we see that when $T(x)$ and $T(y)$ are positive quadrant dependent, then the bounds in Theorem 3 which corresponds to the Fréchet lower bound can be improved by considering the independent case as bound. Remark that in the independent case we have

$$\begin{aligned} \Pr\left[T^{ind}(xy) \le t\right] &= F(t) + G(t) - F(t)\ G(t), \\ \Pr\left[T^{ind}(\overline{xy}) \le t\right] &= F(t)\ G(t). \end{aligned}$$

6 Numerical illustration.

In this section we will illustrate the bounds derived in the previous sections by some numerical examples. The technical interest rate equals 4.75%. Further, (x) and (y) are a male and a female. The marginal distribution functions of the remaining life times of (x) and (y) follow from the Belgian mortality tables MR and FR respectively. For the mortality table MR, the Makeham constants are given by k = 1 000 266.63; s = 0.999 441 703 848; g = 0.999 733 441 115 and c = 1.101 077 536 030. For the mortality table FR, the Makeham constants are given by k = 1 000 048.56; s = 0.999 669 730 966; g = 0.999 951 440 172 and c = 1.116 792 453 830.

We assume that the remaining life times of (x) and (y) are positive quadrant dependent. In each multiple life status, the first age will be the age of the male person.

In Table 1 bounds are given for whole life annuities on (xy) and $(\overline{xy})$ with $x = y$, for different values of x. The bounds follow from Theorems 3 and 4.

Table 1. Bounds for whole life annuities on (xx) and $(\overline{xx})$.

The differences between the upper and lower bounds are relatively small. Remark that the absolute difference between the upper and the lower bound increases with the age.

In Table 2 we compare the single premiums for pure endowment insurances on $(25 : 20)$ and $(\overline{25 : 20})$ respectively, for varying durations of the endowment.

Table 2. Bounds for pure endowment insurances on $(25 : 20)$ and $(\overline{25 : 20})$.

For the joint-life as well as for the last-survivor insurance, the difference between the upper and the lower bound is an increasing function of the duration.

Finally, in Table 3 we compare whole life annuities on $(x:20)$ and $(\overline{x:20})$ with x varying from 20 to 55.

Table 3. Bounds for whole life annuities on $(x:20)$ and $(\overline{x:20})$.

For the last-survivor annuity the lower bound equals a_y and hence is constant. From Table 3 we see that increasing the difference in age between (x) and (y) decreases the absolute difference between the bounds.

Acknowledgement

The authors would like to thank the financial support of grant OT/97/06 of Onderzoeksfonds K.U. Leuven.

References

Aboudi, R. & Thon, D. (1993). Expected utility and the Siegel paradox: a generalization. *Journal of Economics*, 57(1), 69-93.

Aboudi, R. & Thon, D. (1995). Second degree stochastic dominance decisions and random initial wealth with applications to the economics of insurance. *Journal of Risk and Insurance*, 62(1), 30-49.

Barlow R.E. & Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart & Winston, New York.

Bowers, N.L.; Gerber, H.U.; Hickman, J.C.; Jones, D.A. & Nesbitt, C.J. (1997). *Actuarial Mathematics*, second edition. The Society of Actuaries, Itacsa, Illinois.

Carriere, J.F. (1997). Bivariate survival models for coupled lives.*Scandinavian Actuarial Journal,* to appear.

Carriere, J.F. & Chan, (1986). The bounds of bivariate distributions that limit the value of last-survivor annuities. *Transactions of the Society of Actuaries*, XXXVIII, 51-74.

Dall'Aglio, G.; Kotz, S. & Salinetti, G. (Eds.) (1991). *Advances in Probability Distributions with Given Marginals: Beyond the Copulas.* Kluwer

Academic Publishers, Dordrecht, Boston, London.

Denuit, M., Lefèvre, C. & Mesfioui, M.(1997). A class of bivariate stochastic orderings with applications in actuarial sciences, Technical Report 82, I.S.R.O., Université Libre de Bruxelles, Belgium.

Dhaene, J. & Goovaerts, M. (1996). Dependency of Risks and Stop-Loss Order. *ASTIN Bulletin,* 26(2), 201-212.

Dhaene, J. & Goovaerts, M. (1997). On the dependency of risks in the individual life model. *Insurance: Mathematics & Economics,* 19(3), 243-254.

Dhaene, J., Wang, S., Young, V.R. & Goovaerts, M. (1997). Comonotonicty and maximal stop-loss premiums. Research Report 9730, D.T.E.W., Catholic University Leuven, Belgium.

Epstein, L. & Tanny, S. (1980). Increasing generalized correlation: a definition and some economic consequences. *Canadian Journal of Economics*, 8(1), 16-34.

Fréchet, M. (1951). Sur les tableaux de corrélation dont les marges sont données. *Ann. Univ. Lyon Sect.A, Séries 3, 14*, 53-77.

Frees, E.W. & Valdez, E. (1998). Understanding relationships using copulas. *North American Actuarial Journal,* vol.2*,* n°1, 1-26.

Frees, E.W.; Carriere, J.F. & Valdez, E. (1996). Annuity valuation with dependent mortality. *Journal of Risk and Insurance,* vol.3, n°2, 229-261.

Goovaerts, M.J.; Kaas, R.; van Heerwaarden, A.E. & Bauwelinckx, T. (1990). *Effective Actuarial Methods.* Insurance Series 3, Elsevier, The Netherlands.

Kling, B. (1993). *Life Insurance: a Non-Life Approach.* Doctoral thesis, Tinbergen Institute, University of Amsterdam.

Marshall, A.W. & Olkin, I. (1988). Families of multivariate distributions.

Journal of the American Statistical Association, 62 , 30-44.

Müller, A. (1997). Stop-loss order for portfolios of dependent risks. *Insurance: Mathematics & Economics*, to appear.

Norberg, R. (1989). Actuarial analysis of dependent lives. *Mitteilungen der Schweiz. Vereinigung der Versicherungsmathematiker*, 1989(2), 243-255.

Oakes, D. (1989). Bivariate survival models induced by frailties. *Journal of the American Statistical Association,* 84, 487-493.

Shaked, M. & Shanthikumar, J.G. (1994). *Stochastic Orders and their Applications*, Probability and mathematical statistics, Academic Press, Inc.

Tchen, A. (1980). Inequalities for distributions with given marginals. *Annals of Probability,* 8, 814-827.

Wang, S. & Dhaene, J. (1997). Comonotonicity, correlation order and premium principles. *Insurance: Mathematics & Economics*, to appear.

Wolthuis, H. (1994). *Life Insurance Mathematics.* CAIRE Education Series 2, Brussels.

JAN DHAENE and MARLEEN VANNESTE
Departement Toegepaste Economische Wetenschappen
Katholieke Universiteit Leuven
Huis Eighen Heerd
Minderbroederstraat 5
3000 Leuven
Belgium

HENK WOLTHUIS
Actuariaat, Kwantitatieve Methoden en Econometrie
Universiteit van Amsterdam
Roetersstraat 11
1018 WB Amsterdam
The Netherlands