

# STOCHASTIC UPPER BOUNDS FOR PRESENT VALUE FUNCTIONS

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## Abstract

In most practical cases, it is impossible to find an explicit expression for the distribution function of the present value of a sequence of cash flows that are discounted using a stochastic return process. In this paper, we present an easy computable approximation for this distribution function. The approximation is a distribution function which is, in the sense of convex order, an upper bound for the original distribution function. Explicit examples are given for pricing stochastic annuities with stochastic return process, more general stochastic cash flows as well as pricing Asian options. Numerical results seem to indicate that the approximation will often be rather close to the original distribution function.

## 1 Introduction

In several financial-actuarial problems one is faced with the determination of the distribution function of random variables of the form

$$V = \sum_{i=1}^n \alpha_i e^{-X_i}$$

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where  $\alpha_i$  ( $i = 1, \dots, n$ ) represents the deterministic cash flow at time  $i$  and  $e^{-X_i}$  ( $i = 1, \dots, n$ ), is the stochastic discount factor for a payment made at time  $i$ . Hence, the random variable  $V$  can be interpreted as the present value at time 0, of a sequence of default-free payments at times 1, 2,  $\dots$ ,  $n$ . In an actuarial context, such random variables are used for describing the present value of the cash flow of an insurance portfolio, see e.g. Dufresne (1990). They are also useful for the determination of Incurred But Not Reported (IBNR) reserves, see Goovaerts and Redant (1998).

Each  $\alpha_i$  has to be interpreted as an amount that has to be paid at time  $i$ . Equivalently, we can say that there is an income equal to  $-\alpha_i$  at time  $i$ . The random variable  $V$  will be called the loss variable, i.e. the present value of all future (deterministic) payments.

Let us now assume that we know the distribution functions of the random variables  $X_i$  ( $i = 1, \dots, n$ ). One could assume for example that they are normally distributed. In reality, the random variables  $X_i$  will certainly not be mutually independent. This means that besides the distribution functions of the  $X_i$  also the dependency structure of the random vector  $(X_1, \dots, X_n)$  will have to be taken into account in order to determine the distribution function of the loss variable  $V$ . Unfortunately, an expression for the distribution function of  $V$  is not available or hard to obtain in most cases. Thus, we shall introduce a technique which will enable us to find bounds on  $V$ .

In the actuarial literature it is a common feature to replace a loss variable by a "less favorable" loss variable, which has a simpler structure, making it easier to determine the distribution function, see e.g. Goovaerts, Kaas, Van Heerwaarden, Bauwelinckx (1986). In order to clarify what we mean with a less favorable risk, we will make use of the convex order, see e.g. Shaked and Shanthikumar (1994).

Let  $V$  and  $W$  be two random variables (losses) such that

$$E[\phi(V)] \leq E[\phi(W)] \text{ for all convex functions } \phi : R \rightarrow R,$$

provided the expectations exist. Then  $V$  is said to be smaller than  $W$  in the convex order (denoted as  $V \leq_{cx} W$ ).

Roughly speaking, convex functions are functions that take on their largest values in the tails. Therefore,  $V \leq_{cx} W$  means that  $W$  is more likely to take on extreme values than  $V$ . Instead of saying that  $V$  is smaller than  $W$  in the convex order, it is often said that  $-V$  dominates  $-W$  in the sense of second degree stochastic dominance, see e.g. Huang and Litzenberger (1988). In

terms of utility theory,  $V \leq_{cx} W$  means that the loss  $V$  is preferred to the loss  $W$  by all risk averse decision makers, i.e.  $E[u(-V)] \geq E[u(-W)]$  for all concave utility functions  $u$ . This means that replacing the (unknown) distribution function of  $V$  by the distribution function of  $W$ , can be considered as a prudent strategy with respect to setting reserves for example.

Since  $\phi(x) = x$  and  $\phi(x) = -x$  are both convex, it is clear that a convex order can only hold between two random variables with equal mean. Also the function  $\phi$  defined by  $\phi(x) = x^2$  is convex, so it follows that  $V \leq_{cx} W$  implies  $Var[X] \leq Var[Y]$ . We remark that the converse implication does not hold, see e.g. Brockett and Kahane (1992) or Brockett and Garven (1998).

In Shaked and Shanthikumar (1994), the following characterization of convex order is proven:

Let  $V$  and  $W$  be two loss variables such that  $E[V] = E[W]$ . Then  $V \leq_{cx} W$  if, and only if,

$$E[V - d]_+ \leq E[W - d]_+ \quad \text{for all } d.$$

Here, we used the notation  $(x)_+ = \max(0, x)$ .

By using an integration by parts, it is seen that the condition in the theorem can also be written as

$$\int_d^\infty S_V(x) dx \leq \int_d^\infty S_W(x) dx \quad \text{for all } d,$$

provided the integrals exist, and where  $S_V$  denotes the survival function of the random variable  $V$ :  $S_V(x) = \Pr[V > x]$ .

In this paper, we will consider loss variables  $V$  for which the distribution function cannot be determined explicitly. We will construct a new random variable  $W$  which is larger in the convex order sense, meaning that that  $E[V] = E[W]$ , and such that for each retention  $d$ , the stop-loss premium  $E[V - d]_+$  is smaller than or equal to the corresponding stop-loss premium of  $W$ . Replacement of the loss  $V$  by the loss  $W$  is safe in the sense that all risk averse decision makers will consider  $W$  as a less preferable loss. Practically, of course, applying the technique of replacing a loss by a less favorable loss will only make sense if the new loss variable has a simpler dependency structure, making it easier to determine its distribution function.

Finally, we remark that  $V \leq_{cx} W$  is equivalent to  $-V \leq_{cx} -W$ . This means that the convex order is independent of the interpretation of the random variables as loss or gain variables.

## 2 Fréchet Spaces

For any ( $n$ -dimensional) random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , let the distribution function and the survival or tail function be denoted by  $F_{\mathbf{X}}$  and  $S_{\mathbf{X}}$  respectively, i.e.

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) &= \Pr [X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n], \\ S_{\mathbf{X}}(\mathbf{x}) &= \Pr [X_1 > x_1, X_2 > x_2, \dots, X_n > x_n], \quad \mathbf{x} \in R^n. \end{aligned}$$

In general, the distribution function of a univariate random variable  $X$  is not one-to-one (i.e., may have jumps and/or flat parts) so that the inverse functions  $F_X^{-1}$  and  $S_X^{-1}$  have to be defined cautiously. As usual, we define the inverse of the distribution function as follows:

$$F_X^{-1}(p) = \inf \{x \in R \mid F_X(x) \geq p\}, \quad p \in [0, 1].$$

We also define the inverse  $S_X^{-1}$  of the survival function  $S_X$  as

$$S_X^{-1}(p) = \inf \{x \in R \mid S_X(x) \leq p\}, \quad p \in [0, 1].$$

In both definitions, we adopt the convention that  $\inf \emptyset = \infty$ . It is easily seen that

$$F_X^{-1}(p) = S_X^{-1}(1 - p), \quad p \in [0, 1].$$

For all  $x \in R$  and  $p \in [0, 1]$ , the following equivalences hold:

$$F_X(x) \geq p \Leftrightarrow F_X^{-1}(p) \leq x \text{ and } S_X(x) \leq p \Leftrightarrow S_X^{-1}(p) \leq x.$$

A Fréchet space is defined as a class of (distribution functions of) random vectors with fixed marginal distribution functions. Let  $R_n(F_1, F_2, \dots, F_n)$  denote the Fréchet class of all random vectors  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  with marginal distribution functions  $F_1, F_2, \dots, F_n$  respectively, i.e.

$$\Pr [X_i \leq x] = F_i(x), \quad i = 1, \dots, n \text{ for all } \mathbf{X} \in R_n(F_1, F_2, \dots, F_n).$$

We will repeat some well-known results related to Fréchet spaces, which will be needed for deriving our results. Since Hoeffding (1940) and Fréchet (1951), it is well-known that the extremal distribution for all random vectors in  $R_n(F_1, F_2, \dots, F_n)$  is the distribution function  $W_n(\mathbf{x})$  given by

$$W_n(\mathbf{x}) = \min \{F_1(x_1), F_2(x_2), \dots, F_n(x_n)\}$$

in the sense that the joint distribution function  $F_{\mathbf{X}}$  of any  $\mathbf{X}$  in  $R_n(F_1, F_2, \dots, F_n)$  is constrained from above by

$$F_{\mathbf{X}}(\mathbf{x}) \leq W_n(\mathbf{x}) \text{ for all } \mathbf{x} \in R^n.$$

$W_n$  is usually known as the Fréchet upperbound in  $R_n(F_1, F_2, \dots, F_n)$ . We remark that the Fréchet upperbound is reachable within  $R_n(F_1, F_2, \dots, F_n)$ . Indeed, for any uniformly distributed random variable  $U$  on the interval  $[0, 1]$ , we have that the random vector  $(F_1^{-1}(U), F_2^{-1}(U), \dots, F_n^{-1}(U))$  has marginal distributions  $F_1, F_2, \dots, F_n$  and that

$$\Pr [F_1^{-1}(U) \leq x_1, F_2^{-1}(U) \leq x_2, \dots, F_n^{-1}(U) \leq x_n] = W_n(\mathbf{x}), \quad \mathbf{x} \in R^n.$$

Random vectors  $(X_1, X_2, \dots, X_n)$  with the Fréchet upperbound  $W_n$  as their distribution function are said to be *comonotonic*. Comonotonic random variables possess a very strong positive dependency. Indeed, all the  $X_i$  are non-decreasing functions of the same random variable, so that they are indeed 'common monotonic'. Increasing one of the  $X_i$  will lead to an increase of all the other random variables  $X_j$  involved. This means that these random variables cannot compensate each other. They cannot be used as hedges against each other.

Other characterizations of comonotonicity can be found e.g. in Denneberg (1994). The concept of comonotonicity was introduced by Schmeidler (1986) and Yaari (1987), see also Roëll (1987). It has since then played an important role in economic theories of choice under risk and uncertainty. Applications of the concept of comonotonicity in the actuarial literature can be found in Dhaene and Goovaerts (1996), Dhaene, Wang, Young and Goovaerts (1997), Wang and Dhaene (1998) and Wang and Young (1998), amongst others.

### 3 Bounds on Sums of Dependent Risks

Consider a random sum  $V = X_1 + \dots + X_n$  such that  $(X_1, \dots, X_n)$  belongs to the Fréchet space  $R_n(F_1, F_2, \dots, F_n)$ . From now on, we will always silently assume that all the univariate distribution function  $F$  that will be considered are continuous everywhere and strictly increasing on the interval  $[\inf \{x \in R \mid F(x) > 0\}; \sup x \in R \mid F(x) < 1]$ . We will consider the problem of deriving a stochastic upper bound  $W$  for  $V$  such that  $W = Y_1 + \dots + Y_n$  with  $(Y_1, \dots, Y_n)$  belonging to the same Fréchet space and such that the

upper bound  $W$  is larger in the sense of convex order than the original loss  $V$ . A related problem (for non-negative random variables) is considered in Müller (1997), and also in Goovaerts and Dhaene (1999), see also Kemperman (1972).

For a strictly increasing and continuous function  $\phi$  and distribution function  $F_X$ , we have that  $F_{\phi(X)}(x) = (F_X \circ \phi^{-1})(x)$ , from which it follows by inversion that  $F_{\phi(X)}^{-1}(p) = \phi(F_X^{-1}(p))$ . As a special case, consider the strictly increasing and continuous function  $\phi$  defined by  $\phi(p) = \sum_{i=1}^n F_i^{-1}(p)$ , ( $p \in [0, 1]$ ) where the distribution functions  $F_1, F_2, \dots, F_n$  are given, and consider the random variable  $U$ , which is uniformly distributed on the interval  $[0, 1]$ . In this case, we have that  $F_{\phi(U)}^{-1}(p) = \phi(p)$ . Hence, we have proven that the inverse distribution function of a sum of comonotonic risks behaves additively. More specifically, let  $W = F_1^{-1}(U) + F_2^{-1}(U) + \dots + F_n^{-1}(U)$  with  $U$  uniformly distributed on  $[0, 1]$ , then

$$F_W^{-1}(p) = \sum_{i=1}^n F_i^{-1}(p), \quad p \in [0, 1].$$

We remark that this result can be generalized to the case that the distribution functions involved are not one-to-one, see e.g. Denneberg (1994).

In the following theorem, we show that the Fréchet upperbound of a given Fréchet space gives rise to a sum which is larger, in the sense of convex order, than any other random variable which can be written as a sum of the components of an element of the Fréchet space under consideration.

**Theorem 1** *For any  $\mathbf{X}$  in  $R_n(F_1, F_2, \dots, F_n)$  and any uniformly distributed random variable  $U$  on  $[0, 1]$ , we have that*

$$X_1 + X_2 + \dots + X_n \leq_{cx} F_1^{-1}(U) + F_2^{-1}(U) + \dots + F_n^{-1}(U).$$

**Proof.** Let  $V$  and  $W$  be defined by  $V = X_1 + X_2 + \dots + X_n$  and  $W = F_1^{-1}(U) + F_2^{-1}(U) + \dots + F_n^{-1}(U)$  respectively.

It can be shown that  $(x_1 + x_2 + \dots + x_n)_+ \leq (x_1)_+ + (x_2)_+ + \dots + (x_n)_+$  holds for all  $\mathbf{x} \in R^n$ . Hence, for any  $d$  we have

$$E[V - d]_+ = E[V - F_W^{-1}(F_W(d))]_+ \leq \sum_{i=1}^n E[X_i - F_i^{-1}(F_W(d))]_+.$$

On the other hand,

$$\begin{aligned} E[W - d]_+ &= \int_0^1 (F_W^{-1}(p) - d)_+ dp \\ &= \int_{F_W(d)}^1 (F_W^{-1}(p) - F_W^{-1}(F_W(d))) dp \end{aligned}$$

By using the property that the inverse distribution function of a sum of comonotonic risks behaves additively, we find

$$\begin{aligned} E[V - d]_+ &= \sum_{i=1}^n \int_{F_W(d)}^1 (F_i^{-1}(p) - F_i^{-1}(F_W(d))) dp \\ &= \sum_{i=1}^n E[X_i - F_i^{-1}(F_W(d))]_+, \end{aligned}$$

which proves the theorem. ■

From the theorem above, we see that knowledge of the marginal distribution functions of a sum of random variables suffices to find a new loss variable which is larger in convex order sense than the original loss variable. This holds in general, by which we mean that the same bound holds for all elements of a given Fréchet space. Hence, the bound does not depend on the dependency structure between the random variables involved. The special dependency structure giving rise to the the greatest sum (in terms of convex order) in the given Fréchet space, is comonotonicity.

Using the fact that the inverse distribution function of a sum of comonotonic risks behaves additively, we can deduce an algorithm for computing the distribution function of such a sum. Indeed, for  $W = F_1^{-1}(U) + F_2^{-1}(U) + \dots + F_n^{-1}(U)$  with  $U$  uniformly distributed on  $[0, 1]$ , we find

$$\sum_{i=1}^n F_i^{-1}[F_W(x)] = x, \quad x \in R.$$

which implicitly determines the distribution function  $F_W(x)$ .

As we have that  $(X_1, \dots, X_n)$  and  $(F_1^{-1}(U), F_2^{-1}(U), \dots, F_n^{-1}(U))$  have the same marginals, we have that  $X_1 + \dots + X_n$  and  $F_1^{-1}(U) + \dots + F_n^{-1}(U)$  have the same mean. As these random variables are ordered in convex order sense, we also find that the variance of  $X_1 + \dots + X_n$  is smaller than or equal to the variance of  $F_1^{-1}(U) + \dots + F_n^{-1}(U)$ , see e.g. Shaked and Shanthikumar (1994).

Assume that we have to determine  $E[W - d]_+$  for a certain retention  $d$ , we can first determine  $F_W(d)$  from  $\sum_{i=1}^n F_i^{-1}[F_W(d)] = d$ . From the proof of Theorem 1, we find that the stop-loss premium of  $W$  is then given by

$$E[W - d]_+ = \sum_{i=1}^n E\left[X_i - F_i^{-1}(F_W(d))\right]_+.$$

Hence, the stop-loss premium with retention  $d$  of a sum of comonotonic random variables can be written as a sum of stop-loss premiums of the individual random variables involved. The retentions of the individual stop-loss premiums are such that they sum to  $d$ .

## 4 Stochastic Bounds on Discrete Annuities

In this section, we will consider stochastic bounds for random variables of the form

$$\phi_1(X_1) + \phi_2(X_2) + \cdots + \phi_n(X_n)$$

where  $(X_1, \dots, X_n)$  belongs to a given Fréchet space  $R_n(F_1, F_2, \dots, F_n)$ , and where the functions  $\phi_i$  are continuous and strictly decreasing or increasing. As earlier mentioned, we also assume that the marginal distribution functions  $F_i$  are strictly increasing and continuous.

From Theorem 1, we immediately find

$$\phi_1(X_1) + \phi_2(X_2) + \cdots + \phi_n(X_n) \leq_{cx} W$$

where  $W$  is defined by  $W = F_{\phi_1(X_1)}^{-1}(U) + \cdots + F_{\phi_n(X_n)}^{-1}(U)$  with  $U$  uniformly distributed on  $[0, 1]$ . The distribution function of  $W$  follows from

$$\sum_{i=1}^n F_{\phi_i(X_i)}^{-1}(F_W(x)) = x.$$

We remark that if  $\phi_i$  is strictly increasing, then for all  $p \in [0, 1]$  we have that  $F_{\phi_i(X_i)}^{-1}(p) = \phi_i[F_i^{-1}(p)]$ . On the other hand, if  $\phi_i$  is strictly decreasing, then  $F_{\phi_i(X_i)}^{-1}(p) = \phi_i[F_i^{-1}(1-p)]$ .

The stop-loss premium with retention  $d$ , follows from

$$E[W - d]_+ = \sum_{i=1}^n E\left[\phi_i(X_i) - F_{\phi_i(X_i)}^{-1}(F_W(d))\right]_+.$$

As a special case, we now consider the following discounted cash flow

$$V = \sum_{i=1}^n \alpha_i e^{-\delta i - X_i}$$

where the  $X_i$  are assumed to be normally distributed with mean 0 and variance  $\sigma_i^2$ . We first assume that the  $\alpha_i$  are positive.

As  $F_{X_i}^{-1}(p) = \sigma_i \Phi^{-1}(p)$  where  $\Phi$  is the distribution function of a standard normal distributed random variable, we immediately find that  $V \leq_{cx} W$  with  $W$  defined by

$$W = \sum_{i=1}^n \alpha_i \exp \left[ -\delta i - \sigma_i \Phi^{-1}(U) \right],$$

with  $U$  being a uniformly distributed random variable on the interval  $[0, 1]$ . In order to derive the expression above, we have made use of the fact that  $U$  and  $1 - U$  have the same distribution function. The survival function of  $W$  follows from

$$\sum_{i=1}^n \alpha_i \exp \left[ -\delta i - \sigma_i \Phi^{-1}(S_W(x)) \right] = x,$$

or equivalently,

$$S_W(x) = \Phi(\nu_x),$$

with  $\nu_x$  determined by

$$\sum_{i=1}^n \alpha_i \exp \left[ -\delta i - \sigma_i \nu_x \right] = x.$$

The stop-loss premiums can be determined as follows:

$$E[W - d]_+ = \sum_{i=1}^n \alpha_i e^{-\delta i} E \left[ Y_i - e^{-\sigma_i \nu_d} \right]_+$$

where the  $Y_i$  are log-normal distributed random variables with parameters 0 and  $\sigma_i^2$ .

Let us now consider the case that the  $\alpha_i$  are negative. Then we have that  $V \leq_{cx} W$  with  $W$  defined by

$$W = \sum_{i=1}^n \alpha_i \exp \left[ -\delta i - \sigma_i \Phi^{-1}(U) \right].$$

The distribution function of  $W$  follows from

$$\sum_{i=1}^n \alpha_i \exp \left[ -\delta i - \sigma_i \Phi^{-1}(F_W(x)) \right] = x,$$

or equivalently,

$$F_W(x) = \Phi(\nu_x),$$

with  $\nu_x$  determined by

$$\sum_{i=1}^n \alpha_i \exp \left[ -\delta i - \sigma_i \nu_x \right] = x.$$

The stop-loss premiums can be determined as follows:

$$E[W - d]_+ = \sum_{i=1}^n \alpha_i e^{-\delta i} E \left[ Y_i - e^{-\sigma_i \nu_x} \right]_-$$

where the  $Y_i$  are lognormal distributed random variables with parameters 0 and  $\sigma_i^2$  and  $(x)_- = \min(x, 0)$ .

More generally, we can consider the case where the values of the  $\alpha_i$  can take on positive and negative values. In this case, we find that  $V \leq_{cx} W$ , with  $W$  determined by

$$W = \sum_{i=1}^n e^{-\delta i} \left[ (\alpha_i)_+ e^{-\sigma_i \Phi^{-1}(1-U)} + (\alpha_i)_- e^{-\sigma_i \Phi^{-1}(U)} \right].$$

It is straightforward to derive expressions for the distribution function and the stop-loss premiums in this case.

## 5 Further Results and Applications

### 5.1 Continuous Annuities

Our previous results can be used for deriving stochastic bounds for continuous annuities. Consider e.g. the continuous temporary annuity  $V$  defined by

$$V = \int_0^t \alpha(\tau) \exp \left[ -\delta \tau - \sigma X(\tau) \right] d\tau$$

where  $X(\tau)$  represents a standard Brownian motion,  $\delta$  is the risk free interest intensity and  $\alpha(\tau)$  is a non-negative continuous function of  $\tau$ .

We define an appropriate sequence of discrete annuities  $V_1, V_2, V_3, \dots$  with respective stochastic upper bounds  $W_1, W_2, W_3, \dots$ . Taking limits ( $n \rightarrow \infty$ ), we find that  $V \leq_{cx} W$ , where the random variable  $W$  is defined by

$$W = \int_0^t \alpha(\tau) \exp \left[ -\delta\tau - \sigma \sqrt{\tau} \Phi^{-1}(U) \right] d\tau,$$

where, as usual,  $U$  is a random variable which is uniformly distributed on the interval  $[0, 1]$ .

The tail function of  $W$  follows from

$$S_W(x) = \Phi(\nu_x)$$

where  $\nu_x$  is determined by

$$\int_0^t \alpha(\tau) \exp \left[ -\delta\tau - \sigma \sqrt{\tau} \nu_x \right] d\tau = x.$$

## 5.2 Stochastic Cash Flows

Consider the random variable

$$V = \sum_{i=1}^n X_i Y_i$$

where  $X_i$  ( $i = 1, \dots, n$ ) represents a stochastic cash flow at time  $i$  and  $Y_i$  ( $i = 1, \dots, n$ ), is the stochastic discount factor for a payment made at time  $i$ . Hence, the random variable  $V$  can be interpreted as the present value at time 0, of a sequence of random payments to be made at times 1, 2,  $\dots$ ,  $n$ . In general, the random payments  $(X_1, \dots, X_n)$  will not be mutually independent, but even if they were, the discount factors will not be mutually independent. Concerning the dependency structure, we only assume that the vectors  $\mathbf{X}$  and  $\mathbf{Y}$  are mutually independent. We also assume that the  $X_i$  and  $Y_i$  are non-negative random variables with strictly increasing and continuous distribution functions. By conditioning and using the results of the previous sections, we immediatey find that  $V \leq_{cx} W$  with  $W$  defined by

$$W = \sum_{i=1}^n F_{X_i}^{-1}(U) F_{Y_i}^{-1}(V)$$

where  $U$  and  $V$  are mutually independent uniformly distributed random variables.

We have that  $W | V = v$  is the sum of  $n$  comonotonic risks. This implies  $x = F_{W|V=v}^{-1} [F_{W|V=v}(x)] = \sum_{i=1}^n F_{X_i}^{-1}(F_{W|V=v}(x)) F_{Y_i}^{-1}(v)$ . This means that the conditional distribution function of  $W$ , given that  $V = v$ , follows from

$$\sum_{i=1}^n F_{X_i}^{-1}(F_{W|V=v}(x)) F_{Y_i}^{-1}(v) = x.$$

In order to determine the distribution function of  $W$ , the following algorithm can be used:

For any  $x$ , the value of  $F_W(x)$  is given by

$$F_W(x) = \int_0^1 f_x(v) dv$$

where the function  $f_x$  can be determined from

$$\sum_{i=1}^n F_{X_i}^{-1}(f_x(v)) F_{Y_i}^{-1}(v) = x.$$

We also remark that we can derive upper bounds (in terms of convex order) for a sum of products of three random variables  $\sum_{i=1}^n X_i Y_i Z_i$  in a similar way.

### 5.3 Asian Options

Asian options are options where the terminal pay-offs depend of some form of averaging of the price of the underlying asset over a part of the life of the option. In particular, consider an arithmetic Asian call option with price given by

$$e^{-rT} E^Q \left[ \frac{1}{n} \sum_{i=0}^{n-1} S(T-i) - K \right]_+,$$

where  $S(t)$  is the price process of the underlying risky asset,  $T$  is the expiration date,  $K$  is the exercise price,  $r$  is the risk-free interest rate and  $n$  is the number of averaging days.

In general, we are not able to evaluate the expectation in the above pricing formula. Different approaches have been considered for approximating the price of the option, see e.g. Kemna and Vorst (1990), Turnbull and Wakeman (1991), Levy (1992) and Jacques (1996). It is easy to see that the approximation method we presented here also enables us to find an upper bound for the price of the option. For more details, we refer to Simon, Goovaerts and Dhaene (1999).

## 6 Numerical Examples

In the previous sections, we derived a stochastic bound for the sum of random variables with given marginals. We have seen that this upper bound has an easy computable distribution function, whereas the exact distribution function is often not computable. It remains to compare the goodness-of-fit of our proposed approximation. In order to be able to do this, we will have to consider a case where the exact distribution function can be determined. We will then compare the exact distribution with our approximation. Therefore, we will consider the continuous (temporary) annuity with constant payments

$$V = \int_0^t \exp[-\delta\tau - \sigma X(\tau)] d\tau$$

where as before  $X(\tau)$  represents a standard Brownian motion process and  $\delta$  is the risk free interest intensity. For this annuity, an analytic result for the distribution function is known (see e.g. De Schepper et al. (1994)), such that we can compare the distribution of  $V$  with the distribution function of the stochastic upper bound  $W$  defined by

$$W = \int_0^t \exp[-\delta\tau - \sigma \sqrt{\tau} \Phi^{-1}(U)] d\tau.$$

In figures I to III, we present the graphs of both distribution functions for different choices of the volatility so as to see the appropriateness of the upper bound in various situations.

[ Figure I ]

[ Figure II ]

[ Figure III ]

Figure IV shows the graph of the distribution functions (exact and Fréchet bound) for a perpetuity. When the time horizon  $t$  reaches infinity,  $V$  is known to have an inverted Gamma distribution, see e.g. Dufresne (1990) and Milevsky (1997).

[ Figure V ]

From the figures 1-5, it seems that the distribution function of the approximation we propose is rather close to the original distribution function. This result was more or less to be expected, because for realistic models, the dependency structure between  $X(t)$  and  $X(s)$  will resemble comonotonicity, at least if  $t$  and  $s$  are close enough to each other.

## 7 Conclusions

In this paper, we considered the problem of deriving the distribution function of the present value of a sequence of cash flows that are discounted using some given stochastic return process. It is known that it is impossible to find an explicit expression for this distribution function in most cases. We presented a stochastic upper bound for this distribution function, in the sense that this approximation is greater, in convex order sense, than the exact distribution.

Explicit expressions for the stochastic upper bounds were given for stochastic annuities with deterministic and stochastic cash flows and with a stochastic return process. Both the discrete and the continuous case were considered. We also presented an easy computable upper bound for the price of an arithmetic Asian option.

Finally, we considered some cases where the exact distribution function can be obtained, and compared the exact distribution function with our approximation. It turned out that the approximation will be rather close for realistic return processes.

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