An Easy Computable Upper Bound for the Price of an Arithmetic Asian Option

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Abstract

Using some results from risk theory on stop-loss order and comonotone risks, we show in this paper that the price of an arithmetic Asian option can be bounded from above by the price of a portfolio of European call options.

Keywords: Asian options, stop loss order, comonotonicity

1 Introduction

We consider a securities market consisting of one risky asset $S(t)$ and a riskless money-market account in which money can be invested at a fixed spot-rate r. The risky asset $S(t)$ is assumed to be defined on a filtered probability space $(\Omega, \mathcal{F}_t, P)$ with $\mathcal{F}_t$ the filtration generated by $S(t)$. Furthermore, we assume there exists a single equivalent martingale measure Q, i.e. that we are dealing with a complete and arbitrage free market, see Harrison and Kreps (1979) and Harrison and Pliska (1981). An arithmetic Asian call option with exercise date T , n averaging dates and exercise price K generates a pay-off $\left[\frac{1}{n}\sum_{i=0}^{n-1} S(T-i) - K\right]_+$ at T, and will as such trade at t against a price given by:

$$AA(t, S(t), n, K, T, r) = e^{-(T-t)r} E^Q \left(\left[\frac{1}{n} \sum_{i=0}^{n-1} S(T-i) - K \right]_+ |\mathcal{F}_t \right) \quad (1)$$

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However, in most cases this pricing formula is hard to evaluate. For instance, the distribution of $\sum_{i=0}^{n-1} S(T-i)$ is not known when the price process $S(t)$ is an exponential Brownian motion. One can use Monte-Carlo simulation techniques to obtain a numerical estimate of the price, see Kemna and Vorst (1990) and F.J. Vázquez-Abad (1998), or one can numerically solve a parabolic partial differential equation, see Rogers (1995). But as both approaches are rather time consuming, it would be more than interesting to have an accurate, easily computable approximation of this price.

A very accurate lower bound was obtained in Rogers (1995). In Jacques (1996) an approximation is obtained by approximating the distribution of $\sum_{i=0}^{n-1} S(T-i)$ by a more tractable one. We will follow this last approach, using results from actuarial risk theory on comonotone risks to obtain an accurate upper bound for the price of the Asian option.

2 Options and Stop-loss Transforms

In actuarial science one often compares risks, i.e. nonnegative random variables, by means of their stop-loss premiums/stop-loss transforms. We will use stop-loss transforms of distribution functions that are concentrated on the positive half line. For the sake of completeness, we give the following definition.

Definition 1 *For a distribution function $F(x)$ with a support $D \subseteq \mathbb{R}^+$, the stop-loss transform $\Psi_F(r)$ is given by:*

$$\Psi_F : \mathbb{R}^+ \mapsto \mathbb{R}^+ : r \to \Psi_F(r) = \int_{[r,+\infty[\cap D} (x-r)\, dF(x) \tag{2}$$

Now, we can define the stop-loss order for distribution functions, and therefore as well of random variables, as:

Definition 2 *Of distribution functions $F(x)$ and $G(x)$, both with their support in $\mathbb{R}^+$, $F(x)$ is said to precede $G(x)$ in stop-loss order, written $F \leq_{sl} G$, if:*

$$\forall r \in \mathbb{R}^+ : \Psi_F(r) \leq \Psi_G(r). \tag{3}$$

If we combine equalities 1 and 2, we see that the pay-off of an arithmetic Asian option can be written using the stop-loss transform of $W_n(T) = \sum_{i=0}^{n-1} S(T-i)$. First we can rewrite equation 1 as:

$$AA(t, S(t), n, K, T, r) = \frac{e^{-(T-t)r}}{n} E_t^Q \left[\sum_{i=0}^{n-1} S(T-i) - nK \left| \mathcal{F}_t \right. \right]_+ \tag{4}$$

For a given value s of $S(t)$, we have immediately:

$$AA(t, s, n, K, T, r) = \frac{e^{-(T-t)r}}{n} \Psi_{F^s_{W_n(T)}}(nK)$$

With: $F^s_{W_n(T)}(x) = Q\left(W_n(T) \leq x \left| S(t) = s \right.\right)$

The problem of pricing an arithmetic Asian option therefore turns out to be equivalent to calculating the stop-loss transform of a sum of dependent risks. And as such we can apply results on bounds for stop-loss transforms to the option pricing problem.

3 Bounds for Stop-loss Transforms

In this section we will discuss some results from actuarial science on bounds for stop-loss transforms of sums of dependent stochastic variables. Let us first return to the pricing of an arithmetic Asian option. As explained in the introduction, the main problem we are confronted with in pricing this type of options, is that we, in general, do not know the distribution of the sum $\sum_{i=0}^{n-1} S(T-i)$. However, we do know the distribution of every term in this sum, i.e. of every $S(T-i)$. The stop-loss bounds that we will introduce in this section will be based on these marginal distributions. Let us first introduce the concept of a Fréchet class.

Definition 3 *The Fréchet class $R_n(F_1, ..., F_n)$ determined by n (monovariate) distribution functions, $F_1, ...F_n$, is the class of all n-variate distribution functions F with $F_1, ...F_n$ as marginal distributions.*

Notation:

By $(X_1, ..., X_n) \sim R_n(F_1, ..., F_n)$, we mean that the marginal distributions of this random vector are given by F_1 to F_n, i.e.: $F_X \in R_n\left(F_1, ..., F_n\right)$

We will now introduce the concept of comonotone risks, which we will need to construct the upper bound for an Asian option. For more results on comonotone risks in the actuarial field, we refer the interested reader to Dhaene et al.(1997) and Wang and J.Dhaene (1997). We use the following definition from Dhaene et al. (1997):

Definition 4 *A positive random vector* $(X_1, ..., X_n)$ *is said to be comonotone (the positive random variables* $X_1, ..., X_n$ *are said to be mutually comonotone) if any of the following equivalent conditions hold:*

1. *For the n-variate distribution function we have:*

$$F_{X_1,...X_n}(x_1, ..., x_n) = \min\left(F_1(x_1), ..., F_n(x_n)\right) \ \forall x,, ..., x_n \geq 0. \quad (5)$$

2. *There exist a random variable* Z *and non-decreasing functions* $g_1, ..., g_n$ *on* $\mathbb{R}$*, such that:*

$$(X_1, ..., X_n) \stackrel{\mathcal{D}}{=} (g_1(Z), ..., g_n(Z))$$

3. *For any random variable* U *uniformly distributed on* $[0, 1]$*, we have:*

$$(X_1, ..., X_n) \stackrel{\mathcal{D}}{=} \left(F_1^{-1}(U), ..., F_n^{-1}(U)\right)$$

From the first condition in the above definition it is clear that for a given Fréchet class $R_n(F_1, ..., F_n)$ the distribution of a comonotone random vector $(X_1, ..., X_n) \sim R_n(F_1, ..., F_n)$ is uniquely defined.

Remark 1 *For a given Fréchet class* $R_n(F_1, ..., F_n)$*, the n-variate distribution in the right-hand side of equation 5 is called the upper Fréchet bound of* $R_n(F_1, ..., F_n)$*, we write:*

$$W_R(x_1, ..., x_n) = \min\left(F_1(x_1), ..., F_n(x_n)\right)$$

Definition 5 *Let a Fréchet class* $R_n(F_1, ..., F_n)$ *be given and let* $(X_1, ..., X_n) \sim R_n(F_1, ..., F_n)$ *be a given comonotone vector. We define the distribution function* F_R *as:*

$$F_R(x) = P\left(\sum_{i=1}^{n} X_i \leq x\right). \quad (6)$$

Note that as the distribution of the comonotone vector $(X_1, ..., X_n)$ is uniquely defined, equality 6 uniquely defines the distribution $F_R(x)$.

We have the following result, a prove can be found in Dennenberg (1994):

Proposition 1 *For any given Fréchet class* $R_n(F_1, ..., F_n)$ *, we have:*

$$F_R^{-1}(x) = \sum_{i=1}^{n} F_i^{-1}(x) \tag{7}$$

Goovaerts and Dhaene (1999) prove a more general version of the following result on Fréchet classes:

Proposition 2 *For a given Fréchet class* $R_n(F_1, ..., F_n)$ *we have:*

$$\forall r \in \mathbb{R}^+ \; \Psi_{F_R}(r) = \sum_{i=1}^{n} \Psi_{F_i}\left[F_i^{-1}\left(F_R\left(r\right)\right)\right] \tag{8}$$

Note that we only have to calculate F_R for one value r, and that this can be done using expression 7. Combining these two results, we obtain an upper bound for the stop-loss transform of any sum $\sum_{i=1}^{n} X_i$, when only the marginal distributions are known.

Proposition 3 *Let a Fréchet class* $R_n(F_1, ..., F_n)$ *be given. For all non-negative random vectors* $(X_1, ..., X_n) \sim R_n(F_1, ..., F_n)$ *we have:*

$$F_W \leq_{sl} F_R \tag{9}$$

With: $W = \sum_{i=1}^{n} X_i$. *That is:*

$$\forall d \in \mathbb{R}^+ \; E\left[\sum_{i=1}^{n} X_i - d\right]_+ \leq \sum_{i=1}^{n} E\left[X_i - d_i^*\right]_+ \tag{10}$$

With:

$$d_i^* = F_i^{-1}\left(F_R\left(d\right)\right) \tag{11}$$

Proof:

Because of formula 8 the right-hand side of inequality 10 is indeed equal to $\Psi_{F_R}(d)$. And from formula 11 and formula 7 we have that $\sum_{i=1}^{n} d_i^* = d$, and therefore inequality 10 holds.

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Moreover, the choice of the retention levels as in formula 11 turns out to be optimal, as stated in the following theorem.

Theorem 4 *Given a Fréchet class $R_n(F_1, ..., F_n)$ and a non-negative random vector $(X_1, ..., X_n) \sim R_n(F_1, ..., F_n)$ we have that for any retention level $d \in \mathbb{R}^+$ and for any choice of the retention levels $d_1, ..., d_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n d_i = d$ we have:*

$$\sum_{i=1}^{n} E\left[X_i - d_i^*\right]_+ \leq \sum_{i=1}^{n} E\left[X_i - d_i\right]_+ \tag{12}$$

with:

$$d_i^* = F_i^{-1}\left(F_R\left(d\right)\right)$$

Proof:

For $d_1, ..., d_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n d_i = d$ we have:

$$E\left[\sum_{i=1}^{n} X_i - d\right]_+ \leq \sum_{i=1}^{n} E\left[X_i - d_i\right]_+ \tag{13}$$

However, under the assumption of comonotonicity, which leaves the right-hand side unchanged, the left hand side is given by:

$$\sum_{i=1}^{n} E\left[X_i - d_i^*\right]_+$$

Which proves inequality 12.

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4 Application to Asian Options, the General Case

Here we will apply the results of the previous section to the option pricing formula 1 for the price $AA(t, s, n, K, T, r)$ at time t of an arithmetic Asian option with exercise date T, exercise price K and n averaging dates when $S(t) = s$, with the securities market as described above. By $F(x_2, t_2, x_1, t_1)$ we will denote the conditional distribution of $S(t_2)$ under the equivalent martingale measure Q, i.e.:

$$F(x_2, t_2, x_1, t_1) = Q\left(S(t_2) \leq x_2 \left| S(t_1) = x_1\right.\right), \; t_2 \geq t_1$$

Let us first assume that at time t the averaging has not yet started. In this case the n variables $S(T-n+1), ..., S(T)$ are still unknown, i.e. random and the n distribution functions $F(x, T-n+1, s, t), ..., F(x, T, s, t)$ determine a Fréchet class $R_n\left(F(x, T-n+1, s, t), ..., F(x, T, s, t)\right)$. As discussed in section 2 we have:

$$AA(t, s, n, K, T, r) = \frac{e^{-(T-t)r}}{n}\Psi_{F_{W_n}}(nK)$$

By proposition 3 we obtain:

$$AA(t, S(t), n, K, T, r) \leq \frac{e^{-(T-t)r}}{n}\Psi_{F_R}(nK)$$

With F_R being defined with respect to the Fréchet class $R_n\left(F(x, T-n+1, s, t), ..., F(x, T, s, t)\right)$ along definition 5. By proposition 3 we obtain the following inequality:

$$AA(t, n, K, T, r) \leq \frac{e^{-(T-t)r}}{n}\sum_{i=0}^{n-1}\Psi_{F(\cdot, T-i, s, t)}\left[F^{-1}\left(F_R\left(K\right), T-i, s, t\right)\right]$$

Where, $F^{-1}(x_2, t_2, x_1, t_1)$ is the inverse of $F(x_2, t_2, x_1, t_1)$ with respect to x_2.

Until now, we assumed that $t < T-n+1$. We will now turn to the case that $t \geq T-n+1$. Let i^* be such that: $T-i^* \leq t < T-i^*+1$, then we know the first $n-i^*$ prices and only the last i^* prices: $S(T-i^*), ..., S(T)$, remain random. Therefore we obtain:

$$\begin{aligned}
& AA(t, n, K, T, r) \\
= \; & \frac{e^{-(T-t)r}}{n}E^Q\left[\sum_{i=0}^{i^*-1}S(T-i) - \left(nK - \sum_{i=i^*}^{n-1}S(T-i)\right)|\mathcal{F}_t\right]_+ \\
= \; & \frac{e^{-(T-t)r}}{n}E^Q\left[\sum_{i=0}^{i^*-1}S(T-i) - K_{i^*}\,|\mathcal{F}_t\right]_+
\end{aligned}$$

With: $K_j = nK - \sum_{i=j}^{n-1} S(T-i)$ for $j < n$, and $K_n = nK$.

First, let us assume that $K_{i^*} > 0$. Under this assumption we can apply the same method to obtain upper bounds as in the case $t < T-n+1$. But we are now working in the Fréchet class: $R_{i^*}(F_{T-i^*}, ..., F_T)$ and that K has been replaced by the adjusted exercise price K_{i^*}. As such we obtain:

$$\begin{aligned} & AA(t,s,n,K,T,r) \\ \leq \; & \frac{e^{-(T-t)r}}{n} \sum_{i=0}^{i^*-1} \Psi_{F(\cdot,T-i,s,t)} \left[F^{-1}\left(F_R\left(K_{i^*}\right), T-i, s, t\right) \right] \end{aligned}$$

If we define κ_i as:

$$\kappa_i = F^{-1}\left(F_R\left(K_{i^*}\right), T-i, s, t\right), \; i = 0, ...i^*. \tag{14}$$

Then we obtain:

$$AA(t,s,n,K,T,r) \leq \frac{e^{-(T-t)r}}{n} \sum_{i=0}^{i^*-1} \Psi_{F(\cdot,T-i,s,t)}\left(\kappa_i\right)$$

We will now look at what happens when K_{i^*} is equal to or smaller then zero. In this case, we can not use the bounds from the previous section. However we have:

$$\begin{aligned} & E^Q\left[W_{i^*} - K_{i^*}\right]_+ \\ = \; & E^Q\left[W_{i^*}\right] - K_{i^*} \\ = \; & \sum_{i=0}^{i^*-1} E^Q S(T-i) - K_{i^*} \end{aligned}$$

As the discounted price process $e^{-tr}S(t)$ is a martingale under Q, the above expression is equal to:

$$S(t) \sum_{i=0}^{i^*-1} e^{(T-i-t)r} - K_{i^*}$$

If we combine this result with the option pricing formula 1, we obtain, in case of a negative K_{i^*}:

$$\begin{aligned} AA(t,s,n,K,T,r) &= \frac{e^{-(T-t)r}}{n}\left(S(t)\sum_{i=0}^{i^*-1} e^{(T-i-t)r} - K_{i^*}\right) \\ &= \frac{1}{n}\left[S(t)\sum_{i=0}^{i^*-1} e^{-ir} + e^{-(T-t)r}\sum_{i=i^*}^{n-1} S(T-i)\right] - e^{-(T-t)r}K \end{aligned}$$

If we denote with $EC(t, T, K, r)$ the price of a European call option with exercise date T and exercise price K, we obtain the following result:

Theorem 5 *In a securities market as described above, the following result holds for $AA(t, s, n, K, T, r)$*

1. *If $K_{i^*} > 0$:*

$$\begin{aligned} & AA(t, s, n, K, T, r) \\ \leq\ & \frac{e^{-(T-t)r}}{n} \sum_{i=0}^{i^*-1} \Psi_{F(\cdot,T-i,s,t)}(\kappa_i) \\ =\ & \frac{1}{n} \sum_{i=0}^{i^*-1} e^{-ir} EC(t, s, \kappa_i, T-i, r) \end{aligned} \tag{15}$$

   Where: $\kappa_i = F^{-1}(F_R(K_{i^*}), T-i, s, t)$, $i = 0, ...i^*$. With the exercise prices κ_i, $i = 0, ...i^*$ being optimal in the sense of theorem 1.

2. *If $K_{i^*} \leq 0$:*

$$\begin{aligned} & AA(t, s, n, K, T, r) \\ =\ & \frac{1}{n}\left[S(t)\sum_{i=0}^{i^*-1} e^{-ir} + e^{-(T-t)r}\sum_{i=i^*}^{n-1} S(T-i)\right] - e^{-(T-t)r}K \end{aligned} \tag{16}$$

As such, we have shown (for $K_{i^*} > 0$) that the price of an arithmetic Asian option is bounded from above by the price of a portfolio of European call options. And we have found the exercise prices for the European options that optimize this bound.

5 Application in a Black and Scholes Setting

Here we assume that the price process of the underlying asset $S(t)$ follows an exponential Brownian motion with constant coefficients, i.e.:

$$dS(t) = S(t)\left[\mu dt + \sigma dB(t)\right].$$

Which enables us to use the well known Black and Scholes pricing formula for European call options. Combining the Black and Scholes formula with formula 15 yields:

$$\begin{aligned} AA(t,s,n,K,T,r) &\leq \frac{1}{n}\sum_{i=0}^{i^*-1} e^{-ir} EC\left(t,s,\Gamma_i^{-1}(K_{i^*},s),T-i,r\right) \\ &= \frac{1}{n}\sum_{i=0}^{i^*-1} e^{-ir}\left[sN\left(d_{i,1}\right)-\kappa_i\, e^{-r(T-i-t)}N\left(d_{i,2}\right)\right] \end{aligned}$$

where d_1 and d_2 are given by:

$$d_{i,1} = \frac{\log\left(s/\kappa_i\right)+(r+\sigma^2/2)(T-i-t)}{\sigma\sqrt{T-i-t}}$$

and:

$$d_{i,2} = d_{i,1} - \sigma\sqrt{T-i-t}.$$

As such, the main difficulty is to calculate the different strike prices κ_i. The distribution function $F(x_2,t_2,x_1,t_1) = Q\left(S(t_2)\leq x_2\,|S(t_1)=x_1\right)$ is now given by:

$$F(x_2,t_2,x_1,t_1) = \mathrm{LN}\left(x_2;\ln(x_1)+(r-\sigma^2/2)(t_2-t_1),\sigma\sqrt{(t_2-t_1)}\right)$$

Where $\mathrm{LN}(x;\mu,\sigma)$ is the lognormal distribution with mean μ and standard deviation σ. Using formula 7 we can rewrite equation 14 as:

$$K_{i^*} = \sum_{k=0}^{i^*-1} F^{-1}\left(F\left(\kappa_i,T-i,S,t\right),T-k,S,t\right).$$

It is easily verified that this is equivalent to:

$$\sum_{k=0}^{i^*-1} S^{\left(1-\sqrt{\frac{T-k-t}{T-i-t}}\right)}\kappa_i^{\sqrt{\frac{T-k-t}{T-i-t}}}e^{\mu\left(\sqrt{(T-k-t)(T-i-t)}-(T-k-t)\right)} = K_{i^*}. \qquad (17)$$

If we define $\alpha_k(i)$ and $\beta_k(i,s)$ as:

$$\alpha_k(i) = e^{\mu\left(\sqrt{(T-k-t)(T-i-t)}-(T-k-t)\right)} \text{ and } \beta_k(i) = \sqrt{\frac{T-k-t}{T-i-t}},$$

then we can rewrite equation 17 as:

$$\Gamma_i(\kappa_i, S) = K_{i^*} \tag{18}$$

where:

$$\Gamma_i(x, s) = \sum_{k=0}^{i^*-1} \alpha_k(i)\, s^{1-\beta_k(i)} x^{\beta_k(i)}. \tag{19}$$

6 Numerical Example and testing

In this section we will give a numerical example of the Black and Scholes case. We will compare our results with those of Jacques (1996) where the distribution of the arithmetic Asian option was approximated by means of a lognormal (LN) and the inverse Gaussian (IG) distribution. Therefore the parameters that were used to generate the results given in table 1 were given the same values as in Jacques (1996) : an initial stock price $S(0) = 100$, an annual interest rate of 9%, i.e. the daily rate $r = \ln(1+0.09)/365$, a maturity of 120 days, an averaging period of 30 days and three values: 0.2, 0.3 and 0.4 for the volatility b.

6.1 Evaluating the Upper Bound

Here we will look how sharp the upper bound is. Table 1 gives the values obtained by Jacques and compares them with our results. The last two columns give the relative difference between the price obtained by means of the bound and the lognormal and the inverse Gaussian approximation respectively.

We see that the prices obtained by means of the comonotone approximation are relatively close to the values obtained by either the lognormal as well as the inverse Gaussian approximation. Furthermore, the results suggest that the upper bound is sharper for options for which the exercise price K is smaller than the initial stock price. The effect of the volatility level is less clear as this seems to be intertwined with the effect of the exercise price.

Table 1
Comparing copula results with LN and IG

b	K	LN	IG	Upper Bound	Upper Bound vs LN	Upper Bound vs IG
0.2	90	12.68	12.68	12.72	0.32%	0.32%
	100	5.46	5.46	5.56	1.83%	1.83%
	110	1.63	1.63	1.71	4.91%	4.91%
0.3	90	13.85	13.85	13.95	0.72%	0.72%
	100	7.48	7.49	7.62	1.87%	1.74%
	110	3.48	3.49	3.62	4.02%	3.72%
0.4	90	15.36	15.37	15.51	0.98%	0.91%
	100	9.51	9.53	9.70	2.00%	1.78%
	110	5.48	5.49	5.67	3.47%	3.82%

In table 2 we compare the upper bound with a price obtained by generating 10000 paths. This was done for three different options: the first with a maturity of 120 days and 30 averaging dates, the second with a maturity of 60 days and 30 averaging dates and the third one with again a maturity of 120 days but only 10 averaging dates. In each case we considered the 5 following exercise prices: 80, 90, 100, 110 and 120.

Table 2
Comparing the upper bound with the simulated price, $M = 120, N = 30$

K	Bound	Simulation	Relative Error
80	21.9269	21.9315	0.02%
90	12.7207	12.6907	0.24%
100	5.5563	5.4971	1.08%
110	1.7077	1.5839	7.82%
120	0.3675	0.3155	16.48%

TABLE 3
Comparing the upper bound with the simulated price, $M = 60, N = 30$

K	Bound	Estimate	Relative Error
80	20.7845	20.7334	0.25%
90	11.0601	10.9259	1.23%
100	3.3452	3.1357	6.68%
110	0.4084	0.3487	17.12%
120	0.0185	0.0066	178.58%

TABLE 4
Comparing the upper bound with the simulated price, $M = 120, N = 10$

K	Bound	Estimate	Relative Error
80	22.1735	22.1013	0.33%
90	13.0233	13.0363	0.10%
100	5.8934	5.8885	0.08%
110	1.9442	1.8853	3.12%
120	0.4666	0.4292	8.72%

Again, we see that the upper bound is close to the estimated true price for options that are in the money. The price estimate even exceeds the upper bound for $K = 90$ in two cases: for $M = 120, N = 10$ and for $M = 120, N = 30$. However, for options that are out of the money, the upper bound becomes less accurate.

Note that the upper bound performs best for the option with $M = 120$ and $N = 10$ and worst when $M = 60$ and $N = 30$. This is probably due to the fact that the upper Fréchet bound is a better model for the dependency between the averaging values $S(T-i)$, for small values of N and large values of T.

7 Conclusion

Using some results from risk theory on comonotone risks and stop-loss order, we were able to show that the price of an arithmetic Asian option can be bounded from above by the price of a portfolio of European call options. The upper bound appears to be rather sharp if compared to simulated prices. Furthermore, we showed that the exercise prices we derived optimize this portfolio of European options.

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