# Convex Upper and Lower Bounds for Present Value Functions

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**Abstract**

In this paper we present an efficient methodology for approximating the distribution function of the net present value of a series of cash-flows, when the discounting is presented by a stochastic differential equation as in the Vasicek model and in the Ho-Lee model. Upper and lower bounds in convexity order are obtained. The high accuracy of the method is illustrated for cash-flows for which no analytical results are available.

## 1 Introduction

When determining the present value of a series of $n$ payments $c_i$ at times $\tau_i$ $(i = 1, \ldots, n)$, one has to define a discount process $X(\tau)$. The present value of this series is then given by

$$V_0 = \sum_{i=1}^{n} c_i e^{-X(\tau_i)}. \tag{1}$$

To determine the cumulative distribution function (cdf) of this random variable (rv), one has to cope with a standard problem: the summation of rvs with marginal cdfs of the same type need not (and often will not) produce a cdf of that type. Secondly, the dependence structure of the rvs $X(\tau_i)$ is not known or hard to obtain in general. Although we could approximate the cdf via Monte Carlo simulation when the dependence structure of the $X(\tau_i)$ is given, this would be very time-consuming. Moreover, if we want to estimate a high quantile (e.g. Value-at-Risk) accurately, we should increase the sample size – and consequently the computation time – drastically. Using results from actuarial risk theory on comonotonic risks, we can however obtain an easily computable upper bound for $V_0$. In addition, Jensen's inequality combined with the theory on comonotonic risks provides a tool for obtaining a lower bound.

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In this paper we will define the discount factors as follows. We write $X(\tau)$ as

$$X(\tau) = \int_0^{\tau} r(s)ds, \tag{2}$$

hence

$$V_0 = \sum_{i=1}^{n} c_i \exp\left( - \int_0^{\tau_i} r(s)ds \right), \tag{3}$$

and consider two types of models for $r(s)$. In the first model, the stochastic differential equation for describing the behaviour of $r(s)$ is the same as the one for the instantaneous interest rate in the Vasicek (1977) model:

$$dr = (\alpha - \beta r)dt + \gamma dW, \tag{4}$$

where $\alpha$, $\beta$ and $\gamma$ are non-negative constants and $W$ represents a standard Wiener process. Replacing $\alpha$ by a non-negative function $\alpha(t)$ of time, as in the Ho-Lee (1986) model yields a second model:

$$dr = \alpha(t)dt + \gamma dW. \tag{5}$$

In the present paper analytical upper and lower bounds for the distribution function of $V_0$ are obtained. They are shown to be practically applicable due to the very small relative error bounds. Random variables of this type arise in modern actuarial situations where e.g. discounting is taken into account in the evaluation of the distribution of IBNR provisions. In the case of financial reinsurance it provides the distribution of the experience account and as such it enables the determination of the final premium of this type of reinsurance. Knowing the distribution of $V_0$, provides a tool for the determination of the "fair value" as well as the "supervisory value" of a portfolio of risks. Moreover it avoids simulations in solvency calculations and it helps for the determination of embedded value and appraisal value.

Our methodology only requires the knowledge of the distribution functions of the $X(\tau_i)$ and does not take into account the dependence structure between these random variables. Allowing for all kinds of dependence structures, which often cannot be measured because of the incomplete statistical basis, of course has an influence on the distribution function of $V_0$. Replacing the unknown cdf of $V_0$ by the upper bound (in convex order sense) is a safe strategy in the sense that all risk averse decision makers would prefer the original (unknown) cdf. On the other hand, the lower bound gives us an idea of the high accuracy of the approximation.

## 2 Convex Upper Bound

In the actuarial field it is common practice to replace the cdf of $V_0$ by a "less favourable" one. Of course the new cdf should be easier to determine, see e.g. Goovaerts e.a. (1986). To formalise the concept "less favourable", we make use of the convex order.

**Definition 1** *A rv $V$ is smaller than a rv $W$ in the convex order if*

$$E[\phi(V)] \leq E[\phi(W)]$$

*for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$. This is denoted as $V \leq_{cx} W$.*

In terms of utility theory, $V \leq_{cx} W$ means that the rv $V$ is preferred to the rv $W$ by all risk averse decision makers, i.e. $E[u(-V)] \geq E[u(-W)]$ for all concave utility functions $u$. Replacing the cdf of $V$ by the cdf of $W$ can therefore be considered as a prudent strategy. A closely related order is the stop-loss order.

**Definition 2** *A rv $V$ is smaller than a rv $W$ in the stop-loss order if*

$$E[V-d]_+ \leq E[W-d]_+$$

*for all $d$. This is denoted as $V \leq_{s\ell} W$.*

In Shaked & Shanthikumar (1994) it is proven that the convex order incorporates the stop-loss order:

$$V \leq_{cx} W \quad \Longleftrightarrow \quad \begin{cases} V \leq_{s\ell} W \\ EV = EW \end{cases} \tag{6}$$

We will now introduce the concepts of a Fréchet space and comonotonic risks, which will enable us to construct an upper bound for $V_0$.

**Definition 3** *The Fréchet space $R_n(F_1, \ldots, F_n)$ determined by the (univariate) distribution functions $F_1, \ldots, F_n$ is the class of all $n$-variate distribution functions $F$ (or the corresponding rvs) with marginals $F_1, \ldots, F_n$.*

In the Fréchet space $R_n(F_1, \ldots, F_n)$ any rv $\mathbf{X}$ is constrained from above by

$$F_{\mathbf{X}}(\mathbf{x}) \leq \min\{F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\} =: W_n(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$$

A comonotone risk is a rv with cdf $W_n$, see e.g. Dhaene et al (1997):

**Definition 4** *A random vector $(X_1, \ldots, X_n)$ is said to be comonotone (the rvs $X_1, \ldots, X_n$ are said to be mutually comonotone) if any of the following equivalent conditions hold:*

1. *For the n-variate cdf we have*

$$F_{\mathbf{X}}(\mathbf{x}) = \min\{F_1(x_1), F_2(x_2), \dots, F_n(x_n)\}, \quad \forall \mathbf{x} \in \mathbb{R}^n;$$

2. *There exist a rv $Z$ and non-decreasing functions $g_1, \dots, g_n : \mathbb{R} \to \mathbb{R}$ such that*

$$(X_1, \dots, X_n) \stackrel{d}{=} (g_1(Z), \dots, g_n(Z));$$

3. *For any rv $U$ uniformily distributed on $[0,1]$, we have*

$$(X_1, \dots, X_n) \stackrel{d}{=} (F_1^{-1}(U), \dots, F_n^{-1}(U)).$$

As usual, "$\stackrel{d}{=}$" denotes equality in distribution and $F^{-1}$ represents the inverse of the cdf $F$ defined as

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \quad p \in [0,1].$$

It can be seen from condition 2 that comonotonic rvs possess a very strong positive dependence: increasing one of the $X_i$ will lead to an increase of all other rvs $X_j$ involved. These special rvs will provide us with a tool to construct a close upper bound for $V_0$, see Goovaerts et al (2000).

**Theorem 1** *Let $\mathbf{X} = (X_1, \dots, X_n)$ be a n-dimensional rv with marginals $F_1, \dots, F_n$. Further, let $U$ be a rv, uniformly distributed on $[0,1]$. Finally, let $\phi_1, \dots, \phi_n$ be non-negative and non-increasing functions. Then*

$$\phi_1(X_1) + \dots + \phi_n(X_n) \leq_{cx} \phi_1(F_1^{-1}(U)) + \dots + \phi_n(F_n^{-1}(U)). \tag{7}$$

**Proof.** In Goovaerts & Dhaene (1999), it is shown that

$$\sum_{i=1}^{n} \phi_i(X_i) \leq_{s\ell} \sum_{i=1}^{n} \phi_i(F_i^{-1}(U)).$$

Because $(X_1, \dots, X_n)$ and $(F_1^{-1}(U), \dots, F_n^{-1}(U))$ have the same marginals, $\sum_{i=1}^{n} \phi_i(X_i)$ and $\sum_{i=1}^{n} \phi_i(F_i^{-1}(U))$ have the same mean. Equation (6) then completes the proof. $\square$

Setting $\phi_i(X) := c_i \exp(-X(\tau_i))$, we obtain the convex upper bound

$$W = \sum_{i=1}^{n} \phi_i(F_{X(\tau_i)}^{-1}(U)) = \sum_{i=1}^{n} c_i \exp(-F_{X(\tau_i)}^{-1}(U)). \tag{8}$$

To compute the cdf of $W$, we can use the additivity of the inverse cdfs of comonotonic risks.

**Proposition 1** *Let $Y_1, \ldots, Y_n$ be $n$ comonotonic risks with marginals $F_1, \ldots, F_n$. Then*

$$F_S^{-1}(p) = \sum_{i=1}^{n} F_i^{-1}(p), \qquad p \in [0,1],$$

*with $S = Y_1 + \ldots + Y_n$.*

For a proof of this result, we refer the interested reader to Dennenberg (1994). Remark that, for any strictly decreasing function $\phi$ and any cdf $F_X$,

$$\phi(F_X^{-1}(p)) = F_{\phi(X)}^{-1}(1-p), \qquad p \in [0,1].$$

So, for strictly positive cash-flows $c_i$ and strictly increasing $F_{X(\tau_i)}$, the tail function $\overline{F}_W := 1 - F_W$ is implicitely given by

$$\sum_{i=1}^{n} \phi_i(F_{X(\tau_i)}^{-1}(\overline{F}_W(x))) = x. \tag{9}$$

Notice that we only need to know the inverse marginal cdfs $F_{X(\tau_i)}^{-1}$ to compute the upper bound. If all $c_i < 0$, then $F_W$ is implicitely given by

$$\sum_{i=1}^{n} \phi_i(F_{X(\tau_i)}^{-1}(F_W(x))) = x. \tag{10}$$

The case when certain $c_i$ are negative and other are positive is considered in Goovaerts et al (2000). Theorem 1 can also be used to determine an upper bound for the price of an arithmetic Asian option, see Simon et al (2000).

# 3 Convex Lower Bound

Starting from Jensen's inequality for conditional expectations,

$$E[f(V)|Z] \geq f(E[V|Z]), \tag{11}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex function, we can derive a convex lower bound for $V_0$. This inequality has also been used by Rogers & Shi (1995) to obtain a lower bound for the price of an Asian option, while Feynman & Hibbs (1965) applied it to introduce a variational result for essentially the same quantity, c.q. the partition matrix, an important quantity in mathematical physics.

**Proposition 2** *For any two rvs $Y$ and $Z$, let $L := E(Y|Z)$. Then*

$$L \leq_{cx} Y \tag{12}$$

**Proof.** As $(\cdot)_+ = \max(\cdot, 0)$ is a convex function, we find for all $k$

$$\begin{aligned} E[Y-k]_+ &= E[E((Y-k)_+|Z)] \\ &\geq E[E(Y-k|Z)]_+ \\ &= E[L-k]_+ \end{aligned}$$

Furthermore, $L$ and $Y$ have the same mean, so again equation (6) completes the proof. □

Replacing $Y$ by $V_0$ and choosing an appropriate conditioning variable $Z$, we get an expression for the stop-loss transform $E(L-k)_+$ of the convex lower bound $L$. To compute the cdf $F_L$ out of $E(L-k)_+$, remark that

$$E(X-k)_+ = \int_k^{+\infty} (x-k)dF_X(x),$$

hence

$$\frac{d}{dk}E(X-k)_+ = -\int_k^{+\infty} dF_X(x) = F_X(k) - 1. \tag{13}$$

# 4 Application: Vasicek & Ho-Lee Model

Solving the stochastic differential equation for the Vasicek model results in

$$r(s) = e^{-\beta s}r(0) + \frac{\alpha}{\beta}(1-e^{-\beta s}) + \gamma e^{-\beta s}\int_0^s e^{\beta u}dW(u), \tag{14}$$

$$\sim N\left(e^{-\beta s}r(0) + \frac{\alpha}{\beta}(1-e^{-\beta s}), \frac{\gamma^2}{2\beta}(1-e^{-2\beta s}).\right) \tag{15}$$

Straightforward calculus then yields, for $X(\tau) := \int_0^\tau r(s)ds$,

$$X(\tau) = \frac{\alpha}{\beta}\tau + \frac{1}{\beta}(r(0) - \frac{\alpha}{\beta})(1-e^{-\beta\tau}) + \frac{\gamma}{\beta}\int_0^\tau (1-e^{\beta(u-\tau)})dW(u),$$

which in turn has a normal distribution with mean

$$\mu(\tau) = \frac{\alpha}{\beta}\tau + \frac{1}{\beta}(r(0) - \frac{\alpha}{\beta})(1-e^{-\beta\tau})$$

and variance

$$\sigma^2(\tau) = \frac{\gamma^2}{\beta^2}\left(\tau - \frac{2}{\beta}(1-e^{-\beta\tau}) + \frac{1}{2\beta}(1-e^{-2\beta\tau})\right).$$

For the Ho-Lee model we get

$$r(s) = r(0) + \int_0^s \alpha(u)du + \gamma W(s), \tag{16}$$

$$\sim N\left(r(0) + \int_0^s \alpha(u)du, \gamma^2 s\right). \tag{17}$$

Consequently, $X(\tau)$ is normally distributed with mean

$$\mu(\tau) = r(0)\tau + \varphi_\alpha(\tau)$$

and variance

$$\sigma^2(\tau) = \frac{\gamma^2\tau^3}{3},$$

where we used the abbreviation $\varphi_\alpha(\tau) := \int_0^\tau \alpha(u)(\tau - u)du$.

The convex upper bound for $V_0$ for both models follows from

$$\sum_{i=0}^{n} c_i \exp\left\{-\mu(\tau_i) - \sigma(\tau_i)\Phi^{-1}(\overline{F}_W(k))\right\} = k \tag{18}$$

where $\Phi$ denotes the standard normal cdf. Equivalently,

$$F_W(k) = 1 - \Phi(u_k) \tag{19}$$

with $u_k$ determined by

$$\sum_{i=0}^{n} c_i \exp\{-\mu(\tau_i) - \sigma(\tau_i)u_k\} = k. \tag{20}$$

To compute the convex lower bound, we first have to choose a conditioning variable $Z$. Therefore, define

$$I_\delta := -\int_0^\delta X(\tau)d\tau,$$

which is clearly again normally distributed, say, with mean $\mu_\delta$ and variance $\sigma_\delta^2$. Now we choose

$$Z_\delta := \frac{I_\delta - \mu_\delta}{\sigma_\delta} \sim N(0,1), \tag{21}$$

as conditioning variable. Recall that when a normal rv $-X(\tau)$ is conditioned on a standard normal rv $Z_\delta$, it remains normal with mean

$$E(-X(\tau)|Z_\delta) = -E(X(\tau)) + k_{\tau,\delta}Z_\delta$$

and variance

$$Var(-X(\tau)|Z_\delta) = Var(X(\tau)) - k_{\tau,\delta}^2,$$

where

$$k_{\tau,\delta} = Cov(-X(\tau), Z_\delta) = \frac{1}{\sigma_\delta}\int_0^\delta Cov(X(\tau), X(\nu))d\nu.$$

The stop-loss transform of the lower bound $L$ is given by

$$
\begin{aligned}
E(L-k)_+ &= E\left[E\left(\sum_{i=1}^{n} c_i e^{-X(\tau_i)} \mid Z_\delta\right) - k\right]_+ \\
&= E\left[\sum_{i=1}^{n} c_i E\left(e^{-X(\tau_i)|Z_\delta}\right) - k\right]_+ \\
&= E\left[\sum_{i=1}^{n} c_i \exp\left\{-\mu(\tau_i) + k_{\tau_i,\delta} Z_\delta + \frac{1}{2}(\sigma^2(\tau_i) - k^2_{\tau_i,\delta})\right\} - k\right]_+ \\
&= \int_0^1 \left[\sum_{i=1}^{n} c_i \exp\left\{-\mu(\tau_i) + k_{\tau_i,\delta} \Phi^{-1}(u) + \frac{1}{2}(\sigma^2(\tau_i) - k^2_{\tau_i,\delta})\right\} - k\right]_+ du
\end{aligned}
$$

Notice that the integrand is a non-decreasing function of $u$, at least if $c_i \geq 0$ and $k_{\tau_i,\delta} \geq 0$ $(i = 1, \ldots, n)$. This means that the integrand equals zero for all $u \leq u_k$, with $u_k$ determined by

$$
\sum_{i=1}^{n} c_i \exp\left\{-\mu(\tau_i) + k_{\tau_i,\delta} \Phi^{-1}(u_k) + \frac{1}{2}(\sigma^2(\tau_i) - k^2_{\tau_i,\delta})\right\} = k. \tag{22}
$$

Consequently

$$
E(L-k)_+ = \int_{u_k}^1 \sum_{i=1}^{n} c_i \exp\left\{-\mu(\tau_i) + k_{\tau_i,\delta} \Phi^{-1}(u) + \frac{1}{2}(\sigma^2(\tau_i) - k^2_{\tau_i,\delta})\right\} - k \; du
$$

and

$$
\frac{d}{dk} E(L-k)_+ = \int_{u_k}^1 (-1)\; du = u_k - 1.
$$

Finally, using equation (13), we find

$$
F_L(k) = u_k. \tag{23}
$$

If however $c_i < 0, \forall i$, then

$$
\frac{d}{dk} E(L-k)_+ = \int_0^{u_k} (-1)\; du = -u_k,
$$

and $F_L(k) = 1 - u_k$.

For the Vasicek model, some lengthy yet simple calculations yield

$$
\begin{aligned}
k_{\tau,\delta} &= \frac{1}{\sigma_\delta} \int_0^\delta \frac{\gamma^2}{\beta^2} \left\{(\tau \wedge \nu) - \frac{1}{\beta}(e^{\beta(\tau\wedge\nu)} - 1)(e^{-\beta\tau} + e^{-\beta\nu}) + \frac{1}{2\beta}(e^{\beta(\tau\wedge\nu)} - e^{-\beta(\tau\wedge\nu)})\right\} d\nu \\
&= \frac{1}{\sigma_\delta} \frac{\gamma^2}{\beta^2} \left\{\tau\delta - \frac{\tau^2}{2} + \frac{1}{\beta}(\delta + \frac{e^{-\beta\delta}}{\beta})(e^{-\beta\tau} - 1) - \frac{1}{2\beta^2}(e^{-2\beta\tau} + 1)\right\}
\end{aligned}
$$

where $\tau \leq \delta$ and

$$\sigma_\delta = \frac{\gamma}{\beta^2}\left\{\beta\delta^2\left(\frac{\beta\delta}{3}-1\right)-\delta\left(2e^{-\beta\delta}-1\right)-\frac{1}{2\beta}\left(e^{-2\beta\delta}-1\right)\right\}^{\frac{1}{2}}.$$

Remark that

$$Cov(r(u), r(s)) = e^{-\beta(u+s)}\frac{\gamma^2}{2\beta}\left(e^{2\beta(u\wedge s)}-1\right) \geq 0$$

which implies the positivity of $k_{\tau,\delta}$. Analogous, for the Ho-Lee model we get

$$\begin{aligned} k_{\tau,\delta} &= \frac{1}{\sigma_\delta}\int_0^\delta \left\{\frac{\gamma^2}{2}(\tau\wedge\nu)^2(\tau+\nu)-\frac{2}{3}\gamma^2(\tau\wedge\nu)^3\right\}d\nu \\ &= \frac{\gamma^2\tau^2}{2\sigma_\delta}\left\{\frac{\tau^2}{12}-\frac{\tau\delta}{3}+\frac{\delta^2}{2}\right\} \end{aligned}$$

where $\tau \leq \delta$ and

$$\sigma_\delta = \frac{\gamma\delta^2}{2}\sqrt{\delta/5}.$$

The $k_{\tau_i,\delta}$ are here also positive, because

$$Cov(r(u), r(s)) = \gamma^2(u\wedge s) \geq 0.$$

# 5 Accuracy of the bounds

In this section we investigate the accuracy of the proposed bounds for the present value function $V_0$, by comparing their cdf to the empirical cdf obtained with Monte Carlo simulation. We also construct a QQ-plot to visualise the goodness-of-fit. Finally, we determine the maximum stop-loss error, relatively to the expected value of $V_0$, by calculating the stop-loss premiums of the upper and lower bound respectively:

$$\frac{E(W-k)_+ - E(L-k)_+}{E(V_0)}$$

The first case considered is the Vasicek model with parameters $\alpha = 0.0038438$, $\beta = 0.044688$ and $\gamma = 0.0015313$, see De Winne (1995). We set $c_i = 100$, $\tau_i = i$ $(i = 1, \ldots, 30)$ and choose $r(0) = 0.08$, $\delta = 30$.

Figure 1 shows the distribution functions and the corresponding QQ-plots of the upper and lower bounds, compared to the empirical distribution based on 10000 randomly generated, normally distributed vectors. The distribution functions are remarkably close to each other and enclose the simulated cdf nicely. This is confirmed by the QQ-plot where we also see that the comonotonic upper bound has somewhat heavier tails. In figure 2 we plot the upper and lower stop-loss premiums, $E(W-k)_+$ and $E(L-k)_+$ respectively, for several retentions $k$. The vertical line indicates the mean present value $E(V_0) = 1074.987$. For the maximal value of the maximum relative stop-loss error, we find

$$\max_k \left( \frac{E(W-k)_+ - E(L-k)_+}{E(V_0)} \right) \approx 0.08\%.$$

We now construct a Ho-Lee model where, besides a lineair part, $r(\cdot)$ consists of a harmonically damped oscillation and some normally distributed error. Therefore, we define

$$\alpha(\tau) := B + Ae^{-g\tau}\left[\omega\cos(\omega\tau) - g\sin(\omega\tau)\right]$$

with $\gamma = 0.01$, $A = 0.003$, $B = 0.01$, $g = 0.01$ and $w = 3$. Hence,

$$\varphi_\alpha(\tau) = \frac{B\tau^2}{2} + \frac{A\omega}{g^2+\omega^2} - \frac{Ae^{-g\tau}\left[\omega\cos(\omega\tau) + g\sin(\omega\tau)\right]}{g^2+\omega^2}$$

Again, we assume equal payments $c_i = 100$ at times $\tau_i = i$ $(i = 1, \ldots, 30)$ and choose $\delta = 30$. The initial interest rate $r(0)$ is set to 0.5, so $E(V_0) = 839.4933$. Figures 3 and 4 again indicate the high accuracy of the bounds: e.g. the maximum relative stop-loss error stays below 0.6%.

Intuitively, we expect the bounds to perform worse when the payments $c_i$ are no longer constant or when $\gamma$ increases. We therefore revisit the Vasicek model and set $c_i = i$. Moreover, we increase $\gamma$ by a factor 10, so $E(V_0) = 121.4577$. Despite the absence of the $c_i$ in the conditioning variable $Z_\delta$, both upper and lower bounds remain excellent approximations (see figures 5 en 6).

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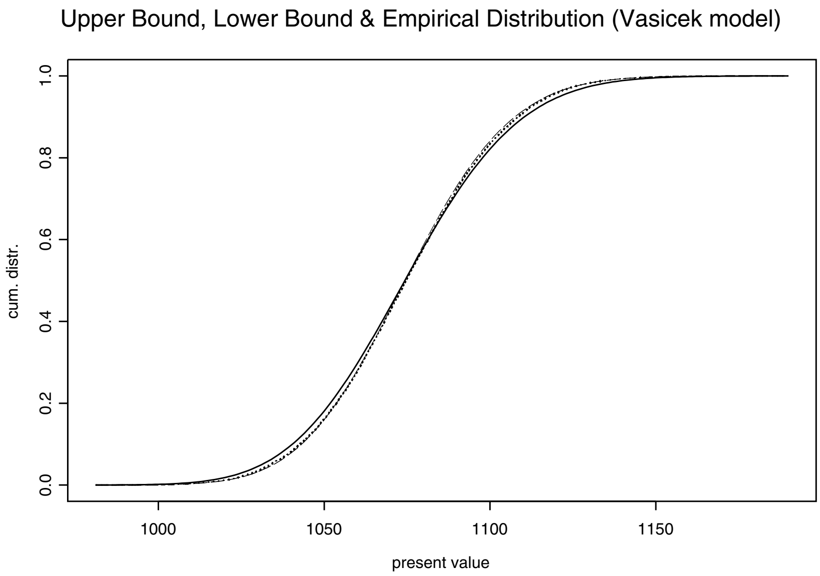
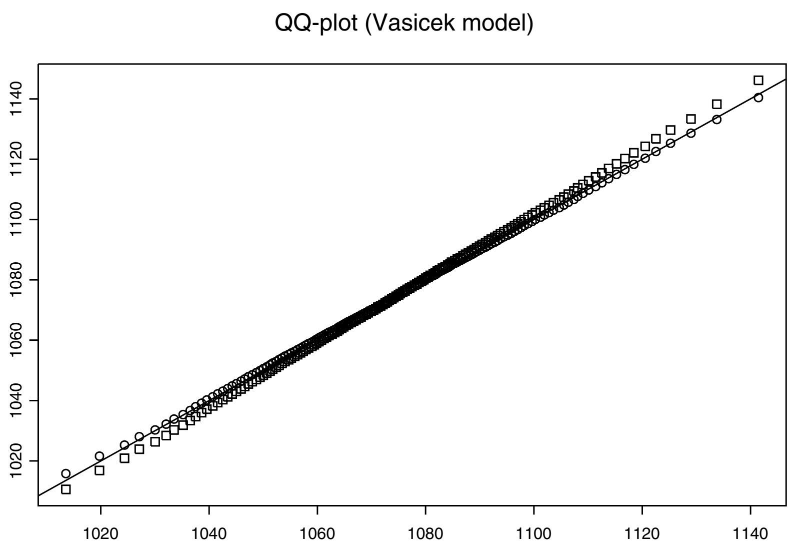
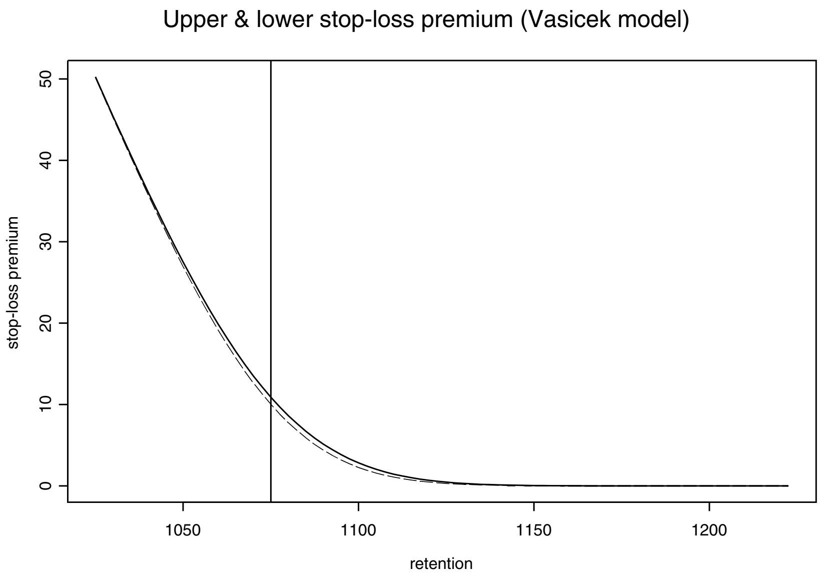



(a) Upper (—) & lower (- -) bound vs. Monte Carlo simulation (···)

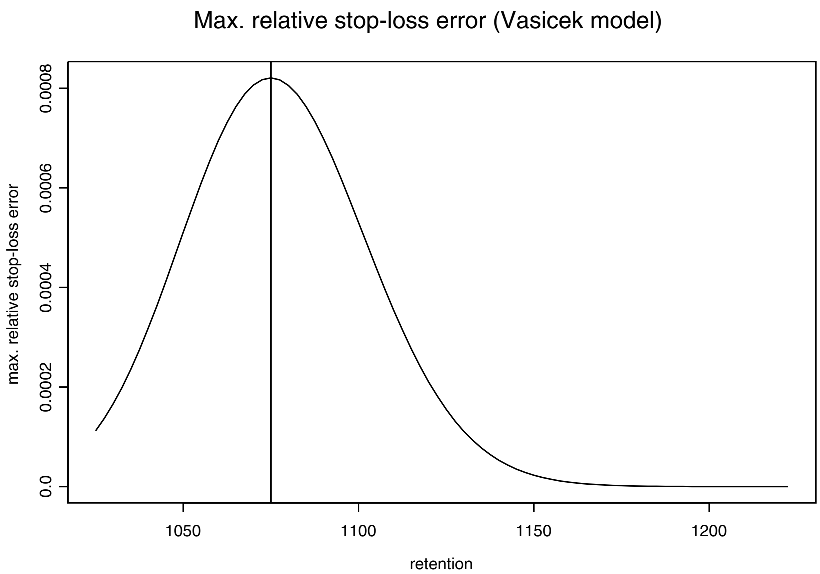



(b) Upper (□) & lower (○) quantiles vs. Monte Carlo quantiles

Figure 1: Distribution function and QQ-plot of the upper & lower bounds (Vasicek model), compared to Monte Carlo simulation.

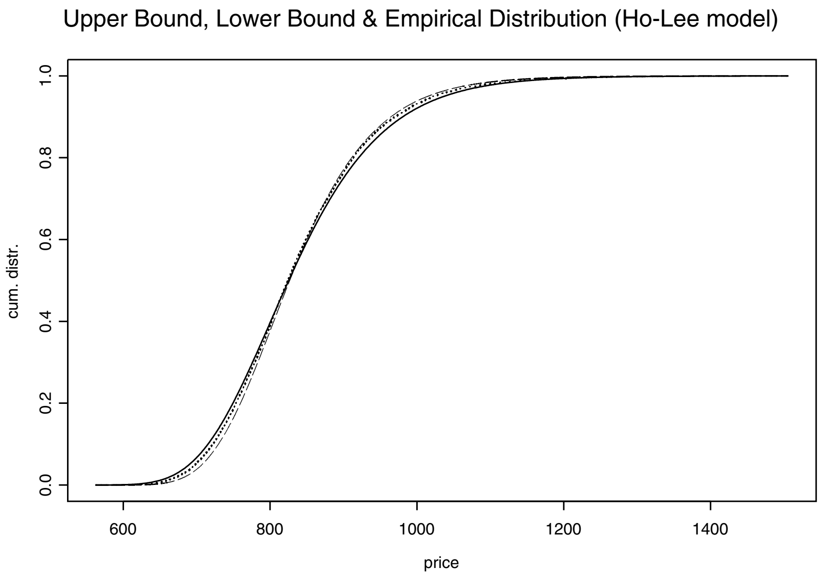



(a) Stop-loss premiums

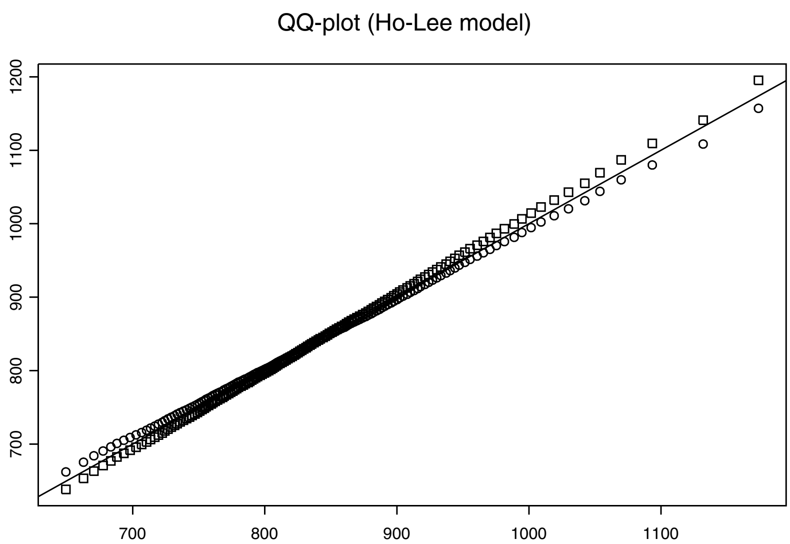



(b) Maximum relative stop-loss error

Figure 2: Stop-loss premiums for the upper & lower bounds and the corresponding maximum relative stop-loss error (Vasicek model)

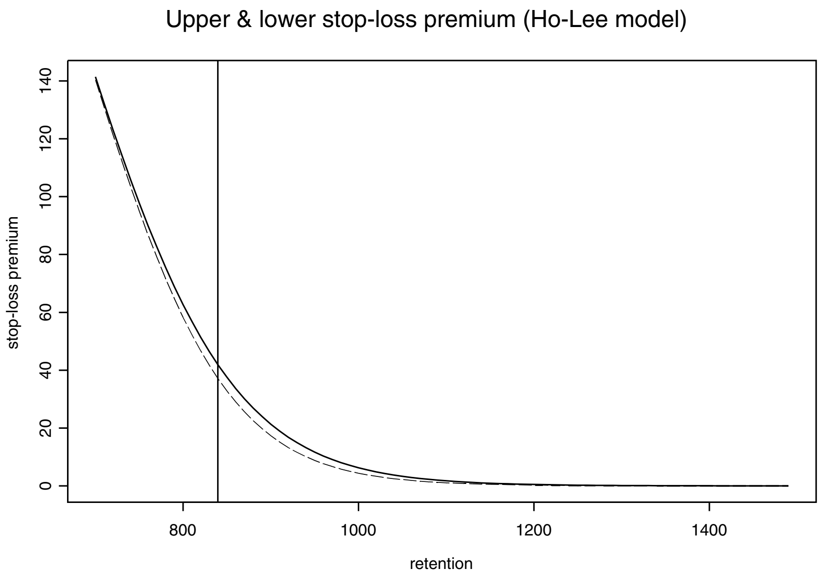



(a) Upper (—) & lower (- -) bound vs. Monte Carlo simulation (···)

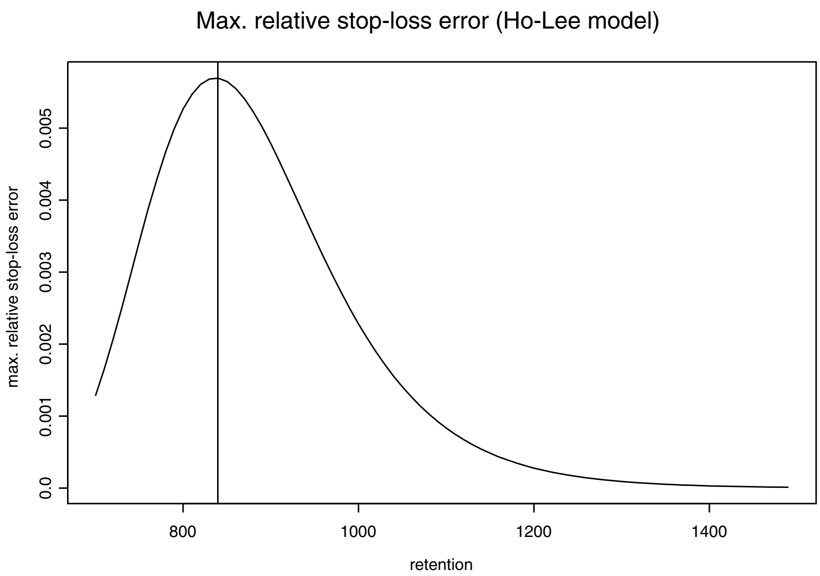



(b) Upper (□) & lower (○) quantiles vs. Monte Carlo quantiles

Figure 3: Distribution function and QQ-plot of the upper & lower bounds (Ho-Lee model), compared to Monte Carlo simulation

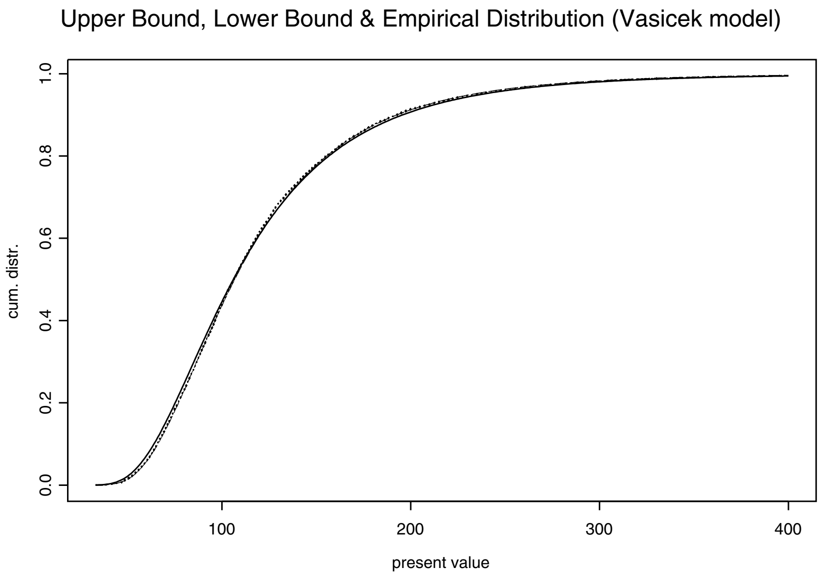



(a) Stop-loss premiums

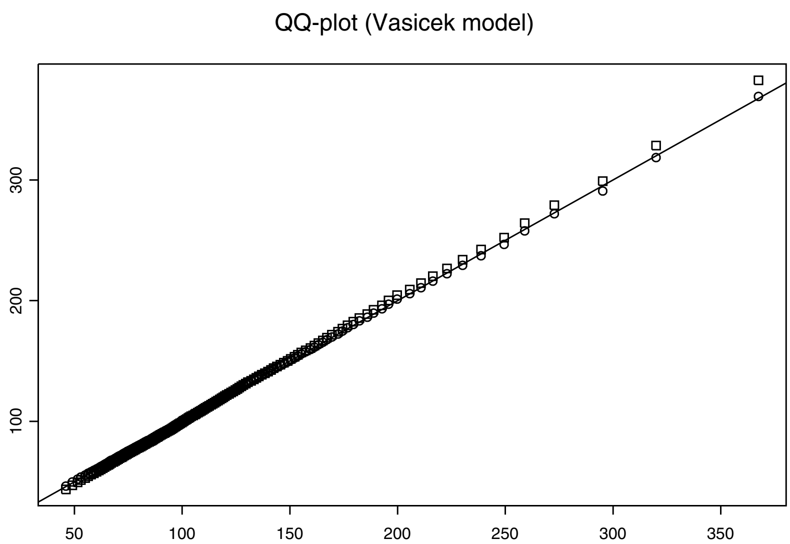



(b) Maximum relative stop-loss error

Figure 4: Stop-loss premiums for the upper & lower bounds and the corresponding maximum relative stop-loss error (Ho-Lee model)

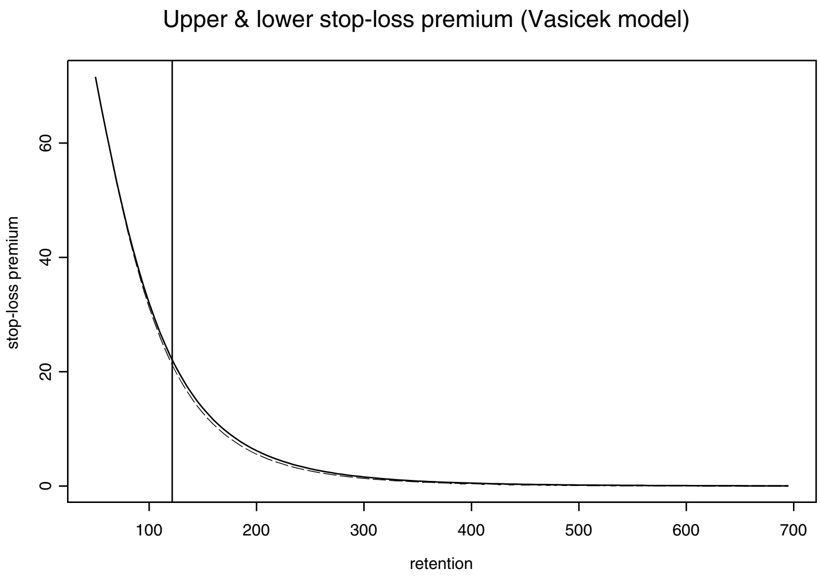



(a) Upper (—) & lower (- -) bound vs. Monte Carlo simulation (· · ·)

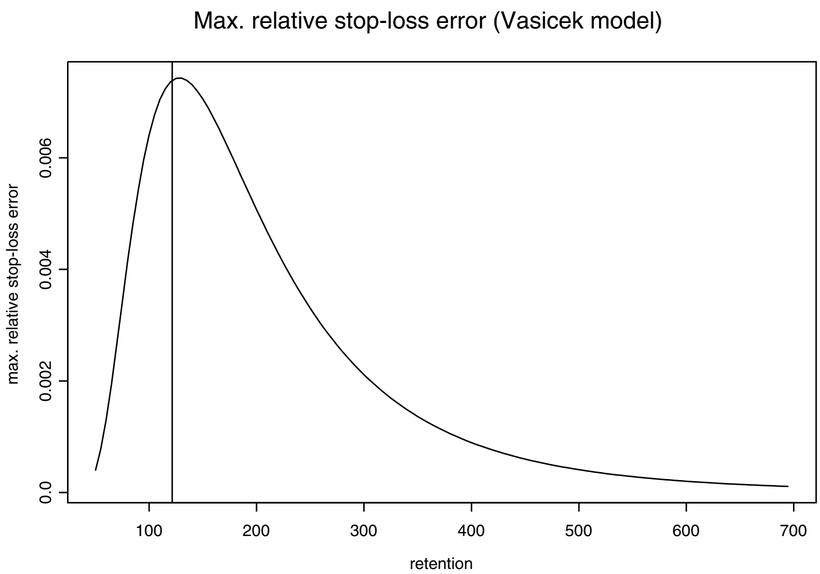



(b) Upper (□) & lower (○) quantiles vs. Monte Carlo quantiles

Figure 5: Distribution function and QQ-plot of the upper & lower bounds (Vasicek model), compared to Monte Carlo simulation




(a) Stop-loss premiums




(b) Maximum relative stop-loss error

Figure 6: Stop-loss premiums for the upper & lower bounds and the corresponding maximum relative stop-loss error (Vasicek model)