Risk and Savings Contracts

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Abstract

Following the "time-capital" approach of De Vylder (1997) it is shown that a fair life insurance contract can uniquely be separated into a fair savings and a fair pure risk contract. It is also shown that a fair life insurance contract can be separated into a fair associated stochastic savings contract and a fair associated pure risk contract.

1 Introduction

Essentially, life actuaries consider "(random) amounts that are payable at (random) times", e.g. a whole life insurance guarantees an amount of 1 payable at the moment of death of the insured. Following De Vylder (1997), we will call the couple which describes a capital and its time of payment a "time-capital". The single premium (also called the actuarial value, the expected present value or the price) of a lot of time-capitals have well-known notations, such as ${}_n\overline{A}_x$ for the continuous term life insurance. De Vylder (1997) introduces a notation for the time-capitals themselves, such as ${}_n\overline{A}_x^{\circ\circ}$. This enables us to write ${}_n\overline{A}_x^{\circ\circ}$ instead of "the insurance which pays an amount

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equal to 1 at the moment of death of the insured provided he dies before time n".

The paper is closely related to the book De Vylder (1997) since both are based on the concept of "time-capital". No knowledge of the book is required, since the concept of time-capital and the necessary related concepts are treated here without assuming any pre-knowledge. The context of the paper is however different in that we direct our attention on savings and risk contracts, which are only partly considered in the Vylder (1997). In this sense, this paper can be considered as a self-contained addendum to the book of De Vylder (1997).

In this paper we will consider insurances on a single life (x) of age x at policy issue. The remaining lifetime X of (x) is assumed to be non-negative and continuous. The time of issue of the policy is the time origin 0. As usual, we introduce the following notation for the distribution function of the remaining lifetime of the insured: ${}_tq_x \doteqdot \Pr[X \leq t]$ and ${}_tp_x \doteqdot \Pr[X > t]$; the symbol "$\doteqdot$" stands for "is defined as".

An amount c to be paid at time t is denoted by the couple (c, t). A deterministic time-capital is a set of couples (c_k, t_k), $(k = 1, ..., n)$. A stochastic time-capital is a set of couples $(c_k(X), t_k(X))$, $(k = 1, ..., n)$ where the $c_k(X)$ and the $t_k(X)$ may be functions of the remaining lifetime X.

A time-capital will often be denoted as a capital letter with the superscript "$^{\circ\circ}$", e.g.

$$\begin{aligned} Q^{\circ\circ} &= (c_1(X),\ t_1(X)) + (c_2(X),\ t_2(X)) + \cdots + (c_n(X),\ t_n(X)) \\ &= \sum_{k=1}^{n} (c_k(X),\ t_k(X)) \qquad (1) \end{aligned}$$

is the time-capital with random payments $c_k(X)$ at random times $t_k(X)$, $(k = 1, ..., n)$. Here the "+"-notation and the "$\sum$"-notation are suggestive. Summation and scalar multiplication of time-capitals are defined in the obvious way.

The present value of a time-capital is defined as the discounted value at the origin of time, i.e. at policy issue, of all future payments. Discounting will always be performed with a deterministic discount factor $v = \frac{1}{1+i} = \frac{1}{u}$. The present value of a time-capital $Q^{\circ\circ}$ will be denoted by Q°. If the time-capital is defined by (1), then we have

$$
\begin{aligned}
Q^{\circ} &= c_1(X)\ v^{t_1(X)} + c_2(X)\ v^{t_2(X)} + \cdots + c_n(X)\ v^{t_n(X)} \\
&= \sum_{k=1}^{n} c_k(X)\ v^{t_k(X)}. \qquad (2)
\end{aligned}
$$

The actuarial value (or the price) of a time-capital is defined as the expected value (with respect to X) of the present value of the time-capital. The actuarial value of a time-capital $Q^{\circ\circ}$ will be denoted by Q. Hence, $Q = E\left[Q^{\circ}\right]$. If the time-capital is defined by (1), then we have

$$
\begin{aligned}
Q &= E\left[c_1(X)\ v^{t_1(X)} + c_2(X)\ v^{t_2(X)} + \cdots + c_n(X)\ v^{t_n(X)}\right] \\
&= \sum_{k=1}^{n} E\left[c_k(X)\ v^{t_k(X)}\right]. \qquad (3)
\end{aligned}
$$

We denote by $\lfloor X \rfloor$ the integer part of X, and $\lceil X \rceil$ the smallest integer greater than or equal to X. The random variable $\hat{X}$ is defined by

$$
\hat{X} \doteqdot \frac{\lfloor X \rfloor + \lceil X \rceil}{2}.
$$

Relations between random variables are regarded as exact if they hold almost–surely. E.g., we will write $\lceil X \rceil = \lfloor X \rfloor + 1$, which means that this relation holds with probability one. Equality of time-capitals has to be interpreted as equality (with probability 1) of the corresponding random variables. E.g. the time-capitals $(c,\ \lceil X \rceil)$ and $(c,\ \lfloor X \rfloor + 1)$ are considered as equal.

In the sequel s and t will always be used to indicate non-negative real numbers; while j, k, m and n will be used to indicate non-negative integers. We also make the convention that $\sum_{i=m}^{n} a_i = 0$ if $m > n$.

Example 1 *Some simple time-capitals*

- n-year temporary annuity-due:

$$
\ddot{a}^{\circ\circ}_{\overline{n}|} \doteqdot \sum_{k=0}^{n-1} (1,\ k).
$$

- t-year pure endowment on a life aged x:

$$ {}_tE_x^{\circ\circ} \doteqdot \left(1_{X>t},\ t\right), $$

where for any event B, the indicator function 1_B equals 1 if B holds true, and 0 otherwise.

- n-year temporary life annuity-due on a life aged x:

$$ {}_n\ddot{a}_x^{\circ\circ} \doteqdot \sum_{k=0}^{n-1} {}_kE_x^{\circ\circ}. $$

- whole life insurance on a life aged x, payable at the moment of death:

$$ \overline{A}_x^{\circ\circ} \doteqdot \left(1,\ X\right). $$

- n-year term life insurance on a life aged x, payable at the moment of death:

$$ {}_n\overline{A}_x^{\circ\circ} \doteqdot \left(1_{X\leq n},\ X\right). $$

- n-year term life insurance, on a life aged x, payable at the end of the year of death:

$$ {}_nA_x^{\circ\circ} \doteqdot \left(1_{X\leq n},\ \lceil X \rceil\right). $$

- n-year term life insurance, on a life aged x, payable in the middle of the year of death:

$$ {}_n\hat{A}_x^{\circ\circ} \doteqdot \left(1_{X\leq n},\ \hat{X}\right). $$

In Section 2 restricted time-capitals are introduced, and present values and (conditional) expectations for those time-capitals are considered. An introduction to the general savings contracts treated in Section 5 is given in Section 3, where we introduce deterministic savings contracts. A straightforward definition of reserves for deterministic savings contracts is given. This will make it easier to comprehend the definitions of reserves for life insurance contracts. Life insurance contracts are defined in Section 4; this section is mainly based on results by De Vylder. In Section 5 we define savings contracts on a life (x) and their associated deterministic savings contract.

(Actuarial) fairness of those contracts is also considered in Section 5. Pure risk contracts are defined in Section 6. The combination of two life insurance contracts is studied in Section 7. In Subsection 7.1 a general life insurance contract is uniquely separated into a fair savings and a fair pure risk contract. The following subsection deals with several types of fair associated savings contracts and a fair associated risk contract. It is shown that a general life policy can be separated into a fair associated savings contract and a fair associated risk contract.

It is important to note that most results presented in this paper are not new. The originality lies in the fact that they are presented in the context of time-capitals and that an integrated approach is given for several types of contracts (deterministic savings contracts and life insurance contracts). The *unicity* of the separation of a fair life insurance contract into a fair savings contract and a fair pure risk contract seems not to have appeared in the literature before. The approach of associated savings and risk contracts is not considered elsewhere in the way presented in the current paper. Furthermore the relationship between deterministic and stochastic savings contracts is made mathematically more clear.

2 Restricted time-capitals

In this section, $Q^{\circ\circ}$ is a general time-capital on a life aged x. It is a linear combination of deterministic time-capitals, pure endowments, life annuities and life insurances on a life aged x. We will say that $Q^{\circ\circ}$ is a time-capital on (x). Note that a notation $Q_x^{\circ\circ}$ would lead to ambiguity in the formulas, see e.g. the left hand side of (9).
The restriction of $Q^{\circ\circ}$ to $[s, s+t[$ is defined as the time-capital obtained from $Q^{\circ\circ}$, by setting all payments outside the interval $[s, s+t[$ equal to 0. The restriction of $Q^{\circ\circ}$ to $[s, s+t[$ is denoted by ${}_{s|t}Q^{\circ\circ}$.

The re-actualized restriction of $Q^{\circ\circ}$ to $[s, s+t[$ is defined as the restriction of $Q^{\circ\circ}$ to $[s, s+t[$, where s is the new origin of time. Hence, all present values are evaluated at time s instead of time 0. The re-actualized restriction of $Q^{\circ\circ}$ to $[s, s+t[$ is denoted by ${}_{\bullet s|t}Q^{\circ\circ}$. The dot indicates that s is the time origin. If their is no explicit indication of the origin of time, then time 0 is the time origin.

At time-capital level, there is no difference between the restriction and the re-actualized restriction of a time-capital:

$$ {}_{s|t}Q^{\circ\circ} = {}_{\bullet s|t}Q^{\circ\circ}. $$

At present value level, the following relation holds:

$$ {}_{s|t}Q^{\circ} = v^{s} \; {}_{\bullet s|t}Q^{\circ}. $$

In the sequel we will often denote ${}_{s|\infty}Q^{\circ\circ}$ as ${}_{s|}Q^{\circ\circ}$, and ${}_{\bullet 0|t}Q^{\circ\circ}$ and ${}_{0|t}Q^{\circ\circ}$ as ${}_{|t}Q^{\circ\circ}$ or ${}_{t}Q^{\circ\circ}$.

The present value of the time-capital ${}_{\bullet t|}Q^{\circ\circ}$ is denoted as ${}_{\bullet t|}Q^{\circ}$. The expectation (with respect to X) of the present value of all payments in $[t, \infty[$, discounted at time t is denoted by ${}_{\bullet t|}Q$. Hence,

$$ {}_{\bullet t|}Q \doteqdot E[{}_{\bullet t|}Q^{\circ}], \tag{4} $$

that is the expected present value at time t of all payments of $Q^{\circ\circ}$ in $[t, \infty[$ if we do not have information about the status of (x) at t (life or death). This expectation is evaluated with the information available at time 0, i.e. the distribution function of X, which was determined at time 0.

Theorem 1 *For any time-capital $Q^{\circ\circ}$, the following expressions hold:*
(a) Decomposition formula:

$$ Q^{\circ} = {}_{0|t}Q^{\circ} + v^{t} \; {}_{\bullet t|}Q^{\circ}, \qquad (t > 0), \tag{5} $$

(b) Generalized decomposition formula:

$$ {}_{\bullet s|}Q^{\circ} = {}_{\bullet s|t-s}Q^{\circ} + v^{t-s} \; {}_{\bullet t|}Q^{\circ}, \quad (0 \le s < t), \tag{6} $$

(c) Fouret's formula (at present value level):

$$ {}_{\bullet k+1|}Q^{\circ} = \left({}_{\bullet k|}Q^{\circ} - {}_{\bullet k|1}Q^{\circ} \right) u. \tag{7} $$

Proof.

(a) At time-capital level we have that $Q^{\circ\circ} = {}_{0|t}Q^{\circ\circ} + {}_{\bullet t|}Q^{\circ\circ}$, from which we find (5).

(b) At time-capital level, we immediately find ${}_{\bullet s|}Q^{\circ\circ} = {}_{\bullet s|t-s}Q^{\circ\circ} + {}_{\bullet t|}Q^{\circ\circ}$, which leads to (6).

(c) Taking $s = k$ and $t = k+1$ in (6) leads to (7). ■

Of course (5), (6) and (7) also hold at expected value level (i.e. without the "°" symbol).

Now assume that at time t, the following information is available concerning (x): "(x) is alive at time t" or "(x) died at time s, for a given

$s < t$". In terms of the remaining lifetime X, this information can be expressed as "$X > t$" or "$X = s < t$". From the Law of Total Probability, we derive the following expression for ${}_{\bullet t|}Q$:

$$ {}_{\bullet t|}Q = \int_0^t E[{}_{\bullet t|}Q^\circ | X = s] \; d_s q_x + E[{}_{\bullet t|}Q^\circ | X > t] \; {}_t p_x. \tag{8} $$

The conditional expectation of ${}_{\bullet t|}Q^\circ$, given that (x) is alive at t, is denoted by ${}_{\bullet t|}Q_x$:

$$ {}_{\bullet t|}Q_x \doteq E\left[{}_{\bullet t|}Q^\circ | X > t \right]. \tag{9} $$

${}_{\bullet t|}Q_x$ is the expected present value at t of all payments of $Q^{\circ\circ}$ in $[t, \infty[$, taking into account the information that (x) is alive at t. The conditional expectation of ${}_{\bullet t|}Q^\circ$, given that (x) died at time s, before time t, is denoted by ${}_{\bullet t|}Q_{x|}(s)$:

$$ {}_{\bullet t|}Q_{x|}(s) \doteq E\left[{}_{\bullet t|}Q^\circ | X = s \right]. \tag{10} $$

${}_{\bullet t|}Q_{x|}(s)$ is the expected present value at t of all payments of $Q^{\circ\circ}$ in $[t, \infty[$, taking into account that (x) died at time s, for some non-negative real number s smaller than t.

Relation (8) can be rewritten as:

$$ {}_{\bullet t|}Q = \int_0^t {}_{\bullet t|}Q_{x|}(s) \; d_s q_x + {}_{\bullet t|}Q_x \; {}_t p_x. \tag{11} $$

Note that if $Q^{\circ\circ}$ is a deterministic time-capital (i.e. it does not depend on the remaining lifetime X), then one has

$$ {}_{\bullet t|}Q^\circ = {}_{\bullet t|}Q = {}_{\bullet t|}Q_x = {}_{\bullet t|}Q_{x|}(s). \tag{12} $$

For a time-capital $Q^{\circ\circ}$ on a life (x) and $s < t$, we have that

$$ {}_{\bullet t|}Q^\circ_{x|}(s) = {}_{\bullet t|}Q_{x|}(s). \tag{13} $$

Expression (13) follows from the fact that all payments of $Q^{\circ\circ}$ in $[t, \infty[$, given that (x) died at $s < t$, are deterministic.

3 Deterministic savings contracts

In this section we will consider deterministic savings contracts, i.e. contracts between two parties, where the payments and the times-of- payment of both parties are deterministic and fixed at contract issue.

Definition 1 *A deterministic savings contract is a couple of two deterministic time-capitals* $(C^{\circ\circ}, P^{\circ\circ})$*, where* $C^{\circ\circ}$ *is the time-capital describing the commitments of the bank (savings institution) and* $P^{\circ\circ}$ *is the time-capital describing the commitments of the client (saver).*

Examples of deterministic saving contracts are $[(1,n)\,,\ (P,0)]$ and $\left[{}_{m|}a^{\circ\circ}_{\overline{n|}},\ p\ \ddot{a}^{\circ\circ}_{\overline{m|}}\right]$. Of course, we have $p\ \ddot{a}^{\circ\circ}_{\overline{m|}} = \sum_{k=0}^{m-1}(p,k)$.

The *reserve time-capital* of a deterministic savings contract is defined by

$$V^{\circ\circ} = C^{\circ\circ} - P^{\circ\circ}. \tag{14}$$

The reserve time-capital describes the global payments of the savings institution related to the contract $(C^{\circ\circ}, P^{\circ\circ})$, where payments of the client are considered as negative payments of the savings institution. From the viewpoint of the client, it describes his income (premium payments are regarded as negative income) related to the savings contract.

The reserve at time t of a deterministic savings contract $(C^{\circ\circ}, P^{\circ\circ})$ is defined as the present value, evaluated at time t, of all payments of the time-capital $V^{\circ\circ}$ in the interval $[t, \infty[$. Hence, the reserve at time t is given by ${}_{\bullet t|}V^{\circ}$ or, equivalently, by ${}_{\bullet t|}V$.

Theorem 2 *Consider a deterministic savings contract* $(C^{\circ\circ}, P^{\circ\circ})$*, then we have:*

(a) Prospective expression of the reserves:

$${}_{\bullet t|}V = {}_{\bullet t|}C - {}_{\bullet t|}P, \qquad (t \geq 0), \tag{15}$$

(b) Retrospective expression of the reserves:

$${}_{\bullet t|}V = (V + {}_{0|t}P - {}_{0|t}C)\ u^{t}, \qquad (t \geq 0), \tag{16}$$

(c) Fouret's formula:

$${}_{\bullet k+1|}V = ({}_{\bullet k|}V + {}_{\bullet k|1}P - {}_{\bullet k|1}C)\ u, \qquad (k = 0, 1, ...). \tag{17}$$

Proof.

(a) Follows from the definition of $V^{\circ\circ}$.

(b) From decomposition formula (5) we find

$$\begin{aligned} V^{\circ} &= {}_{0|t}V^{\circ} + v^t \; {}_{\bullet t|}V^{\circ} \\ &= ({}_{0|t}C^{\circ} - {}_{0|t}P^{\circ}) + v^t \; {}_{\bullet t|}V^{\circ} \end{aligned}$$

from which we find (16).

(c) From Fouret's formula (7), we find

$${}_{\bullet k+1|}V = ({}_{\bullet k|}V - {}_{\bullet k|1}V) \; u, \qquad (k = 0, 1, ...)$$

which leads to (17). ■

In the sequel, we will say that a deterministic savings contract is fair if and only if $V = 0$. Equivalently, a deterministic savings contract is fair if and only if $C = P$.

Example 2 *A deterministic savings contract*

Consider $(C^{\circ\circ}, P^{\circ\circ})$ with $C^{\circ\circ} = \sum_{k=0}^{n}(L_k, k)$ and $P^{\circ\circ} = \sum_{k=0}^{n}(P_k, k)$. Then we have according to Theorem 4

$$\begin{aligned} {}_{\bullet k|}V &= \sum_{j=k}^{n} (L_j - P_j) \; v^{j-k}, \qquad (k = 0, 1, ..., n), && (18) \\ {}_{\bullet k|}V &= V \; u^k + \sum_{j=0}^{k-1} (P_j - L_j) \; u^{k-j}, \qquad (k = 0, 1, ..., n), && (19) \\ {}_{\bullet k+1|}V &= ({}_{\bullet k|}V + P_k - L_k) \; u, \qquad (k = 0, 1, ..., n-1). && (20) \end{aligned}$$

Assume for a moment that $V > 0$, such that the contract is not fair (but advantageous for the client). In this case, in addition to accumulating the yearly residual income $(P_j - L_j)$, the savings institution will have to provide an additional amount V in order to be able to meet its future obligations related to the savings contract. This explains the term $V \; u^k$ in the retrospective expression of the reserves. If the liability of the savings institution at time k is determined by ${}_{\bullet k|}V$, then V must be funded at contract issue.

A deterministic savings contract will be designed such that the reserve will never become negative, implying that the savings institution has a liability vis-a-vis the saver. It is clear that the choice of a low interest rate is a safe strategy, from the viewpoint of the savings institution.

In practice, most savings policies are not savings contracts as defined here, where the obligations of both parties are fixed at contract issue. Instead, most savings policies are what one could call "*flexible savings products*". This means that they are directly based on Fouret's formula (20), where the savings amount P_k and the amount of withdrawal L_k can be chosen freely by the saver at time k. The only restriction is that the reserve must remain non-negative, i.e. $L_k \leq {}_{\bullet k|}V + P_k$. Of course, for flexible savings products, reserves can only be determined retrospectively.

A *loan* can be defined as a deterministic savings contract with a negative reserve at any instant. In this case $- {}_{\bullet k|}V$ is called the remaining debt. A safe strategy for the bank institution is the choice of a sufficiently high interest rate. As an example, consider an annuity-loan, which is a contract $\left[(1,0)\,,\ P\ a^{\circ\circ}_{\overline{n|}}\right]$.

4 Life insurance contracts

In this section, we consider life insurance contracts on a single life (x), with remaining lifetime X. We consider classical life insurance contracts, where the obligations of both parties are fixed at policy issue. The insurer determines the distribution function of X at policy issue. In order to determine his liabilities at time t, the insurer will use the originally chosen life table in addition to the information available concerning (x) at that time. We assume that the information at time t is the status of (x), i.e. life or death at that time, and in the case of death, the moment of death. This implies that at any time t, when the insured is still alive, the survival probabilities will be determined from the distribution function of X, conditional on the available information.

Definition 2 *A life insurance contract on a life (x) is a couple $(C^{\circ\circ}, P^{\circ\circ})$ of time-capitals $C^{\circ\circ}$ and $P^{\circ\circ}$ depending on X, where $C^{\circ\circ}$ is the time-capital describing the commitment of the insurer and $P^{\circ\circ}$ is the time-capital describing the commitment of the insured.*

We will assume that ${}_{\bullet t|}P_{x|}(s) = 0$ for all $0 < s < t$. This means that the premium payments stop at the death of (x), as is the case in practice.

For reasons of simplicity, in the remainder of this paper we will always assume that the benefits at life and the premiums are only payable at integer time points. Taking one year as the time unit, this means that these payments are only due at the end or the beginning of the year. The results hereafter can easily be generalized to the case of more payments than once a year, e.g. by changing the time unit to one month (in case the payments are due monthly).
We will also assume that the insurance period is $[0, n[$. This means that there are no premiums and no benefits payable after time n. The survival benefit and the premium payable at time k, $(k = 0, 1, \cdots, n)$ will be denoted by L_k and P_k respectively. We will assume that $L_0 = P_n = 0$.

The *reserve time-capital* $V^{\circ\circ}$ of a life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ is defined by $V^{\circ\circ} = C^{\circ\circ} - P^{\circ\circ}$. The reserve time-capital describes the global payments of the insurer related to the contract $(C^{\circ\circ}, P^{\circ\circ})$, where payments of the insured are considered as negative payments of the insurer. From the viewpoint of the insured, it describes his future random income (premium payments are regarded as negative income) related to the life insurance contract.

Lemma 3 *Consider a life insurance contract* $(C^{\circ\circ},\ P^{\circ\circ})$*, then we have (at present value level):*

$$\begin{aligned}
{}_{\bullet t|}V^{\circ} &= {}_{\bullet t|}C^{\circ} - {}_{\bullet t|}P^{\circ}, \qquad (t \geq 0), && (21)\\
{}_{\bullet t|}V^{\circ} &= \left(V^{\circ} + {}_{0|t}P^{\circ} - {}_{0|t}C^{\circ}\right)\ u^{t}, \qquad (t \geq 0), && (22)\\
{}_{\bullet k+1|}V^{\circ} &= \left({}_{\bullet k|}V^{\circ} + {}_{\bullet k|1}P^{\circ} - {}_{\bullet k|1}C^{\circ}\right)\ u, \qquad (k = 0, 1, ..., n-1). && (23)
\end{aligned}$$

Proof. The proof is identical to the proof of Theorem 4. ■

¿From (11) we find:

$$ {}_{\bullet t|}V = \int_0^t {}_{\bullet t|}V_{x|}(s)\ d_s q_x + {}_{\bullet t|}V_x\ {}_t p_x, \qquad (t \geq 0). \qquad (24)$$

The reserve at time t of the life insurance contract $(C^{\circ\circ},\ P^{\circ\circ})$ is hereby defined as the conditional expectation of ${}_{\bullet t|}V^{\circ}$, given the information concerning (x) available at that time. When (x) is alive at time t, the reserve is given by ${}_{\bullet t|}V_x$ which is called the reserve at t when (x) is alive at that time. When (x) died at time $s < t$, the reserve is given by ${}_{\bullet t|}V_{x|}(s)$, which is called the reserve at t when (x) died at $s < t$.

We will say that *a contract vanishes with* (x) if ${}_{\bullet t|}V_{x|}(s) = 0$ for all $0 < s < t$. Earlier we assumed that ${}_{\bullet t|}P_{x|}(s) = 0$ for all $0 < s < t$. This implies that a contract vanishes with (x) if ${}_{\bullet t|}C_{x|}(s) = 0$ for all $0 < s < t$. This will be the case if the life insurance components (= the payments at death) of the life insurance contract are all payable immediately at death.

Theorem 4 (De Vylder (1997)) *Let* $(C^{\circ\circ}, P^{\circ\circ})$ *be a life insurance contract on a life* (x)*, then*
(a) Prospective expression of reserves:

$$
\begin{aligned}
{}_{\bullet t|}V_x &= {}_{\bullet t|}C_x - {}_{\bullet t|}P_x, \qquad (t \geq 0), &(25)\\
{}_{\bullet t|}V_{x|} &= {}_{\bullet t|}C_{x|} - {}_{\bullet t|}P_{x|}, \qquad (t > 0), &(26)
\end{aligned}
$$

(b) Retrospective expression of reserves:

$$ {}_{\bullet t|}V_x = \left(V + {}_{0|t}P - {}_{0|t}C\right)\ {}_tE_x^{-1}, \qquad (t \geq 0), \qquad (27) $$

(c) Fouret's formula:

$$ {}_{\bullet k+1|}V_x = \left({}_{\bullet k|}V_x + {}_{\bullet k|1}P_x - {}_{\bullet k|1}C_x\right)\ {}_1E_{x+k}^{-1}, \qquad (k = 0, 1, ..., n-1). \qquad (28) $$

For (b) and (c), we assumed that the contract vanishes with (x)*.*

Proof.
(a) From (9) and (21) we find

$$ {}_{\bullet t|}V_x = E[{}_{\bullet t|}V^{\circ}|X > t] = E[{}_{\bullet t|}C^{\circ}|X > t] - E[{}_{\bullet t|}P^{\circ}|X > t] = {}_{\bullet t|}C_x - {}_{\bullet |t}P_x, $$

which is (25). The proof of (26) is similar.
(b) From (22) and (24) we have

$$
\begin{aligned}
{}_{\bullet t|}V &= \int_0^t {}_{\bullet t|}V_{x|}(s)\ d_sq_x + {}_{\bullet t|}V_x\ {}_tp_x\\
&= {}_{\bullet t|}V_x\ {}_tp_x = \left(V + {}_{0|t}P - {}_{0|t}C\right)\ u^t,
\end{aligned}
$$

which implies

$$ {}_{\bullet t|}V_x = (V + {}_{0|t}P - {}_{0|t}C)\ {}_tE_x^{-1}, $$

since also ${}_{\bullet 0|}V = V = {}_{\bullet 0|}V_x$.
(c) From (23) we find

$$ {}_{\bullet k+1|}V = \left({}_{\bullet k|}V + {}_{\bullet k|1}P - {}_{\bullet k|1}C\right)\ u. $$

As the contract vanishes with (x), we can rewrite this equation as

$$_{\bullet k+1|}V_x \ _{k+1}p_x = \left(_{\bullet k|}V_x \ _kp_x + \ _{\bullet k|1}P_x \ _kp_x - \ _{\bullet k|1}C_x \ _kp_x \right) \ u,$$

which is (28). ■

As we assumed that all payments stop at time n, we find that $_{\bullet n|}V_x$ is equal to the benefit the insurer has to pay if (x) is still alive at that time.

Formulas (27) and (28) are also valid for $t = 0, 1, ...$ and $k = 0, 1, ...$ respectively for contracts that do not vanish with x, but for which $_{\bullet k|1}C_{x|}(s) = 0$ $(k = 0, 1, .., n-1$ and $s < k)$. Contracts that fulfill these conditions are said to "vanish before the end of the year-of-death". This will be the case e.g. if the life insurance components (= the payments at death) are due in the middle of the year of death.

We will say that a life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ on a life (x) is fair if and only if $V = 0$, or equivalently, $C = P$.

Remark that a deterministic savings contract can be considered as a life insurance contract on a status which exists eternally. Such a contract is a fair life insurance contract if and only if it is a fair deterministic savings contract.

5 Savings contracts (on a life (x))

For each life insurance contract $(C^{\circ\circ}, \ P^{\circ\circ})$ on a life (x) we define the *associated deterministic savings contract* $(C^{(ads)\circ\circ}, \ P^{(ads)\circ\circ})$ by replacing the distribution function of X (= the remaining lifetime of (x)) by the distribution function of $X^{(ads)}$ which is the remaining lifetime of a status that will exist eternally. Hence,

$$\Pr \left[X^{(ads)} = \infty \right] = 1. \tag{29}$$

The associated deterministic savings contract is indeed a deterministic savings contract as defined in Section 3.

As an example, consider the life insurance contract

$$(C^{\circ\circ}, \ P^{\circ\circ}) = (_nE_x^{\circ\circ}, \ \pi \ \ddot{a}_{x:\overline{n}|}^{\circ\circ}).$$

The associated deterministic savings contract is given by

$$(C^{(ads)\circ\circ}, \ P^{(ads)\circ\circ}) = \left((1, n), \ \pi \ \ddot{a}_{\overline{n}|}^{\circ\circ}) \right).$$

The associated deterministic savings contract of a fair life insurance contract is in general not a fair savings contract. This will only be the case if, in addition to $C = P$, also the condition $C^{(ads)} = P^{(ads)}$ holds.

Definition 3 *A savings contract (on a life (x)) is a life insurance contract $(C^{\circ\circ},\ P^{\circ\circ})$ on (x) such that the reserves at life ${}_{\bullet k|}V_x$ $(k = 0, 1, ..., n)$ are equal to the corresponding reserves ${}_{\bullet k|}V^{(ads)}$ of the associated deterministic savings contract $(C^{(ads)\circ\circ},\ P^{(ads)\circ\circ})$, i.e.*

$$ {}_{\bullet k|}V_x = {}_{\bullet k|}V^{(ads)}, \qquad (k = 0, 1, ..., n). $$

For any savings contract with given time-capitals $C^{\circ\circ}$ and $P^{\circ\circ}$, the reserves at life can be determined without knowledge of the survival probabilities of (x). Further, note that each deterministic savings contract (as defined in Section 3) is a savings contract.

As introduced above, let P_k and L_k be the premium and life benefit payable at time k $(k = 0, 1, ..., n)$ of the life insurance contract $(C^{\circ\circ},\ P^{\circ\circ})$ in case (x) is alive at that time. Then we have

$$ C^{\circ\circ} = \sum_{j=0}^{n} L_j \ {}_jE_x^{\circ\circ} + benefits\ at\ death \tag{30} $$

and

$$ P^{\circ\circ} = \sum_{j=0}^{n} P_j \ {}_jE_x^{\circ\circ} = \sum_{j=0}^{n} (P_j \ 1_{X>j}, j). \tag{31} $$

Since the insurance period is $[0, n[$ there are no death benefits after time n. We immediately find that the associated deterministic savings contract is given by $(C^{(ads)\circ\circ},\ P^{(ads)\circ\circ})$ with

$$ C^{(ads)\circ\circ} = \sum_{j=0}^{n} (L_j,\ j). \tag{32} $$

and

$$ P^{(ads)\circ\circ} = \sum_{j=0}^{n} (P_j,\ j). \tag{33} $$

As a special case of Theorem 4, we get from (18), (19) and (20)

$$
\begin{aligned}
{}_{\bullet k|}V^{(ads)} &= \sum_{j=k}^{n} (L_j - P_j)\ v^{j-k}, \qquad (k = 0, 1, ..., n), && (34)\\
{}_{\bullet k|}V^{(ads)} &= V^{(ads)}\ u^k + \sum_{j=0}^{k-1} (P_j - L_j)\ u^{k-j}, \qquad (k = 0, 1, ..., n), && (35)\\
{}_{\bullet k+1|}V^{(ads)} &= ({}_{\bullet k|}V^{(ads)} + P_k - L_k)\ u, \qquad (k = 0, 1, ..., n-1) && (36)
\end{aligned}
$$

with $V^{(ads)}$ given by

$$V^{(ads)} = \sum_{j=0}^{n} (L_j - P_j)\ v^j. \qquad (37)$$

The associated deterministic savings contract $(C^{(ads)\circ\circ},\ P^{(ads)\circ\circ})$ is fair if and only if $V^{(ads)} = 0$, or equivalently

$$\sum_{j=0}^{n} (L_j - P_j)\ \ v^j = 0. \qquad (38)$$

A savings contract on a life (x) is fair if and only if $V = 0$. As $V = V^{(ads)} = \sum_{j=0}^{n} (L_j - P_j)\ v^j$, we find that a savings contract is fair if and only if (38) holds.

Theorem 5 *Consider a life insurance contract on a life (x). This contract is a fair savings contract if and only if*

$$ {}_{\bullet k|}V_x = \sum_{j=0}^{k-1} (P_j - L_j)\ u^{k-j}, \qquad (k = 0, 1, ..., n). \qquad (39)$$

Proof.
(a) Assume that $(C^{\circ\circ},\ P^{\circ\circ})$ is a fair savings contract. Then $V^{(ads)} = V = 0$. Hence, (39) follows from (35).
(b) Now assume that $(C^{\circ\circ},\ P^{\circ\circ})$ is a life insurance contract for which (39) holds. We immediately find that $V = {}_{\bullet 0|}V_x = 0$, which means that the contract is fair. By (30), (31) and (39) for $k = n$, we find

$$ {}_{\bullet n|}V_x = L_n - P_n = \sum_{j=0}^{n-1} (P_j - L_j)\ u^{n-j}.$$

This implies

$$\sum_{j=0}^{n} (L_j - P_j)\ v^j = 0,$$

which means that $V^{(ads)} = 0$.
¿From (35) and (39) we then find that ${}_{\bullet k|}V^{ads} = {}_{\bullet k|}V_x, \quad (k = 0, 1, ..., n)$. ■

In the following theorem, we consider life insurance contracts defined by (30) and (31) with as additional requirement that the contracts vanish before the end of the year of death.

Theorem 6 *Consider a life insurance contract on a life* (x)*, which vanishes before the end of the year of death. This contract is a fair savings contract if and only if the following conditions hold:*

$$\sum_{j=0}^{n} P_j\ v^j = \sum_{j=0}^{n} L_j\ v^j \tag{40}$$

and

$${}_{\bullet k|1}C_x = L_k + \left(\sum_{j=0}^{k} (P_j - L_j)\ u^{k+1-j} \right)\ {}_1A_{x+k}, \qquad (k = 0, 1, ..., n-1). \tag{41}$$

Proof.
(a) First assume that $(C^{\circ\circ},\ P^{\circ\circ})$ is a fair savings contract which vanishes at the end of the year of death. Then, (40) must hold and by (28) we find

$$\begin{aligned} {}_{\bullet k+1|}V^{(ads)} &= {}_{\bullet k+1|}V_x = \left({}_{\bullet k|}V_x + {}_{\bullet k|1}P_x - {}_{\bullet k|1}C_x \right)\ {}_1E_{x+k}^{-1} \\ &= \left({}_{\bullet k|}V^{(ads)} + P_k - {}_{\bullet k|1}C_x \right)\ {}_1E_{x+k}^{-1}, \qquad (k = 0, 1, ..., n-1). \end{aligned}$$

Using (36), this leads to

$$\begin{aligned} {}_{\bullet k|1}C_x &= {}_{\bullet k|}V^{(ads)} + P_k - {}_{\bullet k+1|}V^{(ads)}\ {}_1E_{x+k} \\ &= {}_{\bullet k|}V^{(ads)} + P_k - {}_{\bullet k+1|}V^{(ads)}\ v\ (1 - q_{x+k}) \\ &= \left({}_{\bullet k|}V^{(ads)} + P_k - v\ {}_{\bullet k+1|}V^{(ads)} \right) + {}_{\bullet k+1|}V_x\ {}_1A_{x+k} \\ &= L_k + {}_{\bullet k+1|}V_x\ {}_1A_{x+k}, \qquad (k = 0, 1, ..., n-1), \end{aligned}$$

which reduces to (41), by (39).
(b) Now assume that $(C^{\circ\circ},\ P^{\circ\circ})$ is a life insurance contract which vanishes

at the end of the year of death, and for which (40) and (41) hold.
We will prove by induction that (39) holds for $k = 0, 1, ..., n$.
First, (39) holds for $k = 0$, as $(C^{\circ\circ},\ P^{\circ\circ})$ is a fair contract. Indeed,

$$\begin{aligned}
V &= C - P \\
&= \sum_{k=0}^{n} (L_k - P_k)\ {}_kE_x + \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} (P_j - L_j)\ u^{k+1-j} \right)\ {}_{k|1}A_x \\
&= \sum_{k=0}^{n} (L_k - P_k)\ {}_kE_x + \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} (P_j - L_j)\ v^{j} \right)\ ({}_kp_x -\ {}_{k+1}p_x) \\
&= \sum_{k=0}^{n} (L_k - P_k)\ {}_kE_x + \sum_{j=0}^{n-1} (P_j - L_j)\ v^{j} \sum_{k=j}^{n-1} ({}_kp_x -\ {}_{k+1}p_x) \\
&= \sum_{k=0}^{n} (L_k - P_k)\ {}_kE_x + \sum_{j=0}^{n-1} (P_j - L_j)\ v^{j} ({}_jp_x -\ {}_np_x) \\
&= \ {}_np_x \sum_{k=0}^{n} (L_k - P_k)\ v^{k} = 0.
\end{aligned}$$

Now, assume that (39) holds for a particular l in $\{0, 1, ..., n-1\}$.
Then we have by (28) that

$$\begin{aligned}
{}_{\bullet l+1|}V_x &= \left({}_{\bullet l|}V_x +\ P_l -\ {}_{\bullet l|1}C_x \right)\ {}_1E_{x+l}^{-1} \\
&= \left[\sum_{j=0}^{l-1} (P_j - L_j)\ u^{l-j} +\ P_l -\ L_l \right. \\
&\quad \left. - \left(\sum_{j=0}^{l} (P_j - L_j)\ u^{l+1-j} \right)\ {}_1A_{x+l} \right]\ {}_1E_{x+l}^{-1} \\
&= (v -\ {}_1A_{x+l}) \sum_{j=0}^{l} (P_j - L_j)\ u^{l+1-j}\ {}_1E_{x+l}^{-1} \\
&= \sum_{j=0}^{l} (P_j - L_j)\ u^{l+1-j},
\end{aligned}$$

which ends the proof. ■

Example 3 *A fair savings contract*

Consider the life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ with

$$C^{\circ\circ} = \sum_{k=0}^{n} L_k \ {}_kE_x^{\circ\circ} + \sum_{k=0}^{n-1} D_k \ {}_{k|1}\hat{A}_x^{\circ\circ}$$

and

$$P^{\circ\circ} = \sum_{k=0}^{n-1} P_k \ {}_kE_x^{\circ\circ}.$$

For ease of notation, we introduce $L_0 = 0$. Clearly, this contract vanishes before the end of the year of death. We have that

$$_{\bullet k|1}C_x = L_k + D_k \ {}_1\hat{A}_{x+k}, \qquad (k = 0, 1, ..., n-1).$$

Hence, the condition (41) reduces to

$$\sum_{j=0}^{k} (P_j - L_j) \ u^{k+1-j} \ {}_1A_{x+k} = D_k \ {}_1\hat{A}_{x+k}, \qquad (k = 0, 1, ..., n-1),$$

or equivalently,

$$D_k = v^{\frac{1}{2}} \sum_{j=0}^{k} (P_j - L_j) \ u^{k+1-j}, \qquad (k = 0, 1, ..., n-1). \tag{42}$$

6 Pure risk contracts on a life (x)

In a sense, a pure risk contract is the counterpart of a savings contract.

Definition 4 *A pure risk contract on a life (x) is a life insurance contract where the commitment of the insurer only consists of payments-at-death (no survival benefits) and such that*

$$_{\bullet k|}V_x = 0, \qquad (k = 0, 1, ..., n). \tag{43}$$

By its definition, a pure risk contract is always a fair life insurance contract.

Let us now again assume that the premiums are only payable at integer time points (once a year): P_k is the premium paid at time k, in case that (x) is alive at that time, $(k = 0, 1, ..., n-1)$.

Theorem 7 *Consider a life insurance contract on a life (x) with no survival benefits which vanishes before the end of the year of death. This contract is a pure risk contract if and only if*

$$ {}_{\bullet k|1}C_x = P_k, \qquad (k = 0, 1, ..., n-1). \tag{44}$$

Proof. Condition (44) implies that

$$V = \sum_{k=0}^{n-1} {}_kE_x \left({}_{\bullet k|1}C_x - P_k \right) = 0.$$

Further from (28) we find

$$ {}_{\bullet k+1|}V_x = \left({}_{\bullet k|}V_x + \ P_k - \ {}_{\bullet k|1}C_x \right) \ {}_1E_{x+k}^{-1}, \qquad (k = 0, 1, ..., n-1).$$

Using this recursive relation, it is straightforward to prove the theorem. ■

¿From the definition and the theorem above, we see that a "pure risk" contract contains no savings element in it: each yearly premium is used to cover the price of a one year term insurance.

Example 4 *A fair pure risk contract*

Consider a life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ with

$$C^{\circ\circ} = \sum_{k=0}^{n-1} D_k \ {}_{k|1}\hat{A}_x^{\circ\circ}$$

and

$$P^{\circ\circ} = \sum_{k=0}^{n-1} P_k \ {}_kE_x^{\circ\circ}.$$

This contract is a pure risk contract if and only if

$$P_k = D_k \ {}_1\hat{A}_{x+k}. \qquad (k = 0, 1, ..., n-1).$$

7 Combination of contracts

Let $(C^{(1)\circ\circ}, P^{(1)\circ\circ})$ and $(C^{(2)\circ\circ}, P^{(2)\circ\circ})$ be two life insurance contracts. The combined life insurance contract $(C^{(1)\circ\circ}, P^{(1)\circ\circ}) + (C^{(2)\circ\circ}, P^{(2)\circ\circ})$ is defined as the contract $(C^{\circ\circ}, P^{\circ\circ}) = \big(C^{(1)\circ\circ} + C^{(2)\circ\circ}, P^{(1)\circ\circ} + P^{(2)\circ\circ}\big)$.
The reserve at life ${}_{\bullet t|}V_x$ of the combined contract equals the sum of the reserves of the two contracts:

$$ {}_{\bullet t|}V_x = {}_{\bullet t|}V_x^{(1)} + {}_{\bullet t|}V_x^{(2)}, \qquad (t \geq 0). \tag{45} $$

Similarly, the reserve at death ${}_{\bullet t|}V_{x|}$ of the combined contract equals the sum of the corresponding reserves of the two contracts:

$$ {}_{\bullet t|}V_{x|} = {}_{\bullet t|}V_{x|}^{(1)} + {}_{\bullet t|}V_{x|}^{(1)}, \qquad (t > 0). \tag{46} $$

In the following we restrict ourselves to fair life insurance contracts $(C^{\circ\circ}, P^{\circ\circ})$ defined by

$$ C^{\circ\circ} = \sum_{k=1}^{n} L_k \; {}_kE_x^{\circ\circ} + \sum_{k=0}^{n-1} D_k \; {}_{k|1}\hat{A}_x^{\circ\circ} \tag{47} $$

and

$$ P^{\circ\circ} = \sum_{k=0}^{n-1} P_k \; {}_kE_x^{\circ\circ}. \tag{48} $$

Hence, we restrict to the case that the payments-at-death D_k are payable in the middle of the year of death. The contract of course vanishes before the end of the year of death.

7.1 Combination of a fair savings and risk contract

In this subsection we will prove that each fair life insurance contract can be considered as the (unique) combination of a fair savings contract and a pure risk contract.

Theorem 8 *(Payments-at-death due in the middle of the year of death)*
Any fair life insurance contracts $(C^{\circ\circ}, P^{\circ\circ})$ *defined by (47) and (48) is the unique combination of a fair savings contract* $(C^{(s)\circ\circ}, P^{(s)\circ\circ})$ *and a (fair) pure risk contract* $(C^{(r)\circ\circ}, P^{(r)\circ\circ})$.

Proof.

(a) We first prove that there exists such a combination. In (b) we will prove that the combination is unique.

Let ${}_{\bullet k|}V_x$ be the reserve of $(C^{\circ\circ}, P^{\circ\circ})$ at time k $(k = 0, 1, ..., n)$. Consider the contract $(C^{(s)\circ\circ}, P^{(s)\circ\circ})$ defined by

$$P^{(s)\circ\circ} = \sum_{k=0}^{n-1} P_k^{(s)} \ {}_kE_x^{\circ\circ} \tag{49}$$

with

$$P_k^{(s)} = \ {}_{\bullet k+1|}V_x \ v - ({}_{\bullet k|}V_x - L_k), \qquad (k = 0, 1, ..., n-1) \tag{50}$$

and

$$C^{(s)\circ\circ} = \sum_{k=1}^{n} L_k \ {}_kE_x^{\circ\circ} + \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} (P_j^{(s)} - L_j) \ u^{k+1-j} \right) \ v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x^{\circ\circ}. \tag{51}$$

Let us introduce the notation $P_n^{(s)} = 0$. Taking into account that $(C^{\circ\circ}, P^{\circ\circ})$ is a fair life insurance contract, we find from (50) that

$$\begin{aligned} {}_{\bullet k|}V_x &= \sum_{j=0}^{k-1} \left(u^{k-j-1} \ {}_{\bullet j+1|}V_x - u^{k-j} \ {}_{\bullet j|}V_x \right) \\ &= \sum_{j=0}^{k-1} (P_j^{(s)} - L_j) \ u^{k-j}, \qquad (k = 0, 1, ..., n). \end{aligned} \tag{52}$$

As ${}_{\bullet n|}V_x = L_n$, we find from (52) that

$$\sum_{j=0}^{n} (P_j^{(s)} - L_j) \ v^j = 0.$$

On the other hand, from (51), we immediately find that

$${}_{\bullet k|1}C_x^{(s)} = L_k + \sum_{j=0}^{k} (P_j^{(s)} - L_j) \ u^{k+1-j} \ {}_1A_{x+k}, \qquad (k = 0, 1, ..., n-1).$$

Hence, from Theorem 11 it follows that $(C^{(s)\circ\circ}, P^{(s)\circ\circ})$ is a fair savings contract.
Moreover, from (52) and Theorem 10, we find that

$$_{\bullet k|}V_x^{(s)} = \sum_{j=0}^{k-1} (P_j^{(s)} - L_j)\ u^{k-j}, \qquad (k = 0, 1, ..., n). \tag{53}$$

Next, consider the contract $(C^{(r)\circ\circ}, P^{(r)\circ\circ})$ defined by

$$P^{(r)\circ\circ} = \sum_{k=0}^{n-1} P_k^{(r)}\ {}_kE_x^{\circ\circ} \tag{54}$$

with

$$P_k^{(r)} = \left(D_k - \ {}_{\bullet k+1|}V_x\ v^{\frac{1}{2}}\right)\ {}_1\hat{A}_{x+k}, \qquad (k = 0, 1, ..., n-1) \tag{55}$$

and

$$C^{(r)\circ\circ} = \sum_{k=0}^{n-1} \left(D_k - \ {}_{\bullet k+1|}V_x\ v^{\frac{1}{2}}\right)\ {}_{k|1}\hat{A}_x^{\circ\circ}. \tag{56}$$

We immediately find that

$$_{\bullet k|1}C_x^{(r)} = P_k^{(r)}, \qquad (k = 0, 1, ..., n-1). \tag{57}$$

¿From Theorem 14, it follows that $(C^{(r)\circ\circ}, P^{(r)\circ\circ})$ is a pure risk contract.

It remains to prove that $(C^{\circ\circ}, P^{\circ\circ})$ is the combination of $(C^{(s)\circ\circ}, P^{(s)\circ\circ})$ and $(C^{(r)\circ\circ}, P^{(r)\circ\circ})$.
¿From (47), (51), (52) and (56), we immediately find that

$$C^{(s)\circ\circ} + C^{(r)\circ\circ} = C^{\circ\circ}.$$

¿From (50), (55), (47) and (28) it follows that

$$\begin{aligned}
P_k^{(s)} + P_k^{(r)} &= \ {}_{\bullet k+1|}V_x\ v - \left({}_{\bullet k|}V_x - L_k\right) + \left(D_k - \ {}_{\bullet k+1|}V_x\ v^{\frac{1}{2}}\right)\ {}_1\hat{A}_{x+k} \\
&= \ {}_{\bullet k+1|}V_x\ {}_1E_{x+k} - \ {}_{\bullet k|}V_x + L_k + D_k\ {}_1\hat{A}_{x+k} \\
&= \ {}_{\bullet k+1|}V_x\ {}_1E_{x+k} - \ {}_{\bullet k|}V_x + \ {}_{\bullet k|}C_x \\
&= P_k, \qquad (k = 0, 1, ..., n-1).
\end{aligned}$$

(b) Now we prove that the combination is unique.

Therefore, consider a fair savings contract $(C^{(s)\circ\circ}, P^{(s)\circ\circ})$ and a (fair) pure risk contract $(C^{(r)\circ\circ}, P^{(r)\circ\circ})$ defined by

$$C^{(s)\circ\circ} = \sum_{k=1}^{n} L_k^{(s)} \ {}_kE_x^{\circ\circ} + \sum_{k=0}^{n-1} D_k^{(s)} \ {}_{\bullet k|1}\hat{A}_x^{\circ\circ},$$

$$P^{(s)\circ\circ} = \sum_{k=0}^{n-1} P_k^{(s)} \ {}_kE_x^{\circ\circ},$$

$$C^{(r)\circ\circ} = \sum_{k=0}^{n-1} D_k^{(r)} \ {}_{\bullet k|1}\hat{A}_x^{\circ\circ},$$

$$P^{(r)\circ\circ} = \sum_{k=0}^{n-1} P_k^{(r)} \ {}_kE_x^{\circ\circ}$$

and such that

$$(C^{\circ\circ}, P^{\circ\circ}) = \left(C^{(s)\circ\circ} + C^{(r)\circ\circ}, P^{(s)\circ\circ} + P^{(r)\circ\circ}\right)$$

We have that $L_k = L_k^{(s)}$ $(k = 1, ..., n)$. Hence, by Theorem 10,

$$\begin{aligned} {}_{\bullet k|}V_x &= {}_{\bullet k|}V_x^{(s)} + {}_{\bullet k|}V_x^{(r)} \\ &= {}_{\bullet k|}V_x^{(s)} = \sum_{j=0}^{k-1} \left(P_j^{(s)} - L_j\right) \ u^{k-j}, \qquad (k = 0, 1, ..., n). \end{aligned}$$

By (42) we then have

$$\begin{aligned} D_k^{(s)} &= v^{\frac{1}{2}} \sum_{j=0}^{k} \left(P_j^{(s)} - L_j\right) \ u^{k+1-j} \\ &= v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x, \qquad (k = 0, 1, ..., n-1). \end{aligned}$$

Hence,

$$D_k^{(r)} = D_k - D_k^{(s)} = D_k - v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x, \qquad (k = 0, 1, ..., n-1).$$

Further as direct consequence of Theorem 14 (see Example 15),

$$P_k^{(r)} = D_k^{(r)} \ {}_1\hat{A}_{x+k} = \left(D_k - v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x\right) \ {}_1\hat{A}_{x+k}, \qquad (k = 0, 1, ..., n-1).$$

So we have proven that if $(C^{\circ\circ}, P^{\circ\circ})$ can be written as a combination of a fair savings contract and a (fair) risk contract, then the risk contract must be equal to the one defined in part (a) of the proof. This immediately implies that the savings contract is also uniquely determined. ■

For a life insurance contract defined by (47) and (48), the premium $P_k^{(s)}$ as defined in (50) is called the savings premium at time k. The premium $P_k^{(r)}$ as defined in (55) is called the risk premium at time k. The amount $\left(D_k - v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x\right)$ is the amount-at-risk in the k-th year.

An adjusted version of the theorem can be proven for any fair life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ which vanishes before the end of the year-of-death.

Using (52) and the relation between premium, savings and risk premium, it is easy to show that the the accumulated value at time k of the premiums less life benefits of the life insurance contract minus the accumulated value of the annual risk premiums $(j = 0, 1, ..., k-1)$ equals the reserve at time k:

$$\begin{aligned}
{}_{\bullet k|}V_x &= \sum_{j=0}^{k-1} \left(P_j - P_j^{(r)} - L_j\right) \ u^{k-j} \\
&= \sum_{j=0}^{k-1} (P_j \ - L_j) \ u^{k-j} - \sum_{j=0}^{k-1} \left(D_j - v^{\frac{1}{2}} \ {}_{\bullet j+1|}V_x\right) \ {}_1\hat{A}_{x+j} \ u^{k-j} \\
&= \sum_{j=0}^{k-1} (P_j \ - L_j) \ u^{k-j} - \sum_{j=0}^{k-1} q_{x+j} \ D_j^{(r)} \ u^{k-j-\frac{1}{2}} \ , \qquad (k = 0, 1, ..., n).
\end{aligned}$$

For a fair savings contract, this expression of the reserve at life reduces to the first summation.

Example 5 $(C^{\circ\circ}, \ P^{\circ\circ}) = ({}_n\hat{A}_x^{\circ\circ}, \ P \ \ddot{a}_{x:\overline{n}|}^{\circ\circ})$ *with* $P = {}_n\hat{A}_x / \ \ddot{a}_{x:\overline{n}|}$

For this term insurance we have

$$(C^{(s)\circ\circ}, \ P^{(s)\circ\circ}) = \left(\sum_{k=0}^{n-1} \ {}_{\bullet k+1|}V_x \ v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x^{\circ\circ}; \ \sum_{k=0}^{n-1} \ \left({}_{\bullet k+1|}V_x \ v - \ {}_{\bullet k|}V_x\right) \ {}_kE_x^{\circ\circ}\right)$$

and

$$(C^{(r)\circ\circ}, \ P^{(r)\circ\circ}) \ = \ (\sum_{k=0}^{n-1} \left(1 - \ {}_{\bullet k+1|}V_x \ v^{\frac{1}{2}}\right) \ {}_{k|1}\hat{A}_x^{\circ\circ};$$

$$\sum_{k=0}^{n-1} \left(\left(1 - \ {}_{\bullet k+1|}V_x \ v^{\frac{1}{2}} \right) \ {}_1\hat{A}_{x+k} \right) \ {}_kE_x^{\circ\circ}).$$

For the reserve at life, we find

$$_{\bullet k|}V_x = P \ \ddot{s}_{\overline{k|}} - \sum_{j=0}^{k-1} q_{x+j} \left(1 - \ {}_{\bullet j+1|}V_x \ v^{\frac{1}{2}} \right) \ u^{k-j-\frac{1}{2}} \ , \qquad (k = 0, 1, ..., n),$$

which shows that the reserve at life is lower than the accumulated value of the premiums in this case.

Example 6 $(C^{\circ\circ}, \ P^{\circ\circ}) = ({}_nE_x^{\circ\circ}, \ \pi \ \ddot{a}_{x:\overline{n|}}^{\circ\circ})$ *with* $\pi = \ {}_nE_x / \ \ddot{a}_{x:\overline{n|}}$

For this pure endowment policy we have

$$\begin{aligned}(C^{(s)\circ\circ}, \ P^{(s)\circ\circ}) \ &= \ (\ {}_nE_x^{\circ\circ} + \sum_{k=0}^{n-1} \ {}_{\bullet k+1|}V_x \ v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x^{\circ\circ}; \\ &\qquad \sum_{k=0}^{n-1} \left({}_{\bullet k+1|}V_x \ v - \ {}_{\bullet k|}V_x \right) \ {}_kE_x^{\circ\circ})\end{aligned}$$

and

$$(C^{(r)\circ\circ}, \ P^{(r)\circ\circ}) = \left(- \sum_{k=0}^{n-1} \ {}_{\bullet k+1|}V_x \ v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x^{\circ\circ}; \ -v^{\frac{1}{2}} \sum_{k=0}^{n-1} \ {}_{\bullet k+1|}V_x \ {}_1\hat{A}_{x+k} \ {}_kE_x^{\circ\circ} \right).$$

For the reserve at life, we find the following expression:

$$_{\bullet k|}V_x = \ \pi \ \ddot{s}_{\overline{k|}} + \sum_{j=0}^{k-1} q_{x+j} \ {}_{\bullet j+1|}V_x \ u^{k-j-1}, \qquad (k = 0, 1, ..., n),$$

clearly showing that for a pure endowment policy the reserve at life is larger than the accumulated value of the premiums paid. Besides, it shows that the reserves of the persons who die are needed to establish the reserves for persons that stay alive.

Example 7 *Flexible life or universal life insurance.*

¿From the previous deductions, we find two recursions for ${}_{\bullet k|}V_x$ for the life insurance considered in this section (see (47) and (48)). First, from (28) we have

$$ {}_{\bullet k+1|}V_x = \left({}_{\bullet k|}V_x - L_k + P_k - D_k \ {}_1\hat{A}_{x+k} \right) \ {}_1E_{x+k}^{-1}, \ (k = 0, 1, ..., n-1). \quad (58) $$

On the other hand, from (50), we find that

$$ {}_{\bullet k+1|}V_x = \left({}_{\bullet k|}V_x - L_k + P_k^{(s)} \right) \ u, \qquad (k = 0, 1, ..., n-1) \qquad (59) $$

with

$$ P_k^{(s)} = P_k - P_k^{(r)} $$

and

$$ P_k^{(r)} = \left(D_k - v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x \right) \ {}_1\hat{A}_{x+k}. $$

Flexible life or universal life insurance is different from the life insurance studied until now, because the obligations of both parties are not fixed at policy issue. Instead, these kind of insurances are based on (58), or equivalently (59), where the insured is free (to a certain extent) to decide at time k (if still alive) the sizes of P_k, L_k and D_k. Restrictions are imposed so that all parameters remain positive and in order to prevent antiselection. Although equivalent, the interpretations of (58) and (59) are completely different.

Interpretation of universal life with (58):

Assume that the insured is alive at time k. He owns the reserve at life ${}_{\bullet k|}V_x$. He decides to pay P_k, to withdraw L_k, and he wants a death-benefit D_k if he dies during the next year. If he is still alive at time $k+1$, then he owns the reserve at life ${}_{\bullet k+1|}V_x$.

Interpretation of universal life with (59):

Assume that the insured is alive at time k. On his savings account, there is an amount ${}_{\bullet k|}V_x$. He decides to pay P_k, to withdraw L_k, and he wants an amount D_k (the amount on his savings account included) if he dies during the next year. Let $P_k^{(r)}$ be the cost of the one year insurance. As the insured wants to receive D_k at death (assumed to be payable in the middle of the year), while there will be ${}_{\bullet k+1|}V_x$ on the savings account at time $k+1$, the capital to be insured is $D_k - v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x$. Hence, $P_k^{(r)} = \left(D_k - v^{\frac{1}{2}} \ {}_{\bullet k+1|}V_x \right) \ {}_1\hat{A}_{x+k}$. An amount of $P_k^{(s)} = P_k - P_k^{(r)}$ is placed on the savings account. At the end of the year the savings account has grown to ${}_{\bullet k+1|}V_x = \left({}_{\bullet k|}V_x - L_k + P_k^{(s)} \right) \ u$.

7.2 Combination of a fair associated savings contract and a fair associated risk contract

In this subsection we will prove that each fair life insurance contract can be separated into a corresponding fair associated stochastic savings contract and a fair associated stochastic risk contract with similar premium structure as the original contract. The mentioned *associated savings* contract has as life benefits the pure endowment benefits of the original life policy, and death benefits equal to the reserves of the associated deterministic savings contract defined (times $v^{1/2}$) in Section 5. The death benefits of the *associated risk* contract are "sums at risk" equal to the difference of the benefits of the original policy and the end-of-year reserves of the associated deterministic savings contract (also times $v^{1/2}$). To arrive at this separation it is necessary to first consider definitions of two associated deterministic savings contracts of the original fair life insurance. In this subsection it is also shown that the combination of a savings contract and pure risk contract can be considered as a special type of associated contracts. Since we have, in general, two different solutions for the associated contracts, also linear combinations of both solutions satisfy our definition of associated stochastic contracts.

According to Section 5, the associated deterministic savings contract of the fair life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ defined by (47) and (48), is

$$C^{(ads)\circ\circ} = \sum_{k=1}^{n} (L_k, k) \tag{60}$$

$$P^{(ads)\circ\circ} = \sum_{k=0}^{n} (P_k, k), \tag{61}$$

with $P_n = 0$. As has been remarked in Section 5, this associated contract is in general not a fair contract. Hence, we define a corresponding *fair* associated deterministic savings contract $(C^{(fads)\circ\circ}, P^{(fads)\circ\circ})$ by

$$C^{(fads)\circ\circ} = \sum_{k=1}^{n} (L_k, k) \tag{62}$$

$$P^{(fads)\circ\circ} = \sum_{k=0}^{n-1} (P_k^{(fads)}, k), \tag{63}$$

with $P_n^{(fads)} = 0$. The premiums $P_k^{(fads)}$ in (63) are not uniquely defined, but since this contract is assumed to be fair we find from (19)

$$ {}_{\bullet k|}V^{(fads)} = \sum_{j=0}^{k-1} \left(P_j^{(fads)} - L_j \right) u^{k-j}, \qquad (k = 0, 1, ..., n), \tag{64} $$

which leads to

$$ P_k^{(fads)} = {}_{\bullet k+1|}V^{(fads)}\, v - \left({}_{\bullet k|}V^{(fads)} - L_k \right), \qquad (k = 0, 1, ..., n-1). \tag{65} $$

As ${}_{\bullet n|}V^{(fads)} = L_n$ we find from (64)

$$ \sum_{j=0}^{n} \left(P_j^{(fads)} - L_j \right) v^j = 0, \tag{66} $$

confirming the fact that this contract is fair.

To get some grip of the problem we first consider two special cases:

1) In case ${}_{\bullet k|}V^{(fads)} = {}_{\bullet k|}V_x$, $(k = 0, 1, ..., n)$, then the $P_k^{(fads)}$ are uniquely defined for $k = 0, 1, ..., n-1$. In this case we have

$$ P_k^{(fads)} = P_k^{(s)}, \qquad (k = 0, 1, ..., n-1) \tag{67} $$

(see (50) and Theorem 23). In this case we are back to the situation of the previous subsection, which we will consider in more detail in Theorem 28.

2) In case the premiums P_k are level it is reasonable to assume that the same holds for the premiums of the fair associated deterministic savings contract, we then have

$$ P_k^{(fads)} = P^{(fads)} = \sum_{j=1}^{n} L_j\, v^j / \ddot{s}_{\overline{n}|}\,, \qquad (k = 0, 1, ..., n-1). \tag{68} $$

Since the two savings contracts $(C^{(ads)\circ\circ}, P^{(ads)\circ\circ})$ and $(C^{(fads)\circ\circ}, P^{(fads)\circ\circ})$ are *deterministic* contracts they can not immediately be combined with (fair) life insurance contracts. Hence, as a third type of associated savings contract we have to define:

Definition 5 *An associated (stochastic) savings contract of* $(C^{\circ\circ}, P^{\circ\circ})$ *defined by (47) and (48) is a life insurance* $(C^{(fas)\circ\circ}, P^{(fas)\circ\circ})$ *defined by*

$$C^{(fas)\circ\circ} = \sum_{k=1}^{n} L_k^{(fas)} \ {}_kE_x^{\circ\circ} + \sum_{k=0}^{n-1} D_k^{(fas)} \ {}_{k|1}\hat{A}_x^{\circ\circ} \tag{69}$$

and

$$P^{(fas)\circ\circ} = \sum_{k=0}^{n} P_k^{(fas)} \ {}_kE_x^{\circ\circ}, \tag{70}$$

with

$$L_k^{(fas)} = L_k, \qquad (k = 0, 1, ..., n), \tag{71}$$

$$D_k^{(fas)} = \ {}_{\bullet k+1|}V^{(fads)} \ v^{\frac{1}{2}}, \qquad (k = 0, 1, ..., n-1) \tag{72}$$

and

$$P_k^{(fas)} = P_k^{(fads)}, \qquad (k = 0, 1, ..., n). \tag{73}$$

Hence, for the associated stochastic savings contract

1) the pure endowment benefits are the same as that for the original fair life insurance contract (see (71)),

2) the death benefits equal the end-of-year reserve of the fair associated deterministic contract, discounted to the middle of the year of death (see (72)) and

3) the premiums are those from the associated fair deterministic contract (see(73)).

Theorem 9 *The associated* ***stochastic*** *savings contract defined in (69)-(73) is fair and the reserve at integer times of this contract equals that of the fair associated* ***deterministic*** *savings contract defined in (62) and (63):*

$$_{\bullet k|}V_x^{(fas)} = \ {}_{\bullet k|}V^{(fads)}, \qquad (k = 0, 1, ..., n). \tag{74}$$

Proof.

¿From (64), (66) and (69)-(73) it follows that this contract is fair:

$$\begin{aligned} & V^{(fas)} \\ = & \sum_{k=0}^{n} \left(L_k^{(fas)} - P_k^{(fas)} \right) \ {}_kE_x + \sum_{k=0}^{n-1} \ {}_{\bullet k+1|}V^{(fads)} \ v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x \\ = & \sum_{k=0}^{n} \left(L_k - P_k^{(fads)} \right) \ {}_kE_x + \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \left(P_j^{(fads)} - L_j \right) u^{k+1-j} \right) \ {}_{k|1}A_x \end{aligned}$$

$$= \sum_{k=0}^{n} \left(L_k - P_k^{(fads)} \right) {}_kE_x + \sum_{k=0}^{n-1} \left(P_k^{(fads)} - L_k \right) v^k \left({}_kp_x \; - \; {}_np_x \right)$$

$$= - \; {}_np_x \sum_{k=0}^{n} \left(P_k^{(fads)} - L_k \right) v^k = 0. \qquad (75)$$

(In the notation of premiums and benefits we already used the fact that the contract is fair.)

By Theorem 10 and (71) and (73) we now have

$$\begin{aligned} {}_{\bullet k|}V_x^{(fas)} &= \sum_{j=0}^{k-1} \left(P_k^{(fas)} - L_k^{(fas)} \right) u^{k-j} \\ &= \sum_{j=0}^{k-1} \left(P_k^{(fads)} - L_k \right) u^{k-j} \\ &= {}_{\bullet k|}V^{(fads)}, \qquad (k = 0, 1, ..., n), \end{aligned} \qquad (76)$$

so the reserve of the fair associated stochastic savings contract equals that of the reserve of the fair associated deterministic savings contract. ■

Definition 6 *The fair associated risk contract* $(C^{(far)\circ\circ}, P^{(far)\circ\circ})$ *is defined by*

$$C^{(far)\circ\circ} = \sum_{k=0}^{n-1} \left(D_k - \; {}_{\bullet k+1|}V^{(fads)} v^{\frac{1}{2}} \right) \; {}_{k|1}\hat{A}_x^{\circ\circ}, \qquad (77)$$

$$P^{(far)\circ\circ} = \sum_{k=0}^{n-1} P_k^{(far)} \; {}_kE_x^{\circ\circ}, \qquad (78)$$

with

$$\sum_{k=0}^{n-1} P_k^{(far)} \; {}_kE_x = \sum_{k=0}^{n-1} \left(D_k - \; {}_{\bullet k+1|}V^{(fads)} v^{\frac{1}{2}} \right) \; {}_{k|1}\hat{A}_x. \qquad (79)$$

Note that the premiums $P_k^{(far)}$ are not uniquely defined by (79).

Finally the fair life insurance will be separated into an associated stochastic savings and an associated risk contract. We first consider the special type of life insurance policy with level premiums in Theorem 23; immediately afterwards we consider the general case.

Theorem 10 *(Payments-at-death due in the middle of the year of death)*

Any fair life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ *defined by (47) and (48) with* $P_k = P$, $k = 0, 1, ..., n-1$, *is a combination of a fair associated (stochastic) savings contract* $(C^{(fas)\circ\circ}, P^{(fas)\circ\circ})$ *and a (fair) associated risk contract* $(C^{(far)\circ\circ}, P^{(far)\circ\circ})$ *with* $P_k^{(fas)} = P^{(fas)}$ *and* $P_k^{(far)} = P^{(far)}$ *for* $k = 0, 1, ..., n-1$.

Proof.

It is immediately clear from (47), (69), (71), (72) and (77) that $C^{\circ\circ} = C^{(fas)\circ\circ} + C^{(far)\circ\circ}$.

We have from (47) and (48)

$$P = \left(\sum_{k=1}^{n} L_k \; {}_kE_x + \sum_{k=0}^{n-1} D_k \; {}_{k|1}\hat{A}_x \right) / \ddot{a}_{x:\overline{n}|} \, , \tag{80}$$

by (68)

$$P_k^{(fads)} = P^{(fads)} = \sum_{j=1}^{n} L_j \, v^j / \ddot{s}_{\overline{n}|} \, , \qquad (k = 0, 1, ..., n-1), \tag{81}$$

and by (79)

$$\begin{aligned} P_k^{(far)} &= P^{(far)} \\ &= \left(\sum_{j=0}^{n-1} \left(D_j - \; {}_{\bullet j+1|}V^{(fads)} \, v^{\frac{1}{2}} \right) \; {}_{j|1}\hat{A}_x \right) / \ddot{a}_{x:\overline{n}|} \, , \qquad (82) \\ (k &= 0, 1, ..., n-1). \end{aligned}$$

The latter is the premium of a fair life insurance contract but not a **pure** risk contract as defined in Section 6.

¿From (80)-(82) and (75) we get

$$\begin{aligned} & \left(P - P^{(far)} \right) \ddot{a}_{x:\overline{n}|} \\ = & \sum_{k=1}^{n} L_k \; {}_kE_x + \sum_{k=0}^{n-1} \; {}_{\bullet k+1|}V^{fads} v^{\frac{1}{2}} \; {}_{k|1}\hat{A}_x \\ = & \sum_{k=0}^{n} P_k^{(fads)} \; {}_kE_x \end{aligned}$$

$$+ \sum_{k=0}^{n} \left(L_k^{(fas)} - P_k^{(fas)} \right) \ {}_kE_x + \sum_{k=0}^{n-1} \ {}_{\bullet k+1|}V^{(fads)} \, v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x$$

$$= \ P^{(fads)} \, \ddot{a}_{x:\overline{n}|} \, ,$$

Hence,

$$P = P^{(fads)} + P^{(far)}. \tag{83}$$

■

A direct consequence of Theorem 23 is that under the conditions of this theorem we have:

$${}_{\bullet k|}V_x = \ {}_{\bullet k|}V_x^{(fas)} + \ {}_{\bullet k|}V_x^{(far)}, \qquad (k = 0, 1, ..., n). \tag{84}$$

From the proof of (83) it can easily be seen that the theorem can be generalized. The only requirement is that

$$\sum_{k=0}^{n-1} \left(P_k - P_k^{(far)} \right) \ {}_kE_x = \sum_{k=0}^{n-1} P_k^{(fas)} \ {}_kE_x. \tag{85}$$

Assuming $P_0 \neq 0$ we define

$$P_k^{(far)} = (P_k/P_0) P_0^{(far)}, \qquad (k = 1, 2, ..., n-1) \tag{86}$$

and

$$P_k^{(fas)} = (P_k/P_0) P_0^{(fas)}, \qquad (k = 1, 2, .., n-1). \tag{87}$$

According to (66) it is required that

$$P_0^{(fas)} = P_0 \left(\sum_{j=1}^{n} L_j \, v^j / \sum_{j=0}^{n-1} P_j \, v^j \right). \tag{88}$$

Further, we define

$$P_0^{(far)} = P_0 - P_0^{(fas)}. \tag{89}$$

Then we have

$$P_k^{(far)} = P_k - P_k^{(fas)}, \qquad (k = 0, 1, ..., n-1) \tag{90}$$

and (85) is immediately satisfied.

This demonstrates that any fair life insurance contract can be decomposes into a fair associated savings contract and a fair associated risk contract, which leads to Theorem 24:

Theorem 11 *(Payments-at-death due in the middle of the year of death)*
Any fair life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ *defined by (47) and (48) with* $P_0 \neq 0$, *is a combination of a fair associated (stochastic) savings contract* $(C^{(fas)\circ\circ}, P^{(fas)\circ\circ})$ *and a (fair) associated risk contract* $(C^{(far)\circ\circ}, P^{(far)\circ\circ})$ *with* $P_k^{(fas)}$ *and* $P_k^{(far)}$ *defined by (86) and (87) where* $P_0^{(fas)}$ *and* $P_0^{(fas)}$ *are defined by (88) and (89).*

IIllustrations of Theorem 24 are given below.

Example 8 $(C^{\circ\circ},\ P^{\circ\circ}) = ({}_n\hat{A}_x^{\circ\circ},\ P\ \ddot{a}^{\circ\circ}_{x:\overline{n}|})$ *with* $P = {}_n\hat{A}_x/\ddot{a}_{x:\overline{n}|}$

For this term insurance we have $(C^{\circ\circ},\ P^{\circ\circ}) \equiv (C^{(far)\circ\circ},\ P^{(far)\circ\circ})$. For the associated savings contract all premiums and benefits are equal to zero. We have $P^{(far)} = P$.

Example 9 $(C^{\circ\circ},\ P^{\circ\circ}) = ({}_nE_x^{\circ\circ}\ ,\pi\ \ \ddot{a}^{\circ\circ}_{x:\overline{n}|})$, *with* $\pi = {}_nE_x/\ddot{a}_{x:\overline{n}|}$

The associated *deterministic* savings contract of this pure endowment policy is given by

$$(C^{(ads)\circ\circ}, P^{(ads)\circ\circ}) = \left((1,n), \sum_{k=0}^{n-1}\ (\pi, k)\right).$$

The associated *fair deterministic* savings contract with level premiums is equal to

$$(C^{(fads)\circ\circ}, P^{(fads)\circ\circ}) = \left((1,n), \sum_{k=0}^{n-1}\ \left(\frac{1}{\ddot{s}_{\overline{n}|}}, k\right)\right),$$

the associated *fair stochastic* savings contract with level premiums is

$$(C^{(fas)\circ\circ}, P^{(fas)\circ\circ}) = \left({}_nE_x^{\circ\circ} + \sum_{k=0}^{n-1}\ \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}}\ {}_{k|1}\hat{A}_x^{\circ\circ}, \sum_{k=0}^{n-1}\ \frac{1}{\ddot{s}_{\overline{n}|}}\ {}_kE_x^{\circ\circ}\right)$$

and the associated *fair risk* contract with level premiums is

$$\begin{aligned}(C^{(far)\circ\circ}, P^{(far)\circ\circ}) &= \left(\sum_{k=0}^{n-1}\left(-\frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}}\right)\ {}_{k|1}\hat{A}_x^{\circ\circ}, P^{(far)}\ \ddot{a}^{\circ\circ}_{x:\overline{n}|}\right)\\ &= \left(\sum_{k=0}^{n-1}\left(-\ {}_{\bullet k+1|}V^{(fas)} v^{\frac{1}{2}}\right)\ {}_{k|1}\hat{A}_x^{\circ\circ}, P^{(far)}\ \ddot{a}^{\circ\circ}_{x:\overline{n}|}\right).\end{aligned}$$

The associated risk contract is not a pure risk contract, since ${}_{\bullet k|}V_x^{(far)} \neq 0$ for $k = 0, 1, ..., n$.

We have $P^{(fas)} = \frac{1}{\ddot{s}_{\overline{n}|}}$, and negative premiums for the associated risk contract:

$$P^{(far)} = \left(\sum_{k=0}^{n-1} \left(-\frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}} \right) {}_{k|1}\hat{A}_x \right) / \ddot{a}_{x:\overline{n}|} \, .$$

Of course we have $\pi = P^{(fas)} + P^{(far)}$.

Example 10 $(C^{\circ\circ},\ P^{\circ\circ}) = ({}_nE_x^{\circ\circ} + {}_n\hat{A}_x^{\circ\circ}, \Pi\ \ \ddot{a}_{x:\overline{n}|}^{\circ\circ})$, *with* $\Pi = ({}_nE_x + {}_n\hat{A}_x)/\ddot{a}_{x:\overline{n}|}$.

This example is an elaborated special case of Exercise 5.3 of Wolthuis (1994). The associated deterministic savings contract is given by

$$(C^{(ads)\circ\circ}, P^{(ads)\circ\circ}) = \left((1, n), \sum_{k=0}^{n-1} \ (\Pi, k) \right).$$

The associated *fair deterministic* savings contract with level premiums corresponds with that of the pure endowment policy of Example 26 and is given by

$$(C^{(fads)\circ\circ}, P^{(fads)\circ\circ}) = \left((1, n), \sum_{k=0}^{n-1} \ \left(\frac{1}{\ddot{s}_{\overline{n}|}}, k \right) \right),$$

the associated *fair stochastic* savings contract with level premiums is also the same as that of Example 26:

$$(C^{(fas)\circ\circ}, P^{(fas)\circ\circ}) = \left({}_nE_x^{\circ\circ} + \sum_{k=0}^{n-1} \ \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}} \ {}_{k|1}\hat{A}_x^{\circ\circ}, \sum_{k=0}^{n-1} \ \frac{1}{\ddot{s}_{\overline{n}|}} \ {}_kE_x^{\circ\circ} \right)$$

and the associated *fair risk* contract with level premiums is

$$\begin{aligned} (C^{(far)\circ\circ}, P^{(far)\circ\circ}) &= \left(\sum_{k=0}^{n-1} \left(1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}} \right) \ {}_{k|1}\hat{A}_x^{\circ\circ}, P^{(far)} \ \ddot{a}_{x:\overline{n}|}^{\circ\circ} \right) \\ &= \left(\sum_{k=0}^{n-1} \left(1 - \ {}_{\bullet k+1|}V^{(fads)} v^{\frac{1}{2}} \right) \ {}_{k|1}\hat{A}_x^{\circ\circ}, P^{(far)} \ \ddot{a}_{x:\overline{n}|}^{\circ\circ} \right). \end{aligned}$$

The associated risk contract is not a pure risk contract, since ${}_{\bullet k|}V_x^{(ar)} \neq 0$ for $k = 0, 1, ..., n$.

Again we have $P^{(fas)} = \frac{1}{\ddot{s}_{\overline{n}|}}$, and in this case positive premiums for the associated risk contract:

$$P^{(far)} = \left(\sum_{k=0}^{n-1} \left(1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}} \right) {}_{k|1}\hat{A}_x \right) / \ddot{a}_{x:\overline{n}|} .$$

We have by definition $\Pi = P^{(fas)} + P^{(far)}$.

This example is illustrative for a mortgage construction in the Netherlands, where a mortgage is combined with an endowment policy. In case a person (homeowner) dies, the associated savings policy "pays" an amount equal to the accumulated value of the deterministic savings premiums, and the associated risk policy replinishes this to the amount of the original mortgage. If the person still lives at the end of the insurance period the mortgage is paid in full by the associated savings contract. During each year the insured has to pay interest on the original mortgage amount (as long as the policy does not end). In the Netherlands the interest "income" on the savings premiums (included in the associated savings contract) is not charged by the tax authorities, while the annual interest on the original mortgage amount can, up to this moment, be entirely deducted annually from ones personal income. In practice, premiums are normally paid monthly, and interest rates and savings premiums are adjusted every couple of years.

Theorem 12 *(Payments-at-death due in the middle of the year of death)*

Any fair life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$ *defined by (46) and (47) that is the combination of a fair associated savings contract* $(C^{(fas)\circ\circ}, P^{(fas)\circ\circ})$ *and a (fair) associated risk contract* $(C^{(far)\circ\circ}, P^{(far)\circ\circ})$*, where the risk contract is a pure risk contract, reduces to the combination of the fair savings contract and the fair risk contract mentioned in Theorem 16.*

Proof.

It has already been noted in the preceding theorem that $C^{\circ\circ} = C^{(fas)\circ\circ} + C^{(far)\circ\circ}$.

We define

$$P_k^{(far)} = \left(D_k - {}_{\bullet k+1|}V^{(fads)} v^{\frac{1}{2}} \right) {}_1\hat{A}_{x+k}, \qquad (k = 0, 1, ..., n-1). \qquad (91)$$

By (77) we have

$$C^{(far)\circ\circ} = \sum_{k=0}^{n-1} \left(D_k - {}_{\bullet k+1|}V^{(fads)}\, v^{\frac{1}{2}} \right) \; {}_{k|1}\hat{A}_x^{\circ\circ},$$

hence

$${}_{\bullet k|1}C_x^{(far)} = P_k^{(far)}, \qquad (k = 0, 1, ..., n-1),$$

so by Theorem 14 we now have a pure risk process, leading to

$${}_{\bullet k|}V^{(far)} = 0, \qquad (k = 0, 1, ..., n).$$

Hence by (45) it is follows that

$${}_{\bullet k|}V^{(fads)} = {}_{\bullet k|}V_x, \qquad (k = 0, 1, ..., n). \tag{92}$$

¿From (65), (73) and (50) we now have

$$P_k^{(fas)} = P_k^{(fads)} = P_k^{(s)}, \qquad (k = 0, 1, ..., n-1),$$

and from (55) and (91) and (92)

$$P_k^{(far)} = P_k^{(r)}, \qquad (k = 0, 1, ..., n-1).$$

This means we are back to the situation of Theorem 16. ■

Example 11 $(C^{\circ\circ},\ P^{\circ\circ}) = \left({}_nE_x^{\circ\circ} + {}_n\hat{A}_x^{\circ\circ},\ \sum_{k=0}^{n-1} P_k\ {}_kE_x^{\circ\circ} \right)$

For this special type of fair life insurance contract, with still unknown premiums P_k, the associated deterministic savings contract is given by

$$(C^{(ads)\circ\circ}, P^{(ads)\circ\circ}) = \left((1, n), \sum_{k=0}^{n-1} (P_k, k) \right)$$

and the associated *fair deterministic* savings contract with level premiums is

$$(C^{(fads)\circ\circ}, P^{(fads)\circ\circ}) = \left((1, n), \sum_{k=0}^{n-1} \left(\frac{1}{\ddot{s}_{\overline{n}|}}, k \right) \right).$$

The associated *fair stochastic* savings contract (= fair savings contract) with level premiums is

$$
\begin{aligned}
(C^{(fas)\circ\circ}, P^{(fas)\circ\circ}) &= (C^{(s)\circ\circ}, P^{(s)\circ\circ}) \\
&= \left({}_nE_x^{\circ\circ} + \sum_{k=0}^{n-1} \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} v^{\frac{1}{2}} \; {}_{k|1}\hat{A}_x^{\circ\circ}, \sum_{k=0}^{n-1} \frac{1}{\ddot{s}_{\overline{n}|}} \; {}_kE_x^{\circ\circ} \right)
\end{aligned}
$$

and the associated *fair (pure) risk* contract is

$$
\begin{aligned}
(C^{(far)\circ\circ}, P^{(far)\circ\circ}) &= \left(\sum_{k=0}^{n-1} \left(1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} \; v^{\frac{1}{2}} \right) \; {}_{k|1}\hat{A}_x^{\circ\circ}, \sum_{k=0}^{n-1} P_k^{(far)} \; {}_kE_x^{\circ\circ} \right) \\
&= \left(\sum_{k=0}^{n-1} \left(1 - \; {}_{\bullet k+1|}V^{(as)} \, v^{\frac{1}{2}} \right) \; {}_{k|1}\hat{A}_x^{\circ\circ}, \sum_{k=0}^{n-1} P_k^{(far)} \; {}_kE_x^{\circ\circ} \right)
\end{aligned}
$$

with

$$
P_k^{(far)} = P_k^{(r)} = \left(1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} \, v^{\frac{1}{2}} \right) \; {}_1\hat{A}_{x+k}, \qquad (k = 0, 1, ..., n-1).
$$

Since the associated risk contract is in this case a pure risk contract we have ${}_{\bullet k|}V_x^{(ar)} = 0$ for $k = 0, 1, ..., n$. The net premium P_k is in this case

$$
P_k = \frac{1}{\ddot{s}_{\overline{n}|}} + P_k^{(far)}, \; (k = 0, 1, ..., n-1).
$$

It is left to the reader to check that

$$
\sum_{k=0}^{n-1} \; P_k \; {}_kE_x = {}_nE_x \; + {}_n\hat{A}_x.
$$

Note that we also have

$$
P_k^{(fas)} = P_k^{(fads)} = P_k^{(s)} = \frac{1}{\ddot{s}_{\overline{n}|}}, \qquad (k = 0, 1, ..., n-1).
$$

One can, of course, combine Theorems 24 and 28 to obtain other decompositions of the fair life insurance contract $(C^{\circ\circ}, P^{\circ\circ})$. We hereby denote the separation according to Theorem 24 by $(\Gamma^{(fas)\circ\circ}, \Pi^{(fas)\circ\circ})$ and $(\Gamma^{(far)\circ\circ}, \Pi^{(far)\circ\circ})$

and corresponding reserves by ${}_{\bullet k|}\Upsilon_x^{(fas)}$ and ${}_{\bullet k|}\Upsilon_x^{(far)}$. First we consider a simple linear combination; this leads to a decomposition with premiums

$$P_k^{(fas)} = \alpha\, P_k^{(s)} + (1-\alpha)\, \Pi_k^{(fas)}, \qquad (k = 0, 1, ..., n-1), \tag{93}$$

$$P_k^{(far)} = P_k - P_k^{(fas)}, \qquad (k = 0, 1, ..., n-1) \tag{94}$$

with, according to (87) and (88)

$$\Pi_k^{(fas)} = P_k \left(\sum_{j=1}^{n} L_j\, v^j / \sum_{j=0}^{n-1} P_j\, v^j \right), \qquad (k = 0, 1, ..., n-1). \tag{95}$$

The reserve ${}_{\bullet k|}V_x^{(fas)}$ of the fair associated savings contract can for each k be calculated using (64):

$$\begin{aligned} & {}_{\bullet k|}V_x^{(fas)} \\ = \; & \sum_{j=0}^{k-1} \left(P_j^{(fads)} - L_j \right) u^{k-j} \\ = \; & \alpha \sum_{j=0}^{k-1} \left(P_j^{(s)} - L_j \right) u^{k-j} + (1-\alpha) \sum_{j=0}^{k-1} \left(\Pi_j^{(fas)} - L_j \right) u^{k-j} \\ = \; & \alpha\, {}_{\bullet k|}V_x^{(s)} + (1-\alpha)\, {}_{\bullet k|}\Upsilon_x^{(fas)}, \qquad (k = 0, 1, ..., n), \end{aligned} \tag{96}$$

which leads to

$$C^{(fas)\circ\circ} = \sum_{k=1}^{n} L_k + \sum_{k=0}^{n-1} {}_{\bullet k+1|}V^{(fads)}\, v^{1/2}\; {}_{k|1}\hat{A}_x^{\circ\circ},$$

and

$$C^{(far)\circ\circ} = \sum_{k=0}^{n-1} \left(D_k - {}_{\bullet k+1|}V^{(fads)}\, v^{1/2} \right)\, {}_{k|1}\hat{A}_x^{\circ\circ}.$$

Then we have by the additional property (96):

$$C^{(fas)\circ\circ} = \alpha C^{(s)\circ\circ} + (1-\alpha)\Gamma^{(fas)\circ\circ}$$

and

$$C^{(far)\circ\circ} = \alpha C^{(r)\circ\circ} + (1-\alpha)\Gamma^{(far)\circ\circ}.$$

The same can be done for the premiums. This leads to:

Theorem 13 *(Payments-at-death due in the middle of the year of death) Any fair life insurance contract* $(C^{\circ\circ}, P^{\circ\circ})$ *defined by (46) and (47) with* $P_0 \neq 0$ *can be written as a combination of a fair associated savings contract and a fair associated risk contract that are linear combinations of the associated contracts considered in Theorem 24 and Theorem 28.*

It is tempting to consider a generalisation of (93):

$$P_k^{(fas)} = \alpha_k \, P_k^{(s)} + (1-\alpha_k)\, \Pi_k^{(fas)}, \qquad (k = 0, 1, ..., n-1), \tag{97}$$

of which, for instance, relevant values are $\alpha_k = 0$, $(k = 0, 1, ..., m-1)$ and $\alpha_k = 1$, $(k = m, m+1, ..., n-1)$. In this case (94) and (95) remain unchanged. Formula (96) then becomes:

$$\begin{aligned} & {}_{\bullet k|}V_x^{(fas)} \\ = \; & \beta_k \sum_{j=0}^{k-1} \left(P_j^{(s)} - L_j\right) u^{k-j} + (1-\beta_k) \sum_{j=0}^{k-1} \left(\Pi_j^{(fas)} - L_j\right) u^{k-j} \\ = \; & \beta_k \, {}_{\bullet k|}V_x^{(s)} + (1-\beta_k)\, {}_{\bullet k|}\Upsilon_x^{(fas)}, \qquad (k = 0, 1, ..., n). \end{aligned} \tag{98}$$

We define $\beta_0 = \alpha_0$. The β_k $(k = 1, 2, ..., n-1)$ have to satisfy the next two recursion equations

$$\beta_{k+1} \, {}_{\bullet k+1|}V_x^{(s)} = \left(\beta_k \, {}_{\bullet k|}V_x^{(s)} + \alpha_k (P_k^{(s)} - L_k)\right) u \tag{99}$$

and

$$(1-\beta_{k+1})\, {}_{\bullet k+1|}\Upsilon_x^{(fas)} = \left((1-\beta_k)\, {}_{\bullet k|}\Upsilon_x^{(fas)} + (1-\alpha_k)(\Pi_k^{(fas)} - L_k)\right) u. \tag{100}$$

This leads to

$$\beta_{k+1} = \left(\beta_k \, {}_{\bullet k|}V_x^{(s)} + \alpha_k (P_k^{(s)} - L_k)\right) u / \, {}_{\bullet k+1|}V_x^{(s)} \tag{101}$$

and

$$\beta_{k+1} = 1 - \left((1-\beta_k)\, {}_{\bullet k|}\Upsilon_x^{(fas)} + (1-\alpha_k)(\Pi_k^{(fas)} - L_k)\right) u \, / {}_{\bullet k+1|}\Upsilon_x^{(fas)}. \tag{102}$$

In the special case $k = 0$ this reduces to

$$\beta_1 = \alpha_0 \Big((P_0^{(s)} - L_0) \Big) u / {}_{\bullet 1|} V_x^{(s)} \tag{103}$$

$$\beta_1 = 1 - (1 - \alpha_0) \Big((\Pi_0^{(fas)} - L_0) \Big) u / {}_{\bullet 1|} \Upsilon_x^{(fas)} \tag{104}$$

leading in both cases to

$$\beta_1 = \alpha_0. \tag{105}$$

For $k = 1$ we have from (101) and (102)

$$\begin{aligned} \beta_2 &= \Big(\alpha_0 \, {}_{\bullet 1|} V_x^{(s)} + \alpha_1 (P_1^{(s)} - L_1) \Big) u / {}_{\bullet 2|} V_x^{(s)} \\ &= (\alpha_0 - \alpha_1) \, u \, {}_{\bullet 1|} V_x^{(s)} / {}_{\bullet 2|} V_x^{(s)} + \alpha_1 \end{aligned} \tag{106}$$

and

$$\begin{aligned} \beta_2 &= 1 - \Big((1 - \alpha_0) \, {}_{\bullet 1|} \Upsilon_x^{(fas)} + (1 - \alpha_1)(\Pi_1^{(fas)} - L_1) \Big) u / {}_{\bullet 2|} \Upsilon_x^{(fas)} \\ &= 1 - (\alpha_1 - \alpha_0) \, u \, {}_{\bullet 1|} \Upsilon_x^{(fas)} / {}_{\bullet 2|} \Upsilon_x^{(fas)} - (1 - \alpha_1), \end{aligned} \tag{107}$$

which leads to

$$\begin{aligned} & (\alpha_0 - \alpha_1) \; u \, {}_{\bullet 1|} V_x^{(s)} / {}_{\bullet 2|} V_x^{(s)} + \alpha_1 \\ = \; & 1 - (\alpha_1 - \alpha_0) \, u \, {}_{\bullet 1|} \Upsilon_x^{(fas)} / {}_{\bullet 2|} \Upsilon_x^{(fas)} - (1 - \alpha_1), \end{aligned} \tag{108}$$

hence

$$(\alpha_0 - \alpha_1)\ u\ \left({}_{\bullet 1|}V_x^{(s)} / {}_{\bullet 2|}V_x^{(s)} - {}_{\bullet 1|}\Upsilon_x^{(fas)} / {}_{\bullet 2|}\Upsilon_x^{(fas)}\right) = 0. \tag{109}$$

If

$${}_{\bullet 1|}V_x^{(s)} / {}_{\bullet 2|}V_x^{(s)} \neq {}_{\bullet 1|}\Upsilon_x^{(fas)} / {}_{\bullet 2|}\Upsilon_x^{(fas)} \tag{110}$$

then we get

$$\alpha_1 = \alpha_0. \tag{111}$$

This leads to

$$\beta_2 = \alpha_0. \tag{112}$$

If

$${}_{\bullet 1|}V_x^{(s)} / {}_{\bullet 2|}V_x^{(s)} = {}_{\bullet 1|}\Upsilon_x^{(fas)} / {}_{\bullet 2|}\Upsilon_x^{(fas)} \tag{113}$$

then it is not required that $\alpha_1 = \alpha_0$ and β_2 can be calculated using either (106) or (107), leading to the same value in both cases.

Following the above approach we will have

$$\beta_k = \alpha_k = \alpha_0, \qquad (k = 0, 1, ..., n-1) \tag{114}$$

if

$${}_{\bullet k|}V_x^{(s)} / {}_{\bullet k+1|}V_x^{(s)} \neq {}_{\bullet k|}\Upsilon_x^{(fas)} / {}_{\bullet k+1|}\Upsilon_x^{(fas)}, \qquad (k = 1, 2, ..., n-1). \tag{115}$$

In case the equality sign holds in (114) for **each** k then it is, of course, required that

$$P_k^{(s)} = \Pi_k^{(fas)}, \qquad (k = 0, 1, ..., n-1), \tag{116}$$

which is the case considered in Theorem 28.

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