Simple Characterizations of Comonotonicity and Countermonotonicity by Extremal Correlations

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Abstract

In this pedagogical note, it is shown how extremal values of classical measures of association like Pearson's correlation coefficient, Kendall's τ, Spearman's ρ and Gini's γ, characterize comonotonicity and countermonotonicity. The link between zero-correlation and mutual independence is also examined.

Key words and phrases: Comonotonicity, Countermonotonicity, Fréchet spaces, Fréchet bounds, measures of association.

1 Introduction

There are a variety of ways to discuss and to measure dependence. First and foremost is Pearson's product moment correlation coefficient which captures the linear dependence between couples of rv's, but which is not invariant under monotone tranformations of the coordinate axes. As we shall see, other measures are scale-invariant, that is, they remain unchanged under strictly increasing transformations of the rv's; in this category, we find the population versions of Kendall's τ and Spearman's ρ, both of which measure a form of dependence known as "concordance".

Kendall's τ and Spearman's ρ are bivariate measures of dependence for continuous random variables that are invariant with respect to strictly monotone transformations and equal to 1 (resp. -1) for the Fréchet upper (resp. lower) bound, i.e. when one variable is a non-decreasing (resp. non-increasing) transform of the other. In the first case, these variables are said to be comonotonic, while in the second case, they are said to be countermonotonic.

What seems to be less known, however, is that a value of 1 or -1 for these measures of association characterizes the Fréchet bounds. Since Spearman's correlation coefficient r does not enjoy these convenient invariance properties, Kendall's τ and Spearman's ρ are more desirable measures of association for multivariate non-normal distributions. However, if we restrict ourselves to random couples valued in the positive quadrant, an extreme value for r also characterizes the Fréchet bounds. In general, for measures of "concordance" as those defined by Scarsini (1984), such results do not always hold. In passing, we show that for positively or negatively quadrant dependent random couples, joint uncorrelatedness implies mutual independence.

Let F_1 and F_2 be univariate distribution functions. In this paper, we consider the Fréchet space $\mathcal{R}(F_1, F_2)$ consisting of all the (distribution functions $F_{(X_1,X_2)}$ of) random couples (X_1, X_2) with marginals F_1 and F_2, i.e. $F_i(x_i) = P[X_i \leq x_i]$, $x_i \in I\!R$. A celebrated result attributed to Höffding and Fréchet indicates that for any (X_1, X_2) in $\mathcal{R}(F_1, F_2)$ the following inequalities hold:

$$M(x_1, x_2) \leq F_{(X_1,X_2)}(x_1, x_2) \leq W(x_1, x_2) \text{ for all } (x_1, x_2) \in I\!R^2, \tag{1.1}$$

where W is usually referred to as the Fréchet upper bound of $\mathcal{R}(F_1, F_2)$ and is defined by

$$W(x_1, x_2) = \min\{F_1(x_1), F_2(x_2)\}, \ \ (x_1, x_2) \in I\!R^2,$$

while M is usually referred to as the Fréchet lower bound of $\mathcal{R}(F_1, F_2)$ and is defined by

$$M(x_1, x_2) = \max\{F_1(x_1) + F_2(x_2) - 1, 0\}, \ \ (x_1, x_2) \in I\!R^2.$$

Remark that M and W are reachable in $\mathcal{R}(F_1, F_2)$. Indeed, given a random variable U uniformly distributed on $[0, 1]$, it can be shown that W is the distribution function of the random couple

$$(F_1^{-1}(U), F_2^{-1}(U)) \in \mathcal{R}(F_1, F_2),$$

where the generalized inverses of the F_i's are defined as

$$F_i^{-1}(u) = \inf\{x \in I\!R \text{ such that } F_i(x) \geq u\}, \ \ u \in [0, 1], \ \ i = 1, 2.$$

On the other hand, M is the distribution function of the random couple

$$(F_1^{-1}(U), F_2^{-1}(1-U)) \in \mathcal{R}(F_1, F_2).$$

The elements of the Fréchet space $\mathcal{R}(F_1, F_2)$ which have W as distribution function are said to be comonotonic in Economics, Finance and Actuarial Sciences; see e.g. Yaari (1987) or Wang & Dhaene (1997), as well as the recent review papers by Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b). Those corresponding to M are said to be mutually exclusive in Actuarial Sciences; see Dhaene & Denuit (1999).

Tchen (1980) proved that for any (X_1, X_2) and (Y_1, Y_2) in $\mathcal{R}(F_1, F_2)$ such that

$$F_{(X_1,X_2)}(x_1, x_2) \leq F_{(Y_1,Y_2)}(x_1, x_2) \text{ for all } x_1, x_2 \in \mathbb{R},$$

the inequality

$$E\phi(X_1, X_2) \leq E\phi(Y_1, Y_2) \tag{1.2}$$

is satisfied for all the measurable functions ϕ satisfying

$$\phi(x_1, x_2) + \phi(y_1, y_2) - \phi(x_1, y_2) - \phi(y_1, x_2) \geq 0 \text{ for all } x_1 \leq y_1 \text{ and } x_2 \leq y_2,$$

provided the expectations in (1.2) exist. Such functions ϕ are usually called quasi-monotone, superadditive or supermodular in the literature (note that any joint distribution function is supermodular). Combining (1.2) and (1.1), we get the inequalities

$$E\phi(F_1^{-1}(U), F_2^{-1}(1-U)) \leq E\phi(X_1, X_2) \leq E\phi(F_1^{-1}(U), F_2^{-1}(U)) \tag{1.3}$$

for any $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ and supermodular function ϕ, provided the expectations exist.

2 Pearson's correlation coefficient

In this section, we restrict ourselves to random vectors valued in the positive quadrant, i.e.

$$P[X_1 < 0] = P[X_2 < 0] = 0,$$

with a finite variance-covariance matrix. Traditionally, the relationship of two random variables, X_1 and X_2, is usually measured by Pearson's correlation coefficient $r(X_1, X_2)$ given by

$$r(X_1, X_2) = \frac{\text{Cov}[X_1, X_2]}{\sqrt{\text{Var}[X_1]\text{Var}[X_2]}},$$

where $\text{Cov}[X_1, X_2] = E[X_1X_2] - E[X_1]E[X_2]$ is the covariance of X_1 and X_2. It is well-known that $r(X_1, X_2) = \pm 1$ if, and only if, X_1 and X_2 are linearly dependent.

In the practice of data analysis the problem of the estimation of correlations occurs rather often, and it is common to compare the values of correlation coefficients between some variables with the values -1 and 1 as minimal and maximal possible values. But as in the case of empirical data usually the minimal and maximal values ± 1 can never be reached, in the estimation of dependencies between the variables it would be useful to compare the empirical correlations with the minimal and maximal correlation coefficients for the given marginal empirical distributions. We deduce from (1.3) that, for any (X_1, X_2) in $\mathcal{R}(F_1, F_2)$, $r(X_1, X_2)$ is constrained by

$$\frac{\text{Cov}[F_1^{-1}(U), F_2^{-1}(1-U)]}{\sqrt{\text{Var}[X_1]\text{Var}[X_2]}} \leq r(X_1, X_2) \leq \frac{\text{Cov}[F_1^{-1}(U), F_2^{-1}(U)]}{\sqrt{\text{Var}[X_1]\text{Var}[X_2]}}, \tag{2.1}$$

so that a value ± 1 for r is in general not obtainable in $\mathcal{R}(F_1, F_2)$. Shih and Huang (1992) and Schlechtman and Yitzaki (1992) have noticed that (unless the marginal distributions of two random variables differ only in location and/or scale parameters, see below) the range of Pearson's r is narrower than $(-1, 1)$ and depends on the marginal distributions. The following example illustrates this situation.

Example 2.1. Consider the random couple (X_1, X_2) where $\ln X_1$ conforms to a Normal distribution with mean 0 and uit standard deviation and $\ln X_2$ conforms to a Normal distribution with mean 0 and standard deviation σ The extremal correlation occurs when X_1 and X_2 are functionally dependent:

(i) if $X_2 = X_1^{\sigma}$ then the maximal correlation coefficient for these marginals is attained and equal

$$r_{\max}(\sigma) = \frac{\exp(\sigma) - 1}{\sqrt{\exp(\sigma^2) - 1}\sqrt{e - 1}}.$$

(i) if $X_2 = X_1^{-\sigma}$ then the maximal correlation coefficient for these marginals is attained and equal

$$r_{\min}(\sigma) = \frac{\exp(-\sigma) - 1}{\sqrt{\exp(\sigma^2) - 1}\sqrt{e - 1}}.$$

These extremal correlations are shown graphically in Figure 2.1. We observe that

$$\lim_{\sigma \to +\infty} r_{\max}(\sigma) = \lim_{\sigma \to +\infty} r_{\min}(\sigma) = 0.$$

As a consequence, it is possible to have a random couple where the corellation is almost zero even though the components are comonotonic or countermonotonic (and thus exhibit the strongest kind of dependence possible for this pair of marginals). This contradicts the intuition that small correlation implies weak dependence.

Let us now prove that when the bounds in (2.1) are attained, then X_2 is functionnally dependent of X_1. Therefore, we need the following technical lemma, which can be seen as a particular case of Lemma 3.3 in Denuit, Lefèvre & Mesfioui (1999). We provide here an elementary proof of it for the sake of completeness.

Lemma 2.2. *Let* $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$*. Then,*

$$E[X_1 X_2] = \int_{x_1=0}^{+\infty} \int_{x_2=0}^{+\infty} P[X_1 > x_1, X_2 > x_2] dx_1 dx_2.$$

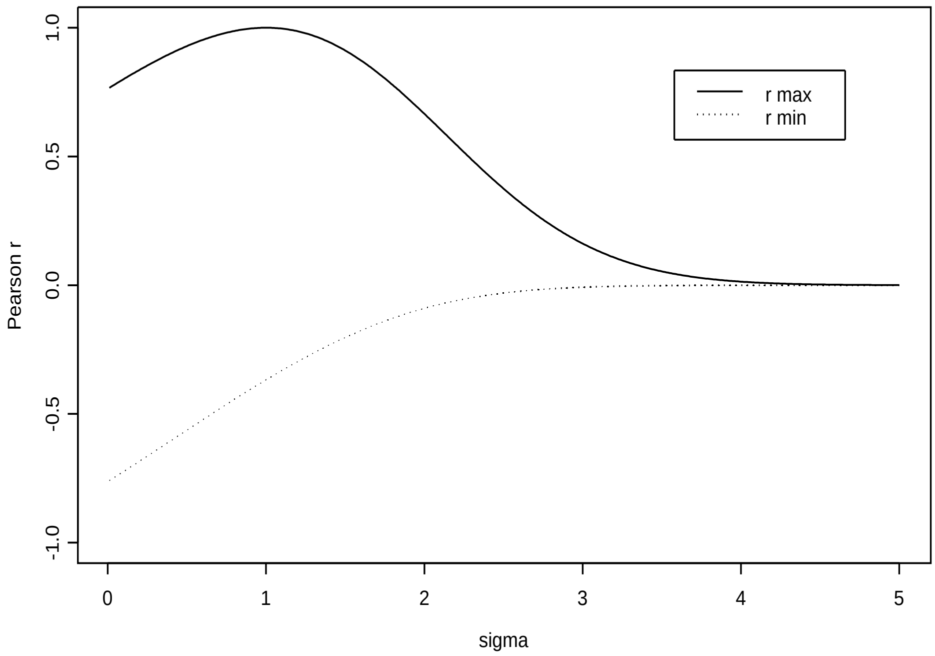

Figure 2.1: *Values for* $r_{\max}(\sigma)$ *and* $r_{\min}(\sigma)$ *as functions of* σ.

Proof. First, write

$$\int_{x_1=0}^{+\infty}\int_{x_2=0}^{+\infty} P[X_1 > x_1, X_2 > x_2]dx_1dx_2 = \int_{x_1=0}^{+\infty}\int_{x_2=0}^{+\infty}\int_{y_1=x_1}^{+\infty}\int_{y_2=x_2}^{+\infty} dF_{(X_1,X_2)}(y_1,y_2)dx_1dx_2.$$

Then, invoke Fubini's theorem to get

$$\begin{aligned}\int_{x_1=0}^{+\infty}\int_{x_2=0}^{+\infty} P[X_1 > x_1, X_2 > x_2]dx_1dx_2 &= \int_{y_1=0}^{+\infty}\int_{y_2=0}^{+\infty}\int_{x_1=0}^{y_1}\int_{x_2=0}^{y_2} dx_1dx_2dF_{(X_1,X_2)}(y_1,y_2)\\ &= \int_{y_1=0}^{+\infty}\int_{y_2=0}^{+\infty} y_1y_2dF_{(X_1,X_2)}(y_1,y_2)\\ &= E[X_1X_2],\end{aligned}$$

and this completes the proof. $\diamond$

Proposition 2.3. *Let* $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ *and* U *be a random variable uniformly distributed over* $[0, 1]$*. Then, the following equivalences hold:*

$$Cov[X_1, X_2] = Cov[F_1^{-1}(U), F_2^{-1}(U)] \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(U)), \tag{2.2}$$

and

$$Cov[X_1, X_2] = Cov[F_1^{-1}(U), F_2^{-1}(1-U)] \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(1-U)), \tag{2.3}$$

where "$=_d$*" stands for the equality in distribution.*

Proof. Let us begin with (2.2). The "$\Leftarrow$"-part is well-known, so that we only consider the "$\Rightarrow$"-implication. It is easily seen that the equality of the covariances yields

$$E[X_1X_2] = E[F_1^{-1}(U)F_2^{-1}(U)].$$

Lemma 2.2 then gives

$$\begin{aligned}0 &= \int_{x_1=0}^{+\infty}\int_{x_2=0}^{+\infty} \left\{P[X_1 > x_1, X_2 > x_2] - P[F_1^{-1}(U) > x_1, F_2^{-1}(U) > x_2]\right\} dx_1dx_2\\ &= \int_{x_1=0}^{+\infty}\int_{x_2=0}^{+\infty} \left\{F_{(X_1,X_2)}(x_1,x_2) - W(x_1,x_2)\right\} dx_1dx_2. \qquad (2.4)\end{aligned}$$

By (1.1), the integrand $\{\ldots\}$ in (2.4) is non-positive for all x_1 and x_2; we then conclude that the equality $F_{(X_1,X_2)}(x_1, x_2) = W(x_1, x_2)$ holds almost everywhere, and this concludes the proof of (2.2). The reasoning to get (2.3) is similar and is therefore omitted. $\diamond$

Note that (2.2) and (2.3) possess another interesting interpretation: $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ has joint distribution function W if, and only if, $\text{Var}[X_1 + X_2]$ is maximal, i.e.

$$\text{Var}[X_1 + X_2] = \text{Var}[X_1] + \text{Var}[X_2] + \text{Cov}[F_1^{-1}(U), F_2^{-1}(U)];$$

$(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ has joint distribution function M if, and only if, $\text{Var}[X_1 + X_2]$ is minimal, i.e.

$$\text{Var}[X_1 + X_2] = \text{Var}[X_1] + \text{Var}[X_2] + \text{Cov}[F_1^{-1}(U), F_2^{-1}(1-U)].$$

Example 2.4. Spreading a portfolio premium to the individual policyholders is often a non-trivial process. For instance, one could spread in proportion to expected losses. However, this method does not differentiate contracts by hazard and is thus probably most appropriate when the riskiness is fairly homogeneous. An elegant suggestion has been presented by Karl Borch who recommended calculating the premium for the risk X by the formula

$$(1+\theta)\mathbb{E}X + \eta\mathbb{C}\mathrm{ov}[X, S],$$

where S is the portfolio aggregate claims. This formula gives premiums that add up to the portfolio premium even when the risks are not independent. Indeed, if the safety loadings θ and η are chosen so that

$$\text{Portfolio premium} = (1+\theta)\mathbb{E}S + \eta\mathbb{V}\mathrm{ar}[S],$$

we get for the portoflio

$$\sum_{i=1}^{n}\Big\{(1+\theta)\mathbb{E}X_i + \eta\mathbb{C}\mathrm{ov}[X_i, S]\Big\} = (1+\theta)\mathbb{E}S + \eta\mathbb{V}\mathrm{ar}[S].$$

In case the risk X is regarded as being strongly positively dependent with S, it may be reasonable to charge an amount of premium equal to

$$(1+\theta)\mathbb{E}X + \eta\mathbb{C}\mathrm{ov}[F_X^{-1}(U), F_S^{-1}(U)],$$

where U is uniformly distributed over $[0,1]$.

In some circumstances, the non-negativity assumption can be dropped. For example, if the marginals F_1 and F_2 belong to the same location scale family of distributions, i.e. if there exist a distribution function G, real constants μ_1, μ_2 and positive real constants σ_1, σ_2 such that the relation

$$F_i(x) = G\left(\frac{x-\mu_i}{\sigma_i}\right) \text{ holds for } i = 1, 2,$$

then

$$r(X_1, X_2) = 1 \Leftrightarrow (X_1, X_2) =_d (\sigma_1 G^{-1}(U) + \mu_1, \sigma_2 G^{-1}(U) + \mu_2).$$

For instance, if G is the distribution function of the standard normal distribution, the Fréchet upper bound is attained for perfectly correlated random couples.

Independence of two random variables implies they are uncorrelated (i.e. $r = 0$) but zero correlation does not in general imply independence. The noticeable exception is the case of the multivariate normal where uncorrelatedness and independence are equivalent. However, the independence structure is sometimes determined by the covariance structure for families of random variables which exhibit certain types of positive or negative dependence. Let us recall Lehmann's definition of positive and negative quadrant dependent (PQD and NQD, in short) random variables: the random couple $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ is said to be PQD if

$$F_1(x_1)F_2(x_2) \leq F_{(X_1,X_2)}(x_1, x_2) \text{ for all } x_1, x_2 \in \mathbb{R}; \tag{2.5}$$

it is said to be NQD if the reverse inequality holds in (2.5). Henceforth, given $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$, we denote as $(X_1^\perp, X_2^\perp)$ its independent version, i.e. the random couple in $\mathcal{R}(F_1, F_2)$ whose joint distribution function factors in F_1F_2. The following result is due to Lehmann (1966); we give a short proof of it for the sake of completeness.

Proposition 2.5. *Let* $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ *be either PQD or NQD. Then, the following equivalence holds:*

$$Cov[X_1, X_2] = 0 \Leftrightarrow (X_1, X_2) =_d (X_1^\perp, X_2^\perp).$$

Proof. The proof follows from

$$0 = \int_{x_1=0}^{+\infty} \int_{x_2=0}^{+\infty} \left\{ F_{(X_1,X_2)}(x_1, x_2) - F_1(x_1)F_2(x_2) \right\} dx_1 dx_2,$$

which is the analog of (2.4). The pointwise non-negativity (resp. non-positivity) of the integrand $\{\ldots\}$ for PQD (resp. NQD) random couples (X_1, X_2) ends the proof. $\diamond$

3 Kendall's τ

Kendall's τ measures a certain form of dependence known as "concordance": roughly speaking, "large" values of one component tend to be associated with "large" values of the other, and "small" values of one with "small" values of the other. To be specific, Kendall's τ is defined as the probability of "concordance" minus the probability of "discordance", i.e. given two independent and identically distributed random couples (X_1, X_2) and (Y_1, Y_2) in $\mathcal{R}(F_1, F_2)$,

$$\tau(X_1, X_2) = P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P[(X_1 - Y_1)(X_2 - Y_2) < 0].$$

Let us now prove the following result which states that $\tau(X_1, X_2) = \pm 1$ if, and only if, the distribution of the random couple (X_1, X_2) coincides with one of the Fréchet bounds. A similar result has been derived by Genest & McKay (1986). Henceforth (in Sections 3, 4 and 5), we assume that F_1 and F_2 are continuous and strictly increasing.

Proposition 3.1. *Let* $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ *and* U *be a random variable uniformly distributed over* $[0, 1]$*. Then, the following equivalences hold:*

$$\tau(X_1, X_2) = 1 \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(U)), \tag{3.1}$$

and

$$\tau(X_1, X_2) = -1 \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(1 - U)). \tag{3.2}$$

Proof. The "$\Leftarrow$"-parts of (3.1) and (3.2) are well-known. Since Kendall's τ is invariant under strictly monotone transformations (see Theorem 5.1.9 in Nelsen (1998)), the equality $\tau(X_1, X_2) = \tau(F_1(X_1), F_2(X_2))$ holds and we may thus consider without loss of generality that the margins F_1 and F_2 are Uniform$[0, 1]$. We have then to show that for any random couple (U_1, U_2) with Uniform$[0, 1]$ marginals

$$\tau(U_1, U_2) = 1 \Rightarrow (U_1, U_2) =_d (U, U).$$

Let C (resp. C_U) denote the joint distribution function of the pair (U_1, U_2) (resp. (U, U)); (1.1) of course ensures that $C \le C_U$. From Theorem 5.1.3 in Nelsen (1998), Kendall's τ can be casted into the form

$$\tau(U_1, U_2) = 4EC(U_1, U_2) - 1.$$

From $C \leq C_U$ and from (1.2) with $\phi = C_U$, we get

$$EC(U_1, U_2) \leq EC_U(U_1, U_2) \leq EC_U(U, U),$$

Now

$$\tau(U_1, U_2) = \tau(U, U) \Leftrightarrow EC(U_1, U_2) = EC_U(U, U)$$

implies

$$EC_U(U_1, U_2) - EC(U_1, U_2) = \int_{u_1=0}^{1} \int_{u_2=0}^{1} \underbrace{\{C_U(u_1, u_2) - C(u_1, u_2)\}}_{\geq 0 \ \forall \ u_1, u_2 \in [0,1]} dC(u_1, u_2) = 0,$$

and this implies that $C \equiv C_U$ and ends the proof of (3.1). Let C_L denotes the joint distribution function of the couple $(U, 1-U)$; (1.1) ensures that $C_L \leq C$. To get (3.2), it suffices to note that from (1.2) with $\phi = C_L$, we get

$$EC_L(U, 1-U) \leq EC_L(U_1, U_2) \leq EC(U_1, U_2),$$

which yields $EC(U_1, U_2) = EC_L(U_1, U_2)$. The proof then follows the same lines. $\diamond$

Let us now prove the analog of Proposition 2.3 for Kendall's τ.

Proposition 3.2. *Let $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ be either PQD or NQD. Then, the following equivalence holds:*

$$\tau(X_1, X_2) = 0 \Leftrightarrow (X_1, X_2) =_d (X_1^{\perp}, X_2^{\perp}).$$

Proof. (i) PQD case. Without loss of generality, let us consider the case of uniform margins. Let $C_I(u_1, u_2) = u_1 u_2$, $0 \leq u_1, u_2 \leq 1$. Saying that (U_1, U_2) is PQD means that its joint distribution function C satisfies $C \geq C_I$. Invoking (1.2) with $\phi = C_I$ then yields

$$EC(U_1, U_2) \geq EC_I(U_1, U_2) \geq EC_I(U_1^{\perp}, U_2^{\perp}),$$

whence it follows that $EC(U_1, U_2) = EC_I(U_1, U_2)$ since $EC(U_1, U_2) = EC_I(U_1^{\perp}, U_2^{\perp})$ by hypothesis. Therefore,

$$EC(U_1, U_2) - EC_I(U_1, U_2) = \int_{u_1=0}^{1} \int_{u_2=0}^{1} \underbrace{\{C(u_1, u_2) - C_I(u_1, u_2)\}}_{\geq 0 \ \forall \ u_1, u_2 \in [0,1]} dC(u_1, u_2) = 0,$$

yields $C \equiv C_I$.
(ii) NQD case. The proof is similar: starting from

$$EC(U_1, U_2) \leq EC_I(U_1, U_2) \leq EC_I(U_1^{\perp}, U_2^{\perp}),$$

we get $EC(U_1, U_2) = EC_I(U_1, U_2)$, which concludes the proof since $C \leq C_I$. $\diamond$

4 Spearman's ρ

As Kendall's τ, Spearman's ρ is based on concordance and discordance. Let us consider $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ and its independent version $(X_1^\perp, X_2^\perp)$. Assume further that (X_1, X_2) and $(X_1^\perp, X_2^\perp)$ are mutually independent. Then,

$$\rho(X_1, X_2) = 3\left\{P[(X_1 - X_1^\perp)(X_2 - X_2^\perp) > 0] - P[(X_1 - X_1^\perp)(X_2 - X_2^\perp) < 0]\right\}.$$

Let us now prove the next result, which is the analog for Spearman's ρ of Proposition 3.1.

Proposition 4.1. *Let $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ and U be a random variable uniformly distributed over $[0,1]$. Then, the following equivalences hold:*

$$\rho(X_1, X_2) = 1 \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(U)), \tag{4.1}$$

and

$$\rho(X_1, X_2) = -1 \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(1-U)), \tag{4.2}$$

Proof. The "$\Leftarrow$"-parts of (4.1) and (4.2) are well-known. Since Spearman's ρ in invariant under strictly monotone transformations (see Theorem 5.1.9 in Nelsen (1998)), it is sufficient to work with uniform random variables and to show that

$$\rho(U_1, U_2) = 1 \Rightarrow (U_1, U_2) =_d (U, U).$$

Since Spearman's ρ can be casted into

$$\rho(U_1, U_2) = 12 \int_{u_1=0}^{1} \int_{u_2=0}^{1} C(u_1, u_2) du_1 du_2 - 3$$

(see Theorem 5.1.6 in Nelsen (1998)), $\rho(U_1, U_2) = 1$ implies

$$\int_{u_1=0}^{1} \int_{u_2=0}^{1} \underbrace{\{C_U(u_1, u_2) - C(u_1, u_2)\}}_{\geq 0 \ \forall \ u_1, u_2 \in [0,1]} du_1 du_2 = 0,$$

and this implies that $C \equiv C_U$ and ends the proof of (4.1). The proof of (4.2) then follows the same lines. $\diamond$

The following result is the analog for Spearman's ρ of Proposition 2.1 concerning Pearson's r and of Proposition 3.1 concerning Kendall's τ. We omit its proof since it is straightly deduced from the reasoning followed to get Proposition 4.1.

Proposition 4.2. *Let $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ be either PQD or NQD. Then, the following equivalence holds:*

$$\rho(X_1, X_2) = 0 \Leftrightarrow (X_1, X_2) =_d (X_1^\perp, X_2^\perp).$$

5 Other concordance measures

Scarsini (1984) defined certain desirable properties for a measure of association between two random variables (see also Definition 5.1.7 and Theorem 5.1.8 in Nelsen (1998)) and introduced the name "concordance measures" for those satisfying these conditions. Kendall's tau and Spearman's ρ are both concordance measures in the sense of Scarsini (1984). Therefore, a natural question is whether the results given in Sections 3-4 are valid for all the concordance measures. The answer is however negative. We give below a concordance measure, Gini's γ, for which analogs of Propositions 3.1 and 4.1 hold, and another one, Blomqvist's β, for which they fail.

Gini's γ is used in Economics to measure the income differences between two populations. Technically, it is a kind of "distance" between the dependence structure of the vector (X_1, X_2) and monotone dependence as represented by M and W. To be specific, given $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$, Gini's γ is defined as

$$\gamma(X_1, X_2) = \gamma(F_1(X_1), F_2(X_2)) = 2 \int_{u_1=0}^{1} \int_{u_2=0}^{1} (|u_1 + u_2 - 1| - |u_1 - u_2|)\, dC(u_1, u_2),$$

where C is the joint distribution function of the couple $(F_1(X_1), F_2(X_2))$. The following results hold for Gini's γ.

Proposition 5.1. *Let* $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ *and* U *be a random variable uniformly distributed over* $[0, 1]$*. Then, the following equivalences hold:*

$$\gamma(X_1, X_2) = 1 \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(U)), \tag{5.1}$$

and

$$\gamma(X_1, X_2) = -1 \Leftrightarrow (X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(1 - U)), \tag{5.2}$$

Proof. The "$\Leftarrow$"-parts of (5.1) and (5.2) are well-known. Since Gini's γ is invariant under strictly monotone transformations (see Theorem 5.1.14 in Nelsen (1998)), it is sufficient to work with uniform random variables and to show that

$$\gamma(U_1, U_2) = 1 \Rightarrow (U_1, U_2) =_d (U, U).$$

Since

$$\gamma(U_1, U_2) = 4 \int_{u_1=0}^{1} \int_{u_2=0}^{1} C(u_1, u_2) dC_U(u_1, u_2) + 4 \int_{u_1=0}^{1} \int_{u_2=0}^{1} C(u_1, u_2) dC_L(u_1, u_2) - 2,$$

$\gamma(U_1, U_2) = \gamma(U, U)$ implies

$$\int_{u_1=0}^{1} \int_{u_2=0}^{1} \underbrace{\{C_U(u_1, u_2) - C(u_1, u_2)\}}_{\geq 0 \ \forall \ u_1, u_2 \in [0,1]} dC_U(u_1, u_2)$$
$$+ \int_{u_1=0}^{1} \int_{u_2=0}^{1} \underbrace{\{C_U(u_1, u_2) - C(u_1, u_2)\}}_{\geq 0 \ \forall \ u_1, u_2 \in [0,1]} dC_L(u_1, u_2) = 0,$$

and this implies that $C \equiv C_U$ and ends the proof of (5.1). The proof of (5.2) then follows the same lines. $\diamond$

Proposition 5.2. *Let $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ be either PQD or NQD. Then, the following equivalence holds:*

$$\gamma(X_1, X_2) = 0 \Leftrightarrow (X_1, X_2) =_d (X_1^\perp, X_2^\perp).$$

Another measure of concordance is Blomqvist's β, also known as the medial correlation coefficient. It is defined as

$$\begin{aligned} \beta(X_1, X_2) &= P[(X_1 - x_{1/2}^{(1)})(X_2 - x_{1/2}^{(2)}) > 0] - P[(X_1 - x_{1/2}^{(1)})(X_2 - x_{1/2}^{(2)}) < 0] \\ &= 4F_{(X_1,X_2)}(x_{1/2}^{(1)}, x_{1/2}^{(2)}) - 1, \end{aligned}$$

where $x_{1/2}^{(i)}$ is the median of X_i, $i = 1, 2$.

Proposition 5.3. *Let $(X_1, X_2) \in \mathcal{R}(F_1, F_2)$ and U be a random variable uniformly distributed over $[0, 1]$. Then, the following implications hold:*

$$(X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(U)) \Rightarrow \beta(X_1, X_2) = 1, \tag{5.3}$$

$$(X_1, X_2) =_d (F_1^{-1}(U), F_2^{-1}(1-U)) \Rightarrow \beta(X_1, X_2) = -1 \tag{5.4}$$

and

$$(X_1, X_2) =_d (X_1^\perp, X_2^\perp) \Rightarrow \beta(X_1, X_2) = 0, \tag{5.5}$$

but the converses of (5.3), (5.4) and (5.5) are in general not true.

Proof. The implications (5.3), (5.4) and (5.5) are well-known. Since Blomqvist's β is invariant under strictly monotone transformations (see Theorem 5.1.14 in Nelsen (1998)), it is sufficient to work with uniform random variables. Then, given a random couple (U_1, U_2) with Uniform$[0, 1]$ marginals and with joint distribution function C, Blomqvist's β can be written as

$$\beta(U_1, U_2) = 4C(1/2, 1/2) - 1.$$

Let us consider the parametric family given in Formula (3.2.2) in Nelson (1998). For

$$C(u_1, u_2) = \begin{cases} \max(0, u_1 + u_2 - \frac{1}{2}) \text{ for } 0 \le u_1, u_2 \le \frac{1}{2}, \\ \max(\frac{1}{2}, u_1 + u_2 - 1) \text{ for } \frac{1}{2} < u_1, u_2 \le 1, \\ C_U(u_1, u_2) \text{ otherwise.} \end{cases}$$

It is then easily seen that $\beta(U_1, U_2) = 1$ whereas C does not coincide with C_U, contradicting (5.3). On the other hand, for

$$C(u_1, u_2) = \begin{cases} \max(0, u_1 + u_2 - \frac{1}{4}) \text{ for } 0 \le u_1, u_2 \le \frac{1}{4}, \\ \max(\frac{1}{4}, u_1 + u_2 - 1) \text{ for } \frac{1}{4} < u_1, u_2 \le 1, \\ C_U(u_1, u_2) \text{ otherwise,} \end{cases}$$

we have $\beta(U_1, U_2) = 0$ but $C \neq C_I$, contradicting (5.5). Finally, let us consider the member of the family in Exercise 3.9 of Nelsen (1998)

$$C(u_1, u_2) = \begin{cases} \min(u_1, u_2 - \theta) \text{ for } (u_1, u_2) \in [0, 1-\theta] \times [\theta, 1], \\ \min(u_1 + \theta - 1, u_2) \text{ for } (u_1, u_2) \in [1-\theta, 1] \times [0, \theta], \\ C_L(u_1, u_2) \text{ otherwise,} \end{cases}$$

coresponding to $\theta = 1/2$. We then have $\beta(U_1, U_2) = -1$ but $C \neq C_L$, contradicting (5.4). $\diamond$

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References

Denuit, M., Lefèvre, Cl., & Mesfioui, M. (1999). A class of bivariate stochastic orderings, with applications in actuarial sciences. *Insurance: Mathematics & Economics* **24**, 31-50.

Dhaene, J., & Denuit, M. (1999). The safest dependence structure among risks. *Insurance: Mathematics & Economics* **25**, 11-21.

Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., & Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and finance: Theory. *Insurance: Mathematics & Economics* **31**, 3-33.

Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., & Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and finance: Applications. *Insurance: Mathematics & Economics* **31**, 133-161.

Genest, Ch., & McKay, J. (1986). The joy of copulas: bivariate distributions with uniform margins. *The American Statistician* **40**, 280-283.

Lehmann, E.L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics* **37**, 1137-1153.

Nelsen, R.B. (1998). *An Introduction to Copulas.* Lecture Notes in Statistics **139**, Springer Verlag, New York.

Scarsini, M. (1984). On measures of concordance. *Stochastica* **8**, 201-218.

Schechtman, E., and Yitzhaki, S. (1999). On the proper bounds of the Gini Correlation. *Economics Letters* **63**, 133-138.

Shih, W. J., and Huang, W. (1992). Evaluating correlation with proper bounds. *Biometrics* **48**, 1207-1213.

Tchen, A. (1980). Inequalities for distributions with given marginals, *The Annals of Probability* **8**, 814-827.

Wang, S., & Dhaene, J. (1997). Comonotonicity, correlation order and premium principles. *Insurance: Mathematics & Economics* **22**, 235-242.

Yaari, M.E. (1987). The Dual Theory of Choice under Risk. *Econometrica* **55**, 95-115.