On the computation of the capital multiplier in the Fortis Credit Economic Capital model

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Abstract

One of the key parameters in the computation of Credit Economic Capital is the so called capital multiplier. In the light of a variance-covariance approach we propose a methodology for computing this parameter rather than using a benchmark number for it. The paper provides an algorithm in doing so.

1 Introduction and motivation

Credit Risk is the risk that a borrower will be unable to pay back his loan. For any individual contract, the future loss (in a one year period) is random, i.e. unknown in advance. The sum of all these losses is called the Portfolio Credit Loss. Fortis quantifies its credit risk through the measurement of the variability of this portfolio credit loss and capital is held to protect against this risk. The amount of this capital has been calibrated to achieve the Fortis target of a S&P rating of 'AA', meaning that the required capital corresponds to a 3 bp. or less default probability over a one-year time horizon.

In order to calculate this capital, the current credit risk framework within Fortis focuses on 2 measures: Expected Loss and Unexpected Loss.

Expected Loss (EL) is the expected annual level of credit losses. Actual losses for any given year will vary from the EL, but EL is the amount that Fortis should expect to lose on average. Expected Loss should be viewed as a cost of doing business rather than as a risk itself.

The real risk arises from the volatility in loss levels. This volatility is called **Unexpected Loss (UL)**. UL is defined statistically as the standard deviation of the credit loss distribution.

Once these two measures are calculated, Fortis determines the **Economic Capital** as a multiple of the Unexpected Loss. This multiple is called the **Capital Multiplier**:

$$\text{EconomicCapital} = \text{Capital Multiplier x Unexpected Loss.}$$

Fortis has models that enable the calculation of the portfolio Unexpected Loss, specifically for its portfolio. For the Capital Multiplier however a benchmark number is being used. Historical data analysis seems to indicate that the loss rate for a large portfolio follows a Beta distribution and based on this observation, the capital multiplier for a "typical and large" bank portfolio, such as Fortis, has been determined. This raises the following questions:

1. What is a 'typical' bank portfolio, i.e. what are the Expected and Unexpected Loss that are implicitly assumed in order to obtain this capital multiplier ?

2. Is the assumption of a Beta distribution correct in all situations, e.g. what if the portfolio is not 'large' enough?

3. What methodology should be used if one deals with small and/or a-typically diversified portfolios?

In close collaboration with the Actuarial Research Group of K.U.Leuven, Fortis developed a model that takes into account for each loan the individual risk parameters: exposure, rating, loss given default and default correlations. This new methodology uses the same parameters as Fortis is using now for computing the "portfolio unexpected loss". Hence it provides a full bottom-up approach as compared to the current approach where a bottom-up model is combined with an external benchmark number.

Given the exact distribution function of the Aggregate Loss S (expressed as a percentage of the "Aggregate-Exposure-At-Default"), it is straightforward to determine its $(1-\epsilon)$ percentile and hence the multiplier "K_ϵ" which is defined as

$$F_S^{-1}(1-\epsilon) = E(S) + K_\epsilon \, \sigma_S.$$

Here S denotes the Loss random variable, $E(S)$ its expection, σ_S its standard deviation and F_S^{-1} its quantile function.

The random variable S is the sum of the losses on the individual policies. Hence, S is a sum of "positive" dependent random variables. In the model that we will present, we assume that we know the distribution functions of the individual losses, as well as the correlations between these individual losses. It is important to note that this information is not enough to determine the distribution of S exactly. In fact, knowledge of the whole multivariate distribution is needed in order to be able to determine the distribution function of the sum. Only in the case of a multivariate normal distribution, the marginal distributions together with the correlation matrix completely determines the distribution function of the sum.

In this document, we propose an appropriate approximation for the distribution function of S, taking into account all the available information (marginal distribution functions and correlation matrix). We will describe an algorithm that can be used to calculate (an approximation for) the distribution function of the Loss. Let S' be a random variable having this approximate distribution function, then, we propose to determine the multiplier "K_ϵ" by

$$F_{S'}^{-1}(1-\epsilon) = E(S^{'}) + K_\epsilon \ \sigma_{S'}.$$

2 Preliminary theoretical results

2.1 The Gamma distribution

If $Y \stackrel{d}{=} \text{Gamma}(\alpha, \beta)$, with $\alpha > 0$ and $\beta > 0$, then the probability density function (pdf) of Y is given by

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \qquad (y > 0).$$

We also have that

$$
\begin{aligned}
E[Y] &= \frac{\alpha}{\beta}, \\
Var[Y] &= \frac{\alpha}{\beta^2}, \\
E\left[Y^k\right] &= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}?? \\
m_Y(t) &= E\left[e^{tY}\right] = \left(\frac{\beta}{\beta-t}\right)^{\alpha}, \qquad (t<\beta),
\end{aligned}
$$

where $m_Y(t)$ denotes the "moment generating function" of Y evaluated at t.

2.2 The Poisson distribution

If $N \overset{d}{=} \text{Poisson}(\lambda)$, with $\lambda > 0$, then the probability function of N is given by

$$
\Pr[N=x] = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad (x = 0, 1, 2, \ \ldots).
$$

We have that

$$
\begin{aligned}
E[N] &= Var[N] = \lambda, \\
m_N(t) &= E\left[e^{tN}\right] = \exp\left[\lambda\left(e^t - 1\right)\right].
\end{aligned}
$$

2.3 The Negative Binomial distribution

If $N \overset{d}{=} \text{NB}(r, p)$, with $r > 0$ and $0 < p \leq 1$, then the probability function of N is given by

$$
\Pr[N=x] = \left(\begin{array}{c} r+x-1 \\ x \end{array}\right) p^r \ (1-p)^x, \qquad (x = 0, 1, 2, \ \ldots).
$$

The first two moments and the moment generating function are given by

$$
\begin{aligned}
E[N] &= \frac{r(1-p)}{p} \\
Var[N] &= \frac{r(1-p)}{p^2}, \\
m_N(t) &= E\left[e^{tN}\right] = \left(\frac{p}{1-(1-p)\ e^t}\right)^r.
\end{aligned}
$$

2.4 The Beta distribution

If $Y \stackrel{d}{=} \text{Beta}(a, b)$, with $a > 0$ and $b > 0$, then the probability density function of Y is given by

$$f_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} y^{a-1}(1-y)^{b-1}, \qquad (0 < y < 1).$$

The first two moments and the moment generating function are given by

$$\begin{aligned} E[Y] &= \frac{a}{a+b}, \\ Var[Y] &= \frac{ab}{(a+b+1)(a+b)^2}, \\ E\left[Y^k\right] &= \frac{a(a+1)\ \cdots\ (a+k-1)}{(a+b)(a+b+1)\ \cdots\ (a+b+k-1)}. \end{aligned}$$

2.5 The Negative Binomial distribution as a Poisson-Gamma mixture

We assume that the random variable $N \mid \Lambda = \lambda$ has a Poisson distribution with parameter λ:

$$(N \mid \Lambda = \lambda) \stackrel{d}{=} \text{Poisson}\,(\ \lambda)\,.$$

Further, we assume that the mixing random variable Λ has a Gamma distribution with parameters α and β:

$$\Lambda \stackrel{d}{=} \text{Gamma}\,(\alpha, \beta)\,.$$

The random variable Λ is also called the structure variable.
We find

$$\begin{aligned} m_N(t) &= E\left[e^{tN}\right] \\ &= E_\Lambda[E\left[e^{tN} \mid \Lambda\right] \\ &= E_\Lambda[e^{\Lambda(e^t-1)}] \\ &= m_\Lambda(e^t - 1) \\ &= \left(\frac{\beta}{\beta - (e^t - 1)}\right)^\alpha \\ &= \left(\frac{p}{1-(1-p)e^t}\right)^\alpha \end{aligned}$$

with $p = \frac{\beta}{\beta+1}$. We can conclude that

$$N \stackrel{d}{=} \mathrm{NB}\left(\alpha,\ \frac{\beta}{\beta+1}\right).$$

2.6 The moment generating function of a compound distribution

Consider a collective model

$$S = \sum_{i=1}^{N} X_i$$

where the severities X_i are i.i.d. and independent of the frequency N. Let

$$X_i \stackrel{d}{=} X, \qquad (i = 1, 2,\ \ldots, n).$$

Then we find

$$\begin{aligned} m_S(t) &= E\left[e^{tS}\right] \\ &= E_N\ E\left[e^{t\sum_{i=1}^{N} X_i} \mid N\right] \\ &= E_N\ \left[(m_X(t))^N \mid N\right] \\ &= E\left[e^{N\ \ln m_X(t)}\right]. \end{aligned}$$

Hence,

$$m_S(t) = m_N\left(\ln m_X(t)\right).$$

It is easy to show that

$$E(S) = E(N)\ E(X)$$

and

$$Var(S) = E(N)\ Var(X) + [E(X)]^2\ Var(N).$$

2.7 The sum of independent Compound Poisson distributions

Consider the sum of n independent Compound Poisson distributions:

$$S = \sum_{j=1}^{N_1} X_{1j} + \cdots + \sum_{j=1}^{N_n} X_{nj}.$$

Hence, we assume that the n random variables $\sum_{j=1}^{N_i} X_{ij}$ are mutually independent. For each i, we assume that the X_{ij} are mutually independent and independent of N_i. We also assume that

$$X_{ij} \stackrel{d}{=} X_i$$

and

$$N_i \stackrel{d}{=} \text{Poisson } \left(\lambda_i\right).$$

Then we find

$$\begin{aligned}
m_S(t) &= E\left[e^{tS}\right] \\
&= E\left[\exp\left(t\sum_{i=1}^{n}\sum_{j=1}^{N_i} X_{ij}\right)\right] \\
&= \Pi_{i=1}^{n}\ E\left\{\exp\left(t\ \sum_{j=1}^{N_i} X_{ij}\right)\right\} \\
&= \Pi_{i=1}^{n}\ m_{N_i}\left(\ln m_{X_i}(t)\right) \\
&= \Pi_{i=1}^{n}\ \exp\left[\lambda_i\left(m_{X_i}(t)-1\right)\right] \\
&= \exp\left[\sum_{i=1}^{n}\lambda_i\left(m_{X_i}(t)-1\right)\right] \\
&= \exp\left[\lambda\left(m_X(t)-1\right)\right]
\end{aligned}$$

with

$$\lambda = \sum_{i=1}^{n}\lambda_i$$

and

$$m_X(t) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} m_{X_i}(t).$$

We can conclude that the sum of the n mutually independent Compound Poisson distributed random variables is again Compound Poisson distributed with Poisson parameter λ and severity distribution F_X given by

$$F_X(x) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} F_{X_i}(x).$$

2.8 The sum of dependent Compound Negative Binomial distributions

Consider the compound distributed random variables $\sum_{j=1}^{N_i} X_{ij}$, $(i = 1, 2, \ldots, n)$.

We assume that the random variables $N_i \mid \Lambda = \lambda$ have a Poisson distribution with parameter $q_i\ \lambda$:

$$(N_i \mid \Lambda = \lambda) \stackrel{d}{=} \text{Poisson}\left(q_i\ \lambda\right).$$

We also assume that the mixing random variable Λ has a Gamma distribution with parameters α and β:

$$\Lambda \stackrel{d}{=} \text{Gamma}\left(\alpha, \beta\right).$$

This implies that the random variables N_i are negative binomial distributed, but not independent.

Further, we assume that for each i, the severities X_{ij}, $(j = 1, 2, \ldots)$, are mutually independent:

$$X_{ij} \stackrel{d}{=} X_i, \qquad (j = 1, 2, \ldots),$$

and also that the X_{ij} are independent of the mixing random variable Λ, and of $N_i \mid \Lambda = \lambda$.
Finally, we assume that the compound sums $\sum_{j=1}^{N_i|\Lambda=\lambda} X_{ij}$, $(i = 1, 2, \ldots, n)$ are mutually independent

Under these assumptions, the sum S defined by

$$S = \sum_{j=1}^{N_1} X_{1j} + \cdots + \sum_{j=1}^{N_n} X_{nj},$$

is a sum of Compound Negative Binomial distributed random variables. At first sight, determining the distribution function of the sum S is not a trivial task, as the random variables $\sum_{j=1}^{N_i} X_{ij}$, $(i = 1, \ldots, n)$ are not mutually independent, the dependency caused by the common mixing random variable Λ. For a general approach of approximating sums of dependent random variables, we refer to Dhaene, Denuit, Kaas, Goovaerts & Vyncke (2002 a, b).

In this particular case however, one can prove that the distribution function of the combined portfolio is Compound Negative Binomial distributed. Indeed, we have that the moment generating function of S is given by

$$\begin{aligned} m_S(t) &= E\left[e^{tS}\right] \\ &= E\left[\exp\left(t\sum_{i=1}^{n}\sum_{j=1}^{N_i} X_{ij}\right)\right] \\ &= E\left\{\Pi_{i=1}^{n}\exp\left(t\sum_{j=1}^{N_i} X_{ij}\right)\right\} \\ &= E_\Lambda\left\{E\left\{\Pi_{i=1}^{n}\exp\left(t\sum_{j=1}^{N_i} X_{ij}\right) \mid \Lambda\right\}\right\} \\ &= E_\Lambda\left\{\Pi_{i=1}^{n}\ E\left\{\exp\left(t\sum_{j=1}^{N_i} X_{ij}\right) \mid \Lambda\right\}\right\} \\ &= E_\Lambda\left\{\Pi_{i=1}^{n}\ m_{N_i|\Lambda}\left(\ln m_{X_i}(t)\right)\right\} \\ &= E_\Lambda\left\{\Pi_{i=1}^{n}\exp\left[q_i\ \Lambda\ \left(m_{X_i}(t)-1\right)\right]\right\} \\ &= m_\Lambda\left\{\sum_{i=1}^{n} q_i\ \left(m_{X_i}(t)-1\right)\right\} \\ &= \left[\frac{\beta}{\beta-\sum_{i=1}^{n} q_i\ \left(m_{X_i}(t)-1\right)}.\right]^{\alpha} \end{aligned}$$

Now, let

$$q = \sum_{i=1}^{n} q_i$$

and let X be a random variable with moment generating function given by

$$m_X(t) = \frac{\sum_{i=1}^{n} q_i \; m_{X_i}(t)}{\sum_{i=1}^{n} q_i}.$$

Then we find

$$\begin{aligned} m_S(t) &= \left[\frac{\beta}{\beta - (m_X(t) - 1)}.\right]^{\alpha} \\ &= \left[\frac{p}{1 - (1-p) \; e^{\ln[m_X(t)]}}\right]^{\alpha} \\ &= m_N \left(\ln m_X(t)\right) \end{aligned}$$

with p given by

$$p = \frac{\beta}{\beta + q}.$$

We can conclude that S is Compound Negative Binomial distributed:

$$S \stackrel{d}{=} \sum_{i=1}^{N} Y_i$$

with

$$N \stackrel{d}{=} NB(\alpha, \frac{\beta}{\beta + q})$$

and where the $Y_i \stackrel{d}{=} Y$ are i.i.d. and independent of N, with the moment generation function of the Y_i given by

$$m_Y(t) = \frac{\sum_{i=1}^{n} q_i \; m_{X_i}(t)}{\sum_{i=1}^{n} q_i},$$

or equivalently,

$$F_Y(x) = \sum_{i=1}^{n} \frac{q_i}{q} F_{X_i}(x).$$

Note that the distribution function of S can also be determined from the results of the previous section on sums of independent compound Poisson distribtions.

3 Description of the model

Consider a portfolio of n credit risks. Let I_i be defined as the indicator variable which equals 1 if risk i leads to failure in the next period, and 0 otherwise. The probability that risk i leads to a failure is denoted by q_i:

$$q_i = \Pr\left[I_i = 1\right].$$

Further, let $(EAD)_i$ denote the "Exposure-At-Default" and $(LGD)_i$ the "Loss-Given-Default" of risk i. The "Exposure-At-Default" is the maximal amount of loss on risk i, given default occurs. The "Loss-Given-Default" is the percentage of the loss on policy i, given default occurs. The "Aggregate Portfolio Loss" (the Loss for short) during the reference period is then given by

$$\text{Loss} = \sum_{i=1}^{n} I_i \ (EAD)_i \ (LGD)_i\,.$$

We will assume that the $(EAD)_i$ and the $(LGD)_i$ are deterministic.

We are interested in the random variable describing the Loss as a percentage of the "Aggregate-Exposure-at-Default", where the "Aggregate-Exposure-at-Default" is given by

$$\text{Aggregate-Exposure-at-Default} = \sum_{i=1}^{n} (EAD)_i$$

which is deterministic because of the assumptions made above. Hence, we are interested in determining the distribution function of

$$S = \frac{\sum_{i=1}^{n} I_i \ (EAD)_i \ (LGD)_i}{\sum_{j=1}^{n} (EAD)_j}.$$

So that the random variable of interest can be written as

$$S = \sum_{i=1}^{n} I_i \ c_i$$

with

$$c_i = \frac{(EAD)_i}{\sum_{j=1}^{n} (EAD)_j} \ (LGD)_i\,.$$

Note that we can write S as a sum of n compound Bernouilli random variables:

$$S = \sum_{i=1}^{n} \sum_{j=1}^{I_i} c_i,$$

where, by convention, $\sum_{i=1}^{0} = 0$.

The Aggregate Loss S is the sum of the (relative) losses on the individual credit risks. In order to compute the distribution function S exactly, knowledge of the multivariate distribution function is required.

We will assume however that our information about the distribution function of S is not "complete". To be more precise, we assume that we know the marginal distribution functions involved, i.e. we assume that the default probabilities q_i are given. Furthermore, we assume that the pairwise default correlations $corr\,[I_i,\ I_j]$, or equivalently, the pairwise correlations between the marginal risks in the sum, are given. In this paper, we will not discuss how to choose or build a model of default correlations.

Note that additional assumptions need to be made concerning the dependency structure between the terms in the sum S in order to be able to determine (approximations) for its percentiles e.g.

4 Approximation for the distribution function of S

The random variable S as defined above can be interpreted as the aggregate claims in an individual risk model, see e.g. Kaas, Dhaene, Goovaerts & Denuit (2001). We will approximate this individual risk model by a collective risk model. One major problem in this respect is the fact that S is the sum of mutually dependent random variables. Indeed, in any realistic model, we will have that the indicator variables I_i all will be positive dependent in some sense, where the positive dependence is caused by a common factor which describes the "global state of the economy".

We propose to approximate each I_i by a random variable N_i. In order to introduce the dependency, we will consider a "Baysian approach". Therefore, let us assume that there exists a random variable Λ such that, conditionally given $\Lambda = \lambda$, the random variables N_i are mutually independent:

$$(N_i \mid \Lambda = \lambda) \text{ are mutually independent.}$$

We further assume that, conditionaly given $\Lambda = \lambda$, the random variables N_i are Poisson distributed with parameters $q_i\ \lambda$:

$$(N_i \mid \Lambda = \lambda) \stackrel{d}{=} \text{Poisson}\left(q_i\ \lambda\right).$$

Furthermore, we assume that the random variable Λ has a Gamma distribution with parameters α and β. We will denote this as

$$\Lambda \stackrel{d}{=} \text{Gamma}\left(\alpha, \beta\right).$$

In order to determine the distribution function of N_i, we will determine its moment generation function. We find

$$\begin{aligned} E\left[e^{tN_i}\right] &= E\left[E\left[e^{tN_i} \mid \Lambda\right]\right] \\ &= E\left[\exp\left(q_i\ \Lambda\ \left(e^t - 1\right)\right)\right] \\ &= m_\Lambda\left(q_i\ \left(e^t - 1\right)\right) \\ &= \left(\frac{\beta}{\beta - q_i\ \left(e^t - 1\right)}\right)^\alpha \\ &= \left(\frac{\frac{\beta}{\beta + q_i}}{1 - \left(1 - \frac{\beta}{\beta + q_i}\right) e^t}\right)^\alpha, \end{aligned}$$

which implies that

$$N_i \stackrel{d}{=} NB\left(\alpha, \frac{\beta}{\beta + q_i}\right).$$

To summarize, we propose to approximate the distribution function of S :

$$S = \sum_{i=1}^{n} \sum_{j=1}^{I_i} c_i,$$

by the distribution function of S' :

$$S' = \sum_{i=1}^{n} \sum_{j=1}^{N_i} c_i.$$

5 Choice of the parameters α and β

In this section, we will explain how to choose the parameters α and β such that the distribution functions of S and S' are "as alike as possible", given the limited information on the random vector $(I_1, I_2, \ldots, I_n)$.

5.1 Determination of α

A first requirement for our approximation to perform well is that the distribution functions of I_i and N_i are "as alike as possible".
In order to have that $E[N_i] = E[I_i] = q_i$, we have to choose α equal to β:

$$\alpha = \beta$$

- Remark 1:
This choice implies that $Var[I_i] = q_i(1 - q_i) \leq Var[N_i] = q_i\left(1 + \frac{q_i}{\beta}\right)$. It can be proven that

$$I_i \leq_{cx} N_i,$$

where $\leq_{cx}$ stands for "smaller in the convex-order sense". This means that it is a safe strategy to replace I_i by N_i, in the sense that any risk-averse decision-maker would prefer claim-numbers I_i to N_i, for more details see Kaas, Goovaerts, Dhaene & Denuit (2001).
- Remark 2:
Under the choice $\alpha = \beta$, the distributions of I_i and N_i will be close to each other (provided $\frac{q_i}{\beta}$ is small enough such that higher order terms can be neglected). Indeed, we have that

$$\Pr[N_i = 0] = \left(\frac{\beta}{\beta + q_i}\right)^{\beta} \approx 1 - q_i = \Pr[I_i = 0],$$

while

$$\Pr[N_i = 1] = \beta\left(\frac{\beta}{\beta + q_i}\right)^{\beta}(1 - \frac{\beta}{\beta + q_i}) \approx q_i = \Pr[I_i = 1].$$

- Remark 3:

It is straightforward to verify that the choice $\alpha = \beta$ implies

$$E[S] = E[S'].$$

5.2 Determination of β

It remains to determine an explicite value for the parameter β.

First note that for $i \neq j$ we have that

$$\begin{aligned} Covar\left[N_i\ ,N_j\right] &= E\left[E\left[N_i\ N_j \mid \Lambda\right]\right] - E\left[N_i\right]\ E\left[N_j\right] \\ &= E\left[E\left[N_i \mid \Lambda\right]\ E\left[N_j \mid \Lambda\right]\right] - q_i\ q_j \\ &= q_i\ q_j \left\{E\left[\Lambda^2\right] - 1\right\} \\ &= q_i\ q_j\ Var\left[\Lambda\right] \\ &= \frac{q_i\ q_j}{\beta}, \end{aligned}$$

while

$$Var\left[N_i\right] = q_i \left(1 + \frac{q_i}{\beta}\right).$$

Hence, for $i \neq j$, the pairwise correlations in the approximated model are given by

$$\begin{aligned} corr\left[N_i\ ,N_j\right] &= \frac{\sqrt{q_i\ q_j}}{\beta} \frac{1}{\sqrt{1+\frac{q_i}{\beta}}\sqrt{1+\frac{q_j}{\beta}}} \\ &\approx \frac{\sqrt{q_i\ q_j}}{\beta}. \end{aligned}$$

In order to fix the parameter β we require that the second moments of S and S' coïncide. We have that

$$Var\left[S\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j\ covar\left[I_i\ ,I_j\right],$$

which is assumed to be known. On the other hand, we have that

$$\begin{aligned} Var\left[S'\right] &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j\ covar\left[N_i\ ,N_j\right] \\ &= \frac{\left(\sum_{i=1}^{n} c_i\ q_i\right)^2}{\beta} + \left(\sum_{i=1}^{n} c_i^2\ q_i\right). \end{aligned}$$

Hence, the condition $Var\,(S) = Var\,(S')$ will be fulfilled if β is chosen as follows:

$$\beta = \frac{\left(\sum_{i=1}^{n} c_i\ q_i\right)^2}{Var(S) - \left(\sum_{i=1}^{n} c_i^2\ q_i\right)}.$$

Note that in the model that we propose, in fact we replace the known correlations $corr\,[I_i\ , I_j]$ by $corr\,[N_i,\ N_j] \approx \frac{\sqrt{q_i\ q_j}}{\beta}$ for $i \neq j$. Hence, our approximation will perform the best if the "exact" correlations $corr\,[I_i\ , I_j]$ are approximately equal to $\frac{\sqrt{q_i\ q_j}}{\beta}$. This correlation structure seems to be consistent with many realistic correlation models.

6 How to compute the d.f. of S'?

From the Section " Preliminary Theoretical Results", we find that S' has a Compound Negative Binomial distribution:

$$S' \overset{d}{=} \sum_{j=1}^{N} Y_i,$$

where

$$N \overset{d}{=} \text{NB}\left(\beta, \frac{\beta}{\beta + \sum_{i=1}^{n} q_i}\right)$$

and where the Y_i are i.i.d. and independent of N, with the moment generation function of the Y_i given by

$$m_{Y_i}(t) = \frac{\sum_{i=1}^{n} q_i\ m_{c_i}(t)}{\sum_{i=1}^{n} q_i}.$$

Without loss of generality we can assume all the c_i to be different. In this case , we find that the random variables Y_i have the following probability function:

$$\Pr\left[Y_i = c_i\right] = \frac{q_i}{\sum_{i=1}^{n} q_i}, \qquad (i = 1, 2, \ldots, n)\,.$$

It remains to present an algorithm with enables to compute the distribution function of a Compound Negative Binomial distribution. This can be performed by a well-known recursion in actuarial sciences, called "Panjer's recursion", see e.g. Kaas, Goovaerts, Dhaene & Denuit (2001). We have that

$$\Pr\left[S' = 0\right] = \Pr\left[N = 0\right]$$

and also

$$\Pr\left[S' = x\right] = \sum_{k=1}^{x} \left(a + \frac{bk}{x}\right) \Pr\left[N = k\right] \Pr\left[S' = x - k\right], \qquad (x = 1, 2, \dots),$$

where

$$a = \frac{\sum_{i=1}^{n} q_i}{\beta + \sum_{i=1}^{n} q_i}$$

and

$$b = a\left(\beta - 1\right).$$

7 Assymptotic behaviour of the proposed approximation

In this section we will show that, under certain assumptions, the distribution function of the aggregate loss S' tends to a Beta distribution when the size of the portfolio becomes sufficiently large.

Assume that all EAD_i and LGD_i are equal to 1, and that all default probabilities q_i are equal to q. These assumptions imply that all c_i are equal to $\frac{1}{n}$. In this case S' has the following Compound Negative Binomial distribution:

$$S' \overset{d}{=} \frac{N}{n}$$

where

$$N \overset{d}{=} \text{NB}\left(\beta, \frac{\beta}{\beta + n\, q}\right)$$

The moment generation function of S' is then given by

$$\begin{aligned} m_{S'}(t) &= m_N\left(\frac{t}{n}\right), \\ &= \left(\frac{\frac{\beta}{\beta+n\ q}}{1-\left(1-\frac{\beta}{\beta+n\ q}\right)e^{t/n}}\right)^{\beta} \\ &= \left(\frac{\beta}{\beta+n\ q-n\ q\ e^{t/n}}\right)^{\beta}. \end{aligned}$$

If the number of contracts n reaches infinity, we find

$$\lim_{n->\infty} m_{S'}(t) = \left(\frac{\beta/q}{\beta/q-t}\right)^{\beta}.$$

This means that if the number of contracts becomes very large, the proposed approximation S' for the aggregate loss will be approximately Gamma distributed:

$$S' \overset{d}{\approx} \text{ Gamma}(\beta, \beta/q) \text{ for } n \text{ sufficiently large.}$$

Note that the "true" outcomes of S are in the region $[0,1]$, while the above mentioned Gamma approximation leads to outcomes in the range $[0,\infty)$. In practice however the probability of exceeding 1, computed with the Gamma$(\beta,\beta/q)$ distribution is almost equal to 0. For $\beta = 1$ and $q = 0.01$ e.g., this probability equals $e^{-100} \approx 0$.

It is interesting to compare this Gamma distribution with a Beta(a,b) distribution. In order to match the first 2 moments of the Beta distribution with the Gamma distribution, we must have that

$$\frac{a}{a+b} = q$$

and

$$\frac{ab}{(a+b+1)(a+b)^2} = \frac{q^2}{\beta}.$$

Hence,

$$\begin{aligned} a &= \beta(1-q)-q, \\ b &= \frac{\beta(1-q)}{q} - 1 - a. \end{aligned}$$

Let us now compare the moments of $X \stackrel{d}{=} \text{Gamma}(\beta, \beta/q)$ and $Y \stackrel{d}{=} \text{Beta}(a, b)$ where the parameters are connected as above. We find

$$E(X^k) = (\frac{q}{\beta})^k \ \beta \ (\beta+1) \ ... \ (\beta+k-1)$$

and

$$E(Y^k) = \left(\frac{\beta(1-q)-q}{\frac{\beta(1-q)}{q}-1}\right)\left(\frac{\beta(1-q)-q+1}{\frac{\beta(1-q)}{q}}\right)...\left(\frac{\beta(1-q)-q+k-1}{\frac{\beta(1-q)}{q}+k-2}\right).$$

After some straightforward computations we find

$$\begin{aligned} E(Y^k) &= (\frac{q}{\beta})^k \left(\frac{\beta(1-q)-q}{1-q-\frac{q}{\beta}}\right)\left(\frac{\beta(1-q)-q+1}{1-q}\right)...\left(\frac{\beta(1-q)-q+k-1}{1-q+\frac{(k-2)q}{\beta}}\right) \\ &= (\frac{q}{\beta})^k \ \beta \left(\frac{1-q\frac{\beta+1}{\beta}}{1-q\frac{\beta+1}{\beta}}\right)(\beta+1)\left(\frac{1-q\frac{\beta+1}{\beta+1}}{1-q\frac{\beta}{\beta}}\right) \ ... \ (\beta+k-1)\left(\frac{1-q\frac{\beta+1}{\beta+k-1}}{1-q\frac{\beta-k+2}{\beta}}\right) \end{aligned}$$

For q small, also $q\frac{\beta+1}{\beta+k-1}$ is small. On the other hand, the factor $\frac{|\beta-k+2|}{\beta}$ is increasing in k. Hence, for $k = 3, \ 4, \ ...$

$$\frac{E(X^k)}{E(Y^k)} \approx 1$$

provided that

$$\frac{|\beta-k+2|}{\beta}q$$

is "small" enough.

We can conclude that for a large portfolio where all risks have the same small default probabilities, the distribution function of the approximation S' tends to be close to a Beta distribution.

8 Conclusion

The Aggregate Loss, (expressed as a percentage of the Aggregate-Exposure-at Default) can be written as

$$S = \sum_{i=1}^{n} \sum_{j=1}^{I_i} c_i .$$

Here the random variables I_i are Bernouilli distributions with given default probabilities $\Pr(I_i = 1) = q_i$. We assume that the covariances $Covar(I_i, \ I_j)$ are given. The c_i are assumed to be deterministic amounts defined by

$$c_i = \frac{(EAD)_i}{\sum_{j=1}^{n} (EAD)_j} \ (LGD)_i \, .$$

We propose to approximate the distribution function of S by the distribution function of S', where

$$S' = \sum_{i=1}^{n} \sum_{j=1}^{N_i} c_i ,$$

with

$$N_i \stackrel{d}{=} \ NB\left(\beta, \frac{\beta}{\beta + q_i}\right)$$

and

$$\beta = \frac{\left(\sum_{i=1}^{n} c_i \ q_i\right)^2}{Var(S) - \left(\sum_{i=1}^{n} c_i^2 \ q_i\right)}.$$

For this approximation, we have that the distribution functions of the N_i and the I_i are very close to each other. Moreover, the first and the second moments of S are equal to the corresponding moments of S'. The distribution function of S' can easily be computed by Panjer's recursion.

If we in addition have that the "real" correlations are such that

$$corr\left[I_i \ , I_j\right] \approx \frac{\sqrt{q_i \ q_j}}{\beta},$$

then also the correlation structure of S' approximately coïncides with the correlation structure of S.

Given the distribution function of S' , the capital multiplier K_ϵ, corresponding to the $(1-\epsilon)$-percentile can then be determined (approximately) by

$$K_\epsilon = \frac{F_{S'}^{-1}(1-\epsilon) - E(S)}{\sigma_S}.$$

For a large portfolio where all risks have the same small default probabilities, the distribution function of the approximation S' tends to be close to a Beta distribution.

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References

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