Stable laws and the present value of fixed cash-flows

Marc Goovaerts[1]
Ann De Schepper[2]
David Vyncke[3]
Jan Dhaene[4]
Rob Kaas[5]

[1] Marc Goovaerts, Professor, University of Leuven (Belgium) and University of Amsterdam (the Netherlands)

[2] Corresponding author : Ann De Schepper, Professor, University of Antwerp, Faculty of Applied Economics, Prinsstraat 13, B-2000 Antwerp, Belgium, tel: ++32-3-220.40.77, fax: ++32-3-220.48.17, email: ann.deschepper@ua.ac.be

[3] David Vyncke, PhD student, University of Leuven (Belgium)

[4] Jan Dhaene, Professor, University of Leuven (Belgium) and University of Amsterdam (the Netherlands)

[5] Rob Kaas, Professor, University of Amsterdam (the Netherlands) and University of Leuven (Belgium)

Abstract

In the current contribution, we consider the present value of a series of fixed cash flows under stochastic interest rates. In order to model these interest rates, we don't use the common lognormal model, but stable laws, which better fit in with reality. For this present value, we want to derive a result about the distribution function. However, due to the dependencies between successive discounted payments, the calculation of an exact analytical distribution for the present value is impossible. Therefore, use is made of the methodology of comonotonic variables and the convex ordering of risks, introduced by the same authors in some previous papers. The present paper starts with a brief overview of properties and qualities of stable laws, and of the possible application of the concept of convex ordering to sums of risks - which is also the situation for a present value of future payments. Afterwards, it is shown how for the present value under investigation an approximation in the form of a convex upper bound can be derived. This upper bound has an easier structure than the original present value, and we derive elegant calculation formulas for the distribution of this bound. Finally, we provide some numerical examples, which illustrate the precision of the approximation. Due to the design of the present value and due to the construction of the upper bound, these illustrations show great promise concerning the accuracy of the approximation.

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1 Introduction

When analyzing future cash flows, one frequently starts with a series of fixed payments together with stochastic interest rates, such that the total value of the cash flow stream on its turn is also stochastic. In many situations where the behaviour and impact of future payments or revenues are examined, it is sufficient to know the expectation of the present value of such a cash flow stream. Nevertheless, for a more complete analysis, the distribution function of the (stochastic) present value can be very useful. Indeed, with the knowledge of the distribution, one can get more information about the variance, the skewness, higher moments, probabilities of reaching certain ranges (e.g. tail probabilities), stop loss expectations and so on. However, due to the interdependence between successive discount factors –the precise correlations are not even known in most cases– this problem cannot be solved by means of the classical convolution theory. An accurate approximation will then be the only solution.

For the investigation of cash flows with stochastic interest rates, a common assumption is that these interest rate changes can be modelled by means of Wiener processes (see e.g. [2, 8, 15]). In order to adopt this model in the context of cash flow analysis, as in some earlier contributions (see [7, 9, 20]) we consider the present value A of a series of n payments at times $t_1 < \ldots < t_n$

$$A = \sum_{j=1}^{n} c_j e^{-Y(t_j)}, \tag{1}$$

where $Y(t_j)$ represents the stochastic continuous compounded rate of return over the period $[0, t_j]$.

This compounded rate of return can be written as the sum of increments over the previous periods,

$$Y(t_j) = \sum_{i=1}^{j} (Y(t_i) - Y(t_{i-1})) \tag{2}$$

where $0 = t_0 < t_1 < \ldots < t_n = t$. The increments $Y(t_i) - Y(t_{i-1})$, denoting the rate of return for the period $[t_{i-1}, t_i]$, are independent and normally distributed. If we use the notation $\stackrel{d}{=}$ for equality in distribution, this can be written as

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \mu(t_i - t_{i-1}) + (t_i - t_{i-1})^{1/2} \sigma Z_i \tag{3}$$

with $Z_i \sim N(0,1)$ independent standard normal variables. As a consequence of the properties of Wiener processes, for the compounded rates of return we have

$$Y(t_j) \stackrel{d}{=} \mu t_j + t_j^{1/2} \sigma X_j \tag{4}$$

with $X_j \sim N(0,1)$ a standard normal variable. Note that $Y(t_j) = Y(t_{j-1}) + (Y(t_j) - Y(t_{j-1}))$, so the variables $Y(t_j)$ and $Y(t_{j-1})$ are not independent.

In several publications, authors give arguments and evidence that such a normal model is far from perfect. We refer among others to [1, 5, 14, 17, 19]. The most important criticism is related to the fact that interest rates often exhibit tails that are too large to come from normal distributions. In other words, a normal model ignores the high positive kurtosis, such that the probability of extreme situations is underestimated. Especially for long term cash flows, this can cause serious problems. When looking for a more accurate model that allows for a presence of larger tails, we do have to take into account some other considerations. Interest rates change due to many small changes. If it is presumed that these small changes are independent (we will return to this hypothesis at the end of this paragraph), the central limit theorem leads to normal distributions – at least if it is assumed that the variance should be finite. Yet, if this last condition is omitted, the class of normal distributions can be extended to general stable distributions. As such general stable distributions allow for larger tails, they provide an interesting improvement of the model. Several authors (see [11, 12, 14]) performed statistical tests with respect to this generalization, and their investigations give strong indications that general stable models indeed provide a good fit for financial data – although also these models still show up some imperfections. Becker (see[1]) also argues that the stable distribution meets a few other shortcomings of the normal distribution: it explains the high positive kurtosis, and it gives an answer to the non-constant variance and to the lack of independence – both due to the fact that now the hypothesis of a finite variance is dropped.

On the base of these considerations, in the present contribution we will model the increments by means of a stable distribution and at the same time, we will introduce a risk parameter Θ. For a particular choice of the distribution of Θ the geometric stable distribution arises. The aim of this paper is the calculation of an accurate approximation for the exact distribution of the random present value of (1) when rates of return are modelled by this generalized stable model.

The paper is organized as follows. First we will give a summary of the concepts, properties and methods that are needed to reach our goal. In section 2 we briefly describe the stable laws and we use them to construct the generalized model for the increments. Section 3 provides the methodology of how an approximation in the form of an upper bound in convexity order can be drawn up. In section 4, we will be able to present the results about the present value in (1). Finally section 5 gives numerical illustrations of the results of section 4.

2 The generalized model

As mentioned in the introduction, the normal distribution is commonly used in financial data modelling. Perhaps the most famous application is the Black-Scholes model for asset logreturns. A nice feature of the normal distribution is its *stability* property.

Definition 2.1. *A random variable X is stable (in the broad sense) if for X_1 and X_2 independent copies of X and any positive constants a and b,*

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{5}$$

for some positive c and some $d \in \mathbb{R}$. The random variable is strictly stable (or stable in the narrow sense) if (5) holds with $d = 0$ for all choices for a and b.

From the Generalized Central Limit Theorem, see [13], we know that the stable distributions are the only possible non-trivial limit of normalized sums of independent and identically distributed terms.

Theorem 2.2 (Generalized Central Limit Theorem). *Let $X_1, X_2, \ldots$ be a series of independent and identically distributed random variables. There exist constants $a_n > 0$, $b \in \mathbb{R}$ and a non-degenerate random variable Z with*

$$a_n(X_1 + \ldots + X_n) - b_n \stackrel{d}{\longrightarrow} Z \tag{6}$$

if and only if Z is α-stable for some $0 < \alpha \leq 2$.

The idea of using stable laws in financial modelling issues is not completely new (see e.g. [11]). A reason why such distributions are not used in practice very often, can be found in the fact that for all but a few stable distributions

(Gaussian, Cauchy, Lévy) there is no closed form available for the density or for the distribution function. Yet, the stable distributions can be characterized by their characteristic function, and fortunately there also exist numerical algorithms, see e.g. [11, 16].
For the definition of the general stable distribution, we first discuss a standard stable distribution.

Definition 2.3. *A variable X is a standard stable variable, or*

$$X \sim S_\alpha(1, \beta, 0) \tag{7}$$

if its characteristic function equals

$$\varphi(t) = \mathrm{E}\left[e^{itX}\right] = \exp\left\{-|t|^\alpha \, \omega_{\alpha,\beta}(t)\right\} \tag{8}$$

where

$$\omega_{\alpha,\beta}(t) = \begin{cases} 1 - i\beta \ \mathrm{sign}(t)\tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ 1 + i\beta\frac{2}{\pi} \ \mathrm{sign}(t)\ln|t| & \text{if } \alpha = 1 \ . \end{cases} \tag{9}$$

Definition 2.4. *A variable Y is a (general) stable variable, or*

$$Y \sim S_\alpha(\gamma, \beta, \delta) \tag{10}$$

if we have the equality in distribution

$$Y \stackrel{d}{=} \begin{cases} \delta + \gamma X & \text{if } \alpha \neq 1 \\ \delta + \gamma X + \gamma\beta\frac{2}{\pi}\ln|\gamma| & \text{if } \alpha = 1 \end{cases} \tag{11}$$

with X a standard stable variable.

A general stable distribution requires four parameters to describe: an index of stability or characterisitic exponent $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$ and a location parameter $\delta \in \mathbb{R}$.

Note that for $\alpha = 2$ the variable X is $\mathrm{N}(0, 2)$ distributed, and the variable Y is normally distributed with mean δ and standard deviation $\gamma\sqrt{2}$. For α decreasing from 2 to 0, the distribution becomes more and more heavy-tailed, and the variance does not exist. For $\alpha > 1$, the mean is always equal to δ; for $\alpha \leq 1$ the mean is infinite.
Note also that the situation $\beta = 0$ corresponds to a perfectly symmetric distribution.

In order not to complicate the formulas, from now on we will assume that $\alpha \neq 1$. The case where $\alpha = 1$ (Cauchy distribution) can be described in an analogous way. To simplify the notation with respect to the time scale, we will write $S_\alpha(\gamma, \beta, \delta; \tau)$ for $S_\alpha(\gamma\tau^{1/\alpha}, \beta, \delta\tau)$.

Returning to the cash-flow under investigation, given in equation (1), we assume that the increments follow a stable law $S_\alpha(\gamma, \beta, \delta; t_i - t_{i-1})$. This implies that (3) is changed into

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \delta(t_i - t_{i-1}) + (t_i - t_{i-1})^{1/\alpha}\gamma Z_i \tag{12}$$

with $Z_i \sim S_\alpha(1, \beta, 0; 1)$ independent standard stable variables.

Since we work with stable processes, for the total rate of return we have

$$Y(t_j) \stackrel{d}{=} \delta t_j + t_j^{1/\alpha}\gamma X_j \tag{13}$$

with $X_j \sim S_\alpha(1, \beta, 0; 1)$ again a standard stable variable. Just as in the Wiener case, the variables $Y(t_1), \ldots, Y(t_n)$ are dependent. For a choice of $\alpha = 2$, the normal model emerges.

Next, we introduce a risk parameter Θ. Conditioning on this risk parameter, the distribution of the increments is the one of a stable law. More concrete, we consider the compounded rate of return

$$Y(t_j)|_{\,\Theta=\theta} \sim S_\alpha(\gamma, \beta, \delta; t_j\theta) \,. \tag{14}$$

In case Θ has all its mass at one, i.e. $\text{Prob}[\Theta = 1] = 1$, $Y(t_j)|_{\,\Theta=\theta}$ reduces to the regular stable law.

The next lemma illustrates the *stability property* of random variables with stable distribution as defined in (14), and at the same time proves the result in (13).

Lemma 2.5. *Let the variables Y_1 and Y_2 be defined as*

$$Y_1|_{\Theta=\theta} \stackrel{d}{=} \delta\tau\theta + (\tau\theta)^{1/\alpha}\gamma X_1 \tag{15}$$

$$Y_2|_{\Theta=\theta} \stackrel{d}{=} \delta(t-\tau)\theta + ((t-\tau)\theta)^{1/\alpha}\gamma X_2 \tag{16}$$

with $0 \leq \tau \leq t$ and with X_1 and X_2 independent standard stable variables. Then, conditionally on Θ, the sum $\tilde{Y} = Y_1 + Y_2$ in distribution equals

$$\tilde{Y}|_{\Theta=\theta} \stackrel{d}{=} \delta t\theta + (t\theta)^{1/\alpha}\gamma\tilde{X} \tag{17}$$

with $\tilde{X}$ a new standard stable variable.

Proof. Although this result is well known, we give a proof for the sake of completeness. Conditionally on Θ, the characteristic function can be written as

$$\begin{aligned}
&\mathrm{E}\left[\exp\left\{ik\tilde{Y}\right\}|\Theta=\theta\right] \\
&\quad=\mathrm{E}\left[\exp\left\{ik\left\{\delta\tau\theta+(\tau\theta)^{1/\alpha}\gamma X_1+\delta(t-\tau)\theta+((t-\tau)\theta)^{1/\alpha}\gamma X_2\right\}\right\}\right] \\
&\quad=\exp\left\{ik\delta t\theta\right\}\cdot\mathrm{E}\left[\exp\left\{ik(\tau\theta)^{1/\alpha}\gamma X_1\right\}\right]\cdot\mathrm{E}\left[\exp\left\{ik((t-\tau)\theta)^{1/\alpha}\gamma X_2\right\}\right]\,.
\end{aligned} \tag{18}$$

Making use of (8) and (9) for both X_1 and X_2, we find

$$\begin{aligned}
&\mathrm{E}\left[\exp\left\{ik\tilde{Y}\right\}|\Theta=\theta\right] \\
&\quad=\exp\left\{ik\delta t\theta\right\}\cdot\mathrm{E}\left[\exp\left\{-k^\alpha\tau\theta\gamma^\alpha\left(1-i\beta\ \mathrm{sign}(k(\tau\theta)^{1/\alpha}\gamma)\tan(\pi\alpha/2)\right)\right\}\right] \\
&\qquad\cdot\mathrm{E}\left[\exp\left\{-k^\alpha(t-\tau)\theta\gamma^\alpha\left(1-i\beta\ \mathrm{sign}(k((t-\tau)\theta)^{1/\alpha}\gamma)\tan(\pi\alpha/2)\right)\right\}\right] \\
&\quad=\exp\left\{ik\delta t\theta\right\}\cdot\mathrm{E}\left[\exp\left\{-k^\alpha t\theta\gamma^\alpha\left(1-i\beta\ \mathrm{sign}(k(t\theta)^{1/\alpha}\gamma)\tan(\pi\alpha/2)\right)\right\}\right]\,.
\end{aligned} \tag{19}$$

From this intermediate result, it is immediately clear that

$$\mathrm{E}\left[\exp\left\{ik\tilde{Y}\right\}|\Theta=\theta\right]\;=\mathrm{E}\left[\exp\left\{ik\left\{\delta t\theta+(t\theta)^{1/\alpha}\gamma\tilde{X}\right\}\right\}\right] \tag{20}$$

with $\tilde{X}$ a standard stable variable. □

3 Convex upper bounds

In many financial and actuarial applications where a sum of stochastic terms is involved, the distribution of the quantity under investigation is too difficult to obtain. In the present case for example, the stochastic variables $Y(t_j)$ in (2) are dependent, since they are constructed as successive partial sums of several independent variables.

In such cases, the method of convex upper bounds is extremely helpful. We will recall the most important results here; for more details, see [3, 4, 10].

The idea consists of replacing the incalculable exact distribution by a simpler approximate distribution of a random variable which is "more dangerous" than the original one. The notion "more dangerous" or "less favourable" variable can be formalized by means of the convex ordering, see [18], with the following definition :

Definition 3.1. *If two random variables V and W are such that for each convex function $u : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto u(x)$ the expected values (provided they exist) are ordered as*

$$E\left[u(V)\right] \leq E\left[u(W)\right], \tag{21}$$

the variable V is said to be smaller in convex ordering than a variable W, which is denoted as

$$V \leq_{cx} W. \tag{22}$$

Since convex functions are functions that take on their largest values in the tails, this means that the variable W is more likely to take on extreme values than the variable V, and thus it can be considered to be more dangerous.

Condition (21) on the expectations can be rewritten as

$$E\left[u(-V)\right] \geq E\left[u(-W)\right] \tag{23}$$

for arbitrary concave utility functions $u : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto u(x)$. Thus, for any risk averse decision maker, the expected utility of the loss W is smaller than the expected utility of the loss V. This means that replacing the unknown distribution function of the variable V by the distribution function of the variable W is a prudent stategy.

The functions $u(x) = x$, $u(x) = -x$ and $u(x) = x^2$ are all convex functions, and thus it follows immediately that $V \leq_{cx} W$ implies $E[V] = E[W]$ as well as $Var[V] \leq Var[W]$.

An equivalent characterisation of convex order is formulated in the following lemma, a proof of which can be found in [18] :

Lemma 3.2. *If two variables V and W are such that $E[V] = E[W]$, then*

$$V \leq_{cx} W \Leftrightarrow E[(V-k)_+] \leq E[(W-k)_+] \text{ for all } k, \tag{24}$$

with $(x)_+ = max(0, x)$.

Since more dangerous risks will correspond to higher stop-loss premiums $E[(V-k)_+]$, again it can be seen that the notion of convex order is very adequate to describe an ordering in dangerousness. Indeed, $E[(V-k)_+]$ denotes the expected loss (in financial terms) of realizations exceeding k.

The notion of convex ordering can be extended from two single variables to two sums of variables. The most important result in this context is summarized in the following theorem, a proof of which can be found in [7]. In this theorem, it is shown how for an arbitrary sum of functions of variables, a convex upper bound can always be constructed, assuming only that the marginal distributions of the underlying variables are known.

Proposition 3.3. *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + \phi_2(X_2) + \ldots + \phi_n(X_n), \tag{25}$$

where the functions ϕ_i are real functions.
The variable

$$W = F^{-1}_{\phi_1(X_1)}(U) + F^{-1}_{\phi_2(X_2)}(U) + \ldots + F^{-1}_{\phi_n(X_n)}(U) \tag{26}$$

with U a standard uniformly random variable, defines an upper bound in convexity order, or

$$V \leq_{cx} W. \tag{27}$$

Remarks

- The notation $F_{X_j}(x)$ is used for the distribution function of X_j, i.e.
$$F_{X_j}(x) = \text{Prob}(X_j \leq x); \tag{28}$$
the inverse function is defined in the classical way as ($p \in (0,1)$):
$$F^{-1}_{X_j}(p) = \inf\{x \in \mathbb{R} : F_{X_j}(x) \geq p\}. \tag{29}$$
- If the function ϕ is strictly increasing, then $F^{-1}_{\phi(X)}(p) = \phi(F^{-1}_X(p))$.
If ϕ is strictly decreasing, then $F^{-1}_{\phi(X)}(p) = \phi(F^{-1}_X(1-p))$.
- The corresponding terms in the original variable V and in the upper bound W are all mutually identically distributed, or
$$\phi_i(X_i) \stackrel{d}{=} F^{-1}_{\phi_i(X_i)}(U). \tag{30}$$

In fact, the convex upper bound is constructed as a sum of variables with the same marginal distributions as the original ones, and with the same global expectation, but with the most dangerous interdependence structure. Indeed, each term in the sum (26) is a non-decreasing function of a common stochastic U, and thus they can not be used as hedges against each other.

The more the original variables are mutually positively correlated, the better the upper bound will accord with the real but unknown sum.

4 Results for cash-flows

We now return to the present value of a series of (positive and/or negative) payments

$$A = \sum_{j=1}^{n} c_j e^{-Y(t_j)}. \tag{31}$$

Remember that the variables $Y(t_j)$ ($j = 1, \ldots, n$), representing the stochastic continuous compounded rates of return over the periods $[0, t_j]$, can be written as

$$Y(t_j) = \sum_{i=1}^{j} (Y(t_i) - Y(t_{i-1})) \qquad (0 = t_0 < t_1 < \ldots < t_n = t) \tag{32}$$

with, conditionally on $\Theta = \theta$,

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \delta(t_i - t_{i-1})\theta + ((t_i - t_{i-1})\theta)^{1/\alpha}\gamma Z_i \ ; \tag{33}$$

the random variables Z_i are independent standard stable variables with distribution $S_\alpha(1, \beta, 0; 1)$, and the risk parameter Θ is independent of the variables Z_i.

As mentioned before, it follows from the model that, conditionally on $\Theta = \theta$,

$$Y(t_j) \stackrel{d}{=} \delta t_j \theta + (t_j \theta)^{1/\alpha} \gamma X_j \tag{34}$$

where now the variables X_j are dependent standard stable variables.

Due to the design of the present value (31), the terms in the sum are highly positively dependent when the payments have the same sign. Indeed, the compounded rates of return $Y(t_j)$ are the sums of increments over the past periods; two successive returns only differ in one such an increment. As a consequence, the convex upper bound as defined in proposition 3.3 –which is the sum of terms with highest possible positive interdependence structure– will be very accurate as an approximation for the original present value. This will be confirmed by the illustrations in the last section.

4.1 General results

We commence by applying proposition 3.3 in order to find a stochastic upper bound for the present value under investigation. The following result holds:

Proposition 4.1. *Let U be a random variable which is uniformly distributed on $[0,1]$. For the present value A in (31), the variable*

$$A_{upp} = \sum_{j=1}^{n} c_j \exp\left\{-\delta t_j \Theta - (t_j\Theta)^{1/\alpha}\gamma \operatorname{sign}(c_j) F^{-1}(U;\alpha,\operatorname{sign}(c_j)\beta)\right\} \quad (35)$$

where $F(x;\alpha,\beta) = \mathrm{Prob}(X_j \leq x)$ denotes the distribution function of a standard stable variable, defines an upper bound in convexity order, or

$$A \leq_{cx} A_{upp} \, . \quad (36)$$

Proof. This follows directly from proposition 3.3 and from the symmetry property $F^{-1}(1-U;\alpha,\beta) = -F^{-1}(U;\alpha,-\beta)$. □

Starting from this result for the boundary variable, we arrive at an expression for the stop-loss premiums.

Proposition 4.2. *The stop-loss premiums of the present value A in (31) are bounded from above by*

$$\mathrm{E}\left[(A-k)_+\right] \leq \int_0^{+\infty} dF_\Theta(\theta) \int_0^{u_\theta(k)} du \quad (37)$$
$$\left(\sum_{j=1}^{n} c_j \exp\left\{-\delta t_j\theta - (t_j\theta)^{1/\alpha}\gamma \operatorname{sign}(c_j) F^{-1}(u;\alpha,\operatorname{sign}(c_j)\beta)\right\} - k\right)$$

where for each value of k and θ the value $x_\theta(k)$ is defined implicitly through the equation

$$\sum_{j=1}^{n} c_j \exp\left\{-\delta t_j\theta - (t_j\theta)^{1/\alpha}\gamma \operatorname{sign}(c_j) F^{-1}(u_\theta(k);\alpha,\operatorname{sign}(c_j)\beta)\right\} = k \, . \quad (38)$$

The function $F_\Theta(\theta)$ denotes the distribution function of the risk parameter Θ.

Proof. Because of proposition 4.1, we know that

$$\mathrm{E}\left[(A-k)_+\right] \leq \mathrm{E}\left[(A_{upp}-k)_+\right] \quad (39)$$

with

$$\mathrm{E}\left[(A_{upp}-k)_+\right] = \int_0^{+\infty} dF_\Theta(\theta) \int_0^{1} du \quad (40)$$
$$\left(\sum_{j=1}^{n} c_j \exp\left\{-\delta t_j\theta - (t_j\theta)^{1/\alpha}\gamma \operatorname{sign}(c_j) F^{-1}(u;\alpha,\operatorname{sign}(c_j)\beta)\right\} - k\right)_+ \, .$$

The desired result follows by observing that the sum in (40) is a decreasing function of u, since each of the terms is. □

Finally, once the stop-loss premiums are found, the distribution function can be easily determined. Indeed, there is a well-known link between stop-loss premiums and distribution, stating that the right-hand derivative of a stop-loss premium $\mathrm{E}\left[(A-k)_+\right]$ with respect to k equals $F_A(k)-1$.

Proposition 4.3. *The cumulative distribution for the quantity A_{upp} mentioned in proposition 4.1 can be calculated as*

$$F_{upp}(k) = \mathrm{Prob}[A_{upp} \leq k] = 1 - \int_0^{+\infty} u_\theta(k) dF_\Theta(\theta) \tag{41}$$

with $u_\theta(k)$ defined implicitly in (38)*.*

Proof. This follows immediately by taking the right-hand derivative of (37). □

Note that if all $c_j > 0$, then

$$F_{upp}(k) = \mathrm{Prob}[A_{upp} \leq k] = 1 - \int_0^{+\infty} F(x_\theta(k); \alpha, \beta) dF_\Theta(\theta) \tag{42}$$

with $x_\theta(k)$ defined implicitly through

$$\sum_{j=1}^{n} c_j \exp\left\{-\delta t_j \theta - (t_j\theta)^{1/\alpha} \gamma x_\theta(k)\right\} = k \ . \tag{43}$$

4.2 Special cases & model modifications

After presenting the general results, we also want to specify the results for three special cases for the distribution of the variable Θ. We will use the same three cases for the numerical illustrations in the next section.

1. *The risk parameter Θ has all its mass in one, or* $\mathrm{Prob}[\Theta = 1] = 1$*.*
   The model degenerates to the regular and unconditional stable model. The distribution function of the upper bound can be written as

$$F_{upp}^{(1)}(k) = 1 - u(k) \tag{44}$$

with the values $u(k)$ defined implicitly through the equation

$$\sum_{j=1}^{n} c_j \exp\left\{-\delta t_j - t_j^{1/\alpha}\gamma \operatorname{sign}(c_j) F^{-1}(u(k); \alpha, \operatorname{sign}(c_j)\beta)\right\} = k \; . \qquad (45)$$

If α is chosen equal to 2, we recover the results as mentioned in [9].

2. *The risk parameter Θ is exponentially distributed with unit mean.*
   The model is said to follow a geometric stable law. The variable $Y(t)$ can be seen as the sum of a stochastic number of independent standard stable variables, where the total number of terms follows a geometric distribution (see [12]).
   Now the distribution function of the upper bound can be written as

$$F_{upp}^{(2)}(k) = 1 - \int_0^{+\infty} e^{-\theta} u_\theta(k)\, d\theta \qquad (46)$$

   with the values $u_\theta(k)$ defined in (38).

3. *The risk parameter Θ only appears in the volatility term.*
   In this case the model slightly differs, and the rate of return $Y(t_j)$ is (conditionally on $\Theta = \theta$) distributed as

$$Y(t_j) \stackrel{d}{=} \delta t_j + (t_j\theta)^{1/\alpha}\gamma X_j \; . \qquad (47)$$

   The distribution function of the upper bound then equals

$$F_{upp}^{(3)}(k) = 1 - \int_0^{+\infty} dF_\Theta(\theta) v_\theta(k) \qquad (48)$$

   with $v_\theta(k)$ defined implicitly through

$$\sum_{j=1}^{n} c_j \exp\left\{-\delta t_j - (t_j\theta)^{1/\alpha}\gamma \operatorname{sign}(c_j) F^{-1}(v_\theta(k); \alpha, \operatorname{sign}(c_j)\beta)\right\} = k \; . \qquad (49)$$

5 Numerical illustration

In this last section, we will present a few figures with graphs of the distribution functions of the upper bounds for the present value (31), as given in (44), (46) and (48).

As stated above, the use of stable laws brings about a difficulty, which has to be found in the fact that we do not have a closed form for their distribution function and that it is very hard to calculate it numerically. In order to solve this problem, we will make use of a recent numerical algorithm proposed by Nolan (see [16]). Note that there are other possible algorithms, see e.g. [6].

For the values of the parameters of the stable law in our numerical illustration, we choose the estimates based on the monthly changes in 30-year US Treasury yields from 1977 to 1990 as calculated in a paper of Klein (see [11]):

- $\alpha = 1.58$
- $\beta = 0$
- $\gamma = 0.021714$
- $\delta = 0$.

In Klein's paper, the values for α and for γ were estimated; β and δ were put equal to zero since their estimates were not significantly different from zero. Note that the value of α significantly differs from 2, rejecting the normal hypothesis. On the other hand, the value of β corresponds to the observation that the hypothesis of symmetry can not be rejected. The zero choice for δ can be justified as the expected value at any time in the future being equal to the starting value.

In order to check the accuracy of our bounds, we compare them with estimations of the real distribution of A, obtained by means of a Monte Carlo simulation. It could be argued that –if such a simulation is possible– an analytical upper bound is not necessary or even not useful. Yet, given the fact that a Monte Carlo simulation is very time consuming, the rather simple formulas for the distribution of the upper bound are very attractive. Moreover, since the approximations seem to show a high level of precision when compared with the exact distributions (see the illustrations below), we think that there are enough arguments in favour of our methodology.

In Figure 1 we plot the distribution function of A_{upp}, in case of a cash-flow $c_t = 10$, $t = 1, \ldots, 10$, and with $\text{Prob}[\Theta = 1] = 1$. The distribution function appears to be rather close to the distribution function of A. In order to compare the accuracy in the tails, we construct a QQ-plot of the corresponding distributions. Figure 2 confirms the heavy-tailedness of the upper bound and indicates that the right quantiles are slightly overestimated. For instance, the relative error of the 99% quantile is approximately 2.6%.

Replacing the distribution of the risk parameter Θ by the Exp(1) distribution yields Figure 3. In Figure 4 we turn to the modified model (47) with $\Theta \sim \chi^2_1$. Again, both upper bounds prove to be good approximations for the corresponding exact distributions.

In Figures 5 and 6, we use the same model as in Figure 1, but we change the cash-flow to $c_t = 1, \ldots, 10$ and $c_t = 10, \ldots, 1$ respectively. In case of an increasing cash-flow, the upper bound seems to approximate the exact distribution slightly better than in case of a decreasing cash-flow.

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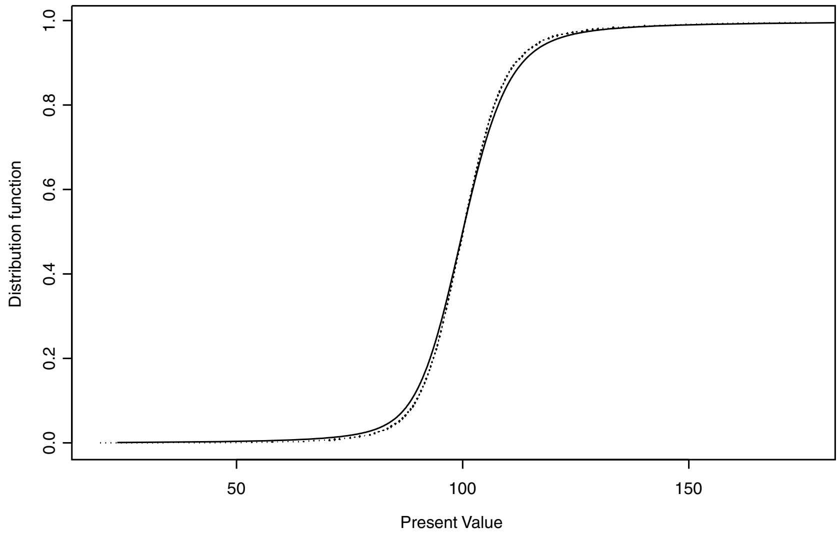

Figure 1: Distribution function of A_{upp} (continuous line) for $c_t = 10$ ($t = 1, \dots, 10$) and Prob[$\Theta = 1$] = 1, compared to a simulated distribution function of A (dotted line).

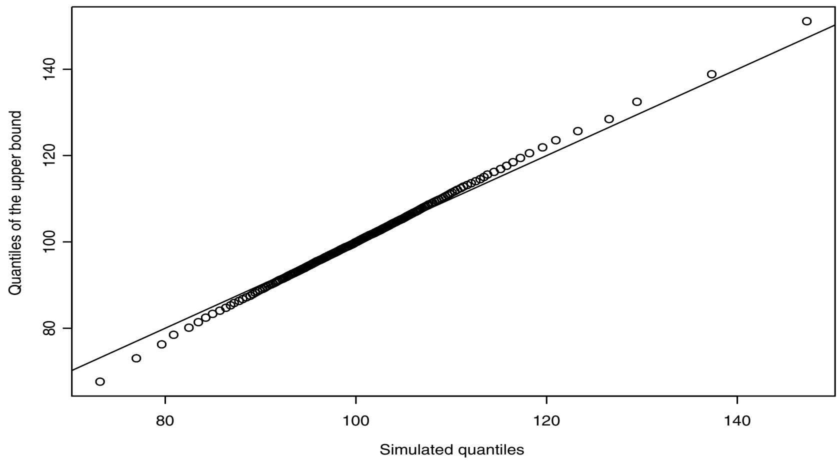

Figure 2: QQ-plot of A_{upp} versus A, for $c_t = 10$ ($t = 1, \dots, 10$) and Prob[$\Theta = 1$] = 1.

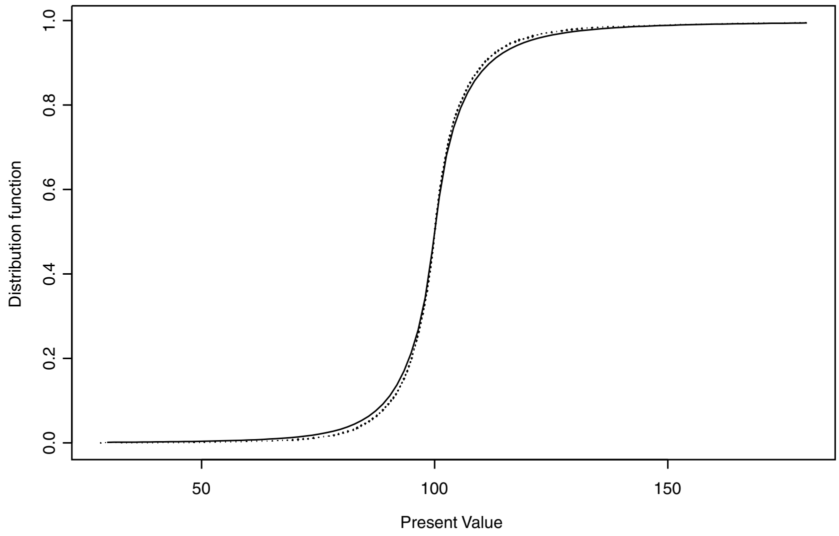

Figure 3: Distribution function of A_{upp} (continuous line) for $c_t = 10$ ($t = 1, \ldots, 10$) and $\Theta \sim \text{Exp}(1)$, compared to a simulated distribution function of A (dotted line).

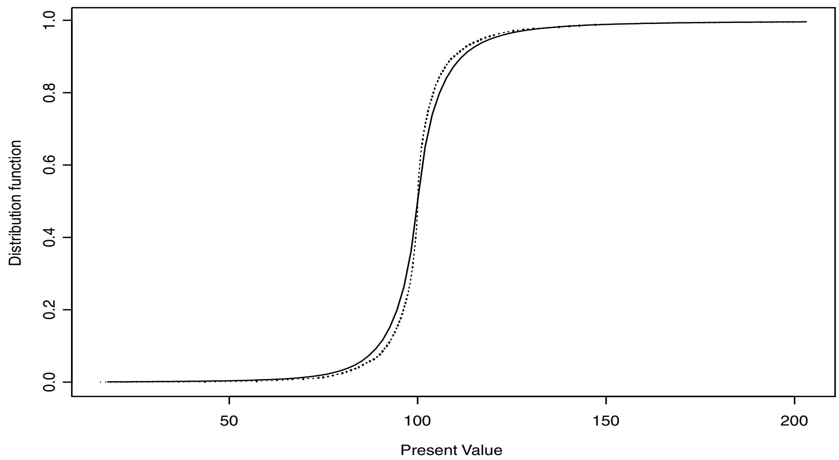

Figure 4: Distribution function of A_{upp} (continuous line) for $c_t = 10$ ($t = 1, \ldots, 10$) in special case 3 with $\Theta \sim \chi^2_1$, compared to a simulated distribution function of A (dotted line).

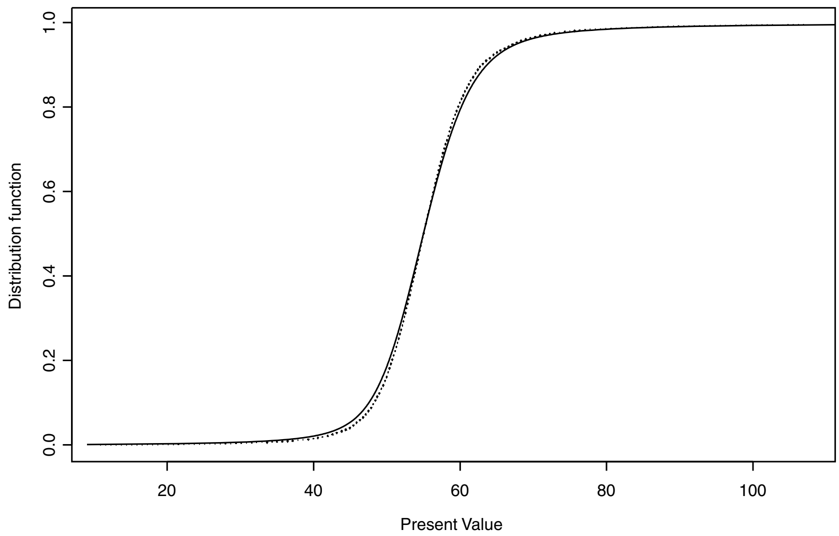

Figure 5: Distribution function of A_{upp} (continuous line) for $c_t = 1, \ldots, 10$ and $\text{Prob}[\Theta = 1] = 1$, compared to a simulated distribution function of A (dotted line).

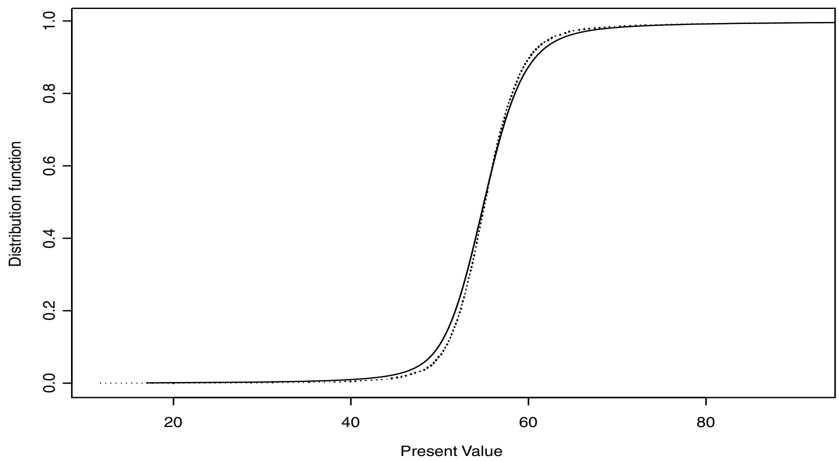

Figure 6: Distribution function of A_{upp} (continuous line) for $c_t = 10, \ldots, 1$ and $\text{Prob}[\Theta = 1] = 1$, compared to a simulated distribution function of A (dotted line).