Confidence Bounds for Discounted Loss Reserves

Tom Hoedemakers*† Jan Beirlant‡ Marc J. Goovaerts*§ Jan Dhaene*§

Abstract

In this paper we give some methods to set up confidence bounds for the *discounted* IBNR reserve. We start with a loglinear regression model and estimate the parameters by maximum likelihood such as given for example in Doray, 1996. The knowledge of the distribution function of the discounted IBNR reserve (S) will help us to determine the initial reserve, for example through the 95th percentile $F_S^{-1}(0.95)$. The results are based on convex order techniques, such that our approximations for the distribution function of S are larger or smaller, in convex order sense, than the true distribution function of S.

Keywords: IBNR, confidence bound, comonotonicity, simulation.

*Department of Applied Economics, K.U.Leuven, Naamsestraat 69, 3000 Leuven, Belgium
†Corresponding author. E-mail address: Tom.Hoedemakers@econ.kuleuven.ac.be
‡Department of Mathematics, K.U.Leuven, Celestijnenlaan 200B, 3001 Heverlee, Belgium
§Institute of Actuarial Science, University of Amsterdam, Roeterstraat 11, 1018 WB Amsterdam, Netherlands

1 Introduction

An important problem in insurance is to determine the provision for claims already incurred but not yet reported (hence IBNR), or not fully paid. The past data used to construct estimates for the future payments consist of a triangle of incremental claims Y_{ij}, as depicted in Figure 1. This is the simplest shape of data that can be obtained and it avoids having to introduce complicated notation to cope with all possible situations.

The random variables Y_{ij} with $i,\ j = 1, 2, \ldots, t$ denote the claim figures for year of origin i and development year j, meaning that the claims were paid in calendar year $i+j-1$. Year of origin, year of development and calendar year act as explanatory variables for the observation Y_{ij}. For (i, j) combinations with $i + j \leq t + 1$, Y_{ij} has already been observed, otherwise it is a future observation. Next to claims actually paid, these figures can also be used to denote quantities such as loss ratios. To a large extent, it is irrelevant whether incremental or cumulative data are used when considering claims reserving in a stochastic context.

The purpose is to complete this run-off triangle to a square, and even to a rectangle if estimates are required pertaining to development years of which no data are recorded in the run-off triangle at hand. To this end, the actuary can make use of a variety of techniques. The inherent uncertainty is described by the distribution of possible outcomes, and one needs to arrive at the best estimate of the reserve. Loss reserving deals with the determination of the uncertain present value of an unknown amount of future payments. Since this amount is very important for an insurance company and its policyholders, these inherent uncertainties are no excuse for providing anything less than a rigorous scientific analysis. In order for the reserve estimate truly to represent the actuary's "best estimate" of the needed reserve, both the determination of the expected value of unpaid losses and the appropriate discount should reflect the actuary's best estimates (i.e. should not be dictated by others or by regulatory requirements). Since the reserve is a provision for the future payment of unpaid losses, we believe the estimate loss reserve should reflect the time value of money. In many situations this *discounted* reserve is useful, for example dynamic financial analysis, assessing profitability and pricing, identifying risk based capital needs, loss portfolio transfers, ... Ideally the discounted loss reserve would also be acceptable for regulatory reporting. However, many current regulations do not permit it. Undiscounted loss reserves include in fact a certain risk margin depending on the level of the interest rate. In this paper we consider the discounted IBNR reserve and impose an explicit margin based on a risk measure (for example VaR) from the distribution of the total discounted reserve.

As a first attempt to analyze the *discounted* IBNR reserve, we consider here a simple loglinear statistical model to describe the past and future payments. So, the total IBNR reserve will be a sum of lognormal random variables which implies that its exact distribution function (d.f.) cannot be determined analytically. Considering the *discounted* IBNR reserve (S), we have to incorporate a certain dependence structure. This will be explained in detail in the next section. In general, it is hard or even impossible to determine the quantiles of S analytically, because in any realistic model for the return process the random variable S will be a sum of strongly dependent random variables. The "true" multivariate distribution function of the lower triangle cannot be determined in most cases, because the mutual dependencies are not

Year of origin	*Development year* 1	2	$\cdots$	j	$\cdots$	$t-1$	t
1	Y_{11}	Y_{12}	$\cdots$	Y_{1j}	$\cdots$	$Y_{1,t-1}$	Y_{1t}
2	Y_{21}	Y_{22}	$\cdots$	Y_{2j}	$\cdots$	$Y_{2,t-1}$	
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		
i	Y_{i1}	$\cdots$	$\cdots$	Y_{ij}			
$\vdots$	$\cdots$	$\cdots$	$\cdots$				
t	Y_{t1}						

Figure 1: Random variables in a run-off triangle

known, or are difficult to cope with. We suggest to solve this problem by calculating upper and lower bounds for this sum of dependent random variables making efficient use of the available information. These bounds are based on a general technique for deriving lower and upper bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas et al. (2000). The first approximation we will consider for the d.f. of the discounted IBNR reserve is derived by approximating the dependence structure between the random variables involved by a comonotonic dependence structure. The second approximation, which is derived by considering conditional expectations, takes part of the dependence structure into account. We will include a numerical comparison of our approximations with a simulation study. The second approximation turns out to perform quite well. For details of this technique we refer to Dhaene et al. (2002a,b) and the references therein.

The choice of an appropriate statistical model is an important matter. Furthermore within a stochastic framework, there is considerable flexibility in the choice of predictor structures. In England and Verrall (2002) the reader finds an excellent review of possible stochastic models. An appropriate model will enable the calculation of the distribution of the reserve that reflects the process variability producing the future payments, and accounts for the estimation error and statistical uncertainty (in the sense given in Taylor and Ashe, 1983). It is necessary to be able to estimate the variability of claims reserves, and ideally to be able to estimate a full distribution of possible outcomes so that percentiles (or other risk measures of this distribution) can be obtained. Next, recognizing the estimation error involved with the parameter estimates, confidence intervals for these measures constitute another desirable part of the output. Here, putting the emphasis on the discounting aspect of the reserve, we consider simple loglinear models. Doray (1996) studied these models extensively, taking into account the estimation error on the parameters and the statistical prediction error in the model. This class of models have some significant disadvantages. We need to impose that each incremental value should be greater than zero. Moreover predictions from this model can yield unusable results. In the future the authors intend to deal with other statistical models as well.

This paper is set out as follows. Section 2 gives a summary of results on loglinear models in claims reserving. In section 3 we state stochastic bounds for the scalar product of two independent random vectors, where the marginal distribution functions of each vector are given, but the dependence structures are unknown. We will describe how these results can be used for discounted IBNR evaluations. Finally, we will calculate the cdf's of these bounds. Some numerical illustrations for a simulated data set are provided in section 4, together with a discussion of the estimation error using a bootstrap approach. We also graphically illustrate the obtained bounds. Most of the proofs are deferred to the appendix.

2 Loglinear Models

We consider the following loglinear regression model

$$\vec{Z}_i = \ln \vec{Y}_i = \mathbf{X}_i \vec{\beta} + \vec{\epsilon}_i, \qquad \vec{Y}_i > 0 \tag{1}$$

where

- $\vec{Y}_i$ is the ith element of the data vector $\vec{Y}$, of dimension $\frac{t(t+1)}{2}$,
- $\mathbf{X}$ is the regression matrix of dimension $[\frac{t(t+1)}{2}] \times p$; the ith row is denoted by $\mathbf{X}_i$, and element (i, j) is denoted X_{ij},
- $\vec{\beta}$ is the vector (of dimension p) of unknown parameters,
- $\vec{\epsilon}_i$ are independent normal random errors with mean 0 and variance σ^2.

In matrix notation this linear model can be represented as

$$\vec{Z} = \ln \vec{Y} = \mathbf{X}\vec{\beta} + \vec{\epsilon}, \quad \vec{\epsilon} \sim N(0, \sigma^2 \mathbf{I}). \tag{2}$$

The normal responses Z_{ij} are assumed to decompose (additively) into a deterministic non-random component with mean $(\mathbf{X}\vec{\beta})_{ij}$ and a homoscedastic normally distributed random error component with zero mean.

For the regression parameters, various choices are possible. A well-known and widely used model is the stochastic chain-ladder model

$$Z_{ij} = \ln Y_{ij} = \alpha_i + \beta_j + \epsilon_{ij}, \tag{3}$$

(α_i is the parameter for each year of origin i and β_j for each development year j). It should be noted that this representation implies the same development pattern for all years of origin, where that pattern is defined by the parameters β_j. The relationship between this loglinear model and the chain-ladder technique was first pointed out by Kremer (1982) and used by Renshaw (1989), Verrall (1989) and Christofides (1990), among others. Using this model gives not exactly the same predictions as those obtained by the chain-ladder technique.

For a general model with parameters in the three directions, we refer to De Vylder and Goovaerts (1979). We give here some frequently used special cases:

- The probabilistic trend family (PTF) of models as suggested in Barnett and Zehnwirth (1998)

$$Z_{ij} = \ln Y_{ij} = \alpha_i + \sum_{k=1}^{j-1} \beta_k + \sum_{t=1}^{i+j-2} \gamma_t + \epsilon_{ij}, \tag{4}$$

where γ denotes the calendar year effect; it combines the effects of monetary inflation and changing jurisprudence.

- The Hoerl curve as in Zehnwirth (1985)

$$Z_{ij} = \ln Y_{ij} = \alpha_i + \beta_i \log(j) + \gamma_i j + \epsilon_{ij} \quad (j > 0). \tag{5}$$

This model has the advantage that you can predict payments by extrapolation for $j > t$, because development year j is considered as a continuous covariate. This is useful in estimating tail factors.

- A mixture of models (3) and (5) as in England and Verrall (2001)

$$Z_{ij} = \ln Y_{ij} = \begin{cases} \alpha_i + \beta_j + \epsilon_{ij} & \text{if } j \leq q; \\ \alpha_i + \beta_i \log(j) + \gamma_i j + \epsilon_{ij} & \text{if } j > q \end{cases} \tag{6}$$

for some integer q specified by the modeller.

The parameters are estimated by maximum likelihood, which in the case of the normal error structure is equivalent to minimizing the residual sum of squares. The unknown variance σ^2 is estimated by the residual sum of squares divided by the degrees of freedom (the number of observations minus the numbers of regression parameters estimated):

$$\tilde{\sigma}^2 = \frac{1}{n-p} (\vec{Z} - \mathbf{X}\hat{\vec{\beta}})'(\vec{Z} - \mathbf{X}\hat{\vec{\beta}}). \tag{7}$$

This is an unbiased estimator of σ^2. The maximum likelihood estimator of σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{n} (\vec{Z} - \mathbf{X}\hat{\vec{\beta}})'(\vec{Z} - \mathbf{X}\hat{\vec{\beta}}), \tag{8}$$

while the maximum likelihood estimator of $\vec{\beta}$ is

$$\hat{\vec{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\vec{Z}. \tag{9}$$

Let $\mathbf{B}$ be the regression matrix corresponding to the lower triangle, of dimension $[\frac{t(t-1)}{2}] \times p$, defined analogously to the regression matrix $\mathbf{X}$.

Now we can forecast the total IBNR reserve with

$$\text{IBNR reserve} = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij} + \epsilon_{ij}}. \tag{10}$$

This definition of the IBNR reserve can, among others, be found in Doray (1996). Here $(\mathbf{B}\hat{\vec{\beta}})_{ij}$ and ϵ_{ij} are independent. Remark that another definition could be $\sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\vec{\beta})_{ij}+\epsilon_{ij}}$. The approach taken in (10) partly uses the information contained in the upper triangle (through $\hat{\vec{\beta}}$), and acknowledges the underlying stochastic structure (through ϵ_{ij}).

We have that

$$\begin{aligned} \epsilon_{ij} &\sim \text{i.i.d } N(0,\sigma^2), && (11)\\ (\mathbf{B}\hat{\vec{\beta}})_{ij} &\sim N\big((\mathbf{B}\vec{\beta})_{ij}, \sigma^2 \left(\mathbf{B}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{B}'\right)_{ij}\big). && (12)\end{aligned}$$

Starting from model (1), we summarize now some properties of the IBNR reserve (10), which can be found in Doray (1996).

1. The mean of the IBNR reserve equals

$$W = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\vec{\beta})_{ij}+\frac{1}{2}\sigma^2\left(1+(\mathbf{B}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{B}')_{ij}\right)}. \tag{13}$$

2. The unique UMVUE of the mean of the IBNR reserve is given by

$$\hat{W}_U = {}_0F_1\Big(\frac{n-p}{2};\frac{SS_z}{4}\Big)\sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij}}, \tag{14}$$

   where ${}_0F_1(\alpha;z)$ denotes the hypergeometric function.

3. The MLE of the mean of the IBNR reserve:

$$\hat{W}_M = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij}+\frac{1}{2}\hat{\sigma}^2\left(1+(\mathbf{B}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{B}')_{ij}\right)}. \tag{15}$$

Verrall (1991) has considered an estimator similar to $\hat{W}_M$, but with $\hat{\sigma}^2$ replaced with $\tilde{\sigma}^2$:

$$\hat{W}_V = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij}+\frac{1}{2}\tilde{\sigma}^2\left(1+(\mathbf{B}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{B}')_{ij}\right)}. \tag{16}$$

The simple estimator

$$\hat{W}_D = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij}+\frac{1}{2}\tilde{\sigma}^2}, \tag{17}$$

was considered in Doray (1996).
Now we have the order relation

$$\hat{W}_U < \hat{W}_D < \hat{W}_V, \tag{18}$$

which implies that

$$W = E[\hat{W}_U] < E[\hat{W}_D] < E[\hat{W}_V]. \tag{19}$$

Hence both the estimators $\hat{W}_D$ and $\hat{W}_V$ exhibit a positive bias.

In the case that the type of business allows for discounting, or in the case that the value of the reserve itself is seen as a risk in the framework of financial reinsurance, we add a discounting process. Of course, the level of the required reserve will strongly depend on how we will invest this reserve. Let us assume that the reserve will be invested such that it generates a stochastic return Y_j in year j, $j = 1, 2, \cdots, t-1$, i.e. an amount of 1 at time $j-1$ will become e^{Y_j} at time j. The discount factor for a payment of 1 at time i is then given by $e^{-(Y_1+Y_2+\cdots+Y_i)}$, because this stochastic amount will exactly grow to an amount 1 at time i. We will assume that the return vector $(Y_1, Y_2, \cdots, Y_{t-1})$ has a multivariate normal distribution, which is independent of $\vec{\epsilon}$. The present value of the payments is then a linear combination of dependent lognormal random variables. We introduce the random variable $Y(i)$ defined by

$$Y(i) = Y_1 + Y_2 + \cdots + Y_i \tag{20}$$

and assume that

$$Y(i) = (\mu - \frac{\delta^2}{2})i + \delta B(i), \tag{21}$$

where $B(i)$ is the standard Brownian motion and where μ is a constant force of interest. In order to obtain a net present value, that is consistent with pricing in the financial environment, we transform the total estimated IBNR-reserve as follows

$$S \stackrel{def}{=} \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1) + \epsilon_{ij}} \tag{22}$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \exp\left((\mathbf{B}\hat{\vec{\beta}})_{ij} - (\mu - \delta^2/2)(i+j-t-1) - \delta B(i+j-t-1) + \epsilon_{ij}\right). \tag{23}$$

With this adaptation, we have that

$$E[e^{Y(i)}]\ .\ e^{-\mu i} = 1. \tag{24}$$

In order to study the distribution of the discounted IBNR reserve (22), we will use recent results concerning bounds for sums of stochastic variables. In the following section, we will explain the methodology we used for finding the desired answers. We will briefly repeat the most important results.

3 Methodology

Because the discounted IBNR reserve is a sum of dependent lognormal random variables, its distribution function cannot be determined analytically. Therefore, instead of calculating the exact distribution, we will look for bounds, in the sense of "more favourable/less dangerous" and "less favourable/more dangerous", with a simpler structure. This technique is common practice in the actuarial literature. When lower and upper bounds are close to each other, together they can provide reliable information about the original and more complex variable. The notion "less favourable" or "more dangerous" variable will be defined by means of the convex order.

3.1 Convex order and comonotonicity

Definition 1 *A random variable V is smaller than a random variable W in convex order if*

$$E[u(V)] \leq E[u(W)], \tag{25}$$

for all convex functions u: $\mathbb{R} \rightarrow \mathbb{R} : x \mapsto u(x)$, provided the expectations exist. This is denoted as

$$V \leq_{cx} W. \tag{26}$$

Roughly speaking, convex functions are functions that take on their largest values in the tails. Therefore, $V \leq_{cx} W$ means that W is more likely to take on extreme values than V. In terms of utility theory, $V \leq_{cx} W$ means that the loss V is preferred to the loss W by all risk averse decision makers, i.e. $E[u(-V)] \geq E[u(-W)]$ for all concave utility functions u. This means that replacing the (unknown) distribution function of V by the distribution function of W, can be considered as a prudent strategy with respect to setting reserves.

It follows that $V \leq_{cx} W$ implies $E[V] = E[W]$ and $\text{Var}[V] \leq \text{Var}[W]$, see for example Dhaene et al. (2002a).

We will now introduce the concepts of a Fréchet space and comonotonic risks, which will enable us to construct an upper bound for the discounted IBNR reserve S.

Definition 2 *The Fréchet space $R_n(F_1, F_2, \dots, F_n)$ determined by the (univariate) distribution functions $F_1, F_2, \dots, F_n$ is the class of all n-variate distribution functions F (or the corresponding random variables) with marginals $F_1, F_2, \dots, F_n$.*

In the Fréchet space $R_n(F_1, F_2, \dots, F_n)$ any random variable $\mathbf{X}$ is constrained from above by

$$F_{\mathbf{X}}(\mathbf{x}) \leq \min\{F_1(x_1), F_2(x_2), \dots, F_n(x_n)\} =: W_n(\mathbf{x}), \qquad \forall\ \mathbf{x} \in \mathbb{R} \tag{27}$$

A comonotone risk is a random variable with cdf W_n, see for example Dhaene et al. (1997).

Definition 3 *A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is said to be comonotone (the random variables $X_1, X_2, \dots, X_n$ are said to be mutually comonotone) if any of the following conditions hold:*

1. *For the n-variate cdf we have*

$$F_{\mathbf{X}}(\mathbf{x}) = \min\{F_1(x_1), F_2(x_2), \dots, F_n(x_n)\}, \qquad \forall\ \mathbf{x} \in \mathbb{R}^n; \tag{28}$$

2. *There exist a random variable Z and non-decreasing functions $g_1, g_2, \dots, g_n$: $\mathbb{R} \rightarrow \mathbb{R}$ such that*

$$(X_1, X_2, \dots, X_n) \stackrel{d}{=} (g_1(Z), g_2(Z), \dots, g_n(Z)); \tag{29}$$

3. *For any random variable U uniformly distributed on $(0, 1)$, we have:*

$$(X_1, X_2, \dots, X_n) \stackrel{d}{=} (F_1^{-1}(U), F_2^{-1}(U), \dots, F_n^{-1}(U)). \tag{30}$$

As usual "$\stackrel{d}{=}$" denotes equality in distribution and F^{-1} represents the inverse of the cdf F defined as

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in [0,1]. \tag{31}$$

It can be seen from condition 2 that comonotonic random variables possess a very strong positive dependence: increasing one of the X_i will lead to an increase of all other random variables X_j involved. These special random variables will provide us with a tool to construct a close upper bound for S.

3.2 Convex bounds for sums of random variables

If a random variable V consists of a sum of random variables $(X_1, \ldots, X_n)$ then replacing the copula of $(X_1, \ldots, X_n)$ by the comonotonic copula yields an upper bound for V in the convex order. On the other hand, applying Jensen's inequality to V provides us with a lower bound. Finally, if we combine both ideas, then we end up with an improved upper bound. This is formalized in the following theorem, which is taken from Dhaene et al. (2002a) and Kaas et al. (2000).

Theorem 1 *Consider an arbitrary sum of random variables,*

$$V = X_1 + X_2 + \ldots + X_n, \tag{32}$$

and define the related stochastic quantities

$$V_l = E[X_1|Z] + E[X_2|Z] + \ldots + E[X_n|Z] \tag{33}$$

$$V'_u = F^{-1}_{X_1|Z}(U) + F^{-1}_{X_2|Z}(U) + \ldots + F^{-1}_{X_n|Z}(U) \tag{34}$$

$$V_u = F^{-1}_{X_1}(U) + F^{-1}_{X_2}(U) + \ldots + F^{-1}_{X_n}(U), \tag{35}$$

with U a uniform$(0,1)$ random variable, and with Z an arbitrary random variable, independent of U. The following relations then hold:

$$V_l \leq_{cx} V \leq_{cx} V'_u \leq_{cx} V_u. \tag{36}$$

For each $j = 1, \ldots, n$, the terms in the original variable V and the corresponding terms in the upper bounds V_u and V'_u are all identically distributed, i.e.

$$X_j \stackrel{d}{=} F^{-1}_{X_j}(U) \stackrel{d}{=} F^{-1}_{X_j|Z}(U). \tag{37}$$

For the lower bound, the equalities of the distributions of X_j and $E[X_j|Z]$ only hold in case all X_j, given $Z = z$, are constant for each z.

These results can be generalized to the case where V consists of a sum of monotonic functions ϕ_j of random variables X_j, simply by substituting Y_j for $\phi_j(X_j)$ and applying Theorem 1, see Kaas et al. (2000).

The next theorem extends the previous results from ordinary sums of variables to sums of scalar products of independent random variables. The proof is deferred to Appendix A.

Theorem 2 *Assume that the vectors* $\mathbf{X}$ *and* $\mathbf{Y}$*, given the random variable* Z*, are mutually independent and that* Z *is independent of* $\mathbf{Y}$*. Consider two mutually independent uniform(0,1) random variables* U *and* V*. If the* X_i *and* Y_i *are non-negative random variables, then we find that the following relations hold:*

$$W_l \leq_{cx} W \leq_{cx} W_u' \leq_{cx} W_u, \tag{38}$$

with

$$\begin{aligned} W &= X_1Y_1 + X_2Y_2 + \ldots + X_nY_n & (39)\\ W_l &= E[X_1|Z]E[Y_1] + E[X_2|Z]E[Y_2] + \ldots + E[X_n|Z]E[Y_n] & (40)\\ W_u' &= F^{-1}_{X_1|Z}(U)F^{-1}_{Y_1}(V) + F^{-1}_{X_2|Z}(U)F^{-1}_{Y_2}(V) + \ldots + F^{-1}_{X_n|Z}(U)F^{-1}_{Y_n}(V) & (41)\\ W_u &= F^{-1}_{X_1}(U)F^{-1}_{Y_1}(V) + F^{-1}_{X_2}(U)F^{-1}_{Y_2}(V) + \ldots + F^{-1}_{X_n}(U)F^{-1}_{Y_n}(V), & (42) \end{aligned}$$

and where U*,* V *and* Z *are mutually independent.*

3.3 Upper and lower bounds for the discounted IBNR reserve

In this subsection we will derive the upper and lower bounds in convex order, as described in the previous theorem, for the discounted IBNR reserve

$$S = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1) + \epsilon_{ij}}. \tag{43}$$

We introduce the random variables V_{ij} and W_{ij} defined by

$$W_{ij} = (\mathbf{B}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1); \qquad V_{ij} = e^{W_{ij}}. \tag{44}$$

Consider now a conditioning normally distributed random variable Z defined as follows:

$$Z = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \nu_{ij} Y_{i+j-t-1}. \tag{45}$$

We will compute the lower and upper bound for the following choice of the parameters

$$\nu_{ij} = \sum_{k=i+1}^{t} \sum_{l=t+2-k}^{t} \exp\Big((\mathbf{B}\vec{\beta})_{kl} - (k+l-t-1)\mu\Big) + \sum_{l=j}^{t} \exp\Big((\mathbf{B}\vec{\beta})_{il} - (i+l-t-1)\mu\Big). \tag{46}$$

This particular random variable Z follows from the same strategy as explained in Kaas et al. (2000). Z is a linear transformation of a first-order approximation of

$$\tilde{S} = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{B}\vec{\beta})_{ij} - Y(i+j-t-1)}. \tag{47}$$

For a multivariate normal distribution, every linear combination of its components has a univariate normal distribution, so Z is normally distributed. Also, (W_{ij}, Z) has a bivariate normal distribution. Conditionally given $Z = z$, W_{ij} has a univariate normal distribution with mean and variance given by

$$E[W_{ij}|Z = z] = E[W_{ij}] + \rho_{ij}\frac{\sigma_{W_{ij}}}{\sigma_Z}(z - E[Z]) \tag{48}$$

and

$$Var[W_{ij}|Z = z] = \sigma^2_{W_{ij}}\left(1 - \rho^2_{ij}\right) \tag{49}$$

where ρ_{ij} denotes the correlation between Z and W_{ij}.

Proposition 1 *Let S, S_l, S'_u and S_u be defined as follows:*

$$S = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} exp\left(W_{ij} + \epsilon_{ij}\right), \tag{50}$$

$$S_l = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} exp\left(E[W_{ij}] + \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(U) + \frac{1}{2}(1-\rho^2_{ij})\sigma^2_{W_{ij}} + \frac{1}{2}\sigma^2_{\epsilon_{ij}}\right), \tag{51}$$

$$S'_u = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} exp\left(E[W_{ij}] + \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(U) + \sqrt{1-\rho^2_{ij}}\sigma_{W_{ij}}\Phi^{-1}(V) + \sigma_{\epsilon_{ij}}\Phi^{-1}(W)\right), \tag{52}$$

$$S_u = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} exp\left(E[W_{ij}] + \sigma_{W_{ij}}\Phi^{-1}(U) + \sigma_{\epsilon_{ij}}\Phi^{-1}(V)\right), \tag{53}$$

where U, V and W are mutually independent uniform $(0,1)$ random variables and Φ is the cdf of the $N(0,1)$ distribution. Then we have

$$S_l \leq_{cx} S \leq_{cx} S'_u \leq_{cx} S_u. \tag{54}$$

Proof.

1. If a random variable X is lognormal (μ, σ^2) distributed, then $E[X] = \exp(\mu + \frac{1}{2}\sigma^2)$. Hence for $Z = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t}\nu_{ij}Y_{i+j-t-1}$, we find, taking $U = \Phi\left(\frac{Z-E[Z]}{\sigma_Z}\right) \sim$ uniform $(0,1)$, that

$$E[V_{ij}|Z]E[e^{\epsilon_{ij}}] = \exp\left(E[W_{ij}] + \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(U) + \frac{1}{2}(1-\rho^2_{ij})\sigma^2_{W_{ij}} + \frac{1}{2}\sigma^2_{\epsilon_{ij}}\right). \tag{55}$$

   From Theorem 2, we find $S_l \leq_{cx} S$.

2. If a random variable X is lognormal (μ, σ^2) distributed, then $F_X^{-1}(p) = \exp(\mu + \sigma\Phi^{-1}(p))$. Hence we find that

$$F^{-1}_{V_{ij}|Z}(p)F^{-1}_{e^{\epsilon_{ij}}}(q) = \exp\left(E[W_{ij}] + \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(U) + \sqrt{1-\rho^2_{ij}}\sigma_{W_{ij}}\Phi^{-1}(p) + \sigma_{\epsilon_{ij}}\Phi^{-1}(q)\right). \tag{56}$$

   From Theorem 2, we find $S \leq_{cx} S'_u$.

3. The stochastic inequality $S'_u \leq_{cx} S_u$ follows from Theorem 2. ■

In Appendix B we provide some more details concerning the (calculation of the) distributions of the different bounds derived in Proposition 1.

3.4 Bounds constructed on the basis of $\hat{W}_D$

The estimator $\hat{W}_D$ (17), for the mean of the IBNR reserve, constitutes a close upper bound for the UMVUE of the mean of the IBNR reserve if $\frac{t(t+1)}{2} - p$ is large and the residual sum of squares is small. It should be noted that $e^{((\mathbf{B}\hat{\vec{\beta}})_{ij}+\tilde{\sigma}^2/2)}$ is the estimator of the mean of a lognormal distribution $LN((\mathbf{B}\vec{\beta})_{ij}, \sigma^2)$ obtained by replacing the parameters $\vec{\beta}$ and σ^2 by their unbiased estimates.

Adding now a discount process to $\hat{W}_D$, like in the previous subsection, gives

$$\hat{W}_{DD} = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1) + \frac{1}{2}\tilde{\sigma}^2}. \tag{57}$$

Now, we can apply the same methodology as explained before. Proposition 1 is still applicable. The only difference is that ϵ_{ij} is changed by $\frac{1}{2}\tilde{\sigma}^2$, with

$$\frac{1}{2}\tilde{\sigma}^2 \sim \text{Gamma}\left(\frac{n-p}{2}, \frac{\sigma^2}{n-p}\right). \tag{58}$$

4 Numerical illustrations

In this section we illustrate the effectiveness of the bounds derived for the discounted IBNR reserve S. We investigate the accuracy of the proposed bounds, by comparing their cdf to the empirical cdf obtained with Monte Carlo simulation, which serves as a close approximation to the exact distribution of S. To analyze the precision of the derived bounds (given the choice of the stochastic model), we built a non-cumulative run-off triangle ourselves based on the chain-ladder model (3). So, the run-off triangle in Table 1 has only trends in the two main directions, namely in the year of origin and in the development year.

	1	2	3	4	5	6	7	8	9	10	11
1	363,346	492,947	322,511	236,555	249,319	151,228	138,373	95,703	71,742	53,788	35,99
2	397,798	543,864	358,855	263,325	276,817	167,045	153,095	106,272	78,515	58,790	
3	806,154	1,096,841	727,977	530,683	557,870	336,716	310,022	213,706	157,504		
4	727,102	995,988	654,059	476,665	502,405	303,132	278,280	192,436			
5	659,846	900,386	591,633	433,425	457,482	276,056	253,301				
6	541,187	736,205	487,730	353,255	373,921	226,091					
7	979,636	1,342,832	882,924	651,920	682,307						
8	890,641	1,219,406	798,007	582,415							
9	486,340	666,405	442,457								
10	445,174	604,206									
11	1,084,253										

Table 1: Run-off triangle with non-cumulative claim figures

In order to illustrate the power of the bounds, namely inspecting the deviation of the cdf of the convex bounds S_l, S_u and S'_u from the true distribution of the total IBNR reserve S, we simulate a triangle from a particular model specified in Table 2. Fitting the loglinear model with a chain-ladder type predictor gives the parameter estimates and standard errors shown in Table 2. A parameter, for example β_1, must be set equal to zero, in order to have a non-singular regression matrix.

Parameter	Model parameter	Estimate	Standard error
α_1	12.8	12.7976	0.0018
α_2	12.9	12.8968	0.0018
α_3	13.6	13.5994	0.0018
α_4	13.5	13.4957	0.0019
α_5	13.4	13.3996	0.0019
α_6	13.2	13.1997	0.0020
α_7	13.8	13.7999	0.0021
α_8	13.7	13.6983	0.0023
α_9	13.1	13.0999	0.0025
α_{10}	13.0	13.0035	0.0029
α_{11}	13.9	13.8964	0.0039
β_2	0.31	0.3109	0.0018
β_3	−0.11	−0.1060	0.0018
β_4	−0.42	−0.4198	0.0019
β_5	−0.37	−0.3677	0.0020
β_6	−0.87	−0.8717	0.0021
β_7	−0.96	−0.9579	0.0022
β_8	−1.33	−1.3267	0.0024
β_9	−1.63	−1.6249	0.0027
β_{10}	−1.92	−1.9100	0.0032
β_{11}	−2.31	−2.3064	0.0043
σ	0.0004	0.0037	

Table 2: Model specification, maximum likelihood estimates and standard errors.

We also specify the multivariate distribution function of the random vector $(Y_1, Y_2, \dots, Y_{t-1})$. In particular, we will assume that the random variables Y_i are i.i.d. and $N(\mu - \frac{1}{2}\delta^2, \delta^2)$ distributed with $\mu = 0.08$ and $\delta = 0.11$. This enables now to simulate the cdf's while there is no way to compute them analytically. The conditioning random variable Z is defined as in (45)-(46).

Fig. 2 shows the cdf's of the upper and lower bounds, compared to the empirical distribution based on 100,000 randomly generated, normally distributed vectors $(Y_1, Y_2, \dots, Y_{t-1})$ and $\vec{\epsilon}$. Since $S_l \leq_{cx} S \leq_{cx} S'_u \leq_{cx} S_u$, the same ordering holds for the tails of their respective distribution functions which can be observed to cross only once. We see that the cdf of S_l is very close to the distribution of S. The "real" standard deviation equals 1,617,912 whereas the standard deviation of the lower bound equals 1,590,233. A lower bound for the 95th percentile is given by 13,638,620. The comonotonous upper bound S_u performs badly in this case. This comes from the fact that in order to determine S_l, we make use of the (estimated values of the) correlations between the cells of the lower triangle, whereas in the case of S_u, the distribution is an upper

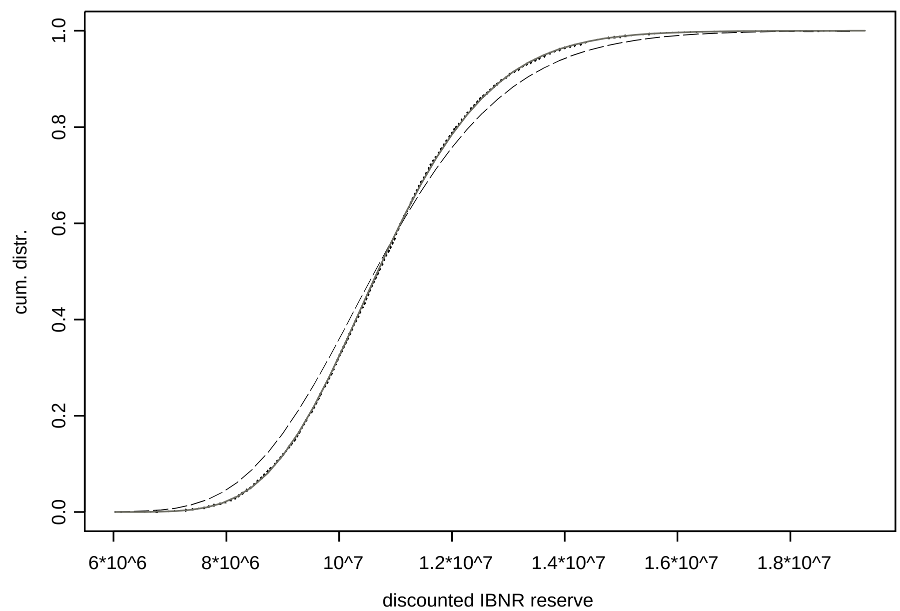

Figure 2: The cdf's of the lower bound S_l (dotted line) and of the improved upper bound S_u' (dashed line) vs. the distribution of the discounted IBNR reserve S approximated by extensive simulation (solid line) for the run-off triangle in Table 1.

	F_{S_l}			F_S			$F_{S_u'}$		
year	95%	mean	st. dev.	95%	mean	st. dev.	95%	mean	st. dev.
2	41,913	36,694	3,043	43,742	36,690	4,072	43,796	36,694	4,096
3	210,781	178,522	18,580	215,958	178,510	21,334	218,463	178,522	22,805
4	339,371	280,596	33,568	344,231	280,570	36,069	350,678	280,596	39,738
5	487,782	396,861	51,644	492,575	396,817	53,804	503,873	396,861	60,357
6	609,034	491,311	66,663	614,094	491,252	68,525	630,052	491,311	77,971
7	1,515,794	1,206,735	174,414	1,526,990	1,206,571	177,891	1,570,251	1,206,735	203,422
8	1,976,955	1,574,772	226,804	1,986,766	1,574,556	230,635	2,053,898	1,574,772	267,668
9	1,392,268	1,095,585	166,894	1,403,295	1,095,420	169,890	1,449,017	1,095,585	196,744
10	1,641,355	1,287,052	199,051	1,657,107	1,286,851	203,005	1,713,161	1,287,052	236,658
11	5,423,367	4,267,416	649,616	5,473,462	4,266,762	662,975	5,674,518	4,267,416	781,003
total	13,638,620	10,815,543	1,590,233	13,718,215	10,814,002	1,617,912	14,207,619	10,815,543	1,890,298

Table 3: 95th percentiles, means and standard deviations of the distributions of S_l and S_u' vs. S. $(\mu = 0.08, \delta = 0.11)$

bound (in the sense of convex order) for any possible dependence structure between the components of the vector $\mathbf{X}$. The improved upper bound performs better, as could be expected. The standard deviation of the improved upper bound is given by 1,890,298. The 95th percentile of the improved upper bound now equals 14,207,619, which is of course much higher than the 95th percentile of S_l.

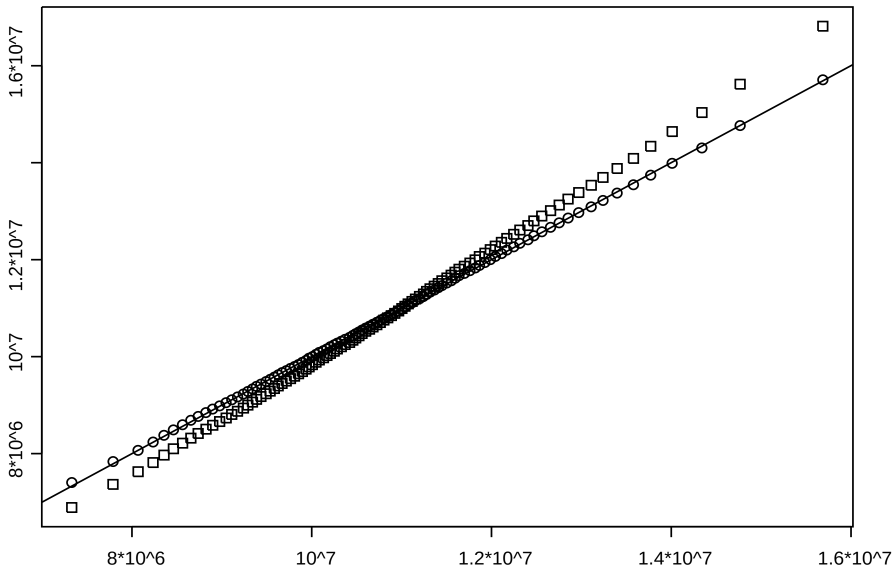

Figure 3: QQ-plot of the quantiles of S_l (O) and S'_u (□) versus those of S.

Table 3 summarizes the numerical values of the 95th percentiles of the two bounds S_l and S'_u, together with their means and standard deviations. This is also provided for the row totals

$$S_i = \sum_{j=t+2-i}^{t} e^{(\mathbf{B}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1)+\epsilon_{ij}}, \qquad i = 2, \cdots, t. \tag{59}$$

We can conclude that the lower bound approximates the "real discounted reserve" very well.

p	$F_{S_l}^{-1}(p)$	$F_S^{-1}(p)$	$F_{S'_u}^{-1}(p)$
0.95	13,638,620	13,718,215	14,207,619
0.975	14,303,311	14,411,869	15,035,380
0.99	15,122,153	15,166,753	16,066,305
0.995	15,709,687	15,710,588	16,813,432
0.999	17,003,250	17,003,255	18,479,550

Table 4: Quantiles of S_l and S'_u versus those of S.

In order to have a better view on the behavior of the improved upper bound S'_u and of the lower bound S_l in the tails, we consider a QQ-plot where the quantiles of S'_u and of the lower bound S_l are plotted against the quantiles of S. The improved upper bound S'_u and the lower bound S_l will be a good approximation for S if the plotted points $(F_S^{-1}(p), F_{S'_u}^{-1}(p))$, respectively

	Distribution of bootstrapped 95th percentiles of S_l	Simulated distribution of $F_S^{-1}(0.95)$
1 st percentile	13,587,825	13,578,331
2.5 th percentile	13,589,852	13,579,131
5 th percentile	13,597,445	13,585,813
10 th percentile	13,616,522	13,598,723
25 th percentile	13,627,692	13,619,389
50 th percentile	13,637,841	13,634,543
75 th percentile	13,647,654	13,651,195
90 th percentile	13,661,140	13,669,104
95 th percentile	13,671,003	13,678,393
97.5 th percentile	13,678,085	13,685,378
99 th percentile	13,680,785	13,688,379

Table 5: Percentiles of the bootstrapped 95th percentile of the distribution of the lower bound $S^B_{l(95)}$ vs. the simulation.

$(F_S^{-1}(p), F_{S_l}^{-1}(p))$, for all values of p in $(0,1)$ do not deviate too much from the line $y = x$. From the QQ-plot in Figure 3, we can conclude that the improved upper bound (slightly) overestimates the tails of S, whereas the accuracy of the lower bond is extremely high for the chosen set of parameter values. Tabel 4 confirms these observations.

Finally, remark that in a practical case study one can bootstrap a high percentile of the distribution of the lower bound in order to describe the estimation error involved. Taylor and Ashe (1983) used the terminology estimation error for $\text{Var}[(\mathbf{B}\hat{\vec{\beta}})_{ij}]$ and statistical or random error for $\text{Var}[\epsilon_{ij}]$. The estimation error arises from the estimation of the vector parameters $\hat{\beta}$ from the data, and the statistical error stems from the stochastic nature of model (1). We bootstrap an upper triangle using the non-parametric bootstrap procedure. This involves resampling, with replacement, from the original residuals and then creating a new triangle of past claims payments using the resampled residuals together with the fitted values. For a description of the bootstrap technique to claims reserving we refer to Lowe (1994), Taylor (2000) and England and Verrall (2002). These authors used this procedure to obtain prediction errors for different claims reserving methods and also to obtain a predictive distribution of reserves.

For each bootstrap sample, we calculate the desired percentile of the distribution of S_l. This two-step procedure is repeated a large number of times. The first column of Table 5 shows the results, concerning the 95th percentile, for 5000 bootstrap samples applied to the run-off triangle in Table 1. When compared with the simulated distribution of $F_S^{-1}(0.95)$ (obtained through 5000 simulated triangles), we can conclude that the bootstrap distribution yields appropriate confidence bounds.

5 Conclusions and possibilities for future research

In this paper, we considered the problem of deriving the distribution function of the present value of a triangle of claim payments that are discounted using some given stochastic return process. Because an explicit expression for the distribution function is hard to obtain, even when starting from a classical loglinear regression model, we presented three approximations for this distribution function, in the sense that these approximations are larger or smaller in convex order sense than the exact distribution.

Extensions could be the generalizations of the underlying regression model for instance to generalized linear models. Incorporating stable laws when modelling the discount factor is another way to generalize the results given above.

The sum of lognormal random variables is not lognormally distributed. However in practice it is often claimed to be approximately lognormally distributed. Perhaps it is useful to quantify the distance between the distribution of S and the lognormal family of distributions by means of the so-called Kullback-Leibler information. The Hellinger distance will allow us to measure the closeness of the derived lower and upper bounds as well as how far these bounds are from S in the distributional sense. So, we can identify which parameters have influence on this distance. This could also be a topic for a next paper.

Acknowledgements

The authors acknowledge financial support by the Onderzoeksfonds K.U.Leuven (GOA/02: Actuariële, financiële en statistische aspecten van afhankelijkheden in Verzekerings- en financiële portefeuilles). The authors like to thank the referee for the valuable comments, which helped to improve the presentation significantly.

Appendix A

In this appendix we will prove Theorem 2, using the next proposition.

Proposition 2 *Assume that the vectors* $\mathbf{X}$ *and* $\mathbf{Y}$ *are mutually independent and also the vectors* $\mathbf{X}$ *and* $\mathbf{Z}$*. If*

$$\sum_{i=1}^{n} x_i Y_i \leq_{cx} \sum_{i=1}^{n} x_i Z_i \tag{60}$$

for all outcomes $\mathbf{x}$ *of* $\mathbf{X}$*, then*

$$\sum_{i=1}^{n} X_i Y_i \leq_{cx} \sum_{i=1}^{n} X_i Z_i. \tag{61}$$

Proof. Let ϕ be a convex function. By conditioning on $\mathbf{X}$ and taking the assumptions into account, we find that

$$E\left[\phi(\sum_{i=1}^{n} X_i Y_i)\right] = E_{\mathbf{X}}\left\{E\left[\phi(\sum_{i=1}^{n} X_i Y_i)|\mathbf{X}\right]\right\} \quad (62)$$

$$\leq E_{\mathbf{X}}\left\{E\left[\phi(\sum_{i=1}^{n} X_i Z_i)|\mathbf{X}\right]\right\} \quad (63)$$

$$= E\left[\phi(\sum_{i=1}^{n} X_i Z_i)\right] \quad (64)$$

holds for any convex function ϕ. ■

A convex upper bound for $\sum_{i=1}^{n} X_i Y_i$

We will first prove the following relation

$$\sum_{i=1}^{n} X_i Y_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U) F_{Y_i}^{-1}(V). \quad (65)$$

Proof. From Kaas, Dhaene and Goovaerts (2000)

$$\sum_{i=1}^{n} X_i y_i \leq_{cx} \sum_{i=1}^{n} F_{X_i y_i}^{-1}(U), \quad (66)$$

where U is a uniformly distributed random variable. Because $F_{\alpha X}^{-1}(p) = \alpha F_X^{-1}(p)$ for α positive, we have that

$$\sum_{i=1}^{n} F_{X_i y_i}^{-1}(U) = \sum_{i=1}^{n} y_i F_{X_i}^{-1}(U). \quad (67)$$

From Proposition 2

$$\sum_{i=1}^{n} X_i Y_i \leq_{cx} \sum_{i=1}^{n} Y_i F_{X_i}^{-1}(U), \quad (68)$$

if U is independent of $\mathbf{Y}$ (*).

Proceeding the same for $\sum_{i=1}^{n} Y_i F_{X_i}^{-1}(U)$ proves the first inequality of the theorem. Note that the independence assumption (*) can be omitted by replacing for example U by U'. ■

Let us now assume that we have complete (or partial) information concerning the dependence structure of the random vector $(X_1Y_1, X_2Y_2, \ldots, X_nY_n)$, but that exact computation of the cdf of the sum $X_1Y_1 + X_2Y_2 + \ldots + X_nY_n$ is very time-consuming or even impossible. In this case we can derive improved upper bounds for S by using part of the information on the dependence structure, by conditioning on some random variable Z which is assumed to be some function of the random vector $\mathbf{X}$. We will assume that we know the conditional cdf's, given $Z = z$, of the random variables X_i.

An improved upper bound for $\sum_{i=1}^n X_i Y_i$

We prove the following relation

$$\sum_{i=1}^n X_i Y_i \leq_{cx} \sum_{i=1}^n F_{X_i|Z}^{-1}(U) F_{Y_i}^{-1}(V) \leq_{cx} \sum_{i=1}^n F_{X_i}^{-1}(U) F_{Y_i}^{-1}(V). \tag{69}$$

Proof. From Kaas, Dhaene and Goovaerts (2000)

$$\sum_{i=1}^n X_i y_i \leq_{cx} \sum_{i=1}^n F_{X_i y_i|Z}^{-1}(U) = \sum_{i=1}^n y_i F_{X_i|Z}^{-1}(U). \tag{70}$$

Proceeding now the same as in the proof of (65), proves the first inequality.
Now, we have that the random vector $\left(F_{X_1|Z}^{-1}(U), \ldots, F_{X_n|Z}^{-1}(U)\right)$ has marginals $F_{X_1}, \ldots, F_{X_n}$, because

$$\begin{aligned} F_{X_i}(x) &= P(X_i \leq x) & (71)\\ &= \int_{-\infty}^{\infty} P\left(X_i \leq x | Z = z\right) dF_Z(z) & (72)\\ &= \int_{-\infty}^{\infty} P\left(F_{X_i|Z=z}^{-1}(U) \leq x\right) dF_Z(z) & (73)\\ &= F_{F_{X_i|Z}^{-1}(U)}(x). & (74) \end{aligned}$$

In view of Theorem 1 (35) this implies the second inequality. ■

Let $\mathbf{X}$ be a random vector with marginals $F_{X_1}, \ldots, F_{X_n}$, and assume that we want to find a lower bound, in the sense of convex order, for $S = X_1 Y_1 + \ldots + X_n Y_n$. We can obtain such a bound by conditioning on some random variable Z, again assumed to be a function of the random vector $\mathbf{X}$.

A lower bound for $\sum_{i=1}^n X_i Y_i$

Finally, we prove the following relation

$$\sum_{i=1}^n E[X_i|Z] E[Y_i] \leq_{cx} \sum_{i=1}^n X_i Y_i \tag{75}$$

Proof. By Jensen's inequality, we find that for any convex function ϕ, the following inequality holds:

$$
\begin{aligned}
E[\phi(X_1Y_1+\ldots+X_nY_n)] &= E_Z E[\phi(X_1Y_1+\ldots+X_nY_n)|Z] &(76)\\
&\geq E_Z[\phi(E[X_1Y_1+\ldots+X_nY_n|Z])] &(77)\\
&= E_Z[\phi(E[X_1Y_1|Z]+\ldots+E[X_nY_n|Z])] &(78)\\
&= E_Z[\phi(E[X_1|Z]E[Y_1]+\ldots+E[X_n|Z]E[Y_n])] &(79)
\end{aligned}
$$

This proves the stated result. ■

Appendix B

In this appendix we derive expressions for the cdf's of S_l, S'_u and S_u.

Consider a random variable W which is defined as the product of two non-negative independent variables X and Y:

$$W = XY \tag{80}$$

The cdf of W follows from

$$F_W(z) = \int_{-\infty}^{\infty} F_Y\left(\frac{z}{x}\right)\, dF_X(x) = \int_0^1 F_Y\left(\frac{z}{F_X^{-1}(u)}\right)\, du. \tag{81}$$

The cdf of S_u

From Theorem 2, we can write the convex upper bound for the discounted IBNR reserve as follows

$$
\begin{aligned}
S_u &= \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)}(U)\right)\exp\left(F^{-1}_{\epsilon_{ij}}(V)\right) &(82)\\
&= e^{\sigma\Phi^{-1}(V)}\sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\vec{\beta})_{ij}-(\mu-\frac{1}{2}\delta^2)(i+j-t-1)+\sqrt{\sigma^2(\mathbf{B}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{B}')_{ij}+\delta^2(i+j-t-1)}\Phi^{-1}(U)}. &(83)
\end{aligned}
$$

There are several possibilities to derive the cdf of S_u. From previous results

$$F_{S_u}(z) = \int_0^1 F_N\left(ln(z) - ln(F^{-1}_{S''_u}(u))\right) du, \tag{84}$$

with $F_N(x)$ the cdf of $N(0,\sigma^2)$ and

$$
\begin{aligned}
S''_u &= \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)}(U)\right) &(85)\\
&= \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(E[W_{ij}]+\sigma_{W_{ij}}\Phi^{-1}(U)\right). &(86)
\end{aligned}
$$

So, we have that $F_{S_u}(z) =$

$$\int_0^1 F_N\left(ln(z) - ln\left(\sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\vec{\beta})_{ij}-(\mu-\frac{1}{2}\delta^2)(i+j-t-1)+\sqrt{\sigma^2(\mathbf{B}(\mathbf{X'X})^{-1}\mathbf{B'})_{ij}+\delta^2(i+j-t-1)}\Phi^{-1}(u)}\right)\right)du. \quad (87)$$

We can also derive an algorithm for the determination of the cdf of S_u. We have that $S_u \mid V = v$ is the sum of $t(t-1)/2$ comonotonic risks. This implies

$$F^{-1}_{S_u|V=v}(p) = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)}(p)\right)\exp\left(F^{-1}_{\epsilon_{ij}}(v)\right) \quad (88)$$

and

$$F_{S_u|V=v}(x) = \left\{p \in [0,1] \middle| \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)}(p)\right)\exp\left(F^{-1}_{\epsilon_{ij}}(v)\right) \le x\right\}. \quad (89)$$

If we in addition assume that the cdf's $F_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)}$ are strictly increasing and continuous, then $F_{S_u|V=v}(x)$ follows from

$$\sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)}(F_{S_u|V=v}(x))\right)\exp\left(F^{-1}_{\epsilon_{ij}}(v)\right) = x. \quad (90)$$

In any case, the cdf of S_u follows from

$$F_{S_u}(x) = \int_0^1 F_{S_u|V=v}(x)dv. \quad (91)$$

The cdf of S_l

From Theorem 2, we obtained the convex lower bound for the discounted IBNR reserve from

$$S_l = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} E\left[\exp\left((\mathbf{B}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1)\right)\middle|\sum_{i=2}^{t}\sum_{j=t+2-i}^{t}\nu_{ij}Y_{i+j-t-1}\right]E\left[\exp\left(\epsilon_{ij}\right)\right]. \quad (92)$$

Taking into account that $Z = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t}\nu_{ij}Y_{i+j-t-1}$ is normally distributed, we find that

$$F_Z^{-1}(1-p) = E[Z] - \sigma_Z\Phi^{-1}(p), \quad (93)$$

and hence

$$\begin{aligned} F^{-1}_{s_l}(p) &= F^{-1}_{\sum_{i=2}^{t}\sum_{j=t+2-i}^{t}E[V_{ij}|Z]E[e^{\epsilon_{ij}}]}(p), \qquad p \in (0,1) & (94)\\ &= \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} F^{-1}_{E[V_{ij}|Z]E[e^{\epsilon_{ij}}]}(p) & (95)\\ &= \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} E[V_{ij}|Z = F_Z^{-1}(1-p)]E[e^{\epsilon_{ij}}] & (96)\\ &= \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(E[W_{ij}] - \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(p) + \frac{1}{2}(1-\rho_{ij}^2)\sigma^2_{W_{ij}} + \frac{1}{2}\sigma^2_{\epsilon_{ij}}\right). & (97)\end{aligned}$$

In order to derive the above result, we used the fact that for a non-increasing continuous function g, we have

$$F_{g(X)}^{-1}(p) = g(F_X^{-1}(1-p)), \qquad p \in (0,1). \tag{98}$$

Here, $g = E[e^{W_{ij}}|Z]$ is a non-increasing function of Z since ρ_{ij} is always negative. So, we have that

$$F_{s_l}^{-1}(p) = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} e^{(\mathbf{B}\vec{\beta})_{ij}-(\mu-\frac{1}{2}\delta^2)(i+j-t-1)+\frac{\delta\sum_{k=2}^{t}\sum_{l=t+2-k}^{\min(i+j-k,t)}\nu_{kl}}{\sqrt{\sum_{k=2}^{t}\sum_{l=t+2-k}^{t}\nu_{kl}^2}}\Phi^{-1}(p)} \tag{99}$$

$$e^{\frac{1}{2}\left(\sigma^2\left(\mathbf{B}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{B}'\right)_{ij}+\delta^2(i+j-t-1)-\frac{\delta^2\left(\sum_{k=2}^{t}\sum_{l=t+2-k}^{\min(i+j-k,t)}\nu_{kl}\right)^2}{\sum_{k=2}^{t}\sum_{l=t+2-k}^{t}\nu_{kl}^2}\right)+\frac{1}{2}\sigma^2}, \qquad p \in (0,1) \tag{100}$$

$F_{S_l}(x)$ can be obtained from solving the equation

$$\sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(E[W_{ij}] - \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(F_{S_l}(x)) + \frac{1}{2}(1-\rho_{ij}^2)\sigma_{W_{ij}}^2 + \frac{1}{2}\sigma_{\epsilon_{ij}}^2\right) = x. \tag{101}$$

The cdf of S'_u

From Theorem 2, we can obtain the improved convex upper bound for the discounted IBNR reserve from

$$S'_u = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)|Z}(U)\right)\exp\left(F^{-1}_{\epsilon_{ij}}(V)\right), \tag{102}$$

with $Z = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t}\nu_{ij}Y_{i+j-t-1}$.
From previous results

$$F_{S'_u}(z) = \int_0^1 F_N\left(ln(z) - ln(F^{-1}_{S'''_u}(y))\right)dy, \tag{103}$$

with $F_N(x)$ the cdf of $N(0,\sigma^2)$ and $S'''_u = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t}\exp\left(F^{-1}_{(\mathbf{B}\hat{\vec{\beta}})_{ij}-Y(i+j-t-1)|Z}(U)\right)$.

Now, we will derive an algorithm for the determination of $F_{S'''_u}(y)$. From Proposition 1 we know that

$$S'''_u = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(E[W_{ij}] + \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(U) + \sqrt{1-\rho_{ij}^2}\sigma_{W_{ij}}\Phi^{-1}(V)\right). \tag{104}$$

Since $F_{S'''_u|U=u}$ is a sum of $t(t-1)$ comonotonous random variables, we have

$$F^{-1}_{S'''_u|U=u}(p) = \sum_{i=2}^{t}\sum_{j=t+2-i}^{t} \exp\left(E[W_{ij}] + \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(u) + \sqrt{1-\rho_{ij}^2}\sigma_{W_{ij}}\Phi^{-1}(p)\right). \tag{105}$$

$F_{S_u'''|U=u}$ also follows implicitly from

$$\sum_{i=2}^{t}\sum_{j=t+2-i}^{t}\exp\left(E[W_{ij}]+\rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(u)+\sqrt{1-\rho_{ij}^2}\sigma_{W_{ij}}\Phi^{-1}(F_{S_u'''|U=u}(y))\right)=y. \tag{106}$$

The cdf of S_u''' then follows from

$$F_{S_u'''}(y)=\int_0^1 F_{S_u'''|U=u}(y)du. \tag{107}$$

References

[1] Barnett, G. and Zehnwirth, B., 1998. Best estimates for reserves. *Casualty Actuarial Society Forum*, 1-54.

[2] Blum K.A. and Otto D.J., 1998. Best estimate loss reserving: an actuarial perspective. *Casualty Actuarial Society Forum.*

[3] Christofides, S., 1990. Regression models based on log-incremental payments. *Claims Reserving Manual*, **Vol 2**, Institute of Actuaries, London.

[4] De Vylder, F. and Goovaerts, M.J., 1979. Proceedings of the first meeting of the contact group "Actuarial Sciences", K.U.Leuven, nr 7904B, wettelijk Depot: D/1979/23761/5.

[5] Dhaene, J., Wang, S., Young, V. and Goovaerts, M.J., 1997. Comonotonicity and maximal stop-loss premiums. Research Report 9730, Department of Applied Economics K.U.Leuven.

[6] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D., 2002. (a) The concept of comonotonicity in actuarial science and finance: Theory. *Insurance: Mathematics and Economics*, **31(1)**, 3-33.

[7] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D., 2002. (b) The concept of comonotonicity in actuarial science and finance: Applications. *Insurance: Mathematics and Economics*, **31(2)**, 133-161.

[8] Doray, L.G., 1996. UMVUE of the IBNR reserve in a lognormal linear regression model. *Insurance: Mathematics and Economics*, **18(1)**, 43-58.

[9] Efron, B., Tibshirani, R. J., 1993. *An Introduction to the Bootstrap.* Chapman and Hall, New York.

[10] England, P.D. and Verrall, R.J., 1999. Analytic and bootstrap estimates of prediction errors in claim reserving. *Insurance: Mathematics and Economics*, **25**, 281-293.

[11] England, P.D. and Verrall, R.J., 2001. A flexible framework for stochastic claims reserving. *Proceedings of the Casualty Actuarial Society*, **88**.

[12] England, P.D. and Verrall, R.J., 2002. Stochastic claims reserving in general insurance. *British Actuarial Journal*, **8(3)**, 443-518.

[13] Goovaerts, M.J. and Redant, H., 1999. On the distribution of IBNR reserves. *Insurance: Mathematics and Economics*, **25(1)**, 1-9.

[14] Goovaerts, M.J., Kaas, R., Dhaene, J. and Denuit, M., 2001. *Modern Actuarial Risk Theory.* Kluwer Academic, The Netherlands.

[15] Kaas, R., Dhaene, J. and Goovaerts, M.J., 2000. Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics*, **27**, 151-168.

[16] Kaas, R., Dhaene, J., Vyncke, D., Goovaerts, M.J. and Denuit, M. 2001. A simple geometric proof that comonotonic risks have the convex-largest sum, *ASTIN Bulletin*, **32(1)**, 71-80.

[17] Kremer, E., 1982. IBNR-Claims and the two-way model of ANOVA. *Scandinavian Actuarial Journal*, 47-55.

[18] Lowe, J., 1994. A practical guide to measuring reserve variability using: Bootstrapping, operational time and a distribution free approach. *Proceedings of the 1994 General Insurance Convention*, Institute of Actuaries and Faculty of Actuaries.

[19] Mack, T., 1993. Distribution free calculation of the standard error of chain ladder reserve estimates. *ASTIN Bulletin*, **23(2)**, 213-225.

[20] Mack, T., 1994. Measuring the variability of chain-ladder reserve estimates. Casualty Actuarial Society, Spring Forum.

[21] Pinheiro, P. J. R., Andrade e Silva, J. M. and Centeno, M., 2001. Bootstrap methodology in claim reserving. *Proceedings of the 6th CEMAPRE Conference on Mathematics Applied to Economics and Management*, 61-73.

[22] Renshaw, A.E., 1989. Chain ladder and interactive modelling (claims reserving and GLIM). *Journal of the Institute of Actuaries*, **116(III)**, 559-587.

[23] Renshaw, A.E., 1994. On the second moment properties and the implementation of certain GLIM based stochastic claims reserving models. Actuarial Research Paper No. 65, Department of Actuarial Science and Statistics, City University, London, EC1V 0HB.

[24] Renshaw, A.E. and Verrall, R.J., 1994. A stochastic model underlying the chain-ladder technique. *Proceedings XXV Astin Colloquium*, Cannes.

[25] Shaked M. and Shanthikumar J.G., 1994. Stochastic orders and their applications. *Academic Press*, pp. 545.

[26] Taylor, G.C. and Ashe, F.R., 1983. Second moments of estimates of outstanding claims. *Journal of Econometrics*, **23**, 37-61.

[27] Taylor, G.C., 2000. *Loss Reserving: An Actuarial Perspective*. Kluwer Academic Publishers.

[28] Verrall, R. J., 1989. A state space representation of the chain-ladder linear model. *J.I.A.*, **116**, 589-610.

[29] Verrall, R. J., 1991. On the unbiased estimation of reserves from loglinear models. *Insurance: Mathematics and Economics*, **10**, 75-80.

[30] Zehnwirth, B., 1985. *ICRFS version 4 manual and users guide*. Benhar Nominees PTy Ltd, Turramurra, NSW, Australia

[31] Zehnwirth, B., 1989. The chain ladder technique - A stochastic model. *Claims Reserving Manual*, **Vol 2**, Institute of Actuaries, London.