Risk Measures and Optimal Portfolio Selection (with applications to elliptical distributions)

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Lecture No. 1

Solvency Capital, Risk Measures and Comonotonicity

Jan Dhaene

Risk measures

- *Risk*: random future loss.
- *Risk Measure*: mapping from the set of quantifiable risks to the real line:

$$X \to \rho(X).$$

- *Actuarial examples*:
  - premium principles,
  - technical provisions (liabilities),
  - solvency capital requirements.
- In sequel: $\rho(X)$ measures the ”upper tails” of the d.f.

Insurance company risk taxonomy

- *Financial risks*:
  - asset risks (credit risks, market risks),
  - liability risks (non-cathastrophic risks, catastrophic risks).
- *Operational risks*:
  - business risks,
  - event risks.

Required vs. available capital

- *Required capital*: required assets $\rho(X)$ minus liabilities $L(X)$, to ensure that obligations can be met:

$$K(X) = \rho(X) - L(X).$$

- *Different kinds of capital*:
    - regulatory capital: you must have,
    - rating agency capital: you are expected to have,
    - economic capital: you should have,
    - available capital: you actually have.

Required vs. available capital

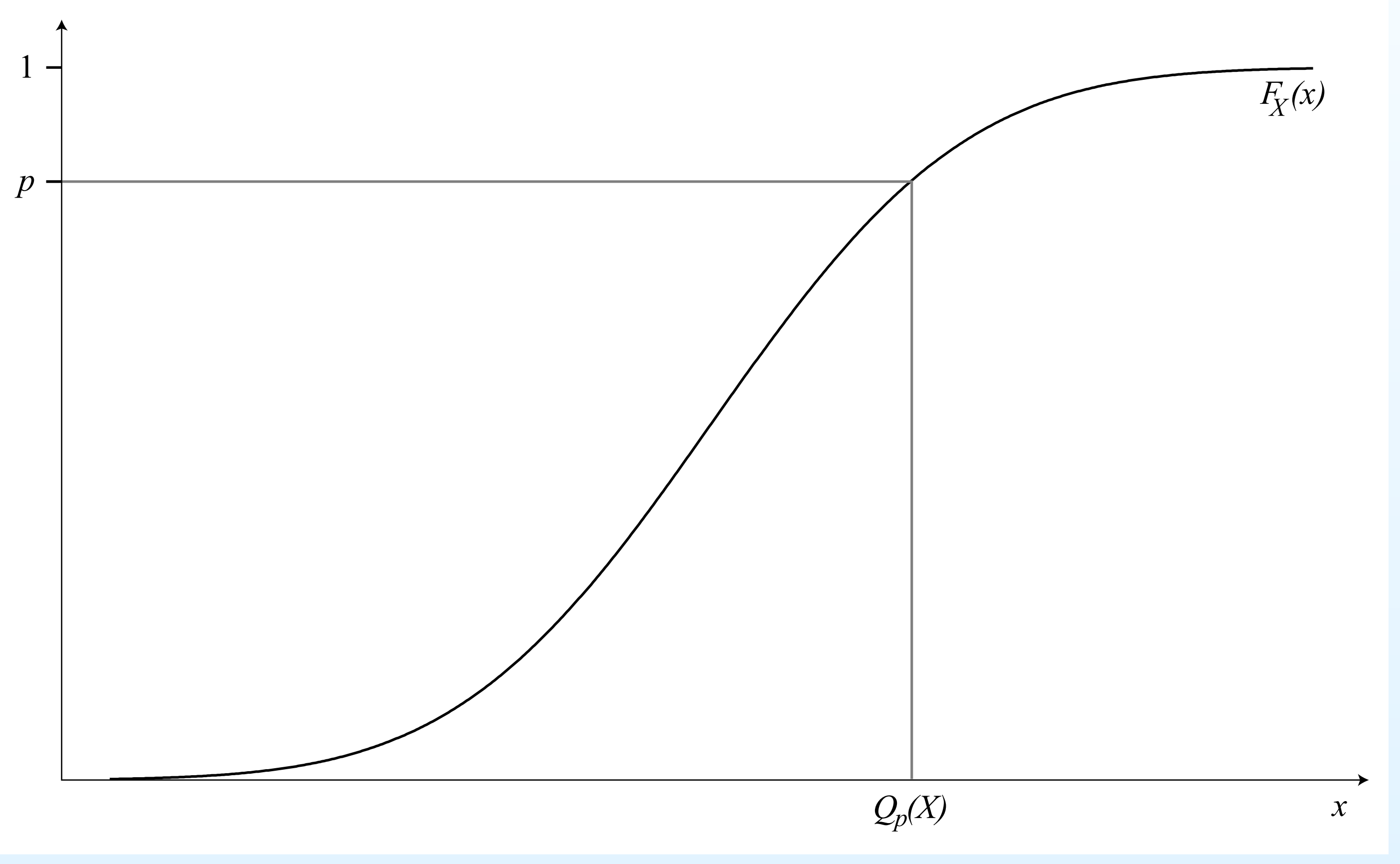
- *Parameters*:
  - default probability,
  - time horizon,
  - run-off vs. wind-up vs. going concern,
  - valuation of liabilities: mark-to-model,
  - valuation of assets: mark-to-market.
- *Total balance sheet capital approach*:

$$\rho(X) = L(X) + K(X).$$

The quantile risk measure

- *Quantiles*:

$$Q_p(X) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in (0,1).$$



The quantile risk measure

- Determining the required capital by

$$K(X) = Q_{0.99}(X) - L(X),$$

  we have

$$K(X) = \inf \{K \mid \Pr[X > L(X) + K] \leq 0.01\}.$$

- $Q_p(X) = F_X^{-1}(p) = VaR_p(X)$.
- Meaningful when only concerned about ”frequency of default” and not ”severity of default”.
- Does not answer the question ”how bad is bad?”

Tail Value-at-Risk and Conditional Tail Expectation

- *Tail Value-at-Risk*:

$$TVaR_p(X) = \frac{1}{1-p}\int_p^1 Q_q(X)\,dq, \qquad p \in (0,1).$$

- Determining the required capital by

$$K(X) = TVaR_{0.99}(X) - L(X),$$

  we define "bad times" if X in "cushion" $[Q_{0.99}(X), TVaR_{0.99}(X)]$.

- *Conditional Tail Expectation*:

$$CTE_p(X) = E\left[X \mid X > Q_p(X)\right], \qquad p \in (0,1)\,.$$

- CTE_p = expectation of the top $(1-p)\%$ losses.

Relations between risk measures

- *Expected Shortfall*:

$$ESF_p(X) = E\left[(X - Q_p(X))_+\right], \qquad p \in (0,1).$$

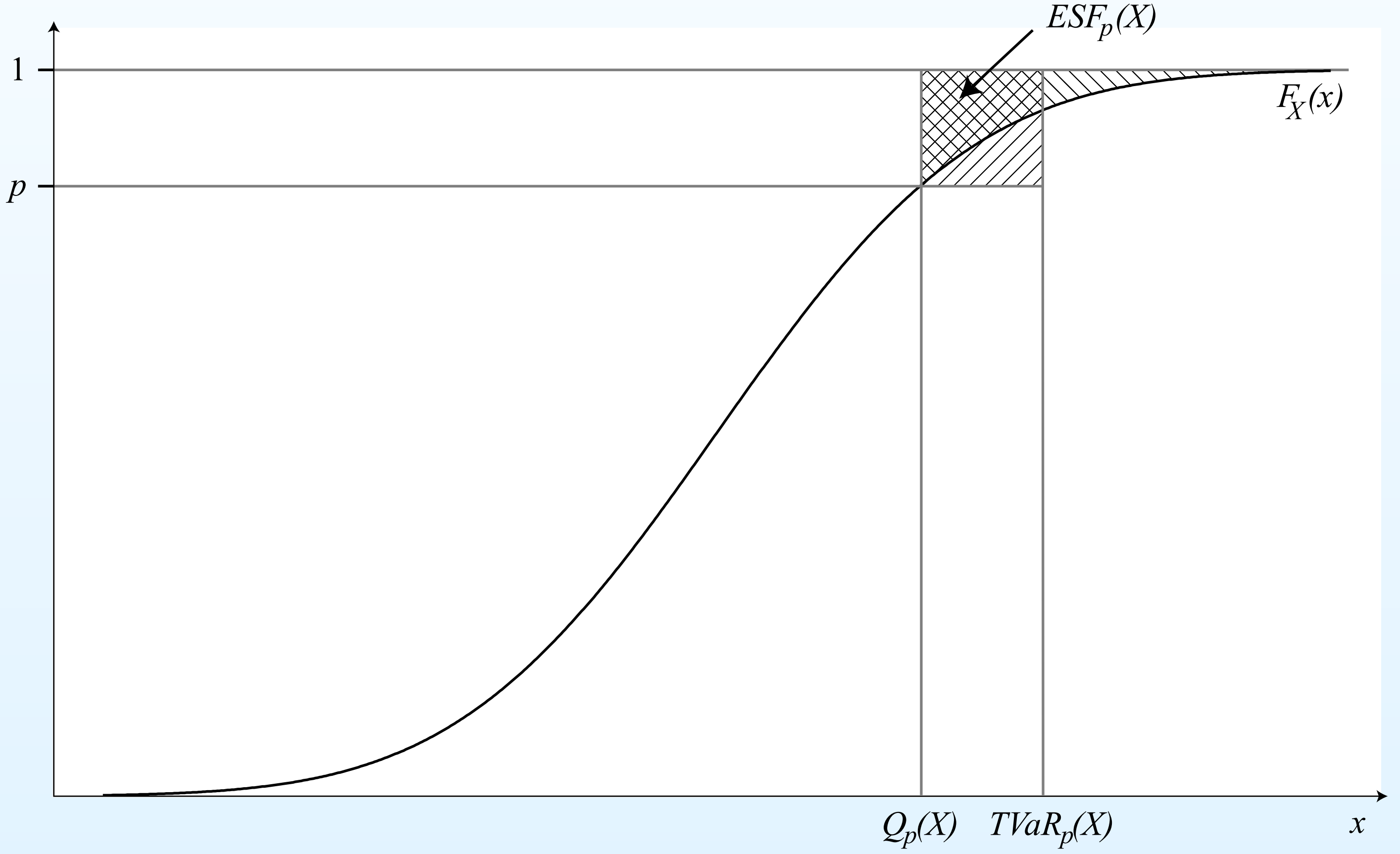
- $ESF_p(X)$ = expectation of shortfall in case required capital $K(X) = Q_p(X) - L(X)$.
- *Relations*:

$$\begin{aligned} TVaR_p(X) &= Q_p(X) + \frac{1}{1-p} ESF_p(X), \\ CTE_p(X) &= Q_p(X) + \frac{1}{1 - F_X(Q_p(X))} ESF_p(X), \\ CTE_p(X) &= TVaR_{F_X(Q_p(X))}(X). \end{aligned}$$

Relations between risk measures

- When F_X is continuous:

$$CTE_p(X) = TVaR_p(X).$$



Normal random variables

- Let $X \sim N\left(\mu,\ \sigma^2\right)$.
- *Quantiles*:

$$Q_p(X) = \mu + \sigma\ \Phi^{-1}\left(p\right).$$

where Φ denotes the standard normal distribution function.

- *Expected Shortfall*:

$$ESF_p(X) = \sigma\ \Phi'\left(\Phi^{-1}\left(p\right)\right) - \sigma\ \Phi^{-1}\left(p\right)\ \left(1-p\right).$$

- *Conditional Tail Expectation*:

$$CTE_p(X) = \mu + \sigma \frac{\Phi'\left(\Phi^{-1}\left(p\right)\right)}{1-p}.$$

Lognormal random variables

- Let $\ln X \sim N\left(\mu,\ \sigma^2\right)$.
- *Quantiles*:

$$Q_p(X) = e^{\mu+\sigma\ \Phi^{-1}(p)}.$$

- *Expected Shortfall*:

$$\begin{aligned} ESF_p(X) &= e^{\mu+\sigma^2/2}\ \Phi\left(\sigma - \Phi^{-1}(p)\right) \\ &\quad -e^{\mu+\sigma\ \Phi^{-1}(p)}\ (1-p)\,. \end{aligned}$$

- *Conditional Tail Expectation*:

$$CTE_p(X) = e^{\mu+\sigma^2/2}\frac{\Phi\left(\sigma - \Phi^{-1}(p)\right)}{1-p}.$$

Risk measures and ordering of risks

- *Ordering of risks*:
  - Stochastic dominance:

$$X \leq_{st} Y \Leftrightarrow F_X(x) \geq F_Y(x) \text{ for all } x.$$

  - Stop-loss order:

$$X \leq_{sl} Y \Leftrightarrow E[(X-d)_+] \leq E[(Y-d)_+] \text{ for all } d.$$

  - Convex order:

$$X \leq_{cx} Y \Leftrightarrow X \leq_{sl} Y \text{ and } E[X] = E[Y].$$

Risk measures and ordering of risks

- *Stochastic dominance vs. ordered quantiles*:

$$X \leq_{st} Y \Leftrightarrow Q_p(X) \leq Q_p(Y) \text{ for all } p \in (0,1).$$

- *Stop-loss order vs. ordered TVaR's*:

$$X \leq_{sl} Y \Leftrightarrow TVaR_p(X) \leq TVaR_p(Y) \text{ for all } p \in (0,1).$$

Comonotonicity

- *A set* $S \subset R^n$ is comonotonic $\Leftrightarrow$
  for all $\mathbf{x}$ and $\mathbf{y}$ in S either $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{x} \geq \mathbf{y}$ holds.

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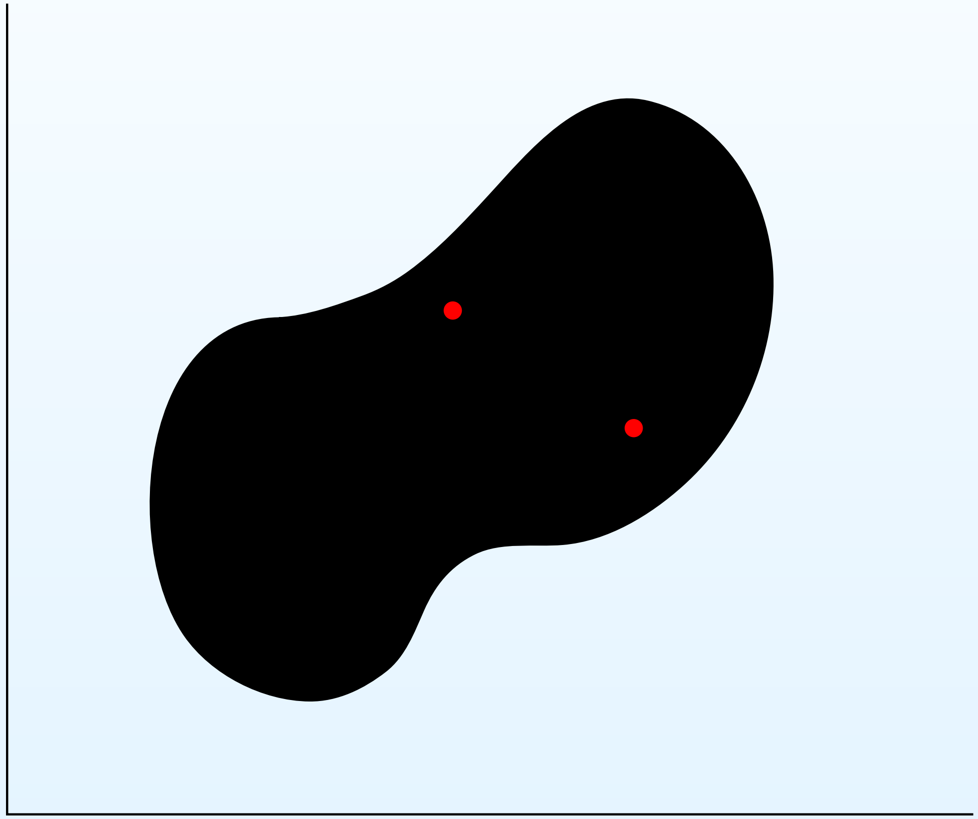

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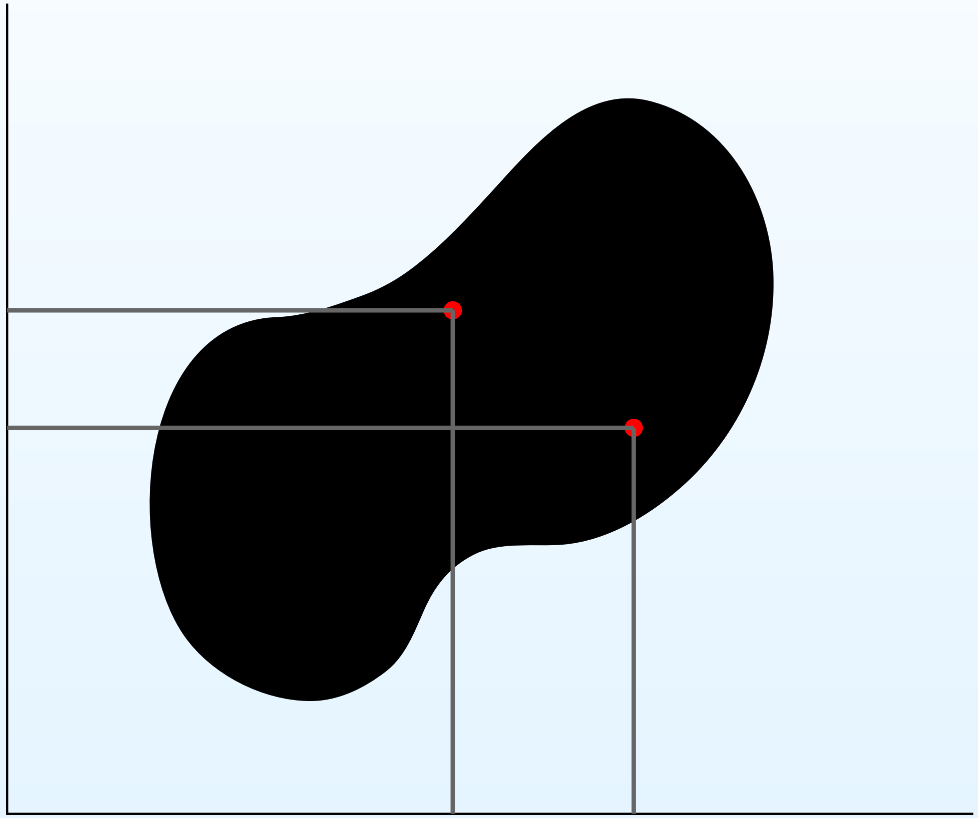

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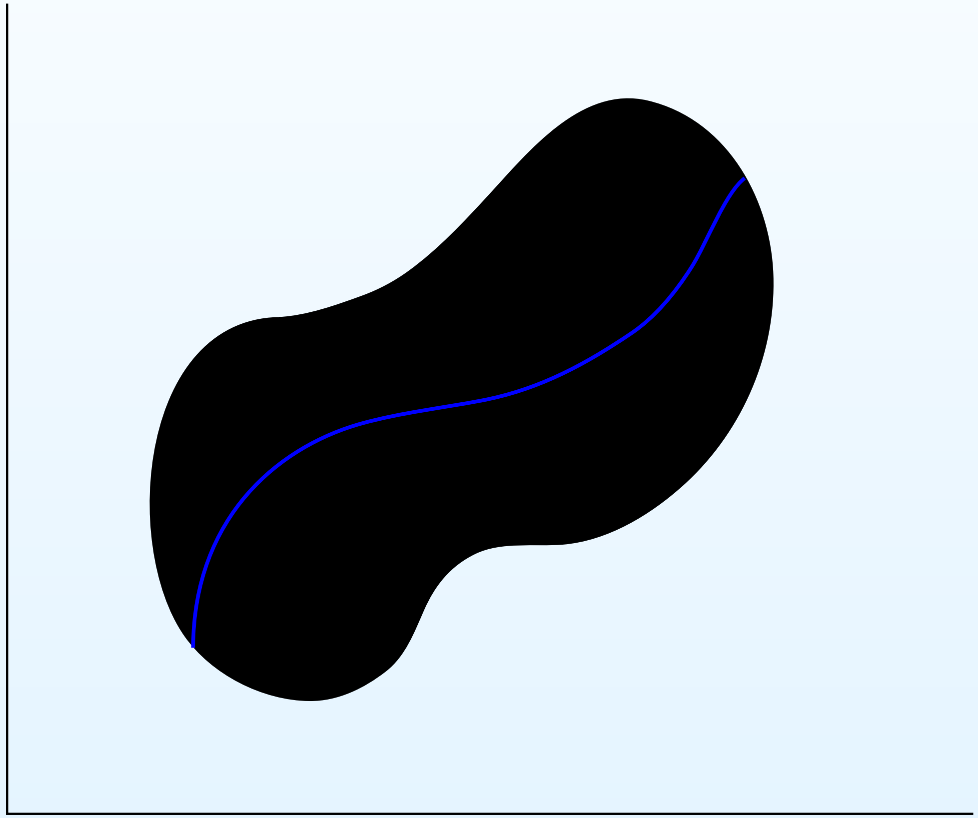

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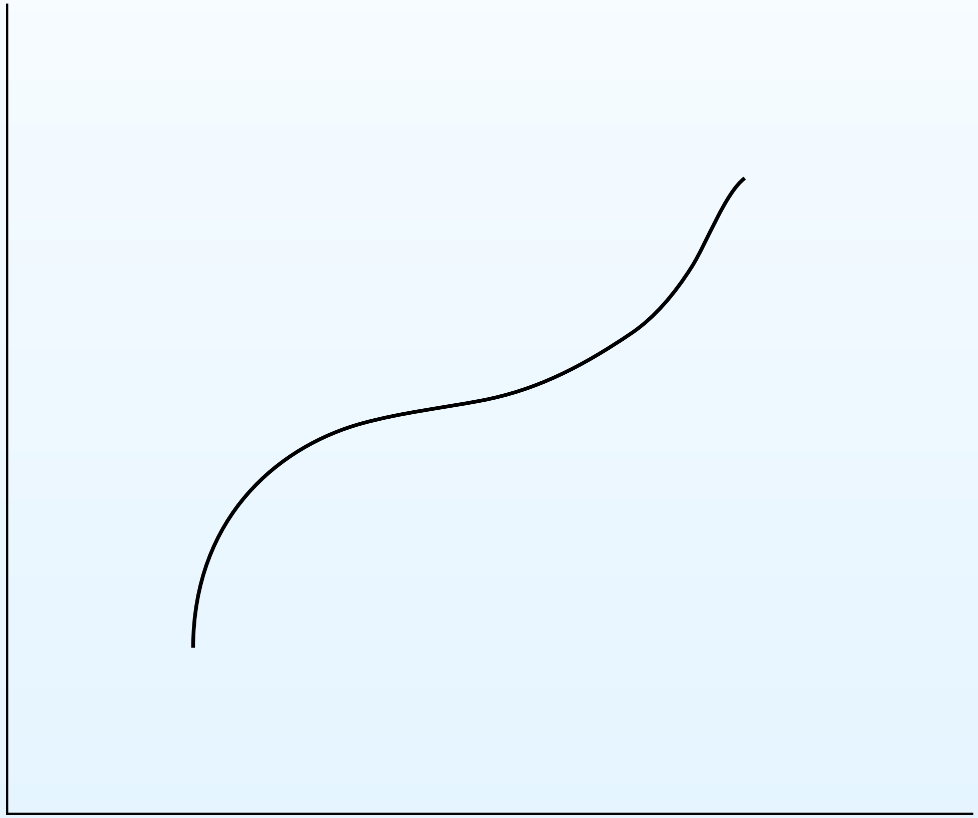

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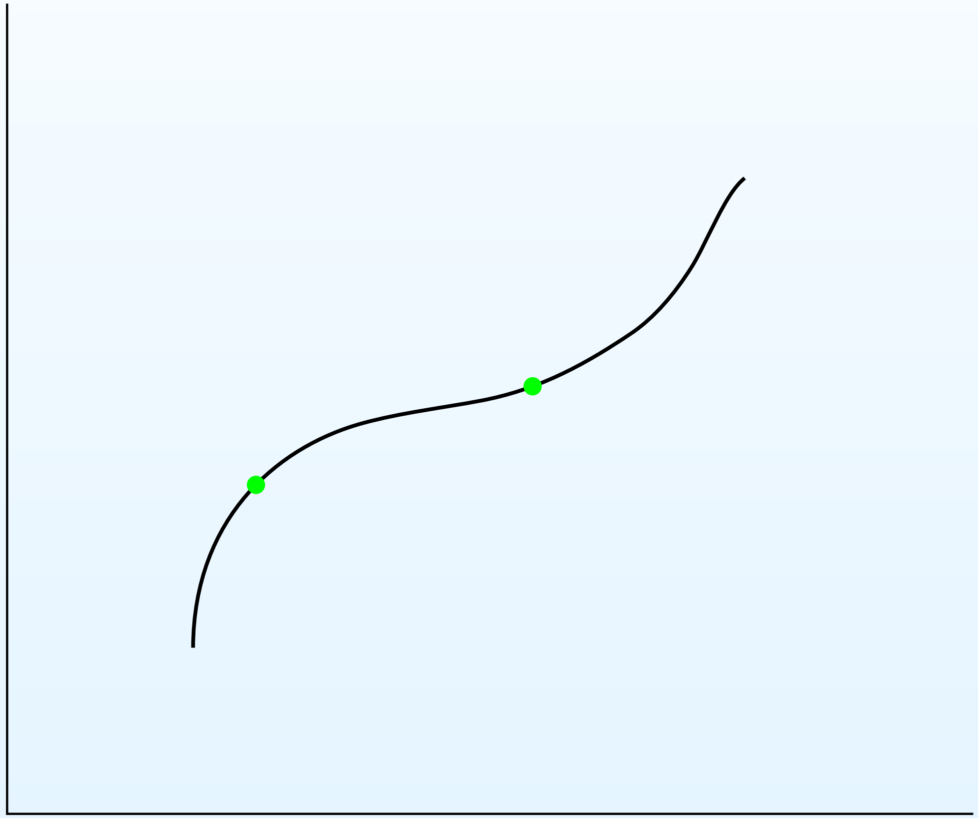

Comonotonicity

- *A set $S \subset R^n$* is comonotonic $\Leftrightarrow$ for all $\mathbf{x}$ and $\mathbf{y}$ in S either $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{x} \geq \mathbf{y}$ holds.
- A comonotonic set is a “thin” set.

Comonotonicity

- *A random vector* $(X_1, \ldots, X_n)$ is comonotonic $\Leftrightarrow$ $(X_1, \ldots, X_n)$ has a comonotonic support.

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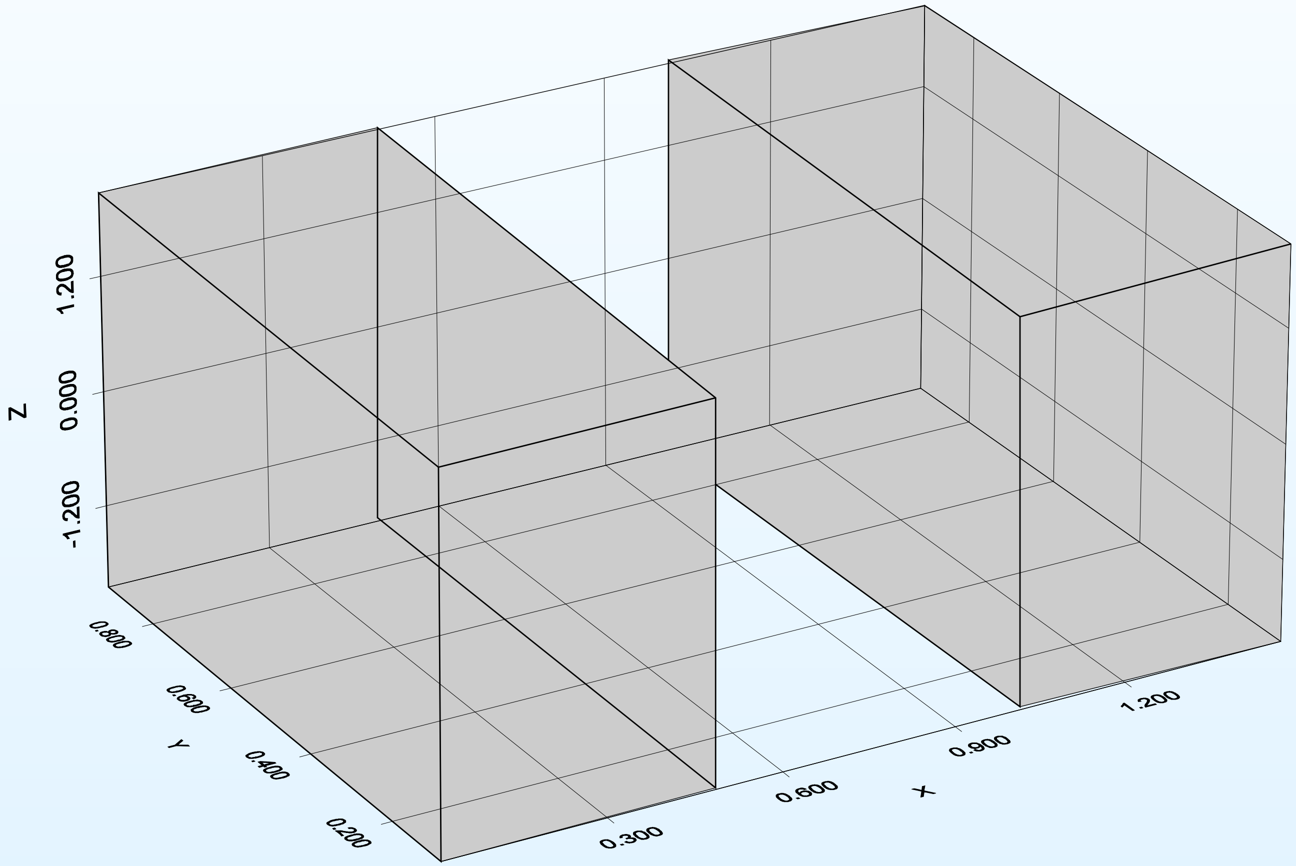

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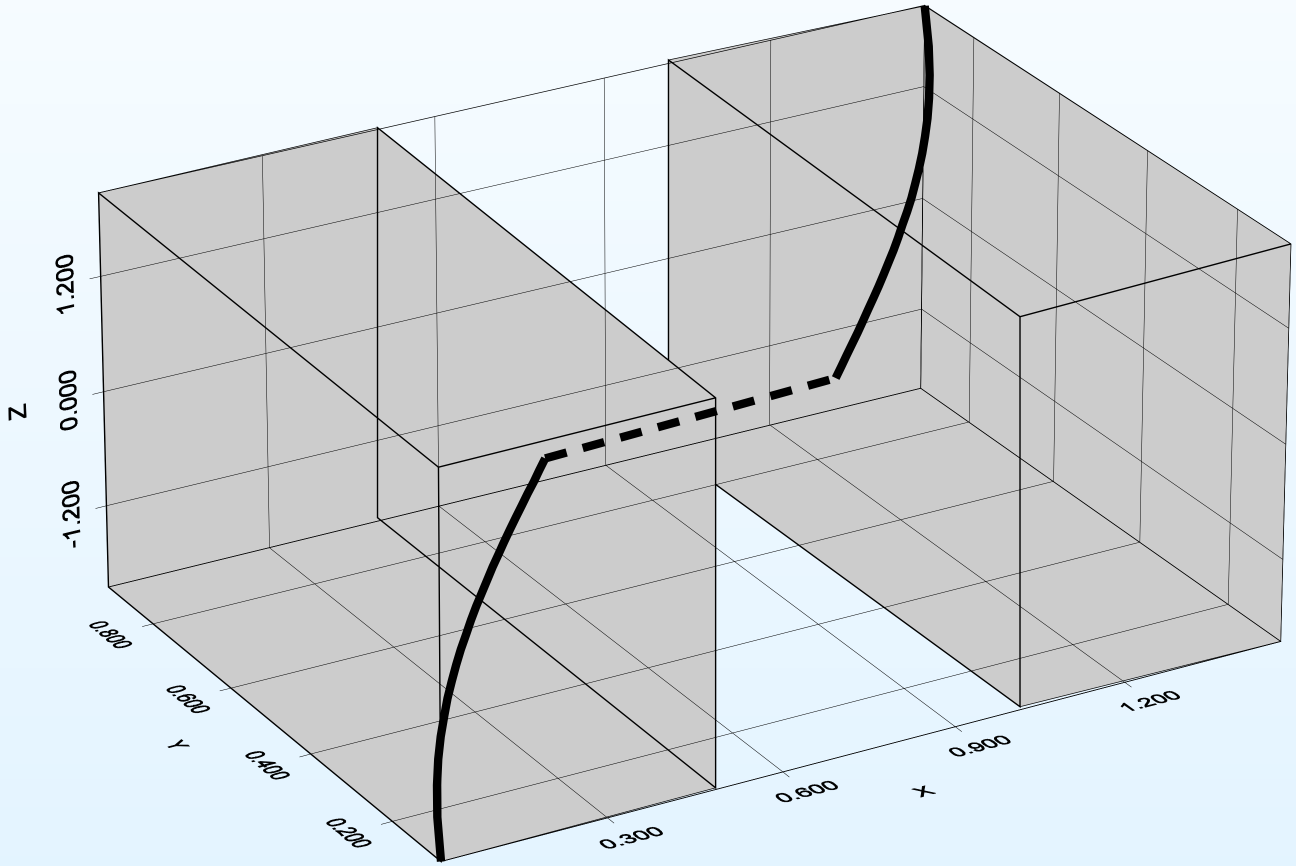


Comonotonicity

- *A random vector* $(X_1, \ldots, X_n)$ is comonotonic $\Leftrightarrow$ $(X_1, \ldots, X_n)$ has a comonotonic support.
- Comonotonicity is very strong positive dependency structure.
- Comonotonic r.v.'s are not able to compensate each other.
- $(Y_1^c, \ldots, Y_n^c)$ is the 'comonotonic counterpart' of $(Y_1, \ldots, Y_n)$.

Characterizations of comonotonicity

- *Notations*:
  - U : uniformly distributed on the $(0,1)$.
  - $\mathbf{X} = (X_1, \ldots, X_n)$.
- *Comonotonicity of a random vector*:
  $\mathbf{X}$ is **comonotonic**
  $\Leftrightarrow$ $\mathbf{X} \overset{d}{=} \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$
  $\Leftrightarrow$ There exists a r.v. Z, and non-decreasing functions $f_1, \ldots, f_n$ such that $\mathbf{X} \overset{d}{=} (f_1(Z), \cdots, f_n(Z))$,
  $\Leftrightarrow$ $\Pr\left[\mathbf{X} \leq \mathbf{x}\right] = \min\left\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right\}$.
- *The Fréchet bound*:
  $\Pr\left[\mathbf{Y} \leq \mathbf{x}\right] \leq \min\left\{F_{Y_1}(x_1), F_{Y_2}(x_2), \ldots, F_{Y_n}(x_n)\right\}$.
  The upper bound is reachable in the class of random vectors with given marginals.

Comonotonicity and correlation

- $Corr[X,Y] = 1 \Rightarrow (X,Y)$ is comonotonic.
- The class of all random couples with given marginals
  - always contains comonotonic couples,
  - does not always contain perfectly correlated couples.
- *Risk sharing schemes*:

$$X = \begin{cases} Z, & Z \leq d \\ d, & Z > d, \end{cases} \qquad Y = \begin{cases} 0, & Z \leq d \\ Z - d, & Z > d. \end{cases}$$

X and Y are comonotonic, but not perfectly correlated.

Comonotonic bounds for sums of dependent r.v.'s

- *Theorem*: For any $(X_1, X_2, \ldots, X_n)$ and any Λ, we have

$$\sum_{i=1}^{n} E\left[X_i \mid \Lambda\right] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U).$$

- *Notations*:
  - $S = \sum_{i=1}^{n} X_i$.
  - $S^l = \sum_{i=1}^{n} E\left[X_i \mid \Lambda\right]$ = lower bound.
  - $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$ = comonotonic upper bound.
- If all $E\left[X_i \mid \Lambda\right]$ are $\nearrow$ functions of Λ, then S^l is a comonotonic sum.

Risk measures and comonotonicity

- *Additivity of risk measures of comonotonic sums*:

$$Q_p(\sum_{i=1}^{n} X_i^c) = \sum_{i=1}^{n} Q_p(X_i).$$

$$TVaR_p(\sum_{i=1}^{n} X_i^c) = \sum_{i=1}^{n} TVaR_p(X_i).$$

- *Sub-additivity of risk measures*: Any risk measure that
  - preserves stop-loss order
  - is additive for comonotonic risks

  is sub-additive: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
- *Examples*:
  - TailVaR$_p$ is sub-additive.
  - CTE$_p$, Q_p and ESF$_p$ are NOT sub-additive.

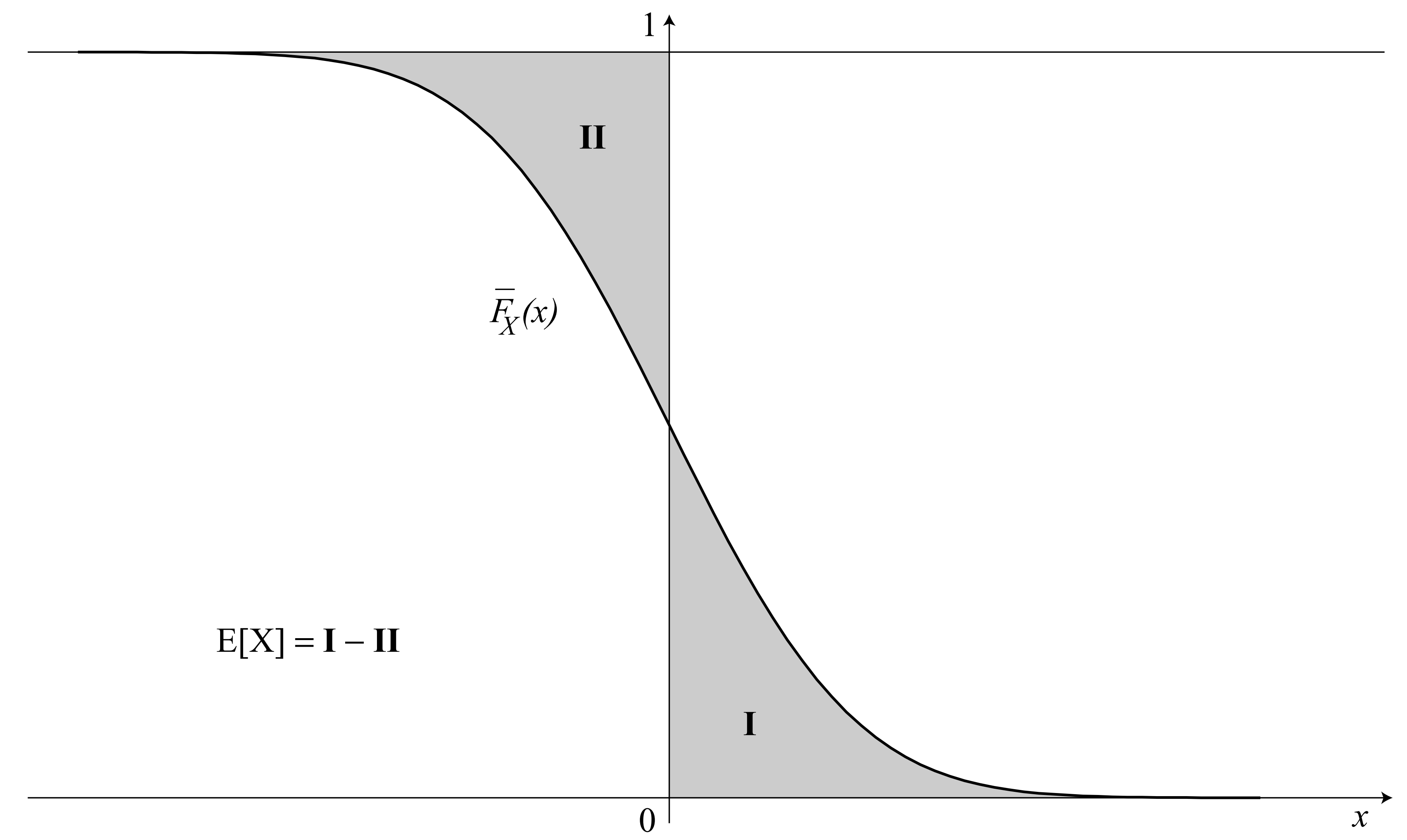
Distortion risk measures

- *Expectation of a r.v.*:

$$E[X] = -\int_{-\infty}^{0} [1 - \bar{F}_X(x)]\, dx + \int_{0}^{\infty} \bar{F}_X(x)\, dx,$$

with $\bar{F}_X(x) = \Pr[X > x]$.

Distortion risk measures



Distortion risk measures

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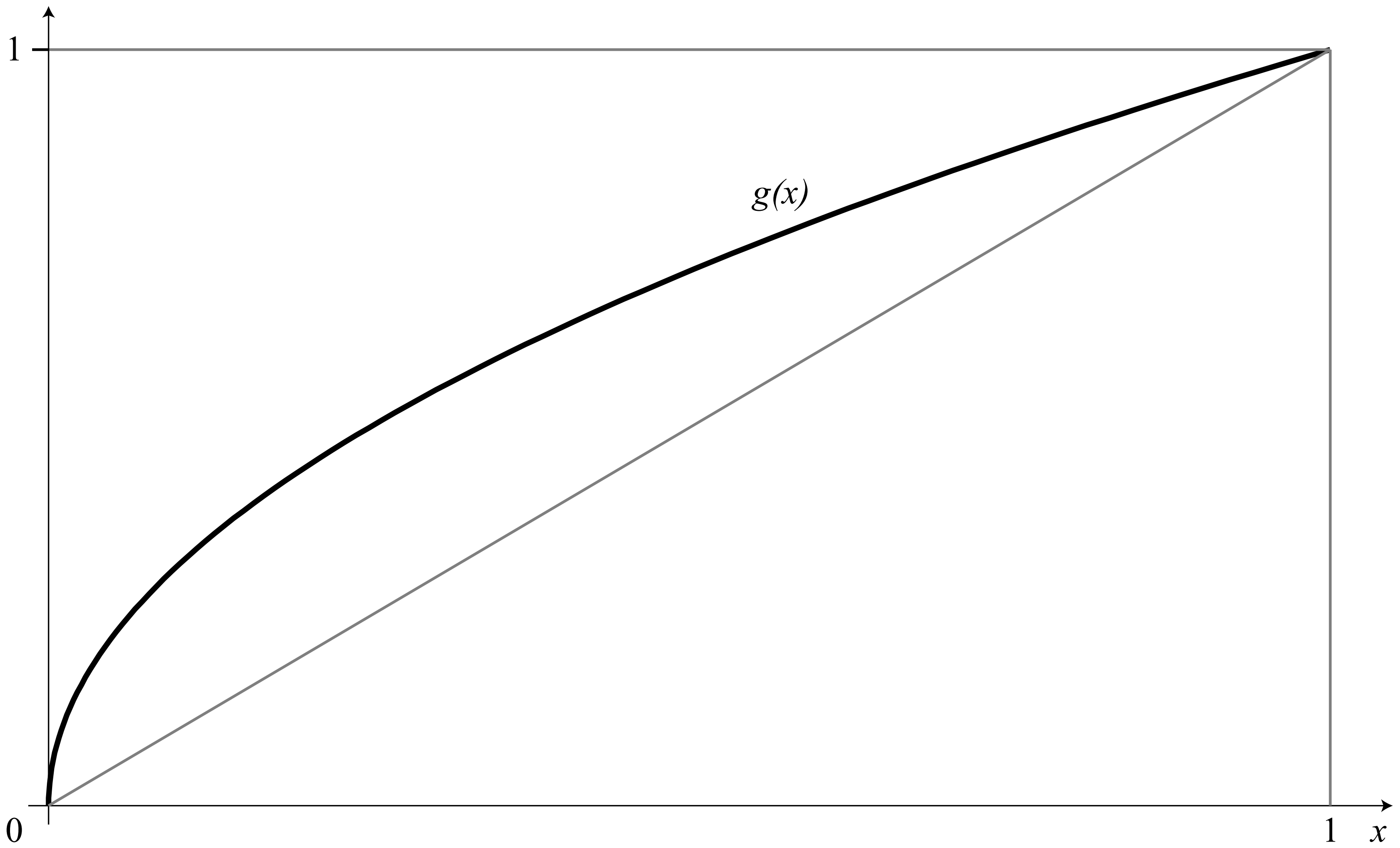
- *Distortion function*:
  $g : [0,1] \to [0,1]$ is a distortion function
  $\Leftrightarrow g$ is $\nearrow$, $g(0) = 0$ and $g(1) = 1$.

Distortion risk measures: $g(x)$ concave $\Rightarrow g(x) \geq x$



Distortion risk measures

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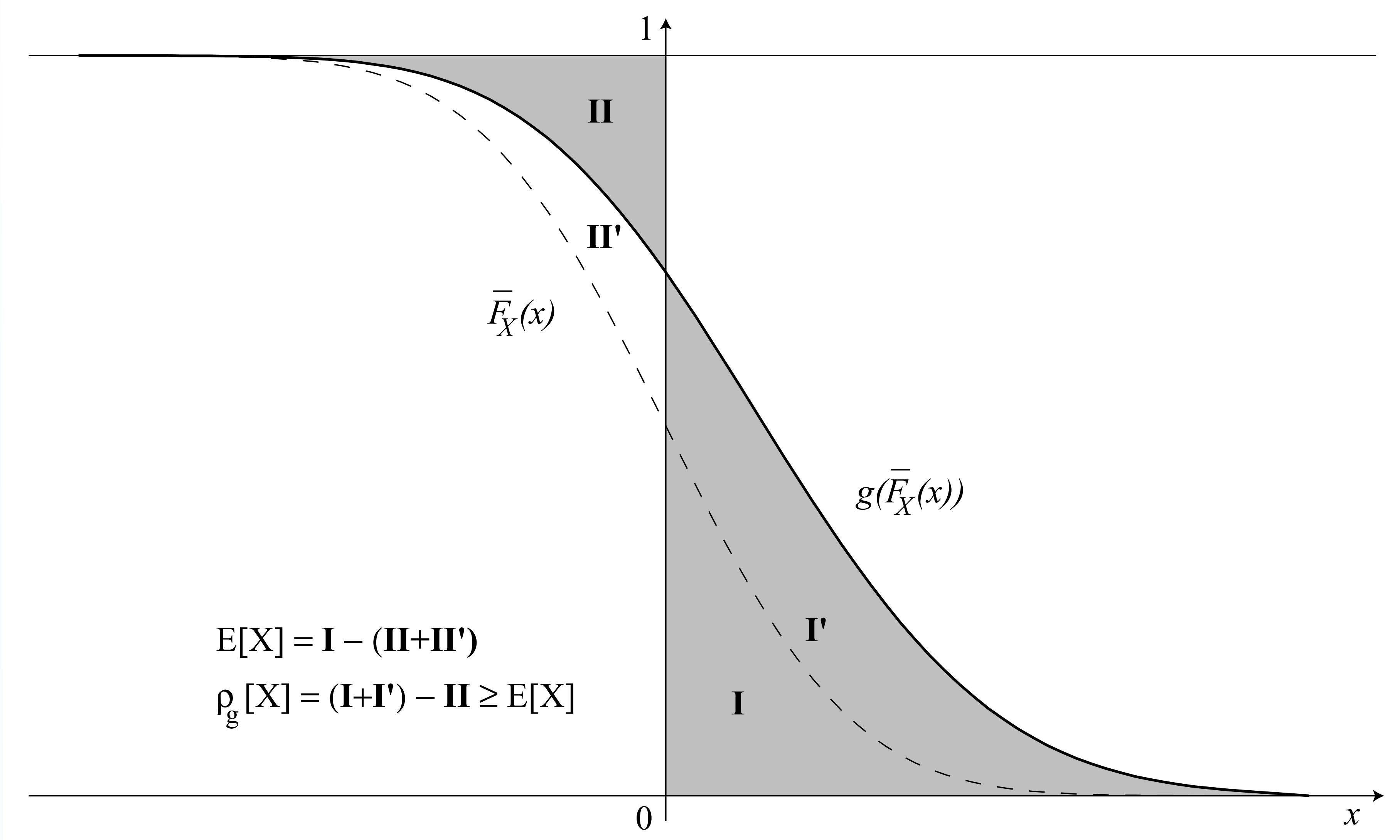
- *Distortion function*:
  $g : [0,1] \to [0,1]$ is a distortion function
  $\Leftrightarrow g$ is $\nearrow$, $g(0) = 0$ and $g(1) = 1$.

- *Distortion risk measure*:

$$\rho_g[X] = -\int_{-\infty}^{0} \left[1 - g\left(\bar{F}_X(x)\right)\right]\, dx + \int_{0}^{\infty} g\left(\bar{F}_X(x)\right)\, dx.$$

  $\rho_g[X]$ = “distorted expectation” of X.

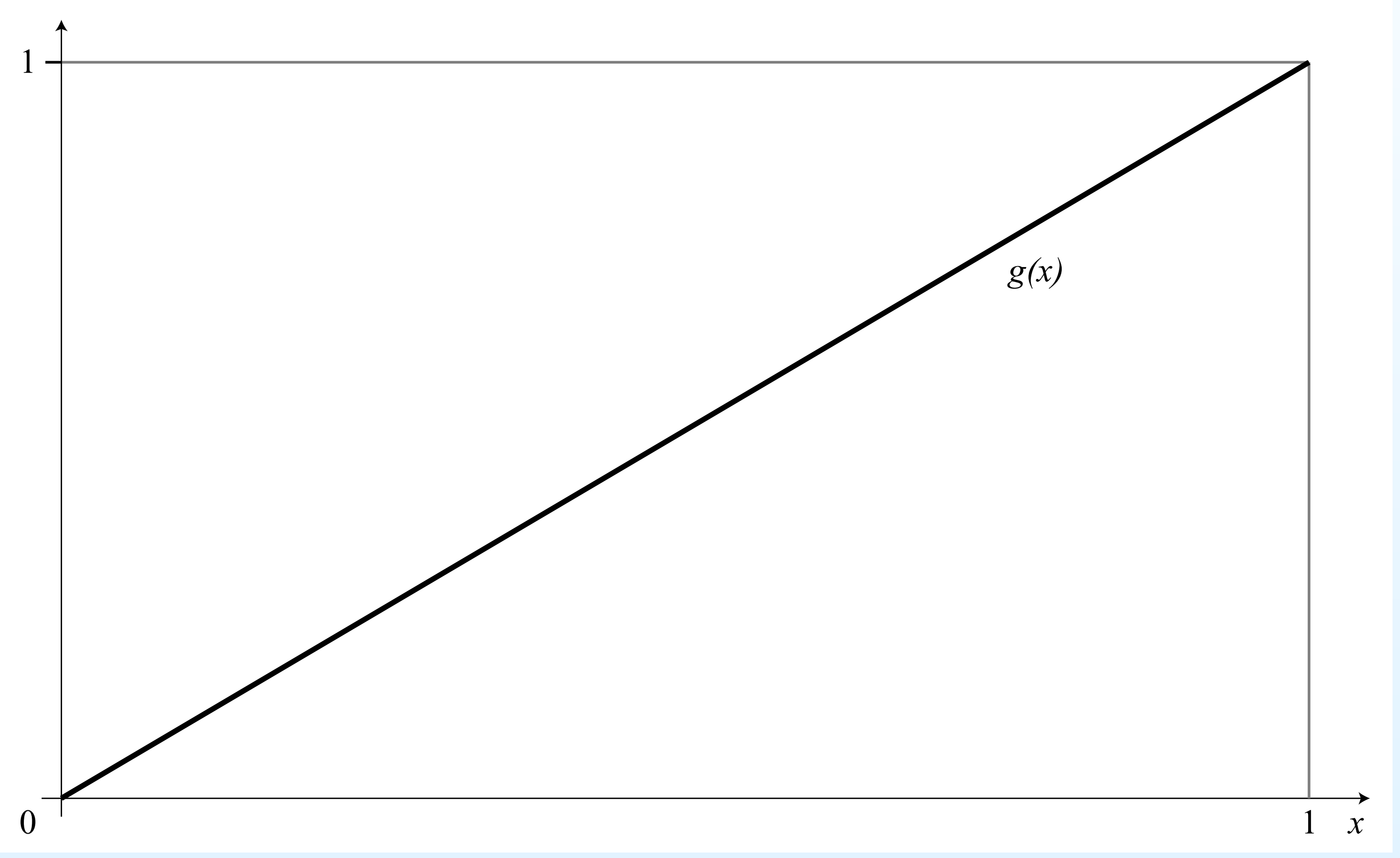
Distortion risk measures: $g(x) \geq x$



Examples of distortion risk measures

- *Expectation*: $X \to E[X]$.

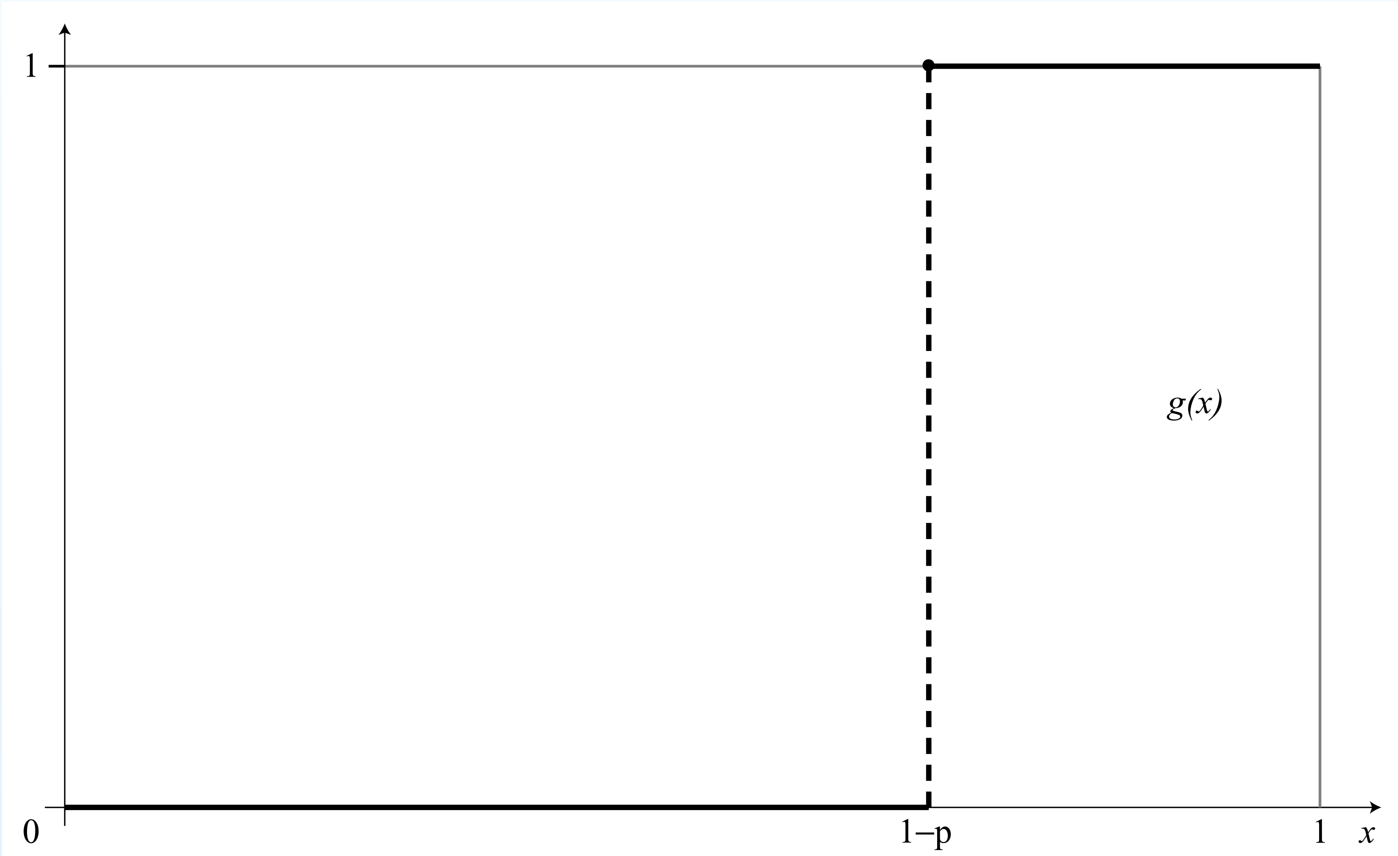
$$g(x) = x, \qquad 0 \leq x \leq 1.$$



Examples of distortion risk measures

- *The quantile risk measure*: $X \to Q_p(X)$.

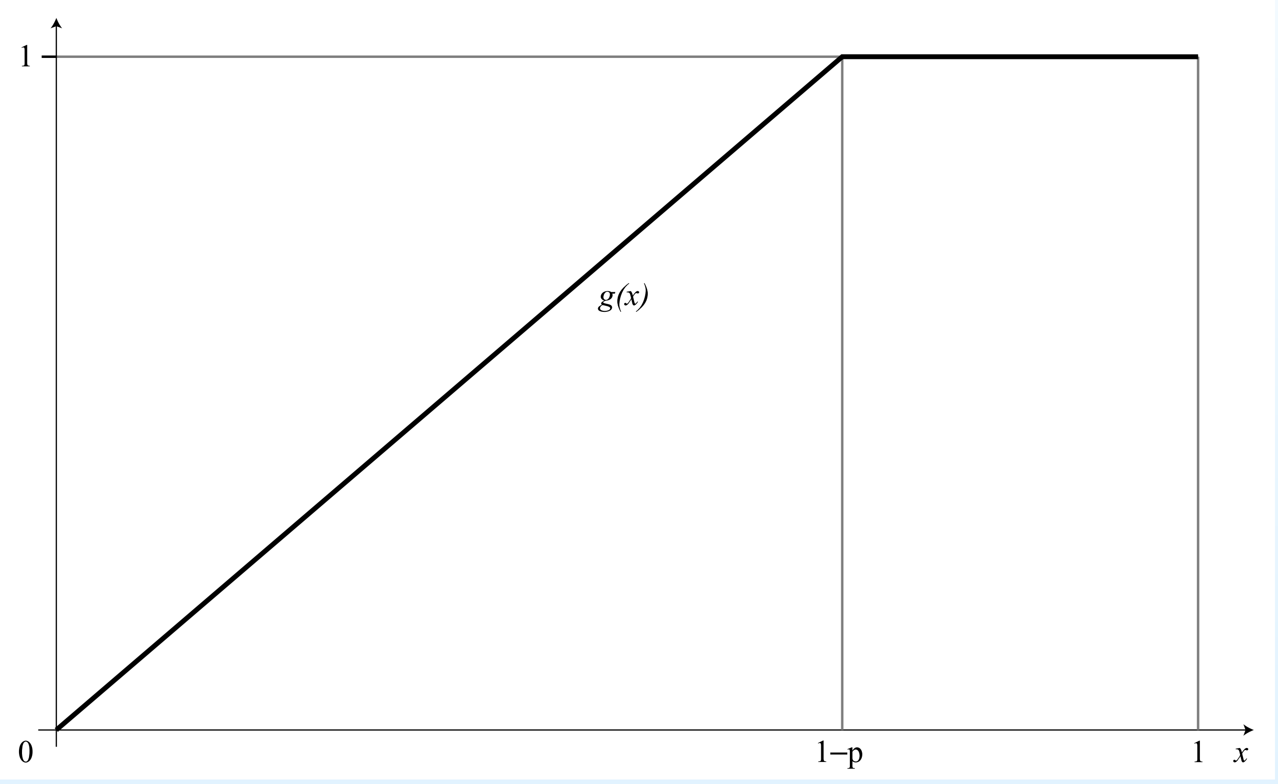
$$g(x) = I\,(x > 1 - p)\,, \qquad 0 \le x \le 1.$$



Examples of distortion risk measures

- *Tail Value-at-Risk*: $X \to TVaR_p(X)$.

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), \qquad 0 \le x \le 1.$$



Examples of distortion risk measures

- *Conditional Tail Expectation*: $X \rightarrow CTE_p(X)$.
  is NOT a distortion risk measure.
- *Expected Shortfall*: $X \rightarrow ESF_p(X)$.
  is NOT a distortion risk measure.
- *Stoch. dominance vs. ordered distortion risk measures*:

$$X \leq_{st} Y \Leftrightarrow \rho_g[X] \leq \rho_g[Y] \text{ for all distortion functions } g.$$

The Wang transform risk measure

- *Problems with TVaR$_p$*:
  - no incentive for taking actions that increase the distribution function for outcomes smaller than Q_p,
  - accounts for the ESF $\Rightarrow$ does not adjust for extreme low-frequency, high severity losses.
- *The Wang transform risk measure* :

$$X \to \rho_{g_p}(X), \qquad 0 < p < 1,$$

with

$$g_p(x) = \Phi\left[\Phi^{-1}(x) + \Phi^{-1}(p)\right], \qquad 0 \leq x \leq 1.$$

offers a possible solution.

The Wang transform risk measure

- *Examples*:
  - if X is normal: $\rho_{g_p}(X) = Q_p(X)$.
  - if X is lognormal: $\rho_{g_p}(X) = Q_{\Phi\left[\Phi^{-1}(p)+\frac{\sigma}{2}\right]}(X)$.

Properties of distortion risk measures

- *Additivity for comonotonic risks*:

$$\rho_g\left[X_1^c + X_2^c + \ldots + X_n^c\right] = \sum_{i=1}^{n} \rho_g(X_i).$$

- *Positive homogeneity*: for any $a > 0$,

$$\rho_g[aX] = a\rho_g[X].$$

- *Translation invariance*:

$$\rho_g[X + b] = \rho_g[X] + b.$$

- *Monotonicity*:

$$X \leq Y \Rightarrow \rho_g[X] \leq \rho_g[Y].$$

Concave distortion risk measures

- *Concave distortion risk measures*:
  - $\rho_g(\cdot)$ is a concave distortion risk measure if g is concave.
  - $TVaR_p(\cdot)$ is concave, $Q_p(\cdot)$ not.
- *SL-order vs. ordered concave distortion risk measures*:

$$X \leq_{sl} Y \Leftrightarrow \rho_g[X] \leq \rho_g[Y] \text{ for all concave } g.$$

The Beta distortion risk measure

- *Problem with $TVaR_p$*: For any concave g, ρ_g strongly preserves stop-loss order $\Leftrightarrow$ g is strictly concave.
  $\Rightarrow TVaR_p$ does not strongly preserve stop-loss order.
- *The Beta distribution*: $(a > 0, b > 0)$

$$F_\beta(x) = \frac{1}{\beta(a,b)} \int_0^x t^{a-1} (1-t)^{b-1} \, dt, \qquad 0 \leq x \leq 1.$$

- *The Beta distortion risk measure*:

$$X \rightarrow \rho_{F_\beta}(X).$$

  ρ_{F_β} strictly preserves stop-loss order provided $0 < a \leq 1$, $b \geq 1$ and a and b are not both equal to 1.
- *A PH-transform risk measure*: Wang (1995).
  $a = 0.1$ and $b = 1$.

Sub-additivity of risk measures

- *Merging decreases the 'insolvency risk'*:

$$(X + Y - \rho[X] - \rho[Y])_+ \leq (X - \rho[X])_+ + (Y - \rho[Y])_+$$

  - Sub-additivity is allowed to some extent.

- *Concave distortion risk measures are sub-additive*:

$$\rho_g[X + Y] \leq \rho_g[X] + \rho_g[Y].$$

  - Q_p is not sub-additive,
  - $TVaR_p$ is sub-additive.

- *Optimality of $TVaR_p$*:

$$TVaR_p(X) = \min\{\rho_g(X) \mid g \text{ is concave and } \rho_g \geq Q_p\}.$$

Axiomatic characterization of risk measures

- *A risk measure is "Artzner-coherent"* if it is sub-additive, monotone, positive homogeneous and translation invariant.
  - Q_p is not ”coherent”.
  - Concave distortion risk measures are ”coherent”.
- *The Dutch risk measure*:

$$\rho(X) = E[X] + E\left[(X - E[X])_+\right].$$

$\rho(X)$ is coherent, but not comonotonic-additive
$\Rightarrow \rho(X)$ is NOT a distortion risk measure.

- *Coherent or not?*
  Markowitz (1959): “We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used.”

Distortion risk measures for sums of dependent r.v.'s

- *Approximations for sums of dependent r.v.'s*:
  $S = \sum_{i=1}^{n} X_i$ with given marginals, but unknown copula.

$$S^l = \sum_{i=1}^{n} E\left[X_i \mid \Lambda\right] \leq_{cx} S \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U) = S^c$$

- *Approximations for $\rho_g[S]$*: (if all $E\left[X_i \mid \Lambda\right]$ are $\nearrow$ in Λ)

$$\begin{aligned} \rho_g\left[S^c\right] &= \sum_{i=1}^{n} \rho_g\left[X_i\right], \\ \rho_g\left[S^l\right] &= \sum_{i=1}^{n} \rho_g\left[E\left(X_i \mid \Lambda\right)\right]. \end{aligned}$$

- If g is concave: $\rho_g\left[S^l\right] \leq \rho_g\left[S\right] \leq \rho_g\left[S^c\right]$.

Application: provisions for future payment obligations

- Problem description
  - Consider a payment obligation of 1 per year, due at times $1, 2, ..., 20$,
  - Let $e^{-Y(i)}$ be the discount factor over $[0, i]$:

$$e^{-Y(i)} \equiv e^{-(Y_1+Y_2+...+Y_i)}.$$

  - Assume the yearly returns Y_j are i.i.d. and normal distributed with parameters $\mu = 0.07$ and $\sigma = 0.1$.
  - The stochastic provision is defined by

$$S = \sum_{i=1}^{20} e^{-(Y_1+Y_2+...+Y_i)}.$$

Provisions for future payment obligations

- Convex bounds for $S = \sum_{i=1}^{20} e^{-Y(i)}$
  Let $\Lambda = \sum_{i=1}^{20} Y_i \sum_{j=i}^{20} e^{-j\mu}$ and $r_i = corr\left[\Lambda, Y(i)\right] > 0$.
  Then
  $$S^l \leq_{cx} S \leq_{cx} S^c$$
  where
  $$S^l = \sum_{i=1}^{n} e^{-E[Y(i)] - r_i\ \sigma_{Y(i)}\ \Phi^{-1}(U) + \frac{1}{2}(1-r_i^2)\sigma_{Y(i)}^2},$$
  $$S^c = \sum_{i=1}^{n} e^{-E[Y(i)] +\ \sigma_{Y(i)}\ \Phi^{-1}(U)}.$$

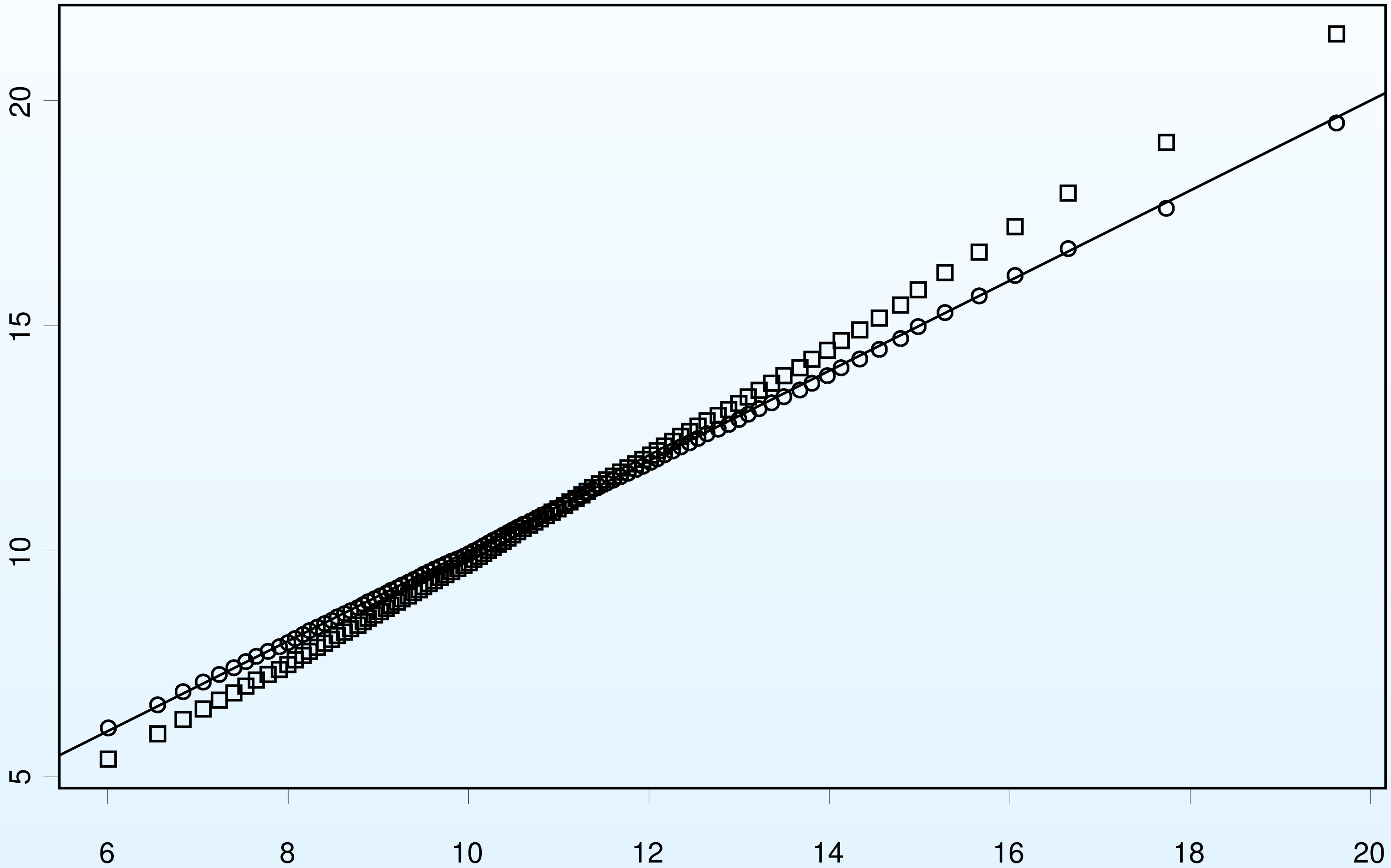
Provisions for future payment obligations

- Provision (or total capital requirement)
  - The provision for this series of future obligations is set equal to $\rho_g[S]$
  - Approximate $\rho_g[S]$ by

$$\begin{aligned} \rho_g\left[S^c\right] &= \sum_{i=1}^{n} \rho_g\left[X_i\right], \\ \rho_g\left[S^l\right] &= \sum_{i=1}^{n} \rho_g\left[E\left(X_i \mid \Lambda\right)\right]. \end{aligned}$$

Provisions for future payment obligations

- The Quantile-provision principle: $\rho_g[S] = Q_p[S]$



Provisions for future payment obligations

- The CTE-provision principle: $\rho_g[S]$ =TVaR$_p[S]$

p	TVAR$_p[S^l]$	'TVAR$_p[S]$'	TVAR$_p[S^c]$
0.950	17.24	17.26	18.61
0.975	18.45	18.50	20.14
0.990	20.03	20.10	22.16
0.995	21.22	21.30	23.69
0.999	23.98	24.19	27.29

Theories of choice under risk

- *Expected utility theory*:
  - von Neumann & Morgenstern (1947).
  - Prefer loss X over loss Y if

$$E\left[u(w-X)\right] \geq E\left[u(w-Y)\right],$$

  - $u(x)$ = utility of wealth-level x, $\nearrow$ function of x.
  - Risk aversion: u is concave.
- *Yaari's dual theory of choice under risk*:
  - Yaari (1987).
  - Prefer loss X over loss Y if

$$\rho_f\left[w-X\right] \geq \rho_f\left[w-Y\right],$$

  - $f(q)$ = distortion function.
  - Risk aversion: f is convex.

Compare theories of choice under risk

- *Transformed expected wealth levels*:

$$
\begin{aligned}
E[w - X] &= \int_0^1 Q_{1-q}(w - X)\, dq, \\
E[u(w - X)] &= \int_0^1 u\left[Q_{1-q}(w - X)\right]\, dq, \\
\rho_f[w - X] &= \int_0^1 Q_{1-q}(w - X)\, df(q).
\end{aligned}
$$

- *Ordering of risks*:
  - In both theories, stochastic dominance reflects common preferences of all decision makers.
  - In both theories, stop-loss order reflects common preferences of all risk-averse decision makers.

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References (see www.kuleuven.ac.be/insurance)

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Lecture No. 2

Comonotonicity and Optimal Portfolio Selection

Jan Dhaene

Introduction

- *Strategic portfolio selection*:
  For a given savings and/or consumption pattern over a given time horizon, identify the best allocation of wealth among a basket of securities.
- *The 'Terminal Wealth' problem*:
  - Saving for retirement.
  - A loan with an amortization fund with random return.
- *The 'Reserving' problem*:
  - The 'after retirement' problem.
  - Technical provisions.
  - Capital requirements.

Introduction

- *The 'Buy and Hold' strategy*:
  - Keep the initial quantities constant.
  - A static strategy.
- *The 'Constant Mix' strategy*:
  - Keep the initial proportions constant.
  - A dynamic strategy.

Comonotonicity

- *Notations*:
  - U: uniformly distributed on $(0,1)$.
  - $\underline{X} = (X_1, \ldots, X_n)$.
  - $F_X^{-1}(p) = Q_p[X] = \mathrm{VaR}_p[X] = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}$.
- *Comonotonicity of a random vector*:
  $\underline{X}$ is **comonotonic** $\Leftrightarrow$ there exist non-decreasing functions $f_1, \ldots, f_n$ and a r.v. Z such that

$$\underline{X} \stackrel{d}{=} [f_1(Z), \ldots, f_n(Z)] \, .$$

- Comonotonicity: very strong positive dependency structure.
- Comonotonic r.v.'s cannot be pooled.

Comonotonic bounds for sums of dependent r.v.'s

- *Theorem*:
  For any $\underline{X}$ and any Λ, we have

$$\sum_{i=1}^{n} \mathrm{E}\left[X_i \mid \Lambda\right] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U).$$

- *Notations*:
  - $S = \sum_{i=1}^{n} X_i$.
  - $S^l = \sum_{i=1}^{n} \mathrm{E}\left[X_i \mid \Lambda\right]$ = lower bound.
  - $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$ = comonotonic upper bound.
- If all $\mathrm{E}\left[X_i \mid \Lambda\right]$ are increasing functions of Λ,
  then S^l is a comonotonic sum.

Performance of the comonotonic approximations

- *Local comonotonicity*:
  Let $B(\tau)$ be a standard Wiener process.
  The accumulated returns

$$\exp\left[\mu\tau + \sigma\ B(\tau)\right],$$
$$\exp\left[\mu\left(\tau + \Delta\tau\right) + \sigma\ B\left(\tau + \Delta\tau\right)\right]$$

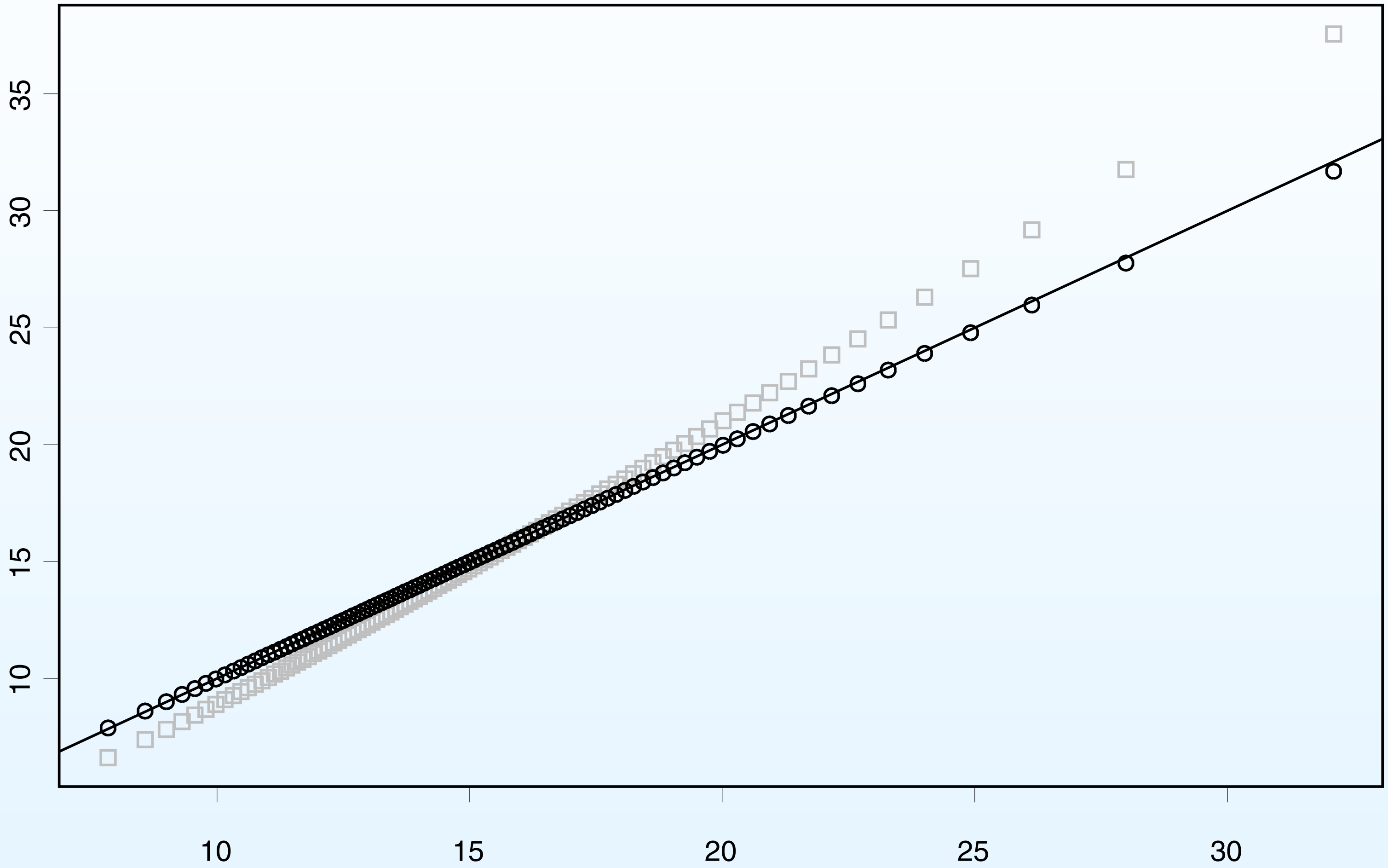
  will be 'almost comonotonic'.

- *The continuous perpetuity*:

$$S = \int_0^\infty \exp\left[-\mu\tau - \sigma\ B(\tau)\right]\ d\tau$$

  has a *reciprocal Gamma distribution.*

Numerical illustration: $\mu = 0.07$ and $\sigma = 0.1.$



Circles: Plot of $(Q_p[S], Q_p[S^l])$

Numerical illustration

p	$Q_p[S^l]$	$Q_p[S]$	$Q_p[S^c]$
0.95	23.62	23.63	25.90
0.975	26.09	26.13	29.34
0.99	29.37	29.49	34.08
0.995	31.90	32.10	37.86

The Black-Scholes setting

- 1 *risk-free and* m *risky assets*:

$$\frac{dP^0(t)}{P^0(t)} = r\,dt$$

$$\frac{dP^i(t)}{P^i(t)} = \mu_i\,dt + \sum_{j=1}^{d} \bar{\sigma}_{ij}\,dW^j(t)$$

with $\left(W^1(\tau),\ \ldots,\ W^d(\tau)\right)$:
independent standard Brownian motions.

The Black-Scholes setting

- *Equivalent formalism*:

$$\frac{dP^0(t)}{P^0(t)} = r\,dt$$

$$\frac{dP^i(t)}{P^i(t)} = \mu_i\,dt + \sigma_i\,dB^i(t)$$

with $\left(B^1(\tau), \ldots, B^m(\tau)\right)$ correlated standard Brownian motions.

The Black-Scholes setting

- *Return of asset i in year k*:

$$P^i(k) = P^i(k-1)\, e^{Y_k^i}$$

- Y_k^i normal distributed with

$$\mathrm{E}\left[Y_k^i\right] = \mu_i - \frac{1}{2}\sigma_i^2 \text{ and } \mathrm{Var}\left[Y_k^i\right] = \sigma_i^2$$

- Independence over the different years:

$$k \neq l \Rightarrow Y_k^i \text{ and } Y_l^j \text{ are independent.}$$

- Dependence within each year: $\mathrm{Cov}\left[Y_k^i,\ Y_k^j\right] = (\underline{\Sigma})_{ij}$
- Assumptions: $\underline{\mu} \neq r\underline{1}$ and $\underline{\Sigma}$ is positive definite.

Investment strategies

- *Constant mix strategies*:

$$\underline{\pi}\left(t\right) = \left(\pi_1, \pi_2, \ldots, \pi_m\right)$$

with

$$\pi_i = \text{ fraction invested in risky asset } i,$$

$$1 - \sum_{i=1}^{m} \pi_i = \text{ fraction invested in riskfree asset.}$$

  - Fractions time-independent.
  - Dynamic trading strategies.
  - Requires continuously rebalancing.

Investment strategies

- *The portfolio return process*: Merton (1971).
  - $P(t)$ = price of one unit of $(\pi_1, \pi_2, \ldots, \pi_m)$.

$$\frac{dP(t)}{P(t)} = \mu\left(\underline{\pi}\right)\ t + \sigma\left(\underline{\pi}\right) dB(t)$$

with $B(\tau)$ a standard Brownian motion and

$$\mu\left(\underline{\pi}\right) = r + \underline{\pi}^T \times \left(\underline{\mu} - r\ \underline{1}\right), \quad \sigma^2\left(\underline{\pi}\right) = \underline{\pi}^T \times \underline{\Sigma} \times \underline{\pi}$$

  - *Yearly portfolio returns*: $P(k) = P(k-1)\ e^{Y_k(\underline{\pi})}$
  - The $Y_k\left(\underline{\pi}\right)$ are i.i.d. normal with

$$\mathrm{E}\left[Y_k\left(\underline{\pi}\right)\right] = \mu\left(\underline{\pi}\right) - \frac{1}{2}\sigma^2\left(\underline{\pi}\right), \quad \mathrm{Var}\left[Y_k\left(\underline{\pi}\right)\right] = \sigma^2\left(\underline{\pi}\right)$$

Markowitz mean-variance analysis

- *The mean-variance efficient frontier*:

$$\max_{\underline{\pi}} \mu\left(\underline{\pi}\right) \text{ subject to } \sigma\left(\underline{\pi}\right) = \sigma$$

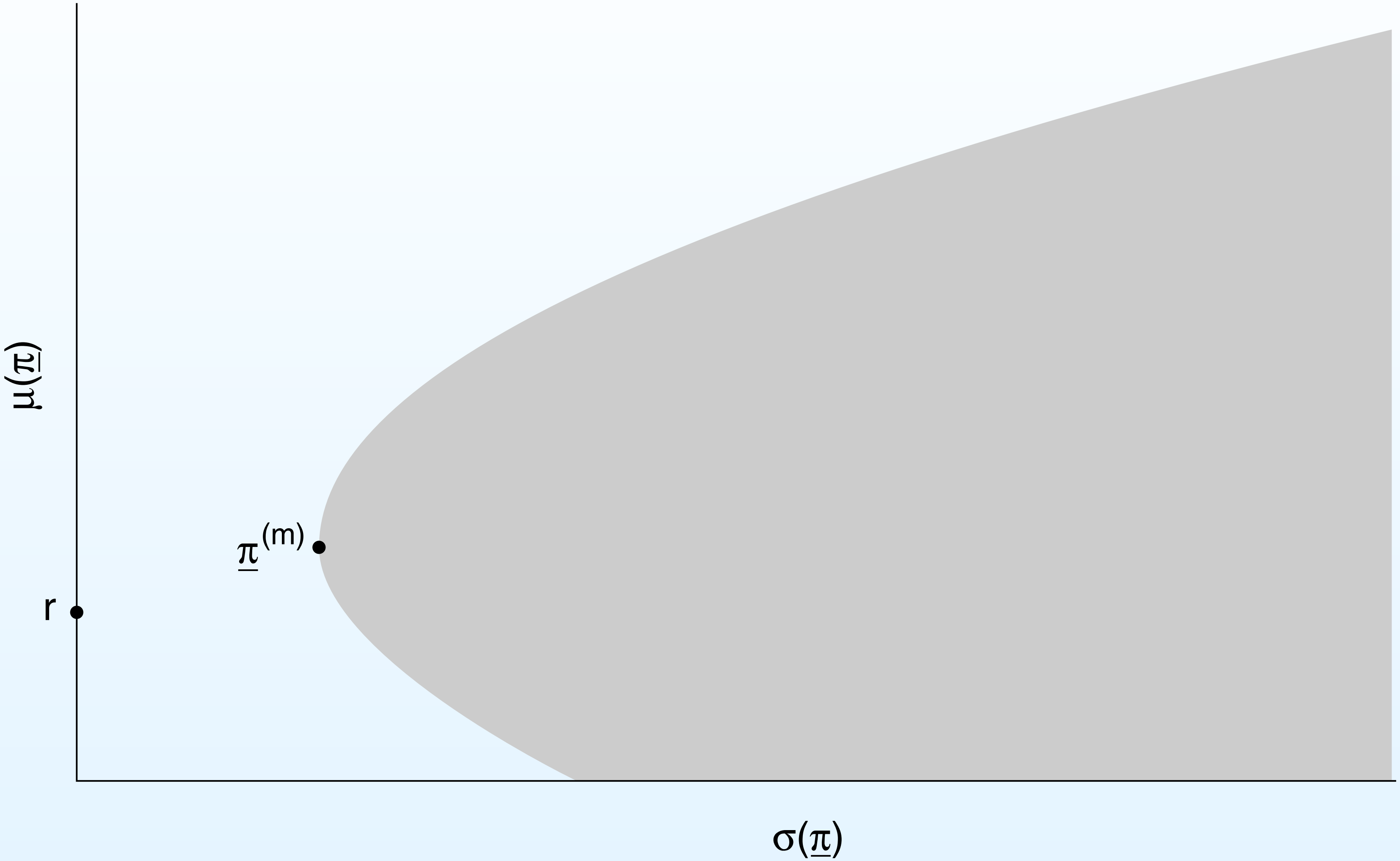
is obtained for the portfolio

$$\underline{\pi}^{\sigma} = \sigma \frac{\underline{\Sigma}^{-1} \cdot \left(\underline{\mu} - r\underline{1}\right)}{\sqrt{\left(\underline{\mu} - r\underline{1}\right)^{T} \cdot \underline{\Sigma}^{-1} \cdot \left(\underline{\mu} - r\underline{1}\right)}}$$

with

$$\mu\left(\underline{\pi}^{\sigma}\right) = r + \sigma\sqrt{\left(\underline{\mu} - r\underline{1}\right)^{T} \cdot \underline{\Sigma}^{-1} \cdot \left(\underline{\mu} - r\underline{1}\right)}$$

Markowitz mean-variance analysis: $r < \mu\left(\underline{\pi}^{(m)}\right)$



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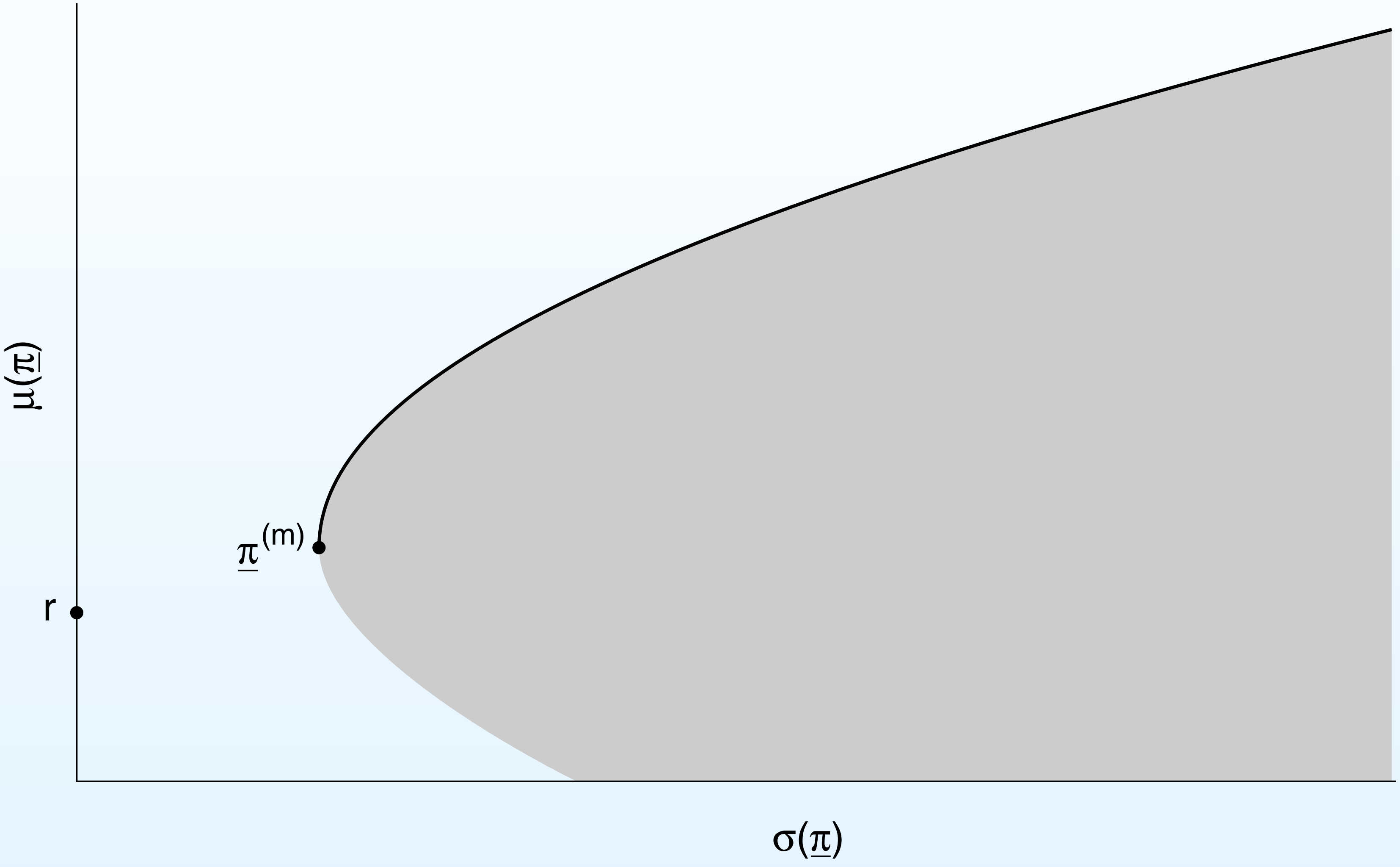
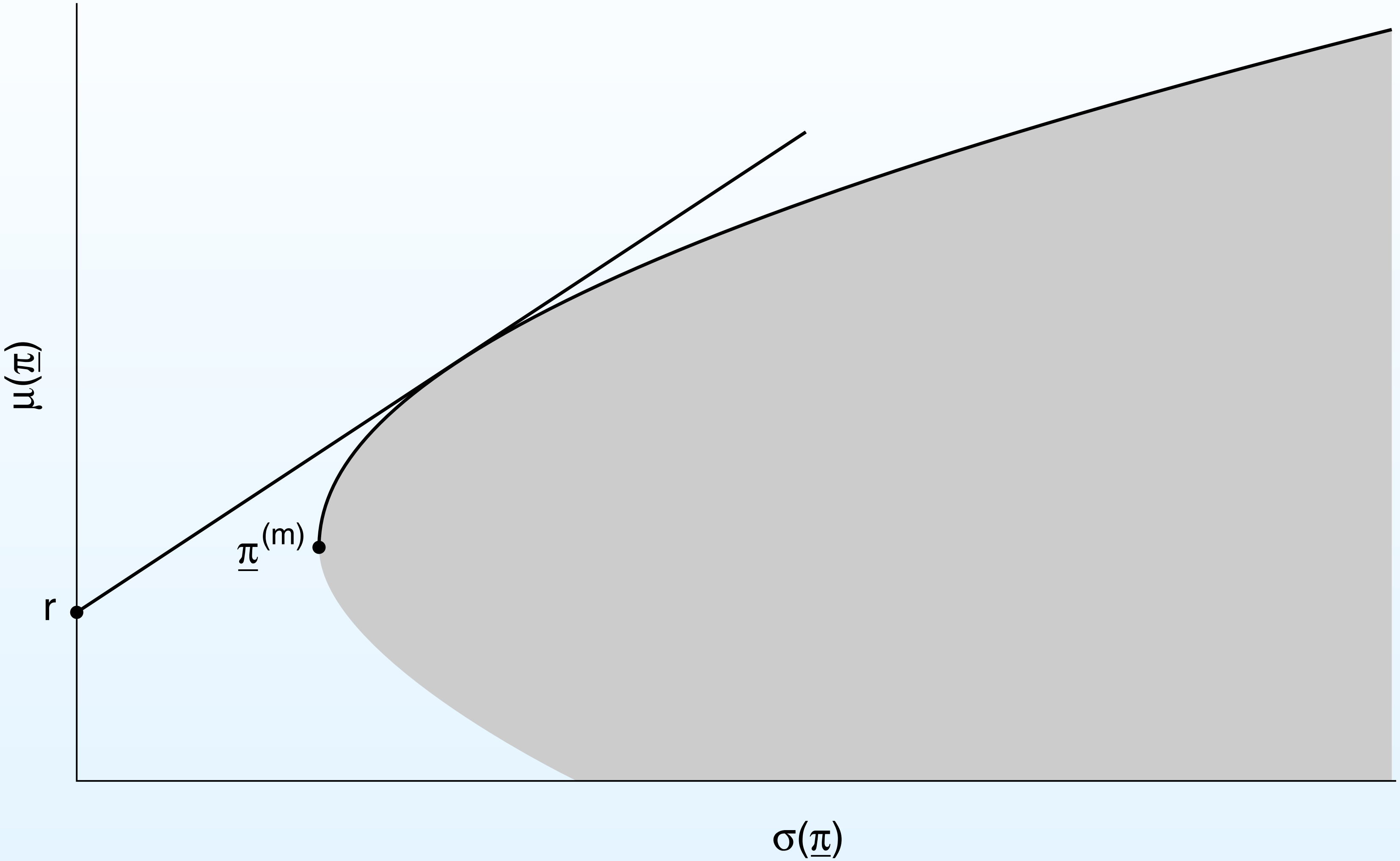
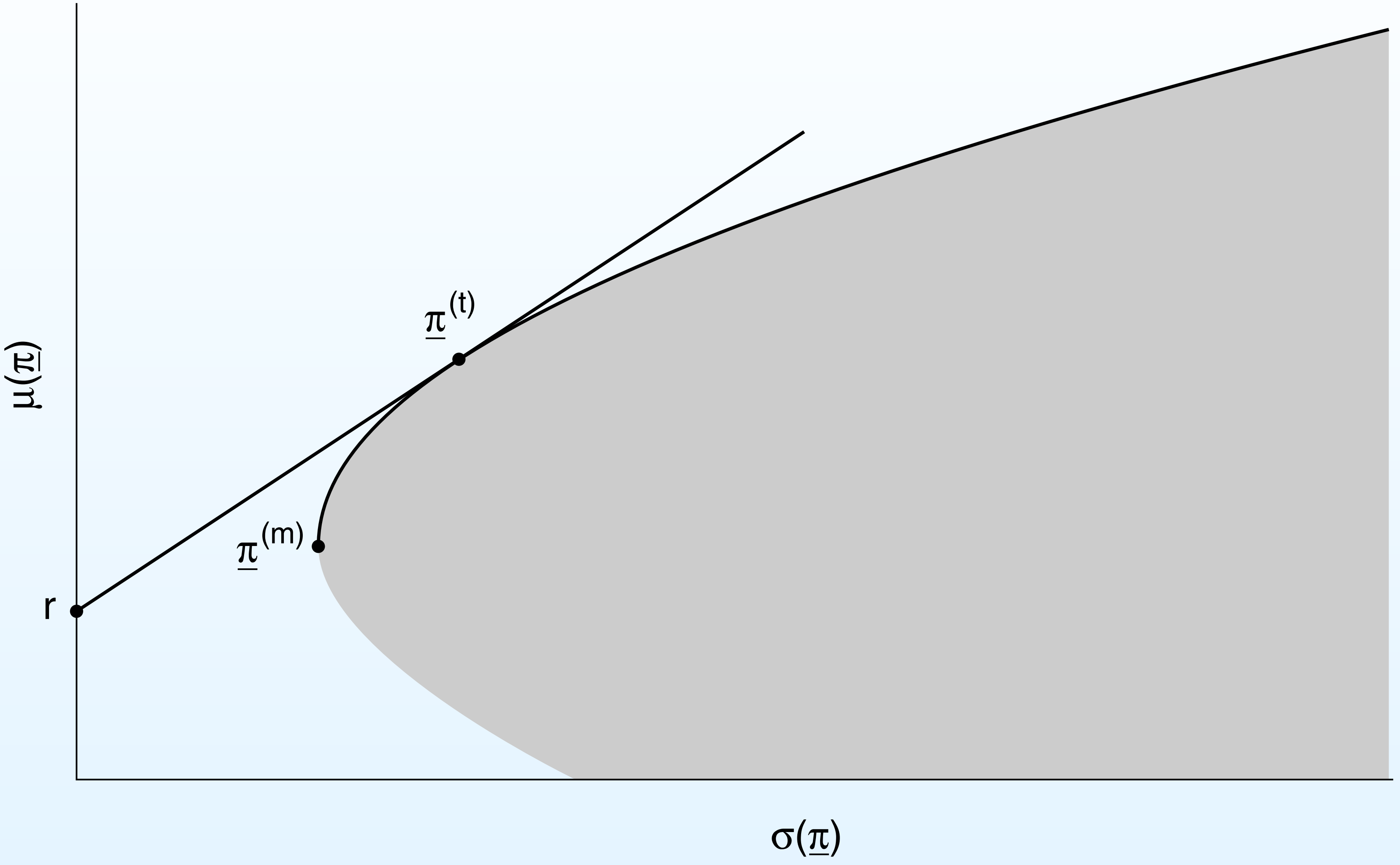


Markowitz mean-variance analysis: $r < \mu\left(\underline{\pi}^{(m)}\right)$



Markowitz mean-variance analysis: $r < \mu\left(\underline{\pi}^{(m)}\right)$



Markowitz mean-variance analysis

- *The Capital Market Line and the Sharpe ratio*:

$$\mu\left(\underline{\pi}^{\sigma}\right) = r + \left(\frac{\mu\left(\underline{\pi}^{(t)}\right) - r}{\sigma\left(\underline{\pi}^{(t)}\right)}\right)\sigma.$$

- *Two Fund Separation Theorem*:

$$\underline{\pi}^{\sigma} = \left(\frac{\mu\left(\underline{\pi}^{\sigma}\right) - r}{\mu\left(\underline{\pi}^{(t)}\right) - r}\right)\underline{\pi}^{(t)}.$$

Saving and terminal wealth

- *Problem description*:
  - $\alpha_0, \alpha_1, \ldots, \alpha_n$: positive savings at times $0, 1, 2, \ldots, n$.
  - *Investment strategy*: $\underline{\pi}(t) = (\pi_1, \pi_2, \ldots, \pi_m)$.
  - *Wealth at time* j:

$$W_j\left(\underline{\pi}\right) = W_{j-1}\left(\underline{\pi}\right) e^{Y_j\left(\underline{\pi}\right)} + \alpha_j$$

    with $W_0\left(\underline{\pi}\right) = \alpha_0$.
  - What is the optimal investment strategy $\underline{\pi}^*$?
  - Depends on 'target capital' and 'probability level'.

Approximating Terminal Wealth

- *Terminal wealth $W_n(\underline{\pi})$*:

$$W_n\left(\underline{\pi}\right) = \sum_{i=0}^{n} \alpha_i \, e^{Y_{i+1}(\underline{\pi})+Y_2(\underline{\pi})+\cdots+Y_n(\underline{\pi})} = \sum_{i=0}^{n} X_i$$

- *The comonotonic upper bound for $W_n\left(\underline{\pi}\right)$*:

$$W_n^c\left(\underline{\pi}\right) = \sum_{i=0}^{n} F_{X_i}^{-1}(U)$$

- *A comonotonic lower bound for $W_n\left(\underline{\pi}\right)$*:

$$W_n^l\left(\underline{\pi}\right) = \sum_{i=0}^{n} E\left[X_i \mid \sum_{j=1}^{n} Y_j\left(\underline{\pi}\right) \sum_{k=0}^{j-1} \alpha_k \, e^{-k\ \mu(\underline{\pi})}\right]$$

- *Convex ordering*: $W_n^l(\underline{\pi}) \ \leq_{cx} \ W_n(\underline{\pi}) \ \leq_{cx} \ W_n^c(\underline{\pi})$

Optimal investment strategies

- *Terminal wealth* $W_n(\underline{\pi})$:

$$W_n(\underline{\pi}) = \sum_{i=0}^{n} \alpha_i \, e^{Y_{i+1}(\underline{\pi})+Y_{i+2}(\underline{\pi})+\cdots+Y_n(\underline{\pi})}$$

- *Utility Theory*: Von Neumann & Morgenstern (1947).

$$\max_{\underline{\pi}} \mathrm{E}\left[u\left(W_n\left(\underline{\pi}\right)\right)\right]$$

- *Yaari's dual theory of choice under risk*: Yaari (1987).

$$\max_{\underline{\pi}} \mathrm{E}^f\left[W_n\left(\underline{\pi}\right)\right]$$

where

- E^f is determined with $f\left(\Pr\left(W_n\left(\underline{\pi}\right) > x\right)\right)$,
- convexity of f corresponds with 'risk aversion'.

Optimal investment strategies

- *Reduced optimization problem*:
  - For $\sigma\left(\underline{\pi}_1\right)=\sigma\left(\underline{\pi}_2\right)$ and $\mu\left(\underline{\pi}_1\right)<\mu\left(\underline{\pi}_2\right)$, we have that
$$W_n\left(\underline{\pi}_1\right)\leq_{st}W_n\left(\underline{\pi}_2\right).$$
  - Hence,
$$\max_{\underline{\pi}}\mathrm{E}\left[u\left(W_n\left(\underline{\pi}\right)\right)\right]=\max_{\sigma}\mathrm{E}\left[u\left(W_n\left(\underline{\pi}^{\sigma}\right)\right)\right]$$
and
$$\max_{\underline{\pi}}\mathrm{E}^f\left[W_n\left(\underline{\pi}\right)\right]=\max_{\sigma}\mathrm{E}^f\left[W_n\left(\underline{\pi}^{\sigma}\right)\right].$$

The Target Capital

- *Distorted expectations*: for

$$f(x) = \begin{cases} 0 & : x \leq p \\ 1 & : x > p, \end{cases}$$

the distorted expection $\mathrm{E}^f \left[W_n \left(\underline{\pi} \right) \right]$ reduces to

$$Q_{1-p} \left[W_n \left(\underline{\pi} \right) \right] = \sup \left\{ x \mid \Pr \left[W_n \left(\underline{\pi} \right) > x \right] \geq p \right\}.$$

- *Problem*: d.f. of $W_n \left(\underline{\pi} \right)$ too cumbersome to work with
  - curse of dimensionality
  - dependencies

Maximizing the Target Capital, for a given p

- *Optimal investment strategy*: $\underline{\pi}^*$ follows from

$$\max_{\underline{\pi}} Q_{1-p}\left[W_n\left(\underline{\pi}\right)\right]$$

- *Approximation*:the approximation $\underline{\pi}^l$ for $\underline{\pi}^*$ follows from

$$\max_{\sigma} Q_{1-p}\left[W_n^l\left(\pi^\sigma\right)\right]$$

with

$$Q_{1-p}\left[W_n^l(\underline{\pi}^\sigma)\right] = \sum_{i=0}^{n} \alpha_i e^{(n-i)\left[\mu(\underline{\pi}^\sigma)-\frac{1}{2}r_i^2(\underline{\pi}^\sigma)\sigma^2\right]-\sqrt{n-i}\; r_i(\underline{\pi}^\sigma)\sigma\Phi^{-1}(p)}$$

Numerical illustration

- *Available assets*:
  - 1 riskfree asset with $r = 0.03$
  - 2 risky assets with

$$\mu_1 = 0.06, \qquad \sigma_1 = 0.10$$
$$\mu_2 = 0.10, \qquad \sigma_2 = 0.20$$

and

$$\mathrm{Corr}\left[Y_k^1, Y_k^2\right] = 0.5$$

- *The tangency portfolio*:

$$\underline{\pi}^{(t)} = \left(\frac{5}{9}, \frac{4}{9}\right), \quad \mu\left(\underline{\pi}^{(t)}\right) = \frac{7}{90}, \quad \sigma\left(\underline{\pi}^{(t)}\right) = \sqrt{\frac{43}{2700}}$$

Numerical illustration

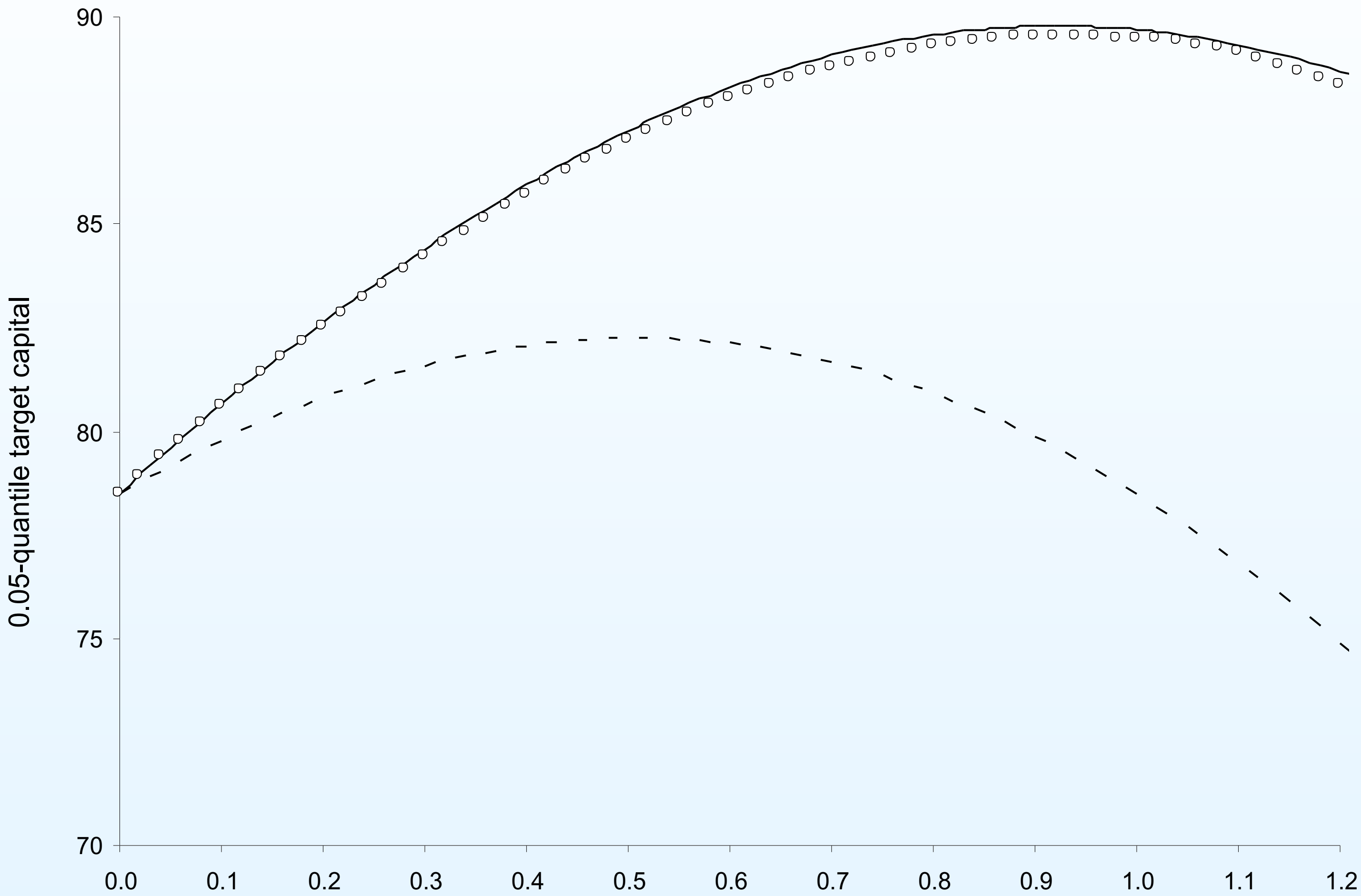
- *Yearly savings*: $\alpha_0 = \ldots = \alpha_{39} = 1$
- *Terminal wealth*:

$$W_{40}\left(\underline{\pi}\right) = \sum_{i=0}^{39} e^{Y_{i+1}(\underline{\pi})+Y_2(\underline{\pi})+\cdots+Y_{40}(\underline{\pi})}$$

- *Optimal investment strategy*:

$$\max_{\underline{\pi}} Q_{0.05}\left[W_{40}\left(\underline{\pi}\right)\right]$$

Numerical illustration



$Q_{0.05}\left[W_n\left(\underline{\pi}^{\sigma}\right)\right]$ as a function of the proportion invested in $\underline{\pi}^{(t)}$

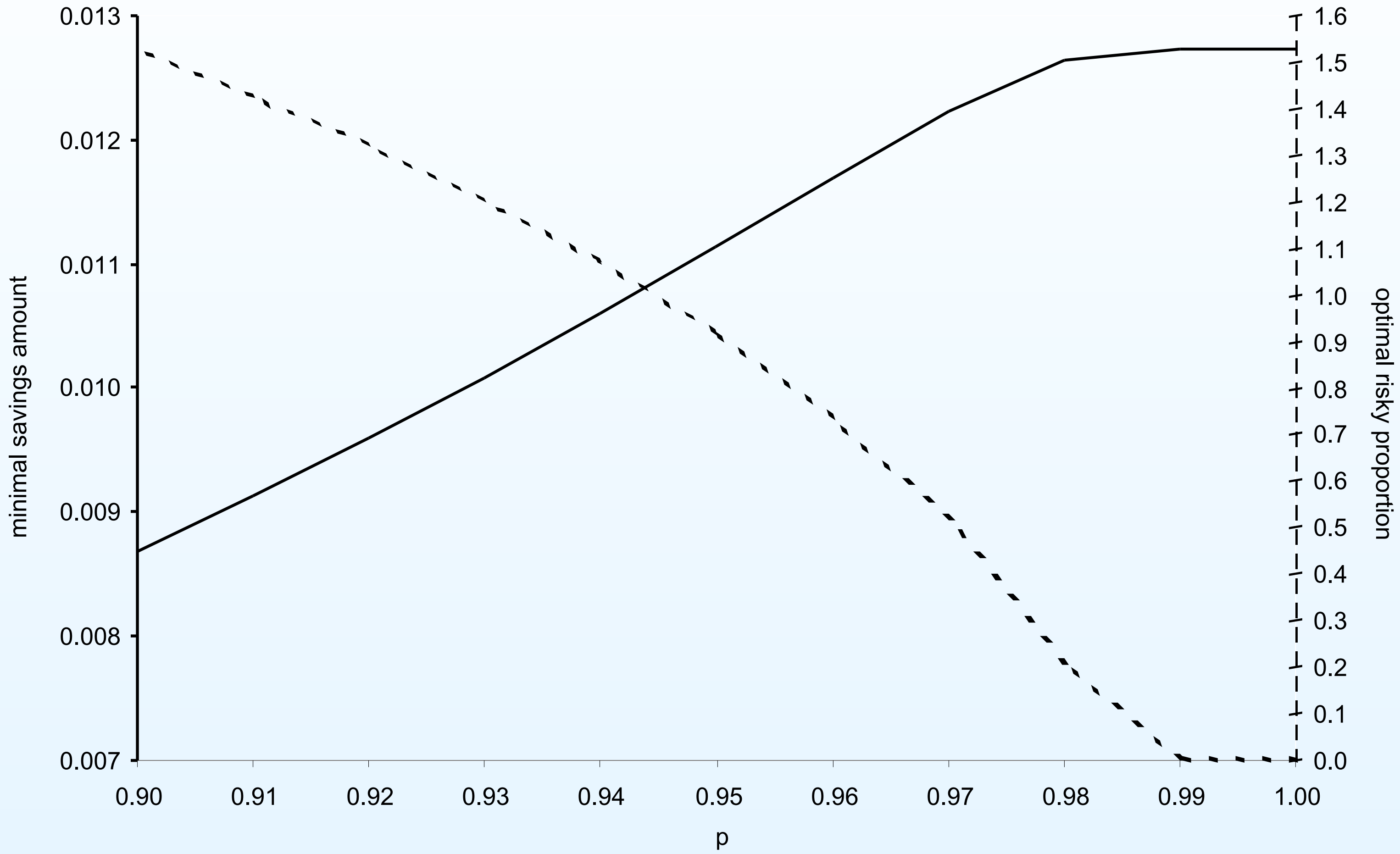
dots: $Q_{0.05}\left[W_n^s\left(\underline{\pi}^{\sigma}\right)\right]$, solid: $Q_{0.05}\left[W_n^l\left(\underline{\pi}^{\sigma}\right)\right]$, dashed: $Q_{0.05}\left[W_n^c\left(\underline{\pi}^{\sigma}\right)\right]$

Numerical illustration

- *Minimizing the savings effort per unit of Target Capital*:
  The optimal investment strategy $\underline{\pi}$ is defined as the one that minimizes $\alpha\left(\underline{\pi}\right)$ in

$$Q_{1-p}\left[\alpha\left(\underline{\pi}\right)\sum_{i=0}^{39} e^{Y_{i+1}\left(\underline{\pi}\right)+Y_{2}\left(\underline{\pi}\right)+\cdots+Y_{40}\left(\underline{\pi}\right)}\right]=1.$$

Numerical illustration



Solid line (left scale): minimal yearly savings amount as a function of p.
Dashed line (right scale): optimal proportion invested in the tangency portfolio.

Other optimization criteria

- *Maximizing the Target Capital for a given probability level* p:

$$\max_{\underline{\pi}} \mathrm{CLTE}_{1-p}\left[W_n\left(\underline{\pi}\right)\right]$$

with

$$\mathrm{CLTE}_{1-p}[X] = \mathrm{E}\left[X \mid X < Q_{1-p}[X]\right]$$

- *Maximizing* p *for a given Target Capital* K:

$$\max_{\underline{\pi}} \Pr\left[W_n\left(\underline{\pi}\right) > K\right]$$

Provisions for future liabilities

- *Problem description*:
  - $\alpha_1, \ldots, \alpha_n$: positive payments, due at times $1, \ldots, n$.
  - R_0 = *initial provision* established at time 0.
  - *Investment strategy*: $\underline{\pi}(t) = (\pi_1,\ \pi_2, \ldots,\ \pi_m)$.
  - *Provision at time j*:

$$R_j\left(R_0,\ \underline{\pi}\right) = R_{j-1}\left(R_0,\ \underline{\pi}\right)\ e^{Y_j(\underline{\pi})} - \alpha_j$$

    with $R_0\left(R_0,\ \underline{\pi}\right) = R_0$.
  - What is the optimal investment strategy $\underline{\pi}^*$?
  - Answer depends on 'initial provision' R_0 and 'probability level' p.

The stochastic provision

- Definition:

$$S\left(\underline{\pi}\right)=\sum_{i=1}^{n}\alpha_i\ e^{-\left(Y_1\left(\underline{\pi}\right)+Y_2\left(\underline{\pi}\right)+\cdots+Y_i\left(\underline{\pi}\right)\right)}.$$

- Relation:

$$R_n\left(R_0,\ \underline{\pi}\right)=\left(R_0-S\left(\underline{\pi}\right)\right)\ e^{\left(Y_1\left(\underline{\pi}\right)+\cdots+Y_n\left(\underline{\pi}\right)\right)}.$$

- An investment strategy $\underline{\pi}$ is only acceptable if $\Pr\left[R_n\left(R_0,\ \underline{\pi}\right)\geq 0\right]$ is "large enough".
- Relation:

$$\Pr\left[R_n\left(R_0,\ \underline{\pi}\right)\geq 0\right]=\Pr\left[S\left(\underline{\pi}\right)\leq R_0\right].$$

- PROBLEM: d.f. of $S\left(\underline{\pi}\right)$ too cumbersome to work with.

Comonotonic approximations for $S\left(\underline{\pi}\right)$

- *The comonotonic upper bound for $S\left(\underline{\pi}\right)$*:

$$S\left(\underline{\pi}\right) \leq_{cx} S^{c}\left(\underline{\pi}\right).$$

- *A comonotonic lower bound for $S\left(\underline{\pi}\right)$*:
  - $S^{l}\left(\underline{\pi}\right) = \mathrm{E}\left[S\left(\underline{\pi}\right) \middle| \sum_{j=1}^{n} Y_j\left(\underline{\pi}\right) \sum_{k=j}^{n} \alpha_k \, e^{-k[\mu(\underline{\pi})-\sigma^2(\underline{\pi})]} \right].$
  - $S^{l} \leq_{cx} S\left(\underline{\pi}\right)$.
  - $S^{l}\left(\underline{\pi}\right)$ is a comonotonic sum.

Optimal investment strategies

- *The Initial Provision*:
  - Definition:
    $$R_0\left(\underline{\pi}\right) = \mathrm{E}^g\left[S\left(\underline{\pi}\right)\right]$$
    where $S\left(\underline{\pi}\right)$ is the Stochastic Provision.
  - $\mathrm{E}^g[\cdot]$ is a 'distortion risk measure'.
  - If g is concave, then $\mathrm{E}^g[\cdot]$ is a 'coherent' risk measure.
- *The optimal investment strategy*: $\left(\underline{\pi}^*, R_0^*\right)$ follows from
  $$R_0^* = \min_{\underline{\pi}} \mathrm{E}^g\left[S\left(\underline{\pi}\right)\right]$$

Reduced optimization problem

- For $\sigma\left(\underline{\pi}_1\right)=\sigma\left(\underline{\pi}_2\right)$ and $\mu\left(\underline{\pi}_1\right)<\mu\left(\underline{\pi}_2\right)$, we have that

$$S\left(\underline{\pi}_2\right)\leq_{st} S\left(\underline{\pi}_1\right).$$

- Hence,

$$\min_{\underline{\pi}} \mathrm{E}^g\left[S\left(\underline{\pi}\right)\right]=\min_{\sigma} \mathrm{E}^g\left[S\left(\underline{\pi}^{\sigma}\right)\right].$$

Minimizing the Initial Provision, for a given p

- *The p - quantile provision principle*:

  If investment strategy $= \underline{\pi}$, then

$$R_0(\underline{\pi}) = Q_p\left[S\left(\underline{\pi}\right)\right] = \inf\left\{x \mid \Pr\left[R_n\left(x,\ \underline{\pi}\right) \geq 0\right] \geq p\right\}.$$

- *Optimal strategy*: $(\underline{\pi}^*, R_0^*)$ follows from

$$R_0^* = \min_{\underline{\pi}} Q_p\left[S\left(\underline{\pi}\right)\right].$$

- *Approximation*: $(\underline{\pi}^l, R_0^l)$ follows from

$$R_0^l = \min_{\sigma} Q_p\left[S^l\left(\underline{\pi}^\sigma\right)\right].$$

Numerical illustration

- *Available assets*:
  - 1 riskfree asset with $r = 0.03$
  - 2 risky assets with

$$\mu_1 = 0.06, \qquad \sigma_1 = 0.10$$
$$\mu_2 = 0.10, \qquad \sigma_2 = 0.20$$

  and

$$\text{Corr}\left[Y_k^1, Y_k^2\right] = 0.5$$

- *The tangency portfolio*:

$$\underline{\pi}^{(t)} = \left(\frac{5}{9}, \frac{4}{9}\right), \quad \mu\left(\underline{\pi}^{(t)}\right) = \frac{7}{90}, \quad \sigma\left(\underline{\pi}^{(t)}\right) = \sqrt{\frac{43}{2700}}$$

Numerical illustration

- *Yearly consumptions*: $\alpha_1 = \ldots = \alpha_{40} = 1.$
- *Stochastic provision*:

$$S\left(\underline{\pi}\right) = \sum_{i=1}^{40} e^{-(Y_1(\underline{\pi})+Y_2(\underline{\pi})+\cdots+Y_i(\underline{\pi}))}.$$

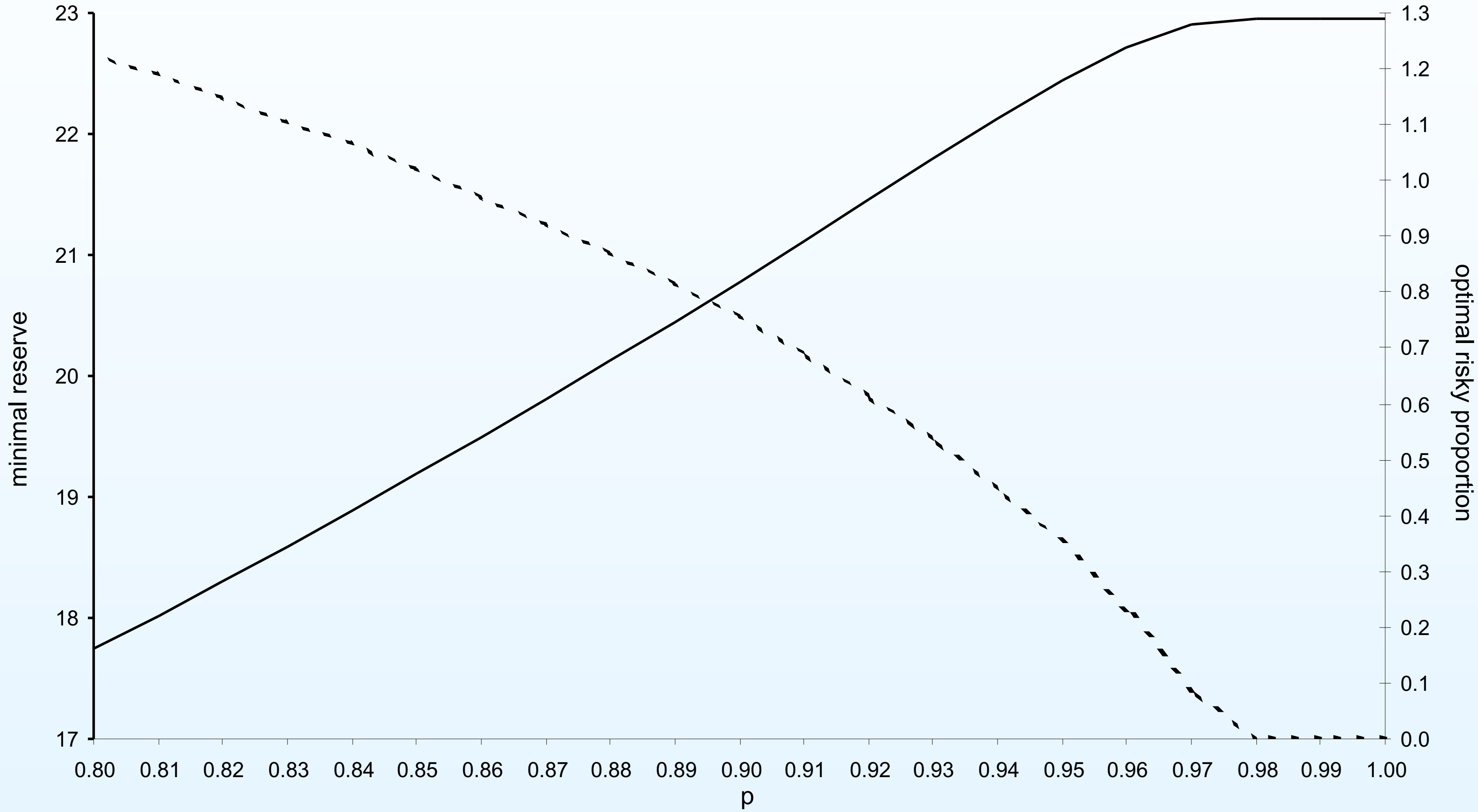
- *Optimal investment strategy*:

$$R_0^* = \min_{\underline{\pi}} Q_p\left[S\left(\underline{\pi}\right)\right].$$

- *Approximation*:

$$R_0^l = \min_{\sigma} Q_p\left[S\left(\underline{\pi}^\sigma\right)\right].$$

Numerical illustration



Solid line (left scale): minimal initial provision R_0^l as a function of p.

Dashed line (right scale): optimal proportion invested in the tangency portfolio.

Other optimization criteria

- *Minimizing the Initial Provision, given p*:

$$R_0^* = \min_{\underline{\pi}} \mathrm{CTE}_p \left[S \left(\underline{\pi} \right) \right]$$

with

$$\mathrm{CTE}_p[X] = \mathrm{E}\left[X \mid X > Q_p[X] \right].$$

- *Maximizing p for a given Initial Provision R_0*:

$$p^* = \max_{\underline{\pi}} \Pr \left[R_n \left(R_0, \underline{\pi} \right) > 0 \right].$$

Generalizations

- *Investment restrictions*: are taken into account by redefining the set of efficient portfolios.
- *Yaari's dual theory*: The 'final wealth problem' can be solved for general distorted expectations.
- *Distortion risk measures*: The initial provision can be defined in terms of general distortion risk measures.
- *Stochastic sums*: 'How to avoid outliving your money?'
- *Positive and negative payments*: 'The savings - retirement problem'.
- *Other distributions*: Lévy-type or Elliptical-type distributions

Some references (www.kuleuven.ac.be/insurance)

[1] Dhaene, Denuit, Goovaerts, Kaas, Vyncke (2002a).
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[2] Dhaene, Denuit, Goovaerts, Kaas, Vyncke (2002b).
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[3] Dhaene, Vanduffel, Goovaerts, Kaas, Vyncke (2004).
Comonotonic approximations for optimal portfolio selection problems. (forthcoming)

[4] Dhaene, Vanduffel, Tang, Goovaerts, Kaas, Vyncke (2003).
Risk measures and comonotonicity. (submitted)

Lecture No. 3

Elliptical Distributions - An Introduction

Emiliano A. Valdez

Elliptical Distributions

- This family coincides with the family of symmetric distributions in the univariate case (e.g. normal, Student-t) and can be characterized using either:
  - characteristic generator
  - density generator
- References:
  - Landsman and Valdez (2003) “Tail Conditional Expectations for Elliptical Distributions”, *North American Actuarial Journal*.
  - Valdez and Dhaene (2004) “Bounds for Sums of Non-Independent Log-Elliptical Random Variables”, work in progress.
  - Valdez and Chernih (2003) “Wang’s Capital Allocation Formula for Elliptically-Contoured Distributions”, *Insurance: Mathematics & Economics*.

Why Elliptical Distributions?

- Provides a rich class of multivariate distributions that share several tractable properties of the multivariate normal.
  - Student t, Laplace, Logistic, etc.
  - Linear combinations of components of multivariate elliptical is again elliptical (Important for modelling yearly returns, and for constructing the conditioning variable.)

- Allows more flexibility to model multivariate extremes and other forms of non-normal dependency structures.
  - Fat extremes, tail dependence.
  - Some studies show that light tailness of normal show its inadequacies to model extreme credit default events.

Some Notation

- Consider an n-dimensional random vector

$$\mathbf{X} = (X_1, X_2, ..., X_n)^T .$$

  - Distribution function:
    $F_{\mathbf{X}}(x_1, x_2, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n)$
  - Density function:
    $$f_{\mathbf{X}}(x_1, x_2, ..., x_n) = \frac{\partial^n F_{\mathbf{X}}(x_1, x_2, ..., x_n)}{\partial x_1 \cdots \partial x_n}$$
  - Characteristic function:
    $\varphi_{\mathbf{X}}(\mathbf{t}) = E\left[\exp\left(i\mathbf{X}^T\mathbf{t}\right)\right] = E\left[\exp\left(i\sum_{k=1}^n X_k t_k\right)\right]$
  - Moment generating function:
    $M_{\mathbf{X}}(\mathbf{t}) = E\left[\exp\left(\mathbf{X}^T\mathbf{t}\right)\right] = \varphi_{\mathbf{X}}(-i\mathbf{t})$
  - Covariance matrix: $Cov(\mathbf{X}) = (Cov(X_i, X_j))$ for $i, j = 1, ..., n$

Multivariate Normal Family

- It is well-known that the joint density of a multivariate normal $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\mathbf{\Sigma}|}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right].$$

- The normalizing constant is given by $c_n = (2\pi)^{-n/2}$.
- Its characteristic function is

$$\begin{aligned} \varphi_{\mathbf{X}}(\mathbf{t}) &= \exp\left(i\mathbf{t}^T\mu - \tfrac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right) \\ &= \exp\left(i\mathbf{t}^T\mu\right)\exp\left(-\tfrac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right) \end{aligned}$$

- And its covariance is

$$Cov(\mathbf{X}) = \mathbf{\Sigma}.$$

Multivariate Normal - continued

- Define the characteristic generator as

$$\psi\left(t\right) = e^{-t}$$

and density generator as

$$g_n\left(u\right) = e^{-u}$$

- The density can then be written as

$$f_{\mathbf{X}}\left(\mathbf{x}\right) = \frac{c_n}{\sqrt{\left|\mathbf{\Sigma}\right|}} g_n\left[-\frac{1}{2}\left(\mathbf{x}-\mu\right)^T \mathbf{\Sigma}^{-1}\left(\mathbf{x}-\mu\right)\right]$$

and its characteristic function as

$$\varphi_{\mathbf{X}}\left(\mathbf{t}\right) = \exp\left(i\mathbf{t}^T\mu\right)\psi\left(\tfrac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right).$$

Class of Elliptical Distributions

- $\mathbf{X}$ has multivariate elliptical distribution, $\mathbf{X} \backsim \mathbf{E}_n(\mu, \mathbf{\Sigma}, \psi)$, if char. function can be expressed as

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(it^T \mu)\psi\left(\tfrac{1}{2}\mathbf{t}^T \mathbf{\Sigma} \mathbf{t}\right)$$

  for some column-vector μ, $n \times n$ positive-definite matrix $\mathbf{\Sigma}$.

- If density exists, it has the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\mathbf{\Sigma}|}} g_n \left[\frac{1}{2}(\mathbf{x} - \mu)^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \mu)\right],$$

  for some function $g_n(\cdot)$ called the density generator.

Elliptical Distributions - continued

- The normalizing constant c_n can be explicitly determined by transforming into polar coordinates and we have

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} g_n(x) dx \right]^{-1}.$$

- Thus, we see the condition

$$\int_0^\infty x^{n/2-1} g_n(x) dx < \infty$$

  guarantees g_n as density generator.

- Note that for a given characteristic generator ψ, the density generator g and/or the normalizing constant c may depend on the dimension of the random vector $\mathbf{X}$.

Some Properties

- If mean exists, it will be

$$E(\mathbf{X}) = \mu.$$

- If covariance exists, it will be

$$Cov(\mathbf{X}) = -\psi'(0)\,\mathbf{\Sigma}.$$

- Let A be some $m \times n$ matrix of rank $m \leq n$ and b some m-dimensional column-vector. Then

$$A\mathbf{X} + b \sim E_m\left(A\mu + b, A\mathbf{\Sigma}A^T, g_m\right).$$

- Define the sum $S = X_1 + X_2 + \cdots + X_n = \mathbf{e}^T\mathbf{X}$, where $\mathbf{e}$ is a column vector of ones with dimension n. Then

$$S \sim E_n\left(\mathbf{e}^T\mu, \mathbf{e}^{\mathbf{T}}\mathbf{\Sigma}\mathbf{e}, g_1\right).$$

Multivariate Student-*t* Family

- Density generator: $g_n(u) = \left(1 + \frac{u}{k_p}\right)^{-p}$ where parameter $p > n/2$ and k_p is some constant.

- Density: $f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\mathbf{\Sigma}|}} \left[1 + \frac{(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1} (\mathbf{x}-\mu)}{2k_p}\right]^{-p}$

- Normalizing constant: $c_n = \frac{\Gamma(p)}{\Gamma(p-n/2)} (2\pi k_p)^{-n/2}$

- If $p = (n+m)/2$ where n, m are integers, and $k_p = m$, we get the traditional form of the multivariate Student *t* with density:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{(\pi m)^{n/2} \Gamma\left(\frac{m}{2}\right) \sqrt{|\mathbf{\Sigma}|}} \left[1 + \frac{(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1} (\mathbf{x}-\mu)}{m}\right]^{-\left(\frac{n+m}{2}\right)}$$

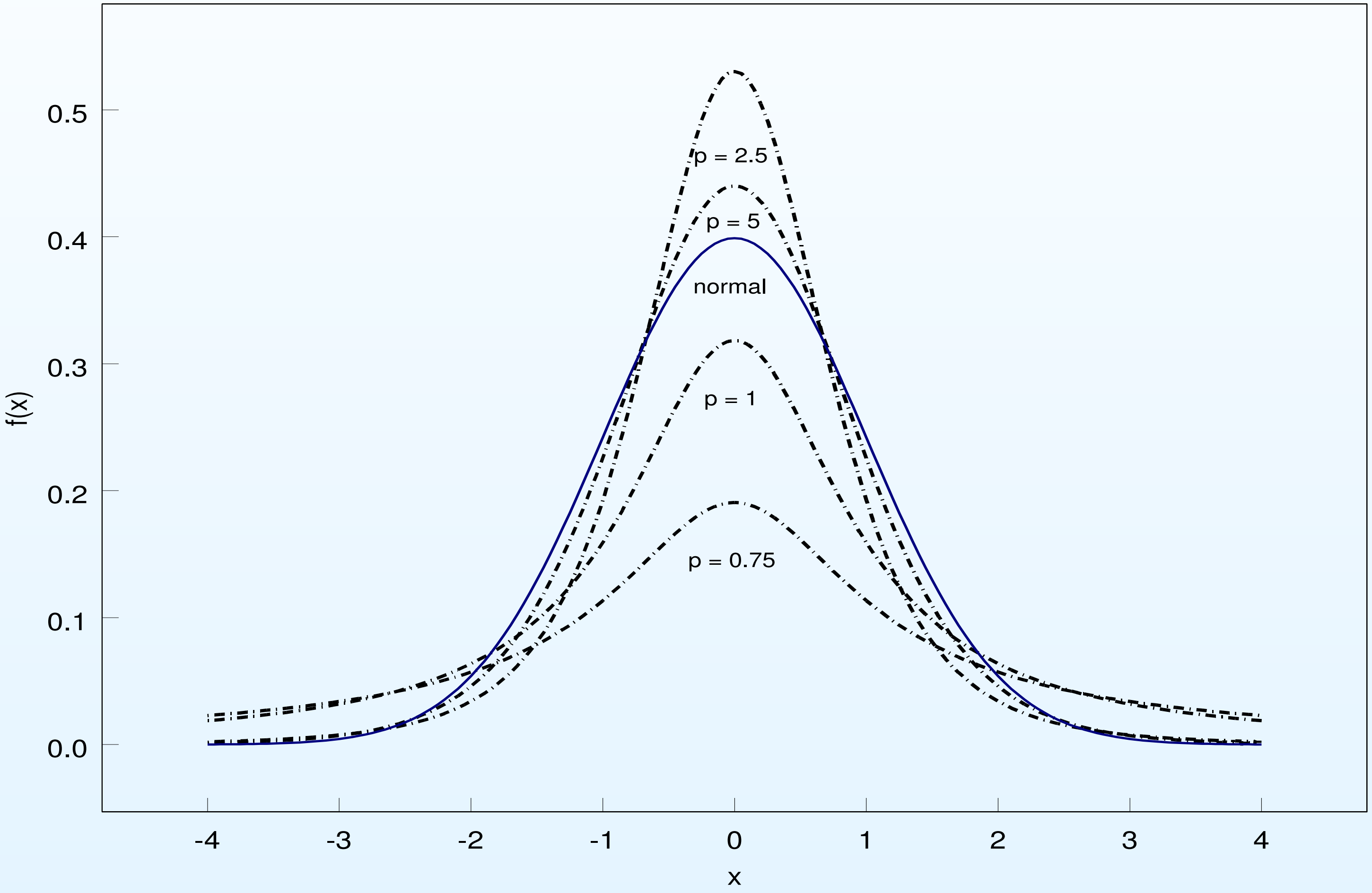
Generalized Student-*t* Distribution

- Density: $f_X(x) = \frac{1}{\sigma\sqrt{2k_p}B(1/2,p-1/2)}\left[1+\frac{(x-\mu)^2}{2k_p\sigma^2}\right]^{-p}$, where $B(\cdot,\cdot)$ is the beta function.
- For $p > 3/2$, usually $k_p = (2p-3)/2$ becaue it leads to the important property that $Var(X) = \sigma^2$.
- For $1/2 < p \leq 3/2$, variance does not exist and $k_p = 1/2$.
- Note for example in the case where $p = 1$, we have standard Cauchy distribution:

$$f_X(x) = \frac{1}{\sigma\pi}\left[1+\frac{(x-\mu)^2}{\sigma^2}\right]^{-1}.$$

It is well-known that mean and variance for this distribution does not exist.

Density Functions of GST - Figure 1



Multivariate Logistic Family

- Density generator: $g(u) = \frac{e^{-u}}{(1+e^{-u})^2}$
- Density:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\mathbf{\Sigma}|}} \frac{\exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right]}{\left\{1+\exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right]\right\}^2}$$

- Normalizing constant:

$$c_n = (2\pi)^{-n/2}\left[\sum_{j=1}^{\infty}(-1)^{j-1}\, j^{1-n/2}\right]^{-1}$$

Multivariate Exponential Power Family

- Density generator: $g(u) = e^{-ru^s}$, for $r, s > 0$
- Density:

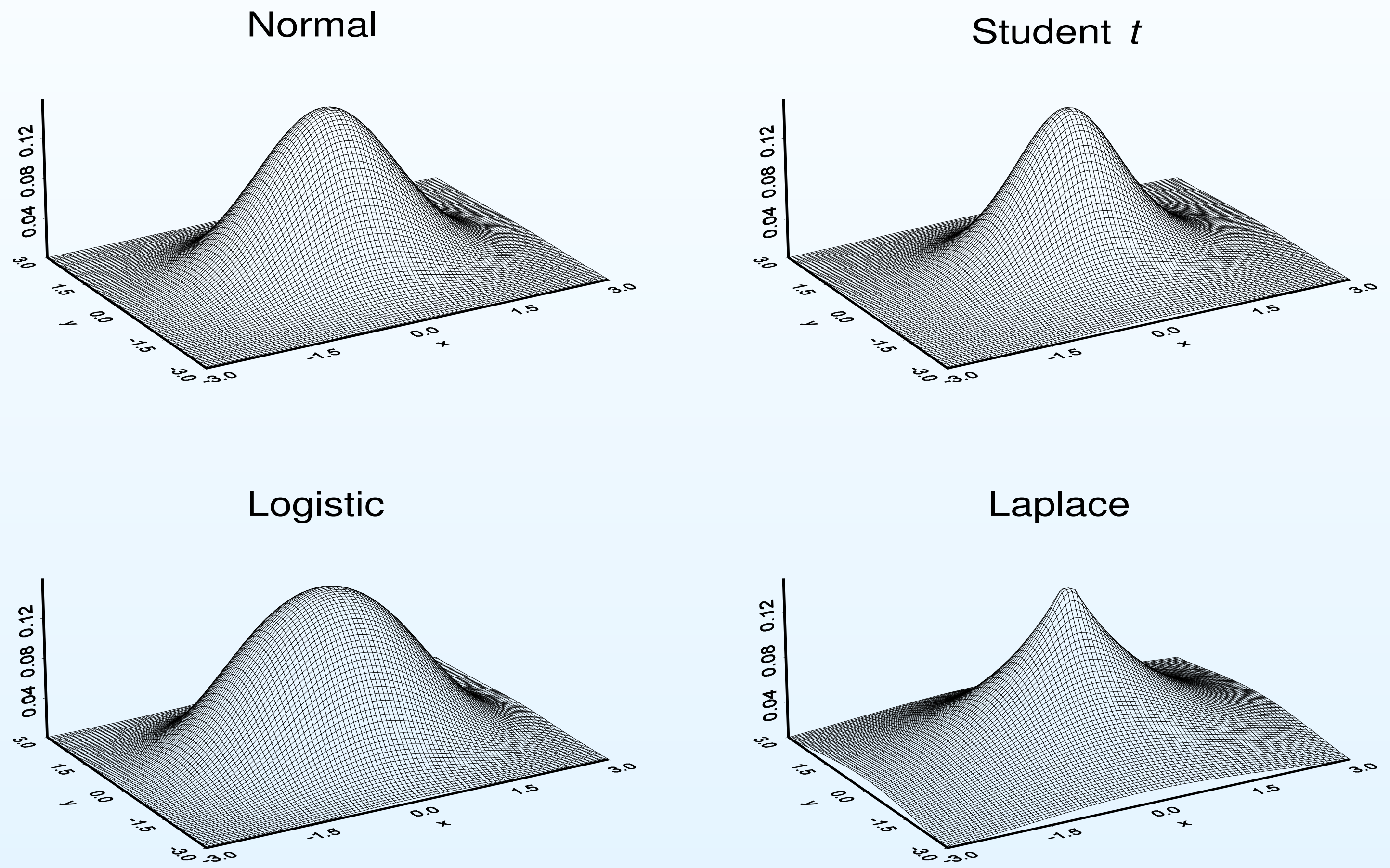
$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left\{-\frac{r}{2}\left[(\mathbf{x}-\mu)^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mu)\right]^s\right\}$$

- Normalizing constant:

$$c_n = \frac{s\Gamma(n/2)}{(2\pi)^{n/2}\,\Gamma(n/2s)} r^{n/2s}$$

- When $r = s = 1$, this reduces to multivariate normal. When $s = 1/2$ and $r = \sqrt{2}$, we have Double Exponential or Laplace distributions.

Bivariate Densities - Figure 2



Lecture No. 4

Tail Conditional Expectations for Elliptical Distributions

Emiliano A. Valdez

Introduction

- Developing a standard framework for risk measurement is becoming increasingly important.
- This paper is about a risk measure called tail conditional expectations and their explicit forms for the family of elliptical distributions.
- This family coincides with the family of symmetric distributions in the univariate case (e.g. normal, Student-t) and can be characterized using either:
    - characteristic generator
    - density generator

Introduction - continued

- We introduce the notion of a cumulative generator which plays a key role in computing tail conditional expectations.
- We extended the ideas into the multivariate framework allowing us to decompose the total of the tail conditional expectations into its various constituents.
  - decomposing the total into an allocation formula
- Landsman and Valdez (2003)

Risk Measure

- A *risk measure* ϑ is a mapping from the space of random variables $\mathfrak{L}$ to the set of real numbers: $\vartheta : X \in \mathfrak{L} \rightarrow \mathbf{R}$.
- Some useful properties of a risk measure:

1. Monotonicity: $X_1 \leq X_2$ with probability $1 \Longrightarrow \vartheta\left(X_1\right) \leq \vartheta\left(X_2\right)$.
2. Homogeneity: $\vartheta\left(\lambda X\right) = \lambda \vartheta\left(X\right)$ for any non-negative λ.
3. Subadditivity: $\vartheta\left(X_1 + X_2\right) \leq \vartheta\left(X_1\right) + \vartheta\left(X_2\right)$.
4. Translation Invariance: $\vartheta\left(X + \alpha\right) = \vartheta\left(X\right) + \alpha$ for any constant α.

- Some consequences:

$$\vartheta\left(0\right) = 0;\ a \leq X \leq b \Longrightarrow a \leq \vartheta\left(X\right) \leq b;\ \vartheta\left(X - \vartheta\left(X\right)\right) = 0.$$

The Tail Conditional Expectation

- Notation: X : loss random variable; $F_X(x)$: distribution function; $\overline{F}_X(x) = 1 - F_X(x)$: tail function; x_q : q-th quantile with $\overline{F}_X(x_q) = 1 - q$
- The *tail conditional expectation* (TCE) is

$$TCE_X(x_q) = E(X \,|X > x_q).$$

- Other names used: tail-VAR, conditional VAR
- Value-at-risk: $x_q = Q_q(X)$
- Expected Shortfall: $E\left[(X - x_q)_+\right] = ESF_q(X)$
- Relationships:

$$TCE_X(x_q) = x_q + E(X - x_q \,|X > x_q) = x_q + \frac{1}{1-q} E\left[(X - x_q)_+\right]$$

TCE for Univariate Elliptical

- Let $X \sim E_1\left(\mu, \sigma^2, g\right)$ so that density $f_X\left(x\right) = \frac{c}{\sigma} g\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ where c is the normalizing constant.
- Since X is elliptical distribution, the standardized random variable $Z = \left(X - \mu\right)/\sigma$ will have a standard elliptical distribution function $F_Z\left(z\right) = c\int_{-\infty}^{z} g\left(\frac{1}{2}u^2\right)du$, with mean 0 and variance $\sigma_Z^2 = 2c\int_0^{\infty} u^2 g\left(\frac{1}{2}u^2\right)du = -\psi'(0)$, if they exist.
- Define the cumulative density generator:

$$G\left(x\right) = c\int_0^x g\left(u\right)du$$

and denote $\overline{G}\left(x\right) = G\left(\infty\right) - G\left(x\right).$

- continued

- The tail conditional expectation of X is

$$TCE_X(x_q) = \mu + \lambda \cdot \sigma^2$$

where λ is $\lambda = \dfrac{\frac{1}{\sigma}\overline{G}\left(\frac{1}{2}z_q^2\right)}{\overline{F}_X(x_q)} = \dfrac{\frac{1}{\sigma}\overline{G}\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)}$ and $z_q = (x_q - \mu)/\sigma$.

- Moreover, if the variance of X exists, then $\frac{1}{\sigma_Z^2}\overline{G}\left(\frac{1}{2}z^2\right)$ has the sense of a density of another spherical random variable Z^* and λ has the form

$$\lambda = \frac{\frac{1}{\sigma}f_{Z^*}(z_q)}{\overline{F}_Z(z_q)}\sigma_Z^2.$$

Some Examples

- Normal Distribution:

$$\lambda = \frac{\frac{1}{\sigma}\varphi\left(z_q\right)}{1-\Phi\left(z_q\right)}$$

  where $\varphi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ denote respectively the density and distribution functions of a standard normal distribution. Notice that Z^* is simply the standard normal variable Z.

- Student-t:

$$\lambda = \frac{\sqrt{\frac{2p-5}{2p-3}} \cdot f_Z\left(\sqrt{\frac{2p-5}{2p-3}} z_q; p-1\right)}{\overline{F}_Z\left(z_q; p\right)}$$

  only for the case where $p > 5/2$. Here, Z^* is simply a scaled GST with parameter $p-1$.

Examples - continued

- Logistic:

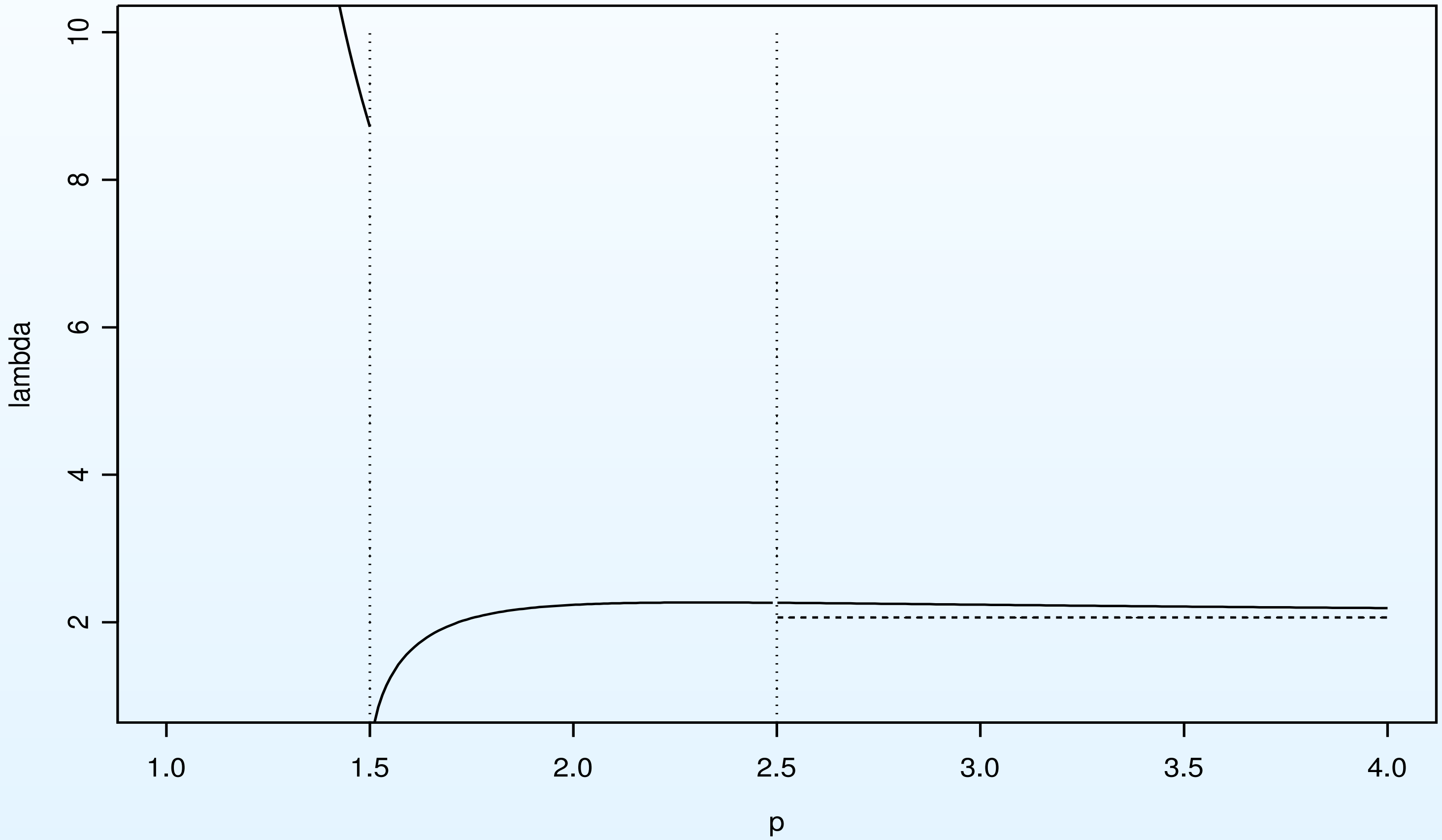
$$\lambda = \left[\frac{1}{2}\frac{1}{\left(\sqrt{2\pi}\right)^{-1} + \varphi\left(z_q\right)}\right]\frac{\frac{1}{\sigma}\varphi\left(z_q\right)}{\overline{F}_Z\left(z_q\right)}$$

which resembles that of a normal distribution, but with a correction factor.

- Exponential Power:

$$\lambda = \frac{1}{\overline{F}_Z\left(z_q\right)}\frac{1}{\sqrt{2}\Gamma\left(1/(2s)\right)\sigma}\left\{\Gamma\left(1/s\right) - \Gamma\left[r\left(\frac{1}{2}z_q^2\right)^s ; 1/s\right]\right\}$$

GST - Figure



TCE for the Marginals

- Let $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g_n)$. Denote the (i,j) element of $\mathbf{\Sigma}$ by σ_{ij} so that $\mathbf{\Sigma} = \|\sigma_{ij}\|_{i,j=1}^{n}$.
- Let $F_Z(z) = c_1 \int_0^z g_1\left(\frac{1}{2}x^2\right) dx$ be the standard d.f. corresponding to this elliptical family and $G(x) = c_1 \int_0^x g_1(u)\, du$ be its cumulative generator.
- The formula for computing TCEs for each component of $\mathbf{X}$ is expressed as

$$TCE_{X_k}(x_q) = \mu_k + \lambda_k \cdot \sigma_k^2$$

where $\lambda_k = \dfrac{\frac{1}{\sigma_k}\overline{G}\left(\frac{1}{2}z_{k,q}^2\right)}{\overline{F}_Z(z_{k,q})}$, $z_{k,q} = \dfrac{x_q - \mu_k}{\sigma_k}$, or $\lambda_k = \dfrac{\frac{1}{\sigma_k} f_{Z^*}(z_q)}{\overline{F}_Z(z_q)} \sigma_Z^2$, if $\sigma_Z^2 < \infty$.

Sums of Elliptical Risks

- The tail conditional expectation of the sum S

$$TCE_S\left(x_q\right) = \mu_S + \lambda_S \cdot \sigma_S^2$$

where

$$\mu_S = \mathbf{e}^T \mu = \sum_{k=1}^{n} \mu_k, \ \sigma_S^2 = \mathbf{e}^T \mathbf{\Sigma} \mathbf{e} = \sum_{i,j=1}^{n} \sigma_{ij},$$

and

$$\lambda_S = \frac{\dfrac{1}{\sigma_S} \overline{G}\left(\frac{1}{2} z_{S,q}^2\right)}{\overline{F}_Z\left(z_{S,q}\right)}$$

with $z_{S,q} = \dfrac{\mu_S - x_q}{\sigma_S}$.

Portfolio Risk Decomposition

- TCE allows for natural decomposition of the total loss:

$$TCE_S\left(x_q\right) = \sum_{k=1}^{n} E\left(X_k \left| S > x_q\right.\right).$$

- This is not in general equivalent to the sum of the tail conditional expectations of the individual components since

$$TCE_{X_k}\left(x_q\right) \neq E\left(X_k \left| S > x_q\right.\right).$$

- Instead, we denote this as $TCE_{X_k|S}\left(x_q\right) = E\left(X_k \left| S > x_q\right.\right)$, the contribution to the total risk attributable to risk k.
- It can be interpreted as follows: in case of a disaster as measured by an amount at least as large as the quantile of the total loss distribution, this refers to the average amount that would be due to the presence of risk k.

Theorem on Risk Decomposition

- Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T \sim E_n(\mu, \mathbf{\Sigma}, g_n)$ such that condition $\int_0^\infty g_1(x)dx < \infty$ holds and let $S = X_1 + \cdots + X_n$.
- Then the contribution of risk $X_k, 1 \leq k \leq n$, to the total TCE

$$TCE_{X_k|S}(x_q) = \mu_k + \lambda_S \cdot \sigma_k \sigma_S \rho_{k,S},$$

for $k = 1, 2, ..., n$, where $\rho_{k,S} = \dfrac{\sigma_{k,S}}{\sigma_k \sigma_S}$ and $\lambda_S = \dfrac{\frac{1}{\sigma_S}\overline{G}\left(\frac{1}{2}z_{S,q}^2\right)}{\overline{F}_Z(z_{S,q})}$.

- Notice that if we take the sum of $TCE_{X_k|S}(x_q)$, we have

$$\sum_{k=1}^{n} TCE_{X_k|S}(x_q) = \mu_S + \lambda_S \sum_{k=1}^{n} \underbrace{\sigma_k \sigma_S \rho_{k,S}}_{\sigma_{k,S}} = \mu_S + \lambda_S \cdot \sigma_S^2$$

Multivariate Normal Case

- Panjer (2002) demonstrated that in the case of a multivariate normal random vector i.e. $\mathbf{X} \sim \mathbf{N}_n(\mu, \mathbf{\Sigma})$, we have

$$E(X_k | S > x_q) = \mu_k + \left[\frac{\frac{1}{\sigma_S} \varphi\left(\frac{x_q - \mu}{\sigma_S}\right)}{1 - \Phi\left(\frac{x_q - \mu}{\sigma_S}\right)} \right] \sigma_k^2 \left(1 + \rho_{k,-k} \frac{\sigma_{-k}}{\sigma_k}\right),$$

  where they have used the negative subscript $-k$ to refer to the sum of all the risks excluding the kth risk, that is, $S_{-k} = S - X_k$.

- Therefore, according to this notation, we have

$$\rho_{k,-k} \frac{\sigma_{-k}}{\sigma_k} = \frac{\sigma_{k,-k}}{\sigma_k \sigma_{-k}} \frac{\sigma_{-k}}{\sigma_k} = \frac{\sigma_{k,-k}}{\sigma_k^2} = \frac{Cov(X_k, S - X_k)}{\sigma_k^2} = \frac{\sigma_{k,S}}{\sigma_k^2} - 1.$$

Multivariate Normal - continued

- Thus, our formula for risk decomposition becomes

$$E\left(X_k \left| S > x_q\right.\right) = \mu_k + \left[\frac{\frac{1}{\sigma_S}\varphi\left(\frac{x_q - \mu}{\sigma_S}\right)}{1 - \Phi\left(\frac{x_q - \mu}{\sigma_S}\right)} \right] \sigma_k \sigma_S \rho_{k,S}$$

  which gives the case of multivariate normal.

- This confirms the formula above for risk decomposition which holds for multivariate elliptical distributions including multivariate normal distributions.

Lecture No. 5

Bounds for Sums of Non-Independent Log-Elliptical Random Variables

Emiliano A. Valdez

Introduction

- This paper is about finding bounds for sums of non-independent log-elliptical random variables.
- Extends the ideas developed in
  - "The Concept of Comonotonicity in Actuarial Science and Finance: Theory" IME, Dhaene, et al.
  - "The Concept of Comonotonicity in Actuarial Science and Finance: Applications" IME, Dhaene, et al.
- These papers considered bounds for the log-normal random variables.

Outline of Talk

- Comonotonicity
- Convex Upper and Lower Bounds
- Elliptical, Spherical, and Log-Elliptical Distributions
- Extension to Log-Elliptical Distributions
- The Results for Log-Normal Distributions

Sums of Dependent Random Variables

- Consider an insurance portfolio $\mathbf{X} = (X_1, X_2, \cdots, X_n)^T$
  - X_i : claim amount of policy i at the end of the period.
  - Assumption: all X_i are i.i.d.
- Introduction of stochastic financial aspects in actuarial models reveals the necessity of determining distributions of sums of dependent random variables.
- Assumption that the X_i are mutually independent
  - is often approximately,
  - leads to easier mathematics,
  - but is sometimes violated.

- continued

- Individual risks X_i may be influenced by the same economic/physical environment:
  - catastrophes (storms, explosions, etc.) cause an accumulation of claims;
  - weather conditions in automobile;
  - fire insurance;
  - pension fund; and
  - lifetimes of a couple.

- The independence assumption probably underestimates:
  - the deviation of the aggregate risk,
  - the probability of large claims,
  - the expected shortfall.

Ordering of Random Variables

- Upper and lower tails
  - $E(X-d)_+$= surface above the d.f., from d on.
  - $E(d-X)_+$= surface below the d.f., from $-\infty$ to d.

- Convex order: $X \leq_{cx} Y$
  - $\Leftrightarrow$ the upper tails as well as the lower tails of Y eclipse the respective tails of X.
    - $\rightarrow$ Extreme values are more likely to occur for Y than for X.
  - $\Leftrightarrow E(X) = E(Y)$ and $E[u(-X)] \geq E[u(-Y)]$ for all non-decreasing concave functions u.
    - $\rightarrow$ Common preferences of risk averse decision makers between rv's with equal means.

- continued

- Sufficient condition:
  - $E(X) = E(Y)$ and the d.f.'s only cross once, (finally, $F_Y \leq F_X$)
  - $\Rightarrow X \leq_{cx} Y.$
- Convexity order and moments:
  - $X \leq_{cx} Y \Rightarrow E(X) = E(Y)$
  - $X \leq_{cx} Y \Rightarrow Var(X) \leq Var(Y)$.
  - $X \leq_{cx} Y$ and $Var(X) = Var(Y) \Rightarrow X \overset{d}{=} Y$.

Comonotonicity

- Suppose $\mathbf{X}$ has joint d.f. F. Well-known Frechet bounds:

$$\max\left[\sum_{k=1}^{n} F_k(x_k) - (n-1), 0\right] \leq F_{\mathbf{X}}(\mathbf{x}) \leq \min\left[F_1(x_1), ..., F_n(x_n)\right].$$

- Hoeffding (1940) and Frechet (1951).

- $\mathbf{X}$ is comonotonic if its joint distribution is the Frechet upper bound:

$$F_{\mathbf{X}}(\mathbf{x}) = \min\left[F_1(x_1), ..., F_n(x_n)\right].$$

Comonotonicity - continued

- Comonotonicity is very strong positive dependency structure.
- Comonotonic rv's are not able to compensate each other, they cannot be used as ”hedge” against each other.
- Quantiles, distribution functions, and tails of sums of comonotonic random variables follow immediately from the respective quantities of the marginals.
  - Notation: $(X_1^c, \cdots, X_n^c)$.

Bounds for Sums

- COMONOTONIC UPPER BOUND:
  - Define the comonotonic vector corresponding to $\mathbf{X}$ by $\mathbf{X}^c = (X_1^c, ..., X_n^c)^T$ where $X_k^c = F_k^{-1}(U)$.
  - Sum: $S^c = X_1^c + \cdots + X_n^c$.
- IMPROVED UPPER BOUND:
  - Define the random vector corresponding to $\mathbf{X}$ by $\mathbf{X}^u = (X_1^u, ..., X_n^u)^T$ where $X_k^u = F_{X_k|\Lambda}^{-1}(U)$.
  - Sum: $S^u = X_1^u + \cdots + X_n^u$.
- LOWER BOUND:
  - Define the vector corresponding to $\mathbf{X}$ by $\mathbf{X}^l = \left(X_1^l, ..., X_n^l\right)^T$ where $X_k^l = E\left(X_k \,|\Lambda\right)$.
  - Sum: $S^l = X_1^l + \cdots + X_n^l$.

Bounds for Sums - continued

- We have the following bounds:

$$S^l \leq_{cx} S \leq_{cx} S^u \leq_{cx} S^c$$

- Proofs can be found in:
  - Tchen (1980)
  - Dhaene, Wang, Young & Goovaerts (1997)
  - Müller (1997); and
  - Kaas, Dhaene, Goovaerts (2000).

Class of Elliptical Distributions

- $\mathbf{Y} \backsim \mathbf{E}_n(\mu, \boldsymbol{\Sigma},\phi)$ if c.f. can be expressed as

$$\varphi_{\mathbf{Y}}\left(\mathbf{t}\right) = \exp(it^T\mu) \cdot \phi\left(\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right)$$

  for some scalar function ϕ and where $\boldsymbol{\Sigma}$ is given by $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^T$ for some matrix $\mathbf{A}(n \times m)$.

- Density: $f_{\mathbf{Y}}\left(\mathbf{y}\right) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} g_n\left[\left(\mathbf{y}-\mu\right)^T \boldsymbol{\Sigma}^{-1}\left(\mathbf{y}-\mu\right)\right]$, for some function $g_n\left(\cdot\right)$ called density generator.

- Normalizing constant: $c_n = \frac{\Gamma(n/2)}{\pi^{n/2}}\left[\int_0^\infty z^{n/2-1} g_n(z)dz\right]^{-1}$. Condition $\int_0^\infty z^{n/2-1} g_n(z)dz < \infty$ guarantees g_n as density generator.

- Kelker (1970); Fang, et al. (1990).

Some Properties

- Mean: $E(\mathbf{Y}) = \mu$.
- Covariance: $Cov(\mathbf{Y}) = -\phi'(0)\,\mathbf{\Sigma}$.
- $\mathbf{Y} \sim E_{\mathbf{n}}(\mu, \mathbf{\Sigma}, \phi)$, iff for any $\mathbf{b}(n \times 1), \mathbf{b}^T\mathbf{Y} \sim E_1\left(\mathbf{b}^T\mu, \mathbf{b}^T\mathbf{\Sigma}\mathbf{b}, \phi\right)$.
- Marginals are also elliptical with the same characteristic generator:,

$$Y_k \sim E_1\left(\mu_k, \sigma_k^2, \phi\right).$$

- For any matrix $\mathbf{B}\,(m \times n)$, any vector $\mathbf{c}\,(m \times 1)$ and any random vector $\mathbf{Y} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$, we have that

$$\mathbf{BY} + \mathbf{c} \sim E_m\left(\mathbf{B}\mu + \mathbf{c}, \mathbf{B\Sigma B}^T, \phi\right).$$

Independence and Elliptical

- Any multivariate elliptical distribution with mutually independent components must necessarily be multivariate normal, see Kelker (1970).

- Let $\mathbf{Y} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$ with mutually independent components Y_k. Assume that the expectations and variances of the Y_k exist and that $var\left(Y_k\right) > 0$. Then it follows that $\mathbf{Y}$ is multivariate normal.

- Thus, it follows that the joint distribution of mutually independent elliptical random variables is not elliptical, unless all the marginals are normal.

Spherical Distributions

- $\mathbf{Z}$ is spherical with c.g. ϕ if $\mathbf{Z} \sim E_n\left(\mathbf{0}_n, \mathbf{I}_n, \phi\right)$.
- Notation: $S_n(\phi)$ for $E_n\left(\mathbf{0}_n, \mathbf{I}_n, \phi\right)$.
- $\mathbf{Z} \sim S_n(\phi)$ iff $E\left[\exp\left(i\mathbf{t}^T\mathbf{Z}\right)\right] = \phi\left(\mathbf{t}^T\mathbf{t}\right)$.
- Suppose m-dim vector $\mathbf{Y}$ is such that $\mathbf{Y} \stackrel{d}{=} \mu + \mathbf{AZ}$, for some $\mu(n \times 1)$, some matrix $\mathbf{A}(n \times m)$ and some m-dim elliptical vector $\mathbf{Z} \sim S_m(\phi)$. Then $\mathbf{Y} \sim E_n\left(\mu, \mathbf{\Sigma}, \phi\right)$ where $\mathbf{\Sigma} = \mathbf{AA}^T$.
- $\mathbf{Z} \sim S_n(\phi)$ iff for any n-dim vector $\mathbf{a}$,

$$\frac{\mathbf{a}^T\mathbf{Z}}{\sqrt{\mathbf{a}^T\mathbf{a}}} \sim S_1(\phi).$$

- Any component Z_i of $\mathbf{Z}$ has a $S_1(\phi)$ distribution.
- Density: $f_{\mathbf{Z}}(\mathbf{z}) = cg\left(\mathbf{z}^T\mathbf{z}\right)$.

Conditional Distributions

- Conditional distributions of bivariate Normal is again Normal.
- GENERALIZATION OF RESULT TO ELLIPTICAL:
  - Let $\mathbf{Y} \sim E_n(\mu, \mathbf{\Sigma}, \phi)$ with d.g. $g_n(\cdot)$. Define Y and Λ to be linear combinations of $\mathbf{Y}$, i.e. $Y = \alpha^T \mathbf{Y}$ and $\Lambda = \beta^T \mathbf{Y}$, for some $\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta^T = (\beta_1, \beta_2, \ldots, \beta_n)$. Then, $(Y, \Lambda) \sim E_2\left(\mu_{(Y,\Lambda)}, \mathbf{\Sigma}_{(Y,\Lambda)}, \phi\right)$.
  - Also, given $\Lambda = \lambda$, Y has a univariate elliptical distribution:
  $$Y \,|\Lambda = \lambda \sim E_1\left(\begin{array}{c} \mu_Y + r(Y,\Lambda)\frac{\sigma_Y}{\sigma_\Lambda}(\lambda - \mu_\Lambda), \\ \left(1 - r(Y,\Lambda)^2\right)\sigma_Y^2, \phi_a \end{array}\right),$$ for some char. gen. $\phi_a(\cdot)$ depending on $a = (\lambda - \mu_\Lambda)^2 / \sigma_\Lambda^2$.

Log-Elliptical Distributions

- $\mathbf{X}$ is multivariate log-elliptical with parameters μ and $\mathbf{\Sigma}$ if $\log \mathbf{X}$ is elliptical:

$$\log \mathbf{X} \sim E_n\left(\mu, \mathbf{\Sigma}, \phi\right).$$

- Notation: $\log \mathbf{X} \sim E_n\left(\mu, \mathbf{\Sigma}, \phi\right)$ as $\mathbf{X} \sim LE_n\left(\mu, \mathbf{\Sigma}, \phi\right)$.
- When $\mu = \mathbf{0}_n$ and $\mathbf{\Sigma} = \mathbf{I}_n$, we write $\mathbf{X} \sim LS_n\left(\phi\right)$.
- If $\mathbf{Y} \sim E_n\left(\mu, \mathbf{\Sigma}, \phi\right)$ and $\mathbf{X} = \exp\left(\mathbf{Y}\right)$, then $\mathbf{X} \sim LE_n\left(\mu, \mathbf{\Sigma}, \phi\right)$.

Some Properties

- If density of $\mathbf{X} \sim LE_n(\mu, \mathbf{\Sigma}, \phi)$ exists, then density of $\mathbf{Y} = \log \mathbf{X}$ also exists with

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c}{\sqrt{|\mathbf{\Sigma}|}} \left(\prod_{k=1}^{n} x_k^{-1} \right) \cdot g \left[(\log \mathbf{x} - \mu)^T \mathbf{\Sigma}^{-1} (\log \mathbf{x} - \mu) \right],$$

  see Fang et al. (1990).

- Any marginal of a log-elliptical distribution is again log-elliptical.
- MEANS:

$$E(X_k) = e^{\mu_k} \phi\left(-\sigma_k^2\right).$$

- COVARIANCES:

$$Cov(X_k, X_l) = e^{(\mu_k + \mu_l)}. \left\{ \phi \left[-(\sigma_k + \sigma_l)^2 \right] - \phi\left(-\sigma_k^2\right) \phi\left(-\sigma_l^2\right) \right\}.$$

Some Risk Measures

- Let $X \sim LE_1\left(\mu,\sigma^2,\phi\right)$ and $Z \sim S_1\left(\phi\right)$ with density $f_Z(x)$.
  - Quantile:

$$F_X^{-1}(p) = \exp\left(\mu + \sigma F_Z^{-1}(p)\right), \quad 0 < p < 1,$$

  - Expected Shortfall:

$$E\left[(X-d)_+\right] = e^{\mu}\phi\left(-\sigma^2\right)F_{Z^*}\left(\frac{\mu-\log d}{\sigma}\right) - dF_Z\left(\frac{\mu-\log d}{\sigma}\right)$$

  - Tail Conditional Expectation:

$$E\left[X \,\middle|\, X > F_X^{-1}(p)\right] = \frac{e^{\mu}}{1-p}\phi\left(-\sigma^2\right)F_{Z^*}\left(F_Z^{-1}(1-p)\right)$$

where the density of Z^* is given by $f_{Z^*}(x) = \frac{f_Z(x)e^{\sigma x}}{\phi(-\sigma^2)}$.

Non-Independent Log-Elliptical Risks

- Payments: $\alpha_1, ..., \alpha_n$
- Rates of return: $Y_i\left(i-1,i\right), i=1,2,\ldots,n$.
- Define $Y\left(i\right)=Y_1+\cdots+Y_i$, the sum of the first i elements of $\mathbf{Y}$.
  - $X_i=\exp\left[-Y\left(i\right)\right]$.
- Present Value:
  $S=\sum_{i=1}^{n}\alpha_i\exp\left[-\left(Y_1+\cdots+Y_i\right)\right]=\sum_{i=1}^{n}X_i$
- Assume return vector $\mathbf{Y}\sim E_n\left(\mu,\boldsymbol{\Sigma},\phi\right)$ with parameters μ and $\boldsymbol{\Sigma}$.
  - Then $\mathbf{X}=\left(X_1,...,X_n\right)^T$ is log-elliptical.

- continued

- We know $Y(i) \sim E_1\left(\mu(i), \sigma^2(i), \phi\right)$ with

$$\mu(i) = \sum_{k=1}^{i} \mu_k \text{ and } \sigma^2(i) = \sum_{k=1}^{i} \sum_{l=1}^{i} \sigma_{kl}.$$

- Conditioning rv: $\Lambda = \sum_{i=1}^{n} \beta_i Y_i$

- Using the property of elliptical, $\Lambda \sim E_1\left(\mu_\Lambda, \sigma_\Lambda^2, \phi\right)$ where

$$\mu_\Lambda = \sum_{i=1}^{n} \beta_i \mu_i \text{ and } \sigma_\Lambda^2 = \sum_{i,.j=1}^{n} \beta_i \beta_j \sigma_{ij}.$$

The Bounds

- COMONOTONIC UPPER BOUND:
  $S^c = \sum_{i=1}^{n} \alpha_i \exp\left[-\mu\left(i\right) + \sigma\left(i\right) F_Z^{-1}(U)\right]$
- IMPROVED UPPER BOUND:

$$S^u = \sum_{i=1}^{n} \alpha_i \exp\left[-\mu\left(i\right) - r_i\, \sigma\left(i\right) F_Z^{-1}(U) + \sqrt{1 - r_i^2}\, \sigma\left(i\right)\; F_Z^{-1}(V)\right]$$

- LOWER BOUND:

$$S^l = \sum_{i=1}^{n} \alpha_i e^{\left[-\mu(i) - r_i\;\sigma(i)\;F_Z^{-1}(U)\right]} \cdot \phi_a\left(-\sigma^2\left(i\right)\left(1 - r_i^2\right)\right)$$

where U and V are mutually indep. $U(0,1)$ rv's, $Z \sim S_1\left(\phi\right)$, and $r_i = \frac{{}_{k=1}^{i}\quad{}_{l=1}^{n}\beta_l\sigma_{kl}}{\sigma(i)\sigma_\Lambda}$.

Sums of Log-Normal RV's

In the case of log-Normal, we have the results from Dhaene, et al. (2002):

$$
\begin{aligned}
S^l &= \sum_{i=1}^{n} \alpha_i \, e^{-E[Y(i)] - r_i \, \sigma_{Y(i)} \, \Phi^{-1}(U) + \frac{1}{2}(1-r_i^2)\sigma_{Y(i)}^2}, \\
S^u &= \sum_{i=1}^{n} \alpha_i e^{-E[Y(i)] - r_i \, \sigma_{Y(i)} \, \Phi^{-1}(U) + \sqrt{1-r_i^2} \, \sigma_{Y(i)} \, \Phi^{-1}(V)}, \\
S^c &= \sum_{i=1}^{n} \alpha_i \, e^{-E[Y(i)] + \sigma_{Y(i)} \, \Phi^{-1}(U)}.
\end{aligned}
$$

Lecture No. 6
Capital Allocation and Elliptical Distributions

Emiliano A. Valdez

Introduction

- Why do we need to allocate capital?
  - Redistribute capital cost equitably
  - Division of capital provides division of risks across business units
  - Allocation of expenses, prioritizing capital budgeting projects
  - Fair assessment of manager performance

- This paper examines what constitutes a fair allocation and studies Wang's allocation within this fair allocation principle and then extends to class of elliptical distributions.

Fair Allocation

- Let $\mathbf{X}^T = (X_1, X_2, ..., X_n)$ denote the vector of losses.
- Define an *allocation* A to be a mapping $A : \mathbf{X}^T \to R^n$ such that $A\left(\mathbf{X}^T\right) = (K_1, K_2, ..., K_n)^T$ where $\sum_{i=1}^n K_i = K = \rho(Z)$.
- Each component K_i of allocation is viewed as the i-th line of business contribution to total capital.
- Because allocation must also reflect the fact that each line operates in the presence of other lines, the notation

$$A(X_i | X_1, ..., X_n) = K_i$$

  is well-suited for this purpose.
- Notice also that the requirement $\sum_{i=1}^n K_i = \rho(Z)$ is sometimes called the "full allocation" requirement.

What constitutes a fair allocation?

- Let $N = \{1, 2, ..., n\}$ be the set of the first n positive integers. An allocation A is said to be a *fair allocation* if:
  - **No Undercut**: For any subset $M \subseteq N$, we have $\sum_{i\in M} A\left(X_i \left| X_1, ..., X_n\right.\right) \leq \rho\left(\sum_{i\in M} X_i\right)$.
  - **Symmetry**: Let $N^* = N - \{i_1, i_2\}$. If $M \subset N^*$ (strict subset) with $|M| = m$, $\mathbf{X}_m^T = (X_{j_1}, ..., X_{j_m})$ and if $A\left(X_{i_1} \left| \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right.\right) = A\left(X_{i_2} \left| \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right.\right)$ for every $M \subset N^*$, then we must have $K_{i_1} = K_{i_2}$.
  - **Consistency**: For any subset $M \subseteq N$ with $|M| = m$, let $\mathbf{X}_{n-m}^T = \left(X_{j_1}, ..., X_{j_{n-m}}\right)$ for all $j_k \in N - M$ where $k = 1, ..., n - m$. Then we have

$$\sum_{i\in M} A\left(X_i \left| X_1, ..., X_n\right.\right) = A\left(\sum_{i\in M} X_i \left| \sum_{i\in M} X_i, \mathbf{X}_{n-m}^T\right.\right).$$

Relative Allocation

- This gives the allocation to the i-th line of business as

$$A(X_i|X_1, \cdots, X_n) = \rho(Z) \frac{\rho(X_i)}{\rho(X_1) + \cdots + \rho(X_n)}.$$

- Simple and appealing, but not a fair allocation.
- Consider 3 indep. risks X_1, X_2, and X_3 with mean $E(X_i) = 0$ and variances $Var(X_i) = \sigma^2(X_i)$ for $i = 1, 2, 3$. Define the risk measure $\rho(X_i) = F_i^{-1}(1-\alpha) \cdot \sigma(X_i)$.
- Now suppose a life company has four lines each facing risks X_1, $-X_1$, X_2, and X_3 so that total risk is $Z = X_2 + X_3$. Consider the subset M consisting of the risks $\{X_1, -X_1, X_2\}$ and observe that $\rho\left(\sum_{i \in M} X_i\right) = \rho(X_2) = F_2^{-1}(1-\alpha) \cdot \sigma(X_2)$.

Relative Allocation - continued

- Because

$$\sum_{i\in M} A\left(X_i \,|X_1, -X_1, X_2, X_3\right)$$

$$= \rho(X_2+X_3)\frac{\rho\left(X_1\right)+\rho(-X_1)+\rho(X_2)}{\rho\left(X_1\right)+\rho(-X_1)+\rho(X_2)+\rho(X_3)},$$

  the “no undercut” cannot be satisfied unless the risks have symmetric distributions.
- The “consistency” property is also not satisfied because

$$A\left(\sum_{i\in M} X_i \,\middle|\, \sum_{i\in M} X_i, X_3\right) = A\left(X_2 \,|X_2, X_3\right)$$

$$= \rho(X_2+X_3)\frac{\rho(X_2)}{\rho(X_2)+\rho(X_3)}$$

Relative Allocation - continued

- Hence

$$\sum_{i \in M} A\left(X_i \left| X_1, -X_1, X_2, X_3 \right.\right) \neq A\left(\sum_{i \in M} X_i \left| \sum_{i \in M} X_i, X_3 \right.\right).$$

- However, it can be shown that the "symmetry" property is satisfied for this allocation formula. Consider for example the case where $A\left(X_1 \left| X_1, -X_1, X_2 \right.\right) = A\left(X_2 \left| X_1, -X_1, X_2 \right.\right)$ and it is straightforward to show that in this case $\rho\left(X_1\right) = \rho\left(X_2\right)$ so that

$$A\left(X_1 \left| X_1, -X_1, X_2, X_3 \right.\right) = A\left(X_2 \left| X_1, -X_1, X_2, X_3 \right.\right)$$

and symmetry is satisfied.

Covariance-Based Allocation

- The allocation formula is based on $A(X_i|X_1, \cdots, X_n) = \lambda_i \rho(Z)$ where $\lambda^T = (\lambda_1, ..., \lambda_n)$ denotes a vector of weights that add up to one so that full allocation is satisfied.
- To determine these weights λ_i, we minimize the following quadratic loss function

$$E\left[\left((\mathbf{X} - \mu) - \lambda (Z - \mu_Z)\right)^T \mathbf{W} \left((\mathbf{X} - \mu) - \lambda (Z - \mu_Z)\right)\right]$$

where the weight-matrix $\mathbf{W}$ is assumed to be positive definite. Differentiating with respect to λ and equating to zero yields

$$\lambda_i = \frac{E\left[(X_i - \mu_i)(Z - \mu_Z)\right]}{E\left[(Z - \mu_Z)^2\right]} = \frac{Cov\,(X_i, Z)}{Var\,(Z)}, \text{ for } i = 1, 2, ..., n.$$

Is the Covariance Principle Fair?

- The capital allocation formula based on the covariance principle satisfies the three properties of a fair allocation:
  - no undercut,
  - symmetry, and
  - consistency.

Wang's Capital Decomposition Formula

- Preserving the notation used by Wang (2002), denote the expectation of $X_{i,Q}$ by

$$H_\lambda[X_i, Z] = E(X_{i,Q}) = \frac{E[X \cdot \exp(\lambda Z)]}{E[\exp(\lambda Z)]}$$

and the expectation of the aggregate loss Z_Q by

$$H_\lambda[Z, Z] = E(Z_Q) = \frac{E[Z \cdot \exp(\lambda Z)]}{E[\exp(\lambda Z)]}.$$

This exactly gives the Esscher transform of Z.

- Price of a random payment X_i traded in the market is $H_\lambda[X_i, Z]$ so that one can think of the difference $\rho(X_i) = E(X_{i,Q}) - E(X_i) = H_\lambda[X_i, Z] - E(X_i)$ as the risk premium.

- continued

- For the aggregate payment Z, its risk premium is given by $\rho(Z) = \rho\left(\sum_{i=1}^{n} X_i\right) = H_\lambda[Z,Z] - E(Z)$.
- It is rather straightforward to show $\rho(X_i) = \frac{Cov(X_i, \exp(\lambda Z))}{E[\exp(\lambda Z)]}$ and $\rho(Z) = \frac{Cov(Z, \exp(\lambda Z))}{E[\exp(\lambda Z)]}$.
- Wang proposes computing the allocation of capital to individual business unit i based on the following formula:

$$K_i = H_\lambda[X_i, Z] - E(X_i).$$

- Assuming an aggregate capital of K for the insurance company as a whole, the parameter λ can be computed using

$$K = H_\lambda[Z,Z] - E(Z).$$

- For $i = 1, 2, ..., n$, it can readily be shown that $K = \sum_{i=1}^{n} K_i$.

Multivariate Normal

- If $X_1, ..., X_n$ follow a multivariate normal, we have that Wang's allocation method reduces to the covariance method.

- Some straightforward calculation yields the results:

$$E\left(Ze^{\lambda Z}\right) = \exp\left(\lambda\mu_Z + \frac{\lambda^2\sigma_Z^2}{2}\right)\cdot(\mu + \lambda\sigma_Z^2)$$

$$E\left(X_i e^{\lambda Z}\right) = \exp\left(\lambda\mu_Z + \frac{\lambda^2\sigma_Z^2}{2}\right)\cdot(\mu_i + \lambda\sigma_{i,Z})$$

- Then it follows that $K = \lambda\sigma_Z^2$ and $K_i = \lambda\sigma_{i,Z}$ which is clearly equivalent to the covariance method.

Some Notation and Assumptions

- Suppose $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g_n)$ and $\mathbf{e} = (1, 1, ..., 1)^T$.
- Assume density generator g_n exists.
- Define

$$Z = X_1 + \cdots + X_n = \sum_{k=1}^{n} X_k = \mathbf{e}^T \mathbf{X}$$

which is the sum of elliptical risks. We know that $Z \sim E_1\left(\mathbf{e}^T \mu, \mathbf{e^T} \mathbf{\Sigma} \mathbf{e}, g_1\right)$.

- Denote by $\mu_Z = \mathbf{e}^T \mu = \sum_{j=1}^{n} \mu_j$ and $\sigma_Z^2 = \mathbf{e}^T \mathbf{\Sigma} \mathbf{e} = \sum_{i,j=1}^{n} \sigma_{ij}$.
- Define the *tail generator* by

$$T_n(u) = \int_{\frac{1}{2}u^2}^{\infty} c_n g_n(x)\, dx.$$

Two Useful Lemmas

- Let $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g_n)$. Then for $1 \leq i \leq n$, the vector $\mathbf{X}_{i,Z} = (X_i, Z)^T$ has an elliptical distribution with the same generator, i.e., $\mathbf{X}_{i,Z} \sim E_2(\mu_{i,Z}, \mathbf{\Sigma}_{i,Z}, g_2)$, where
$\mu_{i,Z} = \left(\mu_i, \sum_{j=1}^n \mu_j\right)^T$, $\mathbf{\Sigma}_{i,Z} = \begin{pmatrix} \sigma_i^2 & \sigma_{i,Z} \\ \sigma_{i,Z} & \sigma_Z^2 \end{pmatrix}$, and
$\sigma_i^2 = \sigma_{ii}, \sigma_{i,Z} = \sum_{j=1}^n \sigma_{ij}, \sigma_Z^2 = \sum_{j,k=1}^n \sigma_{jk}$.

- Let $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g_n)$ and assume condition for existence of density generator holds. Let T be the tail generator as defined above and associated with Z. Then

$$H_\lambda[X_i, Z] = \mu_i + \lambda \rho_{i,Z} \sigma_i \sigma_Z \frac{\exp(\lambda \mu_Z)}{M_Z(\lambda)} \int_{-\infty}^{\infty} T_n(w) \exp(\lambda \sigma_Z w)\, dw$$

Wang's Allocation for Elliptical

- Let $\mathbf{X} \sim E_n(\mu, \mathbf{\Sigma}, g_n)$ and assume conditions for existence of density generator and $|\psi'(0)| < \infty$ hold. Then Wang's capital allocation formula can be expressed as

$$K_i = -\lambda \psi'(0)\, \rho_{i,Z} \sigma_i \sigma_Z$$

- The result immediately follows from the previous lemma:

$$\begin{aligned} K_i &= H_\lambda [X_i, Z] - E(X_i) \\ &= \lambda \rho_{i,Z} \sigma_i \sigma_Z \frac{1}{M_Z(\lambda)} \int_{-\infty}^{\infty} \exp\left[\lambda(\mu_Z + \sigma_Z w)\right] T_n(w)\, dw \\ &= \lambda \rho_{i,Z} \sigma_i \sigma_Z \frac{1}{M_Z(\lambda)} \left[-\psi'(0) M_Z(\lambda)\right] \\ &= -\lambda \psi'(0)\, \rho_{i,Z} \sigma_i \sigma_Z. \end{aligned}$$

Panjer's Example

- Panjer (2002) example to illustrate the capital allocation.
- Insurer has 10 lines of business is faced with risks represented by vector $\mathbf{X}^T = (X_1, ..., X_{10})$ where each X_i represents the P.V. of losses over a specified time horizon.
- The estimated covariance structure, $\widehat{\mathbf{\Sigma}}$, (in millions-squared) is given by [see paper for variance matrix] and the estimated mean vector $\widehat{\mu}^T$ (in millions) is given by

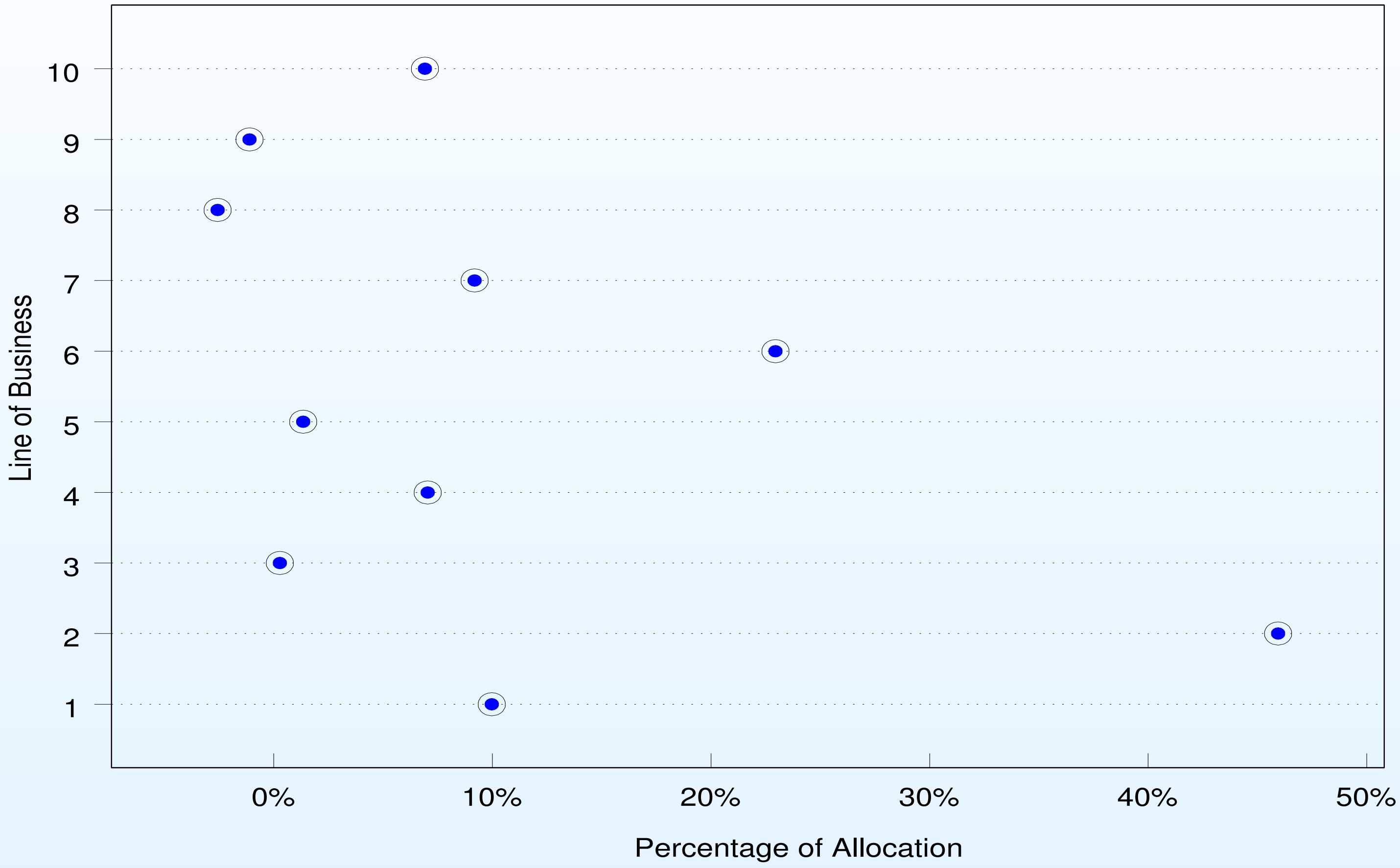
$$(25.69, 37.84, 0.85, 12.70, 0.15, 24.05, 14.41, 4.49, 4.39, 9.56) \,.$$

- The resulting allocation $\mathbf{K}^T$ is given by

$$(2.72, 12.55, 0.08, 1.92, 0.37, 6.27, 2.51, -0.70, -0.30, 1.89) \,,$$

expressed in millions, with total capital equal to 27.31 million.

Graph of Allocation



Lecture No. 7

Convex Bounds for Scalar Products of Random Variables (With Applications to Loss Reserving and Life Annuities)

Tom Hoedemakers

Outline

- Introduction to comonotonicity
- Comonotonic bounds for dependent random variables
- Generalization to scalar products of random variables
- Discounting with Gaussian returns
- Moments based approximations
- Part I Applications: Life Annuities
- Part II Applications: Loss Reserving

Convex order and comonotonic risks

- *Convex order:* Consider two random variables X and Y. Then X is said to precede Y in the **convex order** sense, notation $X \leq_{cx} Y$, if and only if

$$\mathrm{E}[X] = \mathrm{E}[Y] \text{ and } \mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+] \qquad \forall\, d$$

- *Property:* $\boxed{X \leq_{cx} Y \Rightarrow \text{Var}[X] \leq \text{Var}[Y]}$
- *Comonotonicity:* very strong positive dependence structure $\rightarrow$ each two possible outcomes $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of $\vec{X} = (X_1, \cdots, X_n)$ are ordered componentwise

Comonotonicity

<u>*Characterizations:*</u> $\vec{X}$ is **comonotonic** if any of the following conditions holds:

1. For $U \sim Uniform(0,1)$ we have

$$\vec{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)),$$

2. $\exists$ a random variable Z and non-decreasing functions $f_1, f_2, \ldots, f_n$, (or non-increasing functions) such that

$$\vec{X} \stackrel{d}{=} (f_1(Z), f_2(Z), \ldots, f_n(Z)),$$

3. For the n-variate cdf we have

$$F_{\vec{X}}(\vec{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\}, \qquad \forall\, \vec{x} \in \mathbb{R}^n.$$

Quantiles and stop-loss premiums

- *Notations:*

$$\begin{aligned} \Phi &= \text{cdf of } N(0,1) \\ F_X(x) &= \Pr[X \leq x] \\ \overline{F}_X(x) &= 1 - F_X(x) \\ (x-d)_+ &= \max(x-d, 0) \end{aligned}$$

- *Quantiles:*

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in (0,1).$$

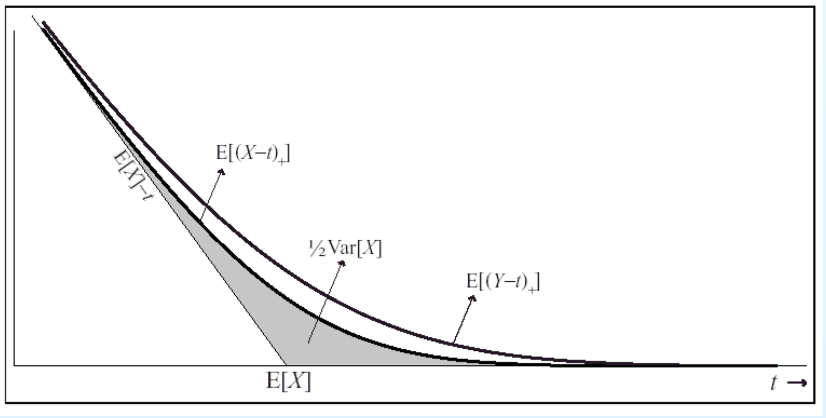
- *Stop-loss premiums:*

$$\mathrm{E}[(X-d)_+] = \int_d^\infty \overline{F}_X(x)\,dx, \qquad -\infty < d < \infty.$$

Quantiles and stop-loss premiums

- *Relations:*
  - $\frac{1}{2}\text{Var}[X] = \int_{-\infty}^{+\infty} \{E[(X-t)_+] - (E[X]-t)_+\}\, dt$,
  - if $X \leq_{cx} Y$, thus $E[(Y-t)_+] \geq E[(X-t)_+]$ for all t, then

$$\frac{1}{2}\{\text{Var}[Y] - \text{Var}[X]\} = \int_{-\infty}^{+\infty} \{E[(Y-t)_+] - E[(X-t)_+]\}\, dt.$$



Comonotonic bounds for sums of dependent r.v.'s

- *General result:* (Kaas et al., 2000)
  Let U be a uniform(0,1) random variable. For any random vector $\vec{X} = (X_1, X_2, \cdots, X_n)$ with marginal cdf's $F_{X_1}, F_{X_2}, \cdots, F_{X_n}$, we have

$$\sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U)$$

- *Notations:*
  - $S = \sum_{i=1}^{n} X_i$.
  - $S^l = \sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda]$ =lower bound.
  - $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$ =comonotonic upper bound.
- If all $\mathrm{E}[X_i|\Lambda]$ are $\nearrow$ functions of Λ, then S^l is a comonotonic sum.

Comonotonic sums

- Kaas et al. (2000):
  - The quantile function is **additive** for **comonotonic** risks

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \qquad p \in (0,1)$$

  - In case of strictly increasing and continuous marginals, the cdf $F_{S^c}(x)$ is uniquely determined by

$$F_{S^c}^{-1}\left(F_{S^c}(x)\right) = \sum_{i=1}^{n} F_{X_i}^{-1}\left(F_{S^c}(x)\right) = x, \qquad \left(F_{S^c}^{-1}(0) < x < F_{S^c}^{-1}(1)\right)$$

Comonotonic sums

- Dhaene et al. (2002):
  - Let $(X_1, \ldots, X_n)$ denote a comonotonic vector with strictly increasing marginal distributions and let $S^c = X_1 + \cdots + X_n$. Then the stop-loss premium of S^c can be computed as follows:

$$\mathrm{E}[(S^c - d)_+] = \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}\left(d\right)\right)\right)_+\right],$$

$$\left(F_{S^c}^{-1}(0) < d < F_{S^c}^{-1}(1)\right)$$

Lower bound $S^l = \sum_{i=1}^n \mathrm{E}[X_i|\Lambda]$: comonotonic

If Λ is such that all $g_i(\Lambda) = \mathrm{E}[X_i|\Lambda]$ are non-decreasing and continuous functions of Λ

$$\begin{aligned} F^{-1}_{S^l}(p) &= \sum_{i=1}^n F^{-1}_{\mathrm{E}[X_i|\Lambda]}(p) = \sum_{i=1}^n F^{-1}_{g_i(\Lambda)}(p) \\ &= \sum_{i=1}^n \mathrm{E}[X_i|\Lambda = F^{-1}_\Lambda(p)], \qquad p \in (0,1) \end{aligned}$$

If the cdf's of $g_i(\Lambda)$ are strictly increasing and continuous

$$\begin{aligned} \mathrm{E}[(S^l - d)_+] &= \sum_{i=1}^n \mathrm{E}\left[\left(\mathrm{E}\left[X_i \mid \Lambda\right] - F^{-1}_{\mathrm{E}[X_i|\Lambda]}\left(F_{S^\ell}(d)\right)\right)_+\right] \\ &= \sum_{i=1}^n \mathrm{E}\left[\left(\mathrm{E}\left[X_i \mid \Lambda\right] - \mathrm{E}\left[X_i \mid \Lambda = F^{-1}_\Lambda\left(F_{S^\ell}(d)\right)\right]\right)_+\right] \end{aligned}$$

Lower bound $S^l = \sum_{i=1}^n \mathrm{E}[X_i|\Lambda]$: not comonotonic

$$F_{S^l}(x) = \int_{-\infty}^{+\infty} \text{Pr}\left[\sum_{i=1}^n \mathrm{E}[X_i \mid \Lambda] \leq x | \Lambda = \lambda\right] dF_\Lambda(\lambda)$$

$$\mathrm{E}[(S^l - d)_+] = \int_{-\infty}^{+\infty} \left(\sum_{i=1}^n \mathrm{E}\left[X_i \mid \Lambda = \lambda\right] - d\right)_+ dF_\Lambda\left(\lambda\right)$$

If F_Λ is strictly $\nearrow$ and continuous: Define U as follows
$U \equiv F_\Lambda(\Lambda) \sim$ Unif$(0,1)$, then $U = u \Leftrightarrow \Lambda = F_\Lambda^{-1}(u), \forall\, 0 < u < 1$

$$F_{S^l}(x) = \int_0^1 \text{Pr}\left[\sum_{i=1}^n \mathrm{E}[X_i \mid \Lambda] \leq x | U = u\right] du$$

$$\mathrm{E}[(S^l - d)_+] = \int_0^1 \left(\sum_{i=1}^n \mathrm{E}\left[X_i \mid \Lambda = F_\Lambda^{-1}(u)\right] - d\right)_+ du$$

Moments based approximations

- Convex order relation: $S^l \leq_{cx} S \leq_{cx} S^c$

$$\Downarrow$$

$$\boxed{\mathrm{E}[(S^l - d)_+] \leq \mathrm{E}[(S - d)_+] \leq \mathrm{E}[(S^c - d)_+]}$$

$$\boxed{\begin{array}{ccccc} \mathrm{E}[S^l] & = & \mathrm{E}[S] & = & \mathrm{E}[S^c] \\ \mathrm{Var}(S^l) & \leq & \mathrm{Var}(S) & \leq & \mathrm{Var}(S^c). \end{array}}$$

- Define the random variable S^m by its stop-loss premiums

$$\mathrm{E}[(S^m - d)_+] = z\mathrm{E}[(S^l - d)_+] + (1 - z)\mathrm{E}[(S^c - d)_+], \qquad 0 \leq z \leq 1,$$

$$\Downarrow$$

$$\mathrm{E}[S^m] = z\mathrm{E}[S^l] + (1 - z)\mathrm{E}[S^c] = \mathrm{E}[S]$$

Moments based approximations

- By taking the (right-hand) derivative we find

$$F_{S^m}(x) = zF_{S^l}(x) + (1-z)F_{S^c}(x), \qquad 0 \leq z \leq 1$$

$\longrightarrow$ the d.f. of the approximation can be calculated fairly easily

- Determine z such that S^m is as close as possible to S. In Vyncke et al. (2004) z is chosen as

$$z = \frac{\mathrm{Var}(S^c) - \mathrm{Var}(S)}{\mathrm{Var}(S^c) - \mathrm{Var}(S^l)}$$

This choice doesn't depend on the retention d and it leads to **equal variances**

$$\boxed{\mathrm{Var}[S^m] = \mathrm{Var}[S]}$$

Generalization to scalar products

- Consider sums of the form: $S = X_1Y_1 + X_2Y_2 + \ldots + X_nY_n$ with $\vec{X} = (X_1, X_2, \ldots, X_n)$ and $\vec{Y} = (Y_1, Y_2, \ldots, Y_n)$ assumed to be **mutually independent**
- One can take $V_j = X_jY_j$ and apply the techniques for sums of dependent random variables $\rightarrow$ not practical !
  - it is not always easy to find the marginal distributions of V_j
  - it is usually very difficult to find a suitable conditioning random variable Λ, which will be a good approximation to the whole scalar product, taking into account the riskiness of the random vector $\vec{X}$ and $\vec{Y}$ simultaneously.

Generalization to scalar products

Lemma 1 *Assume that* $\vec{X} = (X_1, \ldots, X_n)$*,* $\vec{Y} = (Y_1, \ldots, Y_n)$ *and* $\vec{Z} = (Z_1, \ldots, Z_n)$ *are non-negative random vectors and that* $\vec{X}$ *is mutually independent of the vectors* $\vec{Y}$ *and* $\vec{Z}$*.*
If for all possible outcomes $x_1, \ldots, x_n$ *of* $\vec{X}$*:*

$$\sum_{i=1}^{n} x_i Y_i \leq_{cx} \sum_{i=1}^{n} x_i Z_i,$$

then the corresponding scalar products are ordered in the convex order sense, i.e.

$$\sum_{i=1}^{n} X_i Y_i \leq_{cx} \sum_{i=1}^{n} X_i Z_i.$$

Generalization to scalar products

Proof. Let ϕ be a convex function. By conditioning on $\vec{X}$ and taking the assumptions into account, we find that

$$\begin{aligned}
\mathrm{E}\Big[\phi\Big(\sum_{i=1}^{n} X_i Y_i\Big)\Big] &= \mathrm{E}_{\vec{X}}\Big[\mathrm{E}\Big[\phi\Big(\sum_{i=1}^{n} X_i Y_i\Big)|\vec{X}\Big]\Big] \\
&\leq \mathrm{E}_{\vec{X}}\Big[\mathrm{E}\Big[\phi\Big(\sum_{i=1}^{n} X_i Z_i\Big)|\vec{X}\Big]\Big] \\
&= \mathrm{E}\Big[\phi\Big(\sum_{i=1}^{n} X_i Z_i\Big)\Big]
\end{aligned}$$

holds for any convex function ϕ.

■

Generalization to scalar products

- *General result:* Let U and V be two uniform(0,1) r.v.'s. Assume that the vectors $\vec{X} = (X_1, X_2, \ldots, X_n)$ and $\vec{Y} = (Y_1, Y_2, \ldots, Y_n)$ are mutually independent. Then

$$\boxed{\sum_{i=1}^{n} \mathrm{E}[X_i|\Gamma]\mathrm{E}[Y_i|\Lambda] \leq_{cx} \sum_{i=1}^{n} X_i Y_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U) F_{Y_i}^{-1}(V)}$$

with $\begin{cases} \Gamma & \text{a r.v. independent of } \vec{Y} \text{ and } \Lambda \\ \Lambda & \text{a r.v. independent of } \vec{X} \text{ and } \Gamma \end{cases}$

- *Notations:*
  - $S = \sum_{i=1}^{n} X_i Y_i$.
  - $S^l = \sum_{i=1}^{n} \mathrm{E}[X_i|\Gamma]\mathrm{E}[Y_i|\Lambda] =$ lower bound.
  - $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U) F_{Y_i}^{-1}(V) =$ comonotonic upper bound.

Generalization to scalar products: (**Proof.**)

1. $\boxed{\sum_{i=1}^n X_iY_i \leq_{cx} \sum_{i=1}^n F_{X_i}^{-1}(U)F_{Y_i}^{-1}(V)}$

   - For all possible outcomes $(x_1, x_2, \ldots, x_n)$ of $\vec{X}$:
     $\sum_{i=1}^n x_iY_i \leq_{cx} \sum_{i=1}^n F_{x_iY_i}^{-1}(V) = \sum_{i=1}^n x_iF_{Y_i}^{-1}(V)$
     $\overset{Lemma}{\Longrightarrow} \sum_{i=1}^n X_iY_i \leq_{cx} \sum_{i=1}^n X_iF_{Y_i}^{-1}(V)$
   - The same reasoning can be applied to show that
     $\sum_{i=1}^n X_iF_{Y_i}^{-1}(V) \leq_{cx} \sum_{i=1}^n F_{X_i}^{-1}(U)F_{Y_i}^{-1}(V)$

2. $\boxed{\sum_{i=1}^n \mathrm{E}[X_i|\Gamma]\mathrm{E}[Y_i|\Lambda] \leq_{cx} \sum_{i=1}^n X_iY_i}$

   - $\sum_{i=1}^n \mathrm{E}[X_i|\Gamma]\mathrm{E}[Y_i|\Lambda] \leq_{cx} \sum_{i=1}^n X_i\mathrm{E}[Y_i|\Lambda]$
   - $\sum_{i=1}^n X_i\mathrm{E}[Y_i|\Lambda] \leq_{cx} \sum_{i=1}^n X_iY_i$

■

How to deal with two-dimensionality?

- Assume that $\boxed{X = \sum_{i=1}^{n} f_i(\Theta) g_i(V)}$

$$\begin{cases} V & \sim \text{Unif}(0,1) \text{ and independent of } \theta \\ f_i & \text{non-decreasing} \\ g_i & \text{non-negative and non-decreasing} \end{cases}$$

- Distribution function: 3-step calculation:

1. $F^{-1}_{X|\Theta=\theta}(p) = \sum_{i=1}^{n} f_i(\theta) g_i(p)$
2. Obtain $F_{X|\Theta=\theta}$ from $\sum_{i=1}^{n} f_i(\theta) g_i \left(F_{X|\Theta=\theta}(y) \right) = y$;
3. Compute $F_X(y) = \int_{-\infty}^{\infty} F_{X|\Theta=\theta}(y) dF_\Theta(\theta)$

How to deal with two-dimensionality?

- *Convex bounds:*
  - In the case of the upper bound one can always use the described procedure. Indeed, notice that $\Theta = U$, $f_i(u) = F_{X_i}^{-1}(u)$ and $g_i(p) = F_{Y_i}^{-1}(p)$ for which the conditions are naturally satisfied.
  - In the case of the lower bound one takes $\Theta = \Lambda$, $f_i(\gamma) = \mathrm{E}[X_i \mid \Gamma = \gamma]$ and $g_i(p) = \mathrm{E}[Y_i \mid \Lambda = F_\Lambda^{-1}(p)]$
- *In general:*
  The conditions of the previous slide are not always satisfied! However, in our applications they are satisfied.

Stop-loss premia for scalar products of r.v.'s

Upper bound: $\mathrm{E}[(S^c - d)_+]$:

1. Consider the comonotonic sum

$$S^c|_{U=u} = \sum_{i=1}^{n} F_{X_i}^{-1}(u) F_{Y_i}^{-1}(V)$$

2. Apply the basic theorem for stop-loss premia
3. Condition on U: $\Rightarrow \mathrm{E}[(S^c - d)_+] = \mathrm{E}\big[\mathrm{E}[(S^c - d)_+ | U]\big] =$

$$\int_0^1 \sum_{i=1}^{n} F_{X_i}^{-1}(u) \mathrm{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^c|U=u}(d)\right)\right)_+\right] du$$

Stop-loss premia for scalar products of r.v.'s

Lower bound: $\mathrm{E}[(S^l-d)_+]$:

1. Assume that Γ and Λ can be chosen in such a way that for any fixed $\gamma \in \mathrm{supp}(\Gamma)$ all components $\mathrm{E}[X_i|\Gamma=\gamma]\mathrm{E}[Y_i|\Lambda=\lambda]$ are non-decreasing (or equivalently non-increasing) in λ.
2. The vector $\Big(\mathrm{E}[X_1|\Gamma=\gamma]\mathrm{E}[Y_1|\Lambda],\ldots,\mathrm{E}[X_n|\Gamma=\gamma]\mathrm{E}[Y_n|\Lambda]\Big)$ is comonotonic
3. Apply the basic theorem for stop-loss premia
4. Condition on $\Gamma \Rightarrow \mathrm{E}[(S^l-d)_+]=\mathrm{E}[\mathrm{E}[(S^l-d)_+|\Gamma]]=$

$$\int_0^1 \sum_{i=1}^n \mathrm{E}[X_i|\Gamma=F_\Gamma^{-1}(u)]\mathrm{E}\left[\left(\mathrm{E}[Y_i|\Lambda]-F^{-1}_{\mathrm{E}[Y_i|\Lambda]}\left(F_{S^l|\Gamma=F_\Gamma^{-1}(u)}(d)\right)\right)_+\right]du$$

Gaussian returns

- Suppose that one invests the value 1 at time 0. Then at time t it accumulates to the random value $e^{Y(t)}$. The collection of r.v.'s $\{Y(t)\}_{t\geq 0}$ is called a **stochastic return process**.
- We assume that the return process $Y(t)$ is **Gaussian**, i.e. such that $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ is normally distributed $\forall\, 0 < t_1 < t_2 < ... < t_n$.
- Note that any Gaussian process is determined unequivocally by its mean and covariance functions: $m(t) = \mathrm{E}[Y(t)]$ and $c(s,t) =$ Cov$(Y(s), Y(t))$.

Two examples

- *The Black & Scholes model:*

$$Y(t) = \mu t + \sigma B_t$$

with B_t: Brownian motion process.

$$\begin{aligned} \mathrm{E}[Y(t)] &= \mu t \\ \mathrm{Cov}(Y(s), Y(t)) &= \sigma^2 \min(s, t) \end{aligned}$$

Two examples

- *The Ornstein-Uhlenbeck process:*

$$Y(t) = \mu t + X(t)$$

with $dX(t) = -aX(t)dt + \sigma dB_t$

$$\begin{aligned} \mathrm{E}[Y(t)] &= \mu t \\ \mathrm{Cov}(Y(s), Y(t)) &= \frac{\sigma^2}{2a}\big(\exp(-a|t-s|) - \exp(-a(t+s))\big) \end{aligned}$$

$\Longrightarrow$ in both cases: $\mathrm{Cov}(Y(s), Y(t)) > 0$ for any $t, s > 0$.

Two examples

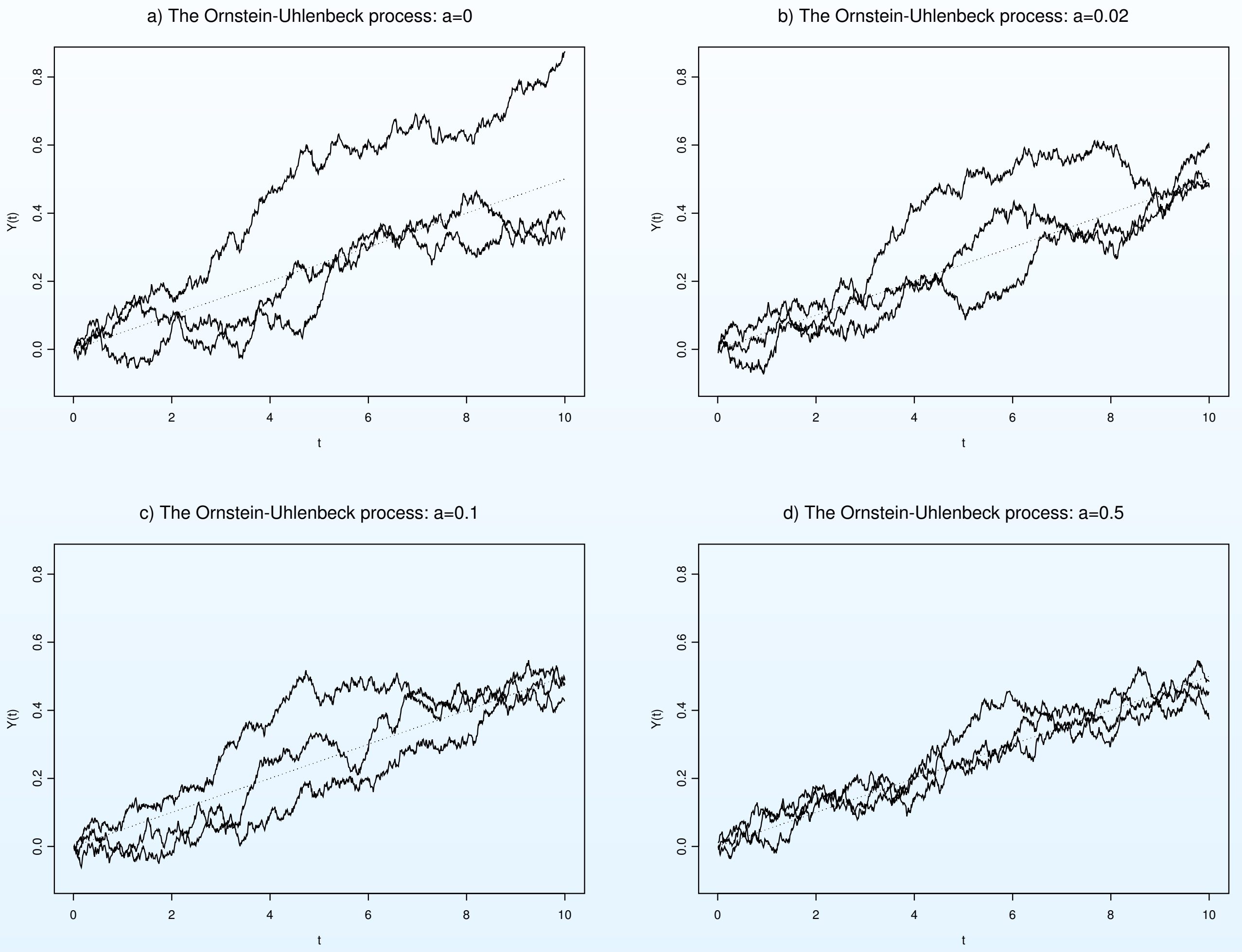


Figure1: Typical paths for the Ornstein-Uhlenbeck process with the mean parameter $\mu = 0.05$, volatility parameter $\sigma = 0.07$ and different values of parameter a.

Two examples

Remark:

- For $a = 0$ the Ornstein-Uhlenbeck process degenerates to an ordinary Brownian motion with drift and is equivalent to the Black & Scholes setting.
- When $a > 0$, the process $Y(t)$ has no independent increments any more. Moreover, it becomes mean reverting.
- $\Rightarrow a$ measures how strong the process $Y(t)$ is attracted by its mean function.
  ($a = 0$: no attraction $\Rightarrow$ increments are independent)

Discounting with Gaussian returns

$$DS = \sum_{i=1}^{n} X_i e^{-Y(i)}$$

- $\vec{Y} = (Y(1), Y(2), \ldots, Y(n)) \sim N(\vec{\mu}, \mathbf{\Sigma})$ with

$$\begin{aligned} \vec{\mu} &= (\mu_1, \ldots, \mu_n) = (\mathrm{E}[Y(1)], \mathrm{E}[Y(2)], \ldots, \mathrm{E}[Y(n)]) \\ \mathbf{\Sigma} &= [\sigma_{ij}]_{1 \le i,j \le n} = \left[\mathrm{Cov}(Y(i), Y(j))\right]_{1 \le i,j \le n} \\ & \quad (\sigma_{ii} \text{ will be denoted by } \sigma_i^2) \end{aligned}$$

- $\vec{X} = (X_1, X_2, \ldots, X_n)$: a vector of non-negative r.v.'s

$\hookrightarrow$ DS: discounted value of future benefits X_i with return process described by one of the well-known Gaussian models

Discounting with Gaussian returns: convex bounds

$$
\begin{aligned}
DS^c &= \sum_{i=1}^{n} F_{X_i}^{-1}(U) F_{e^{-Y(i)}}^{-1}(V) \\
&= \sum_{i=1}^{n} F_{X_i}^{-1}(U) e^{-\mu_i + \sigma_i \Phi^{-1}(V)}, \\
DS^l &= \sum_{i=1}^{n} \mathrm{E}[X_i|\Gamma]\mathrm{E}[e^{-Y(i)}|\Lambda],
\end{aligned}
$$

-U and V are independent Unif$(0,1)$ r.v.'s

-Γ is independent of Λ and $\vec{Y}$

-Λ is independent of Γ and $\vec{X}$

Remark: the quality of the lower bound heavily depends on the choice of the conditioning random variables!

Discounting with Gaussian returns: CDF DS^c

1. Suppose that $U = u$ is fixed $\Rightarrow$ conditional quantiles:

$$F^{-1}_{DS^c|U=u}(p) = \sum_{i=1}^{n} F^{-1}_{X_i}(u) e^{-\mu_i + \sigma_i \Phi^{-1}(p)};$$

2. $F^{-1}_{DS^c|U=u}(p)$ is continuous and strictly $\nearrow \forall\, u \Rightarrow F_{DS^c|U=u}(y)$ can be computed as a solution of

$$\sum_{i=1}^{n} F^{-1}_{X_i}(u) e^{-\mu_i + \sigma_i \Phi^{-1}(F_{DS^c|U=u}(y))} = y;$$

3. The cumulative distribution function of DS^c can be now derived as

$$F_{DS^c}(y) = \int_0^1 F_{DS^c|U=u}(y) du.$$

Discounting with Gaussian returns: stop-loss premia

Lemma 2 *Let X be a lognormal random variable of the form αe^Z with $Z \sim N(\mathrm{E}[Z], \sigma_Z)$ and $\alpha \in \mathbb{R}$. Then the stop-loss premium with retention d equals for $\alpha d > 0$*

$$\mathrm{E}[(X-d)_+] = \operatorname{sign}(\alpha)\, e^{\mu+\frac{\sigma^2}{2}}\ \Phi(\operatorname{sign}(\alpha)\, b_1) - d\Phi(\operatorname{sign}(\alpha)\, b_2),$$

where

$$\mu = \ln|\alpha| + \mathrm{E}[Z] \qquad \sigma = \sigma_Z$$

$$b_1 = \frac{\mu + \sigma^2 - \ln|d|}{\sigma} \qquad b_2 = b_1 - \sigma$$

The cases $\alpha d < 0$ are trivial.

Discounting with Gaussian returns: SL DS^c

$$\boxed{\mathrm{E}[(e^{-Y(i)} - d_{u,i})_+] = e^{-\mu_i + \frac{\sigma_i^2}{2}} \Phi\left(b_{u,i}^{(1)}\right) - d_{u,i}\Phi\left(b_{u,i}^{(2)}\right),}$$

with

$$d_{u,i} = F^{-1}_{\exp(-Y(i))}\left(F_{DS^c|U=u}(d)\right) = e^{-\mu_i + \sigma_i \Phi^{-1}(F_{DS^c|U=u}(d))}$$

$$b_{u,i}^{(1)} = \frac{-\mu_i + \sigma_i^2 - \ln(d_{u,i})}{\sigma_i}, \qquad b_{u,i}^{(2)} = b_{u,i}^{(1)} - \sigma_i$$

$$\begin{aligned}
\mathrm{E}[(DS^c - d)_+] &= \int_0^1 \sum_{i=1}^n F_{X_i}^{-1}(u)\mathrm{E}[(e^{-Y(i)} - d_{u,i})_+]du \\
&= \sum_{i=1}^n e^{-\mu_i + \frac{1}{2}\sigma_i^2} \int_0^1 F_{X_i}^{-1}(u)\Phi\Big(\sigma_i - \Phi^{-1}(F_{DS^c|U=u}(d))\Big)du \\
&\quad -d(1 - F_{DS^c}(d)).
\end{aligned}$$

A comonotonic approximation for cumulative returns

- The exact random variable

$$S = \sum_{i=1}^{n} \alpha_i e^{-Y(i)}$$

- Approximation: replace

$$[Y(1), Y(2), \ldots, Y(n)]$$

  by

$$[Y(1)^c, Y(2)^c, \ldots, Y(n)^c]$$

  where the
  - **marginals are the same,**
  - copula is replaced by the **comonotonic copula**.

The best comonotonic approximation

- The exact random variable

$$S = \mathrm{E}[S|S] = \sum_{i=1}^{n} \alpha_i \mathrm{E}[e^{-Y(i)}|S]$$

- Approximation: replace S by

$$S = \mathrm{E}[S|\Lambda] = \sum_{i=1}^{n} \alpha_i \mathrm{E}[e^{-Y(i)}|\Lambda]$$

or equivalent, replace $[Y(1), Y(2), \ldots, Y(n)]$ by

$$[Y(1)^l, Y(2)^l, \ldots, Y(n)^l]$$

where the **marginals are replaced** and the copula is replaced by the **comonotonic copula**.

Choice of the conditioning variable: return component

a) $\Rightarrow$ choose Λ such that $\Lambda \approx S$ (Var$(S) \approx$ Var(S^l))

$$\boxed{\Lambda = \sum_{i=1}^{n} \beta_i Y(i)}$$

- Taylor based (Kaas et al., 2000): $\underline{\beta_i = \alpha_i e^{-\mu_i}}$
  $\rightarrow \Lambda$: linear transformation of a first order approximation to S

$$\begin{aligned} S &= \sum_{i=1}^{n} \alpha_i e^{-\mu_i + (Y(i)+\mu_i)} \approx \sum_{i=1}^{n} \alpha_i e^{-\mu_i} \left(1 + Y(i) + \mu_i\right) \\ &\approx C + \sum_{i=1}^{n} \alpha_i e^{-\mu_i} Y(i), \end{aligned}$$

Choice of the conditioning variable: return component

- Maximal variance (Vanduffel et al., 2004): $\underline{\beta_i = \alpha_i e^{-\mu_i+\frac{1}{2}\sigma_i^2}}$
  $\rightarrow$ the first order approximation of Var(S^ℓ) is maximized

$$\begin{aligned}
\text{Var}(S^l) &\approx \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j e^{-\mu_i-\mu_j+\frac{1}{2}(\sigma_i^2+\sigma_j^2)}(r_i r_j \sigma_i \sigma_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j e^{-\mu_i-\mu_j+\frac{1}{2}(\sigma_i^2+\sigma_j^2)} \left(\frac{\text{Cov}[Y(i),\Lambda]\text{Cov}[Y(j),\Lambda]}{\text{Var}(\Lambda)}\right) \\
&= \frac{(\text{Cov}(\sum_{i=1}^{n} \alpha_i e^{\mu_i+\frac{1}{2}\sigma_i^2} Y(i),\Lambda))^2}{\text{Var}(\Lambda)} \\
&= (\text{Corr}(\sum_{i=1}^{n} \alpha_i\ e^{\mu_i+\frac{1}{2}\sigma_i^2} Y(i),\Lambda))^2 \text{Var}(\sum_{i=1}^{n} \alpha_i e^{-\mu_i+\frac{1}{2}\sigma_i^2} Y(i)).
\end{aligned}$$

Choice of the conditioning variable: return component

b) $\Rightarrow$ based on the standardized logarithm of the geometric average $\mathbb{G} = (\prod_{i=1}^{n} \alpha_i e^{-Y(i)})^{1/n}$ (Nielsen and Sandman, 2002)

$$\Lambda = \frac{\ln \mathbb{G} - \mathrm{E}[\ln \mathbb{G}]}{\sqrt{\mathsf{Var}[\ln \mathbb{G}]}} = \frac{\sum_{i=1}^{n}(\mu_i - Y(i))}{\sqrt{\mathsf{Var}(\sum_{i=1}^{n} Y(i))}}$$

Discounting with Gaussian returns: lower bound DS^l

- $\boxed{\Lambda = \sum_{i=1}^n \beta_i Y(i)} \Rightarrow Y(i)|\Lambda = \lambda \sim N(\mu_{i,\lambda}, \sigma^2_{i,\lambda})$

  $\mu_{i,\lambda} = \mu_i + \frac{\text{Cov}(Y(i),\Lambda)}{\text{Var}(\Lambda)}(\lambda - \text{E}[\Lambda])$ and $\sigma^2_{i,\lambda} = \sigma^2_i - \frac{\text{Cov}(Y(i),\Lambda)^2}{\text{Var}[\Lambda]}$

$$\begin{aligned}
\Longrightarrow DS^l &= \sum_{i=1}^n \text{E}[X_i|\Gamma]\text{E}[e^{-Y(i)}|\Lambda] \\
&= \sum_{i=1}^n \text{E}[X_i|\Gamma]e^{-\mu_{i,\Lambda}+\frac{\sigma^2_{i,\Lambda}}{2}} \\
&= \sum_{i=1}^n \text{E}[X_i|\Gamma]e^{-\mu_i+\frac{1}{2}\sigma_i^2(1-r_i^2)-\sigma_i r_i \Phi^{-1}(U)},
\end{aligned}$$

with $U \sim$ Unif$(0,1)$

Discounting with Gaussian returns: lower bound DS^l

and correlations given by

$$r_i = \text{Corr}(Y(i), \Lambda) = \frac{\text{Cov}(Y(i), \Lambda)}{\sqrt{\text{Var}[Y(i)]}\sqrt{\text{Var}[\Lambda]}}.$$

- Note that when the β_i's and X_i's are non-negative, also the r_i's are non-negative and the r.v. DS^l is (given a value $\Gamma = \gamma$) the sum of the components of a comonotonic vector.

Discounting with Gaussian returns: CDF DS^l

1. The conditional quantiles (given $\Gamma = \gamma$) can be computed as

$$F^{-1}_{DS^l|\Gamma=\gamma}(p) = \sum_{i=1}^{n} \mathrm{E}[X_i|\Gamma = \gamma] e^{-\mu_i + \frac{1}{2}\sigma_i^2(1-r_i^2)+\sigma_i r_i \Phi^{-1}(p)};$$

2. The conditional distribution function is computed as the solution of

$$\sum_{i=1}^{n} \mathrm{E}[X_i|\Gamma = \gamma] e^{-\mu_i + \frac{1}{2}\sigma_i^2(1-r_i^2)+\sigma_i r_i \Phi^{-1}(F_{DS^l|\Gamma=\gamma}(y))} = y;$$

3. Finally, the cumulative distribution function of DS^l can be derived as

$$F_{DS^l}(y) = \int_0^1 F_{DS^l|\Gamma=F_\Gamma^{-1}(u)}(y) du.$$

Discounting with Gaussian returns: SL DS^l

$$\mathrm{E}\left[\left(\mathrm{E}[e^{-Y(i)}|\Lambda]-d_{\gamma,i}\right)_+\right]=e^{-\mu_i+\frac{1}{2}\sigma_i^2}\Phi\left(b_{\gamma,i}^{(1)}\right)-d_{\gamma,i}\Phi\left(b_{\gamma,i}^{(2)}\right),$$

with

$$d_{\gamma,i}=F^{-1}_{\mathrm{E}[e^{-Y(i)}|\Lambda]}\left(F_{DS^l|\Gamma=\gamma}(d)\right)=e^{-\mu_i+\frac{1}{2}\sigma_i^2(1-r_i^2)+\sigma_i r_i\Phi^{-1}(F_{DS^l|\Gamma=\gamma}(d))}$$

$$b_{\gamma,i}^{(1)}=\frac{-\mu_i+\frac{1}{2}\sigma_i^2(1-r_i^2)+\sigma_i^2 r_i^2-\ln(d_{\gamma,i})}{\sigma_i r_i},\qquad b_{\gamma,i}^{(2)}=b_{\gamma,i}^{(1)}-\sigma_i r_i$$

$$\begin{aligned}\mathrm{E}[S^l-d]_+ &= \int_0^1\sum_{i=1}^n \mathrm{E}[X_i|\Gamma=F_\Gamma^{-1}(u)]\mathrm{E}\left[\left(\mathrm{E}[e^{-Y(i)}|\Lambda]-d_{\gamma,i}\right)_+\right]du\\ &= \sum_{i=1}^n e^{-\mu_i+\frac{1}{2}\sigma_i^2}\int_0^1 \mathrm{E}[X_i|\Gamma=F_\Gamma^{-1}(u)]\\ &\quad\times\Phi\left(r_i\sigma_i-\Phi^{-1}(F_{DS^l|\Gamma=\gamma}(d))\right)du-d(1-F_{DS^l}(d))\end{aligned}$$

Discounting with Gaussian returns

Model	Variable	Formula
B-SM	$\mathrm{E}[Y(i)] = \mu_i$	$i\mu$
	$\mathrm{Var}[Y(i)] = \sigma_i^2$	$i\sigma^2$
	$\mathrm{Var}[\Lambda] = \sigma_\Lambda^2$	$\sum_{j=1}^n j\beta_j^2\sigma^2 + \sum_{1\leq j<k\leq n} 2j\beta_j\beta_k\sigma^2$
	$\mathrm{Cov}[Y(i), \Lambda]$	$\sum_{j=1}^n \min(i,j)\beta_j\sigma^2$
O-UM	$\mathrm{E}[Y(i)] = \mu_i$	$i\mu$
	$\mathrm{Var}[Y(i)] = \sigma_i^2$	$\frac{\sigma^2}{2a}(1 - e^{-2ia})$
	$\mathrm{Var}[\Lambda] = \sigma_\Lambda^2$	$\frac{\sigma^2}{2a}\Big(\sum_{j=1}^n \beta_j^2(1 - e^{-2ja}) +$
		$+ \sum_{1\leq j<k\leq n} 2\beta_j\beta_k(e^{-(k-j)a} - e^{-(j+k)a})\Big)$
	$\mathrm{Cov}[Y(i), \Lambda]$	$\frac{\sigma^2}{2a}\sum_{j=1}^n \beta_j(e^{-\lvert i-j\rvert a} - e^{-(i+j)a})$

Moments based approximations

How to calculate the variances of DS^c and DS^l?

In general: $$X = \sum_{i=1}^{n} f_i(U) g_i(V)$$

$-f_i$ and g_i : non-negative functions

$-U$ and V : independent standard uniform r.v.'s

- DS^c: $f_i(U) = F_{X_i}^{-1}(U)$ and $g_i(V) = F_{e^{-Y(i)}}^{-1}(V)$
- DS^l: $f_i(U) = \mathrm{E}\left[X_i | \Gamma\right]$ and $g_i(V) = \mathrm{E}\left[e^{-Y(i)} | \Lambda\right]$

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Moments based approximations

$$
\begin{aligned}
\mathrm{Var}[X] \\
&= \mathrm{E}[\mathrm{Var}[X|U]] + \mathrm{Var}[\mathrm{E}[X|U]] \\
&= \int_0^1 \mathrm{Var}_V\Big[\sum_{i=1}^n g_i(u) f_i(V)\Big]\, du \\
&+ \int_0^1 \Big(\mathrm{E}_V\Big[\sum_{i=1}^n g_i(u) f_i(V)\Big]\Big)^2 du - \Big(\int_0^1 \mathrm{E}_V\Big[\sum_{i=1}^n g_i(u) f_i(V)\Big]\, du\Big)^2 .
\end{aligned}
$$

Moments based approximations

$$\boxed{S = \sum_{i=1}^{n} \alpha_i g_i(V)}$$

for any vector of non-negative numbers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$

- The **upper** bound: $g_i(V) = e^{-\mu_i + \sigma_i \Phi^{-1}(V)}$

$$\Rightarrow \text{Var}[S^c] = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j e^{-\mu_i - \mu_j + \frac{\sigma_i^2 + \sigma_j^2}{2}} \left(e^{\sigma_i \sigma_j} - 1 \right)$$

- The **lower** bound: $g_i(V) = e^{-\mu_i + \frac{1}{2}\sigma_i^2 (1 - r_i^2) + \sigma_i r_i \Phi^{-1}(V)}$

$$\Rightarrow \text{Var}[S^l] = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j e^{-\mu_i - \mu_j + \frac{\sigma_i^2 + \sigma_j^2}{2}} \left(e^{r_i r_j \sigma_i \sigma_j} - 1 \right).$$

$$\text{E}[S] = \text{E}[S^c] = \text{E}[S^l] = \sum_{i=1}^{n} \alpha_i e^{-\mu_i + \frac{1}{2}\sigma_i^2}$$

Part I Applications

Life Annuities

Stochastic returns in life insurance

- Traditionally actuaries have used deterministic interest rates in life insurance;
- However the investment risk, unlike the insurance risk, cannot be diversified with an increase in the number of policies;
- In this approach conservative assumptions for the technical interest rate aim to protect against poor investments results in some periods;
- A risk-based approach however requires to take the random nature of returns into account;
- However, then there are no closed-form expressions for traditional actuarial functions;
- We show how to apply the comonotonicity theory to get very accurate approximations of typical present value functions in life annuity business.

Decrements

- *Life annuity:* a series of periodic payments where each payment will actually be made only if a designated life is alive at the time the payment is due

- *Notation:*
  - T: total lifetime with limiting age ω
  - T_x: future lifetime of (x) (a person aged x years)
    - $G_x(t) = \Pr[T_x \leq t] = {}_tq_x, \;\; t \geq 0$ ($G_x^{-1}(1) = \omega - x$)
    - $\overline{G}_x(t) = \Pr[T_x > t] = {}_tp_x, \;\; t \geq 0$
  - $K_x = \lfloor T_x \rfloor$: curtate future lifetime of (x)
    - $\Pr(K_x = k) = \Pr(k \leq T_x < k+1) = {}_{k+1}q_x - {}_kq_x = {}_{k|}q_x, \;\; k = 0, 1, \ldots$
  - $T_x^{(j)}$: future lifetime of the j-th insured (assumed to be mutually independent)

Three types of life annuities

- The present value of a **single whole life annuity immediate** paying α_i at the end of year i:

$$S_x^{policy} = \sum_{i=1}^{K_x} \alpha_i e^{-Y(i)} = \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i I_{(T_x > i)} e^{-Y(i)}$$

- The present value of a **homogeneous portfolio of** N_0 **whole life annuity contracts** paying at the end of year i a fixed amount α_i: (N_i: # survivals in year i)

$$\begin{aligned} S_x^{portfolio} &= \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i \left(I_{(T_x^{(1)} > i)} + \ldots + I_{(T_x^{(N_0)} > i)} \right) e^{-Y(i)} \\ &= \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i N_i e^{-Y(i)}, \end{aligned}$$

Three types of life annuities

- Consider a portfolio of N_0 homogeneous life annuity contracts. From the Law of Large Numbers for sufficiently large N_0:

$$\sum_{i=1}^{\lfloor \omega-x \rfloor} \alpha_i N_i e^{-Y(i)} = N_0 \left(\sum_{i=1}^{\lfloor \omega-x \rfloor} \alpha_i \frac{N_i}{N_0} e^{-Y(i)} \right) \approx N_0 \left(\sum_{i=1}^{\lfloor \omega-x \rfloor} \alpha_i \; {}_ip_x e^{-Y(i)} \right)$$

$\Longrightarrow$ in the case of large portfolios of life annuities it suffices to compute risk measures of an 'average' portfolio:

$$\boxed{\begin{aligned} S_x^{average} &= \sum_{i=1}^{\lfloor \omega-x \rfloor} \alpha_i \; {}_ip_x e^{-Y(i)} \\ &= \mathrm{E}[S_x^{policy} | Y(1), \cdots, Y(\lfloor \omega - x \rfloor)] \end{aligned}}$$

The Gompertz-Makeham law

- Force of mortality at age ξ:

$$\mu_\xi = \alpha + \beta c^\xi$$

-$\alpha > 0$: constant component $\rightarrow$ capturing accident hazard
-βc^ξ: variable component $\rightarrow$ capturing the hazard of aging
($\beta > 0$, $c > 1$)

- Survival probability:

$${}_tp_x = \Pr(T_x > t) = \exp\left(-\int_x^{x+t} \mu_\xi d\xi\right) = s^t g^{c^{x+t}-c^x},$$

where $s = \exp(-\alpha)$ and $g = \exp\left(-\frac{\beta}{\log c}\right)$

The Gompertz-Makeham law

- Denote by T'_x the future lifetime of (x) from the Gompertz family with force of mortality $\mu'_\xi = \beta c^\xi$
- $T_x \stackrel{d}{=} \min(T'_x, E/\alpha)$ and $E \sim \exp(1)$

$$\begin{aligned}
\Pr(\min(T'_x, E/\alpha) > t) &= \Pr(T'_x > t)\Pr(E > \alpha t) \\
&= \exp\left(-\int_x^{x+t} \mu'_\xi d\xi\right) e^{-\alpha t} \\
&= \exp\left(-\int_x^{x+t} \mu_\xi d\xi\right) \\
&= \Pr(T_x > t).
\end{aligned}$$

Simulation from Makeham's law :

(a) Generate G from the Gompertz's law by the inversion method

(b) Generate E from the exp(1) distribution

(c) Retain $T = \min(G, E/\alpha)$

Personal finance problem

- Suppose that (x) disposes a lump sum L.
  What is the amount that (x) can yearly consume to be almost sure (i.e. sure with a sufficiently high probability e.g. $p = 99\%$) that the money will not be run out before death?
  A solution to the latter problem is crucial to determine the fair value of future liabilities and the solvency margin.
- Notice that the presented methodology is appropriate not only in the case of large portfolios when the limiting distribution can be used on the basis of the law of large numbers but also for portfolios of average size (e.g. 1000-5000) which are typical for the life annuity business.

Single life annuity: CDF upper bound SLA_x^c

- $X_i = I_{(T_x > i)} \sim \text{Bern}({}_ip_x) \Rightarrow F_{X_i}^{-1}(p) = \begin{cases} 1 & \text{for } p > {}_iq_x \\ 0 & \text{for } p \leq {}_iq_x. \end{cases}$

$$\begin{aligned} SLA_x^c &= \sum_{i=1}^{\infty} F_{X_i}^{-1}(U) F_{\alpha_i e^{-Y(i)}}^{-1}(V) = \sum_{i=1}^{\lfloor F_{T_x}^{-1}(U) \rfloor} F_{\alpha_i e^{-Y(i)}}^{-1}(V) \\ F_{SLA_x^c}(y) &= \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x F_{SLA_x^c | K_x = k}(y) \end{aligned}$$

-conditional quantiles: $F_{SLA_x^c | K_x = k}^{-1}(p) = \sum_{i=1}^{k} \alpha_i e^{-\mu_i + \text{sign}(\alpha_i)\sigma_i \Phi^{-1}(p)}$

-conditional df: $\sum_{i=1}^{k} \alpha_i \exp\left(-\mu_i + \text{sign}(\alpha_i)\sigma_i \Phi^{-1}(F_{SLA_x^c | K_x = k}(y))\right) = y$

Single life annuity: SL upper bound SLA_x^c

$$\begin{aligned}\mathrm{E}[(SLA_x^c - d)_+] &= \mathrm{E}_{K_x}\left[\mathrm{E}\left[(SLA_x^c - d)_+ \,|K_x\right]\right] \\ &= \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \left(\sum_{i=1}^{k} \mathrm{E}[(\alpha_i e^{-Y(i)} - d_{k,i})_+] \right),\end{aligned}$$

with $d_{k,i} = \alpha_i \exp\left(-\mu_i + \mathrm{sign}(\alpha_i)\sigma_i \Phi^{-1}(F_{\tilde{S}_k^c}(d))\right)$

$$\boxed{\begin{aligned}&\mathrm{E}[(SLA_x^c - d)_+] = \\ &\sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \sum_{i=1}^{k} \alpha_i \; e^{-\mu_i + \frac{\sigma_i^2}{2}} \Phi \left[\mathrm{sign}(\alpha_i)\sigma_i - \Phi^{-1}(F_{\tilde{S}_k^c}(d))\right] - d\left(1 - F_{\tilde{S}_k^c}(d)\right)\end{aligned}}$$

$(SLA_x^c|_{K_x=k} \overset{not}{=} \tilde{S}_k^c)$

Single life annuity: CV lower bound SLA_x^l

- $\Gamma = T_x \Rightarrow \mathrm{E}[I_{(T_x>i)}|T_x] = I_{(T_x>i)}$
- Λ?

a) $\Lambda^{(a)} = \sum_{i=1}^{\lfloor\omega-x\rfloor} \alpha_i \ {}_ip_x e^{-\mu_i+\frac{1}{2}\sigma_i^2} Y(i) \rightarrow$ first order approximation to the PV of the limiting portfolio

b) $\Lambda^{(M)} := \Lambda_{j_0}$ with

$$j_0 = \arg\max_j \{\text{Var}(SLA_x^{l,j}), j = 1, \ldots, \lfloor\omega - x\rfloor\}$$

-$SLA_x^{l,j} = \sum_{i=1}^{K_x} \mathrm{E}[\alpha_i e^{-Y(i)}|\Lambda_j]$

-$\Lambda_j = \sum_{i=1}^{j} \alpha_i e^{-\mu_i+\frac{1}{2}\sigma_i^2} Y(i)$

$$SLA_x^l|_{K_x=k} = \sum_{i=1}^{k} \alpha_i e^{-\mu_i+\frac{1}{2}\sigma_i^2(1-r_i^2)-\sigma_i r_i \Phi^{-1}(V)}$$

Single life annuity: CDF lower bound SLA_x^l

$$SLA_x^l = \sum_{i=1}^{K_x} \mathrm{E}[\alpha_i e^{-Y(i)}|\Lambda]$$

$$F_{SLA_x^l}(y) = \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x F_{SLA_x^l|K_x=k}(y)$$

-conditional quantiles: $F^{-1}_{SLA_x^l|K_x=k}(p) = \sum_{i=1}^{k} \alpha_i e^{-\mu_i + \frac{1}{2}\sigma_i^2(1-r_i^2)+\sigma_i r_i \Phi^{-1}(p)}$

-conditional df: $\sum_{i=1}^{k} \alpha_i \exp\left(-\mu_i + \frac{1}{2}\sigma_i^2(1-r_i^2) + \sigma_i r_i \Phi^{-1}(F_{SLA_x^l|K_x=k}(y))\right) = y$

Single life annuity: SL lower bound SLA_x^l

$$\begin{aligned} \mathrm{E}[(SLA_x^l - d)_+] &= \mathrm{E}_{K_x}\left[\mathrm{E}\left[\left(SLA_x^l - d\right)_+ |K_x\right]\right] \\ &= \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \left(\sum_{i=1}^{k} \mathrm{E}\left[\left(\mathrm{E}[\alpha_i e^{-Y(i)}|\Lambda] - d_{k,i}\right)_+\right]\right) \end{aligned}$$

with $d_{k,i} = \alpha_i \exp\left(-\mu_i + \frac{1}{2}\sigma_i^2(1 - r_i^2) + \sigma_i r_i \Phi^{-1}(F_{SLA_x^l|K_x=k}(d))\right)$

$$\mathrm{E}[(SLA_x^l - d)_+] =$$

$$\sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \sum_{i=1}^{k} \alpha_i \, e^{-\mu_i + \frac{\sigma_i^2}{2}} \Phi\left[r_i\sigma_i - \Phi^{-1}\left(F_{\tilde{S}_k^l}(d)\right)\right] - d\left(1 - F_{\tilde{S}_k^l}(d)\right)$$

$(SLA_x^l|_{K_x=k} \stackrel{not}{=} \tilde{S}_k^l)$

Single life annuity: an alternative approximation

- Take as conditioning variable:

$$\Lambda_{K_x} = \sum_{i=1}^{K_x} \alpha_i e^{-\mu_i + \frac{1}{2}\sigma_i^2} Y(i)$$

- The lower bound is then given by

$$\sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \sum_{i=1}^{k} \alpha_i e^{-\mu_i + \frac{1}{2}\sigma_i^2 (1 - r_{i,k}^2) - \sigma_i r_{i,k} \Phi^{-1}(U_k)},$$

with

  - correlations: $r_{i,k} = \frac{\text{Cov}(Y(i), \Lambda_k)}{\sqrt{\text{Var}[Y(i)]}\sqrt{\text{Var}[\Lambda_k]}}$
  - $\{U_k\}_{k=1,\ldots,\lfloor \omega - x \rfloor} \sim$ Unif$(0,1)$ $\Rightarrow$ multidimensional lower bound

Single life annuity: an alternative approximation

A new approximation based upon this lower bound:

- $$SLA_x^{cl} = \sum_{k=1}^{\lfloor \omega - x \rfloor} {}_{k|}q_x \sum_{i=1}^{k} \alpha_i e^{-\mu_i + \frac{1}{2}\sigma_i^2(1-r_{i,k}^2) - \sigma_i r_{i,k} \Phi^{-1}(U)}$$

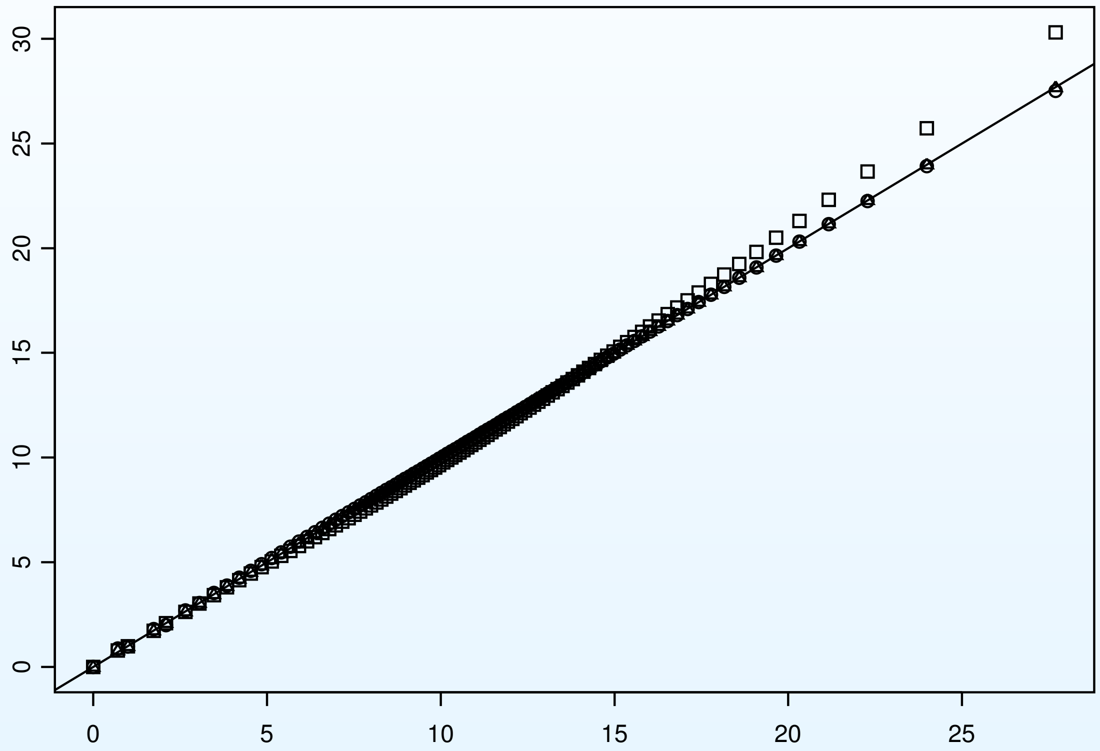
- The "comonotonic upper bound of the lower bound"
- $SLA_x^{cl} \not\leq_{cx} SLA_x$

A numerical illustration: Quantiles

-Return process: Black & Scholes model $\mu = 0.05, \sigma = 0.1$
-Mortality process: Makeham's model, 65 years, males with coefficients Belgian analytical life table MR:
$(\text{m} : a = 1000266.63, s = 0.999441703848, g = 0.999733441115, c = 1.101077536030)$
-Monte-Carlo (MC) simulation: $500 \times 100\,000$ paths
-Payments: $\alpha_i = 1 \; \forall i$

p	SLA^{l}_{65}	SLA^{cl}_{65}	SLA^{c}_{65}	MC (s.e. $\times 10^3$)
0.995	27.5124	27.6700	30.2983	27.6933 (6.324)
0.975	22.2495	22.2875	23.6574	22.2839 (2.816)
0.95	19.9565	19.9713	20.8754	19.9731 (1.896)
0.90	17.5905	17.5972	18.0797	17.5969 (1.420)
0.75	14.1741	14.1887	14.1867	14.1887 (0.978)

A numerical illustration: QQ-plot



QQ-plot of the quantiles of SLA^{l}_{65} ($\circ$), SLA^{cl}_{65} ($\triangle$) and SLA^{c}_{65} ($\Box$) versus those of 'SLA_{65}' (MC).

A numerical illustration: Stop-loss premia

d	SLA^{l}_{65}	SLA^{cl}_{65}	SLA^{c}_{65}	MC (s.e. $\times 10^4$)
0	11.0944	11.0944	11.0944	11.0937 (9.43)
5	6.3715	6.3756	6.3792	6.3748 (8.67)
10	2.5956	2.6071	2.6900	2.6068 (5.89)
15	0.7151	0.7201	0.8629	0.7201 (0.34)
20	0.1628	0.1664	0.2536	0.1668 (0.21)
25	0.0357	0.0379	0.0758	0.0382 (0.10)
30	0.0080	0.0091	0.0239	0.0093 (0.02)
35	0.0019	0.0023	0.0081	0.0024 (0.004)

Homogeneous portfolio of life annuities

$$
\begin{aligned}
PLA_x &= \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i \left(I_{\left(T_x^{(1)} > i\right)} + \ldots + I_{\left(T_x^{(N_0)} > i\right)} \right) e^{-Y(i)} \\
&= \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i N_i e^{-Y(i)}, \\
&= \sum_{j=1}^{N_0} SLA_x^{(j)} \\
&= \sum_{j=1}^{N_0} \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i I_{\left(T_x^{(j)} > i\right)} e^{-Y(i)}
\end{aligned}
$$

$N_i \sim$ binomial$(N_0, {}_ip_x) \Rightarrow$ difficult to deal with !

$\longrightarrow$ **Normal Power Approximation (NPA)**

Homogeneous portfolio of life annuities: NPA

Approximate the distribution of N_i by the NPA $\tilde{N}_i$

$$F_{\tilde{N}_i}(x) = \Phi\left(-\frac{3}{\gamma_{N_i}} + \sqrt{\frac{9}{\gamma_{N_i}^2} + \frac{6(x-\mu_{N_i})}{\gamma_{N_i}\sigma_{N_i}} + 1}\right)$$

with

$$\begin{aligned}
\mu_{N_i} &= \mathrm{E}[N_i] = N_0\ {}_ip_x \\
\sigma_{N_i}^2 &= \mathrm{Var}[N_i] = N_0\ {}_ip_x\,{}_iq_x \\
\gamma_{N_i} &= \frac{\mathrm{E}[N_i-\mu_{N_i}]^3}{\sigma_{N_i}^3} = \frac{1-2\,{}_ip_x}{\sqrt{N_0\ {}_ip_x\,{}_iq_x}}
\end{aligned}$$

The p-th quantile of $\tilde{N}_i$:

$$F_{\tilde{N}_i}^{-1}(p) = \mu_{N_i} + \sigma_{N_i}\Phi^{-1}(p) + \frac{\gamma_{N_i}\sigma_{N_i}}{6}\left((\Phi^{-1}(p))^2 - 1\right)$$

homogeneous portfolio of life annuities: convex bounds

- The upper bound is straightforward, from

$$PLA_x^c|_{U=u} = \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i F_{\tilde{N}_i}^{-1}(u) e^{-\mu_i + \text{sign}(\alpha_i)\sigma_i \Phi^{-1}(V)}$$

- Conditioning variables of the lower bound
  - $\Gamma = N_{i_0} \rightarrow$ the number of policies-in-force in the year i_0

$$\mathrm{E}[N_i | N_{i_0} = n_0] = {}_{i-i_0}p_{x+i_0} n_0 \quad \text{for } i \geq i_0$$

$$\mathrm{E}[N_i | N_{i_0} = n_0] \overset{(Bayes)}{=} \sum_{k=n_0}^{N_0} k \frac{\Pr(N_{i_0} = n_0 | N_i = k)\Pr(N_i = k)}{\Pr(N_{i_0} = n_0)}$$

$$= \sum_{k=n_0}^{N_0} k \binom{N_0 - n_0}{k - n_0} {}_ip_x^{k-n_0} \frac{{}_{i_0-i}q_{x+i}^{k-n_0} \, {}_iq_x^{N_0-k}}{{}_{i_0}q_x^{N_0-n_0}} \quad \text{for } i < i_0$$

homogeneous portfolio of life annuities: convex bounds

- Conditioning variables of the lower bound
  - Γ : take for simplicity $\Gamma = N_1 \Rightarrow \mathrm{E}[N_i|N_1] = {}_{i-1}p_{x+1}N_1$
  - $\Lambda = \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i \, {}_ip_x e^{-\mu_i + \frac{1}{2}\sigma_i^2} Y(i)$
- The lower bound is then straightforward, from

$$PLA_x^l|_{U=u} = \sum_{i=1}^{\lfloor \omega - x \rfloor} \alpha_i \, {}_{i-1}p_{x+1} F_{\tilde{N}_1}^{-1}(u) e^{-\mu_i + \frac{1}{2}\sigma_i^2(1-r_i^2) - \sigma_i r_i \Phi^{-1}(V)}$$

- Moments based approximation PLA_x^m

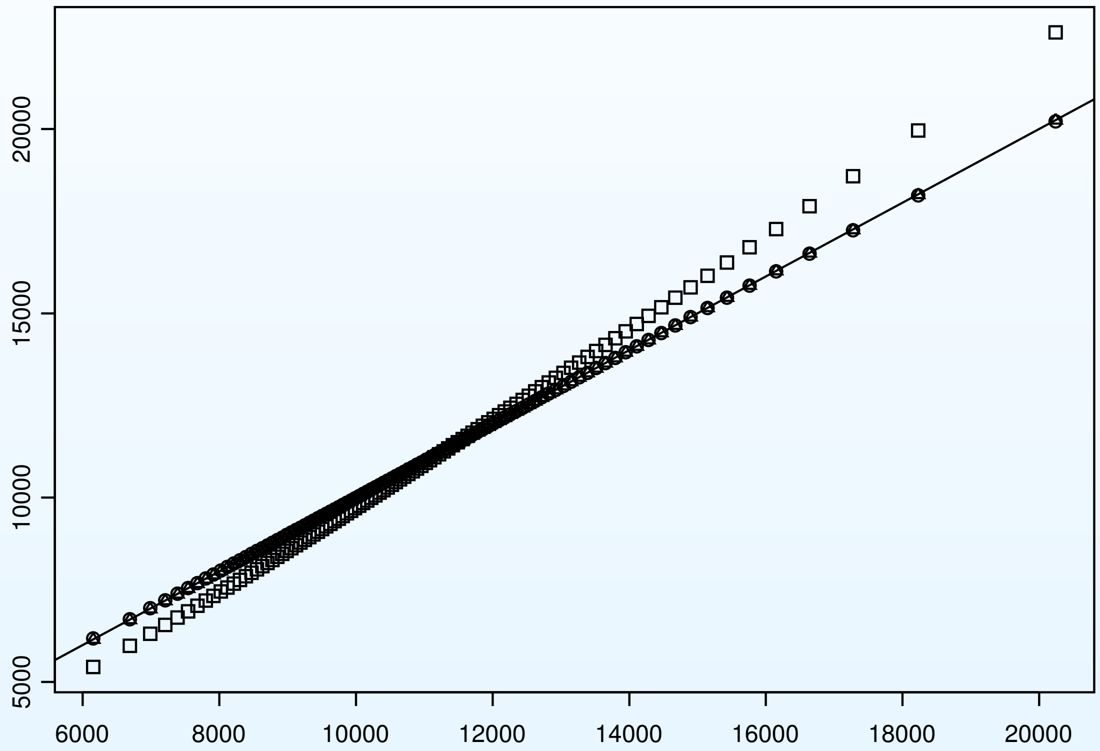
$$\begin{aligned} \mathrm{Var}[PLA_x] &= \mathrm{E}[\mathrm{Var}[PLA_x|\vec{Y}]] + \mathrm{Var}[\mathrm{E}[PLA_x|\vec{Y}]] \\ &= N_0 \mathrm{E}[\mathrm{Var}[SLA_x|\vec{Y}]] + N_0^2 \mathrm{Var}[\mathrm{E}[SLA_x|\vec{Y}]] \\ &= N_0 \mathrm{Var}[SLA_x] + (N_0^2 - N_0) \mathrm{Var}[\mathrm{E}[SLA_x|\vec{Y}]] \end{aligned}$$

A numerical illustration: Quantiles

-Return process: Black & Scholes model $\mu = 0.05, \sigma = 0.1$
-Mortality process: Makeham's model MR, 65 years, males
-Portfolio: 1000 policies
-Payments: $\alpha_i = 1 \ \forall i$

p	PLA_{65}^{l}	PLA_{65}^{m}	PLA_{65}^{c}	MC (s.e.)
0.995	20209	20250	22620	20242 (22.09)
0.975	17252	17272	18722	17276 (8.80)
0.95	15937	15951	17029	15947 (8.15)
0.90	14565	14574	15290	14568 (5.08)
0.75	12574	12577	12821	12577 (3.90)

A numerical illustration: QQ-plot



QQ-plot of the quantiles of PLA^l_{65} ($\circ$), PLA^m_{65} ($\triangle$) and PLA^c_{65} ($\square$) versus those of 'PLA_{65}' (MC).

A numerical illustration: Stop-loss premia

d	PLA^l_{65}	PLA^m_{65}	PLA^c_{65}	MC (s.e.)
0	11094	11094	11094	11098 (2.11)
5000	6094	6094	6095	6098 (2.10)
10000	1608	1610	1793	1611 (1.95)
15000	153.7	155.3	278.4	155.3 (1.78)
20000	10.23	10.57	36.02	10.67 (1.26)
25000	0.680	0.734	4.816	0.743 (0.09)
30000	0.051	0.059	0.711	0.036 (0.02)

Part II Applications

Loss Reserving

Loss Reserving: general framework

- **Stochastic** liability payments: $L_i \geq 0$ at times $i = 1, 2, \ldots, n$ (modified by certain forces that influence the liability over time)
- $L_i^t = L_i^{t-1} R_{Lt}, \qquad t = 1, \ldots, i$
  - L_i^t : amount of liability expressed in money values of time t
  - $R_{Lt} = 1 + r_{Lt}$
  - r_{Lt} : inflation of claim costs over interval $(t-1, t]$
- $A^t = A^{t-1} R_{At}$
  - A^t : holding of assets of value A^t at time t
  - $R_{At} = 1 + r_{At}$
- Assume R_{Xt} $(X = A, L)$ follows **CAPM**:

$$\boxed{r_{Xt} = r_{Ft} + \beta_X \Delta_t + \epsilon_{Xt}}$$

Loss Reserving: general framework

- $\boxed{r_{Xt} = r_{Ft} + \beta_X \Delta_t + \epsilon_{Xt}}$
  - $\Delta_t = r_{Mt} - r_{Ft}$ (distribution independent of t)
  - r_{Ft}: risk-free rate in period t
  - r_{Mt}: periodic increase in value of the economy wide portfolio of assets
  - β_X: CAPM beta associated with X
  - $\epsilon_{Xt} \sim$ i.i.d. and $\mathrm{E}[\epsilon_{Xt}] = 0$ and Var$(\epsilon_{Xt}) := \omega_X^2$
  - $\epsilon_{At}, \epsilon_{Lt}, \Delta_t$ independent
- Assume $R_{Xt} \sim$ i.i.d logN(μ_X, σ_X^2) and $L_s^0 \sim$ logN(ν_{0s}, τ_{0s}^2)
- L_s^0 and R_{Xt} independent $\forall s, t, X$
- $\rho =$ Corr$(\log R_{At}, \log R_{Lt})$ and $\kappa^{(rs)} =$ Corr$(\log L_r^0, \log L_s^0)$
- $\bar{R}_X = \mathrm{E}[R_{Xt}] =$ exp$(\mu_X + \frac{1}{2}\sigma_X^2)$

Discounted loss reserve

Discounted loss reserve:

$$\begin{aligned} V &= \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} L_t^t R_A^{-1}(t) \\ &= \sum_{i=1}^{n} L_t^0 R_L(t) R_A^{-1}(t) \end{aligned}$$

- $R_X(i) = R_{X1} + \cdots + R_{Xi} \sim \text{logN}(i\mu_X, i\sigma_X^2)$
  $\Longrightarrow V_i \sim \text{logN}(\alpha^{(i)}, \delta^{2(i)})$
  - $\alpha^{(i)} = \nu_{0i} + i(\mu_L - \mu_A)$
  - $\delta^{2(i)} = \tau_{0i}^2 + i(\sigma_L^2 + \sigma_A^2 - 2\rho\sigma_L\sigma_A)$

Loss Reserving: general framework

Three relevant values of the loss reserve:

- $\sum_{i=1}^{n} \mathrm{E}[L_i^0]$: CAPM-based economic value of the liability
- $\mathrm{E}[V]$: expected value of the discounted liability cash-flows
- $F_V^{-1}(p)$: $100p\%$ confidence loss reserve

Convex bounds discounted loss reserve

$$\boxed{V = \sum_{i=1}^{n} V_i \overset{not}{:=} \sum_{i=1}^{n} e^{Z_i}}$$

$$V^l := \sum_{i=1}^{n} \mathrm{E}[V_i|\Lambda] \leq_{cx} V \leq_{cx} V^c := \sum_{i=1}^{n} F_{V_i}^{-1}(U)$$

$$Q_p[V^l] = \sum_{i=1}^{n} e^{\mathrm{E}[Z_i]+\frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2+r_i\sigma_{Z_i}\Phi^{-1}(p)}, \qquad p \in (0,1)$$

$$Q_p[V^c] = \sum_{i=1}^{n} e^{E[Z_i]+\sigma_{Z_i}\Phi^{-1}(p)}, \qquad p \in (0,1)$$

$$\mathrm{E}[V] = \mathrm{E}[V^l] = \mathrm{E}[V^c] = \sum_{i=1}^{n} e^{\mathrm{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2}$$

Convex bounds discounted loss reserve

$\Lambda = \sum_{i=1}^{n} \beta_i Z_i$ with $\beta_i = \exp(\mathrm{E}[Z_i] + \frac{1}{2}\sigma^2_{Z_i})$

$$\begin{aligned}
\mathrm{E}[Z_i] &= \nu_{0i} + \log\left\{\left(\frac{\bar{R}_L}{\bar{R}_A}\left(\frac{1+(\beta_A^2\sigma_M^2+\omega_A^2)/\bar{R}_A^2}{1+(\beta_L^2\sigma_M^2+\omega_L^2)/\bar{R}_L^2}\right)^{1/2}\right)^i\right\} \\
\mathrm{Var}(Z_i) &= \sigma^2_{Z_i} = \tau^2_{0i} + i\hat{\sigma}^2
\end{aligned}$$

The variability of the discounting structure

$\hat{\sigma}^2 \overset{not}{=} \sigma_L^2 + \sigma_A^2 - 2\rho\sigma_L\sigma_A$ is given by

$$\log\left\{\frac{[1+(\beta_A^2\sigma_M^2+\omega_A^2)/\bar{R}_A^2][1+(\beta_L^2\sigma_M^2+\omega_L^2)/\bar{R}_L^2]}{[1+\beta_A\beta_L\sigma_M^2/\bar{R}_A\bar{R}_L]^2}\right\}$$

Convex bounds discounted loss reserve

The correlation between Z_i and Λ is given by

$$r_i = \frac{\text{Cov}(Z_i, \Lambda)}{\sigma_{Z_s}\sigma_\Lambda} = \frac{\sum_{k=1}^{n} \beta_k \left(\hat{\sigma}^2 \min(i,k) + \eta^{(i,k)}\right)}{\sigma_{Z_i}\sqrt{\sum_{k=1}^{n}\sum_{l=1}^{n} \beta_k\beta_l(\hat{\sigma}^2 \min(k,l) + \eta^{(k,l)})}}$$

with

$$\eta^{(k,s)} = \text{Cov}\left(\log L_k^0, \log L_s^0\right) = \kappa^{(ks)}\tau_{0k}\tau_{0s}$$

Note that if the liability cash-flows are independent $\eta^{(k,s)} = \tau_{0k}^2 I_{(k=s)}$ and $I_{(.)}$ the indicator function.

Moment matching vs. convex bounds

Security margin for confidence level p (Taylor, 2004):

$$SM_p[V] \stackrel{not}{=} (Q_p[V]/\mathrm{E}[V]) - 1$$

LB $\stackrel{not}{=} \dfrac{SM_p[V^l] - SM_p[V^{MC}]}{SM_p[V^{MC}]} \times 100\%$ and LN $\stackrel{not}{=} \dfrac{SM_p[V^{LN}] - SM_p[V^{MC}]}{SM_p[V^{MC}]} \times 100\%$,

(MC: Monte Carlo simulation - LN: lognormal moment matching)

Stochastic liability cash-flow structure: $(n = 30)$

-$\nu_{0i} = -4.46$ for $i = 1, \ldots, 30$

$$\tau_{0i} = \begin{cases} 5\% & s \leq 5; \quad 10\% \quad 5 < i \leq 15; \quad 15\% \quad 15 < i \leq 25 \\ 20\% & 25 < i \leq 28; \quad 25\% \quad 28 < i \leq 30 \quad . \end{cases}$$

- $\sum_{i=1}^{30} \mathrm{E}[L_i] = 100\%$ and $\mathrm{E}[L_i^0] = 35.51\%$
- $\omega_L = 10\%$ and $\omega_A = 5\%$

Moment matching vs. convex bounds

$p = 0.975$	$\sigma_M = 0.05$	$\sigma_M = 0.10$	$\sigma_M = 0.15$	$\sigma_M = 0.20$	$\sigma_M = 0.25$	$\sigma_M = 0.30$
LB	$-\mathbf{0.19}\%$	$-\mathbf{0.15}\%$	$-\mathbf{0.23}\%$	$-\mathbf{0.16}\%$	$-\mathbf{0.11}\%$	$-\mathbf{0.17}\%$
LN	-4.94%	-3.92%	-3.17%	-2.49%	-1.95%	-1.56%
MC	0.4390	0.5250	0.6528	0.8103	0.9924	1.1970
s.e.$(\times 10^5)$	(0.15)	(0.29)	(0.41)	(0.69)	(1.22)	(3.78)

$\sigma_M = 0.25$	$p = 0.995$	$p = 0.975$	$p = 0.95$	$p = 0.90$	$p = 0.80$	$p = 0.70$	$p = 0.60$
LB	$-\mathbf{0.93}\%$	$-\mathbf{0.04}\%$	$-\mathbf{0.02}\%$	$-\mathbf{0.18}\%$	$-\mathbf{0.03}\%$	$-\mathbf{0.6}\%$	$+\mathbf{0.86}\%$
LN	-3.94%	$+3.78\%$	$+7.22\%$	$+11.29\%$	$+19.68\%$	$+53.46\%$	-15.50%
MC	4.4521	2.2264	1.4998	0.8814	0.3508	0.0761	-0.1069
s.e.$(\times 10^5)$	(37.63)	(2.99)	(7.44)	(2.79)	(0.78)	(0.27)	(0.08)

Loss Reserving: overview

3 categories of reserves in non-life:

1. Reserves with respect to unexpired or unearned exposure
   - Unearned Premium Reserve (UPR)
   - Additional Unexpired Risk Reserve (AURR): correction on UPR if loss ratio higher than expected
2. Catastrophe Reserves
   (Also ‘claims equalisation reserves’; ‘adverse deviation reserves’, ‘fluctuation reserves’, ...)
   $\hookrightarrow$ To smooth the influence of perils such as hurricanes, floods, earthquakes, ... on the result
3. Reserves with respect to earned exposures (loss reserves)
   - Outstanding claims reserves (‘also case reserves’): for reported losses that are not yet settled
   - IBNR: Incurred But Not Reported

IBNR reserves

The settlement of claims is always subject to delay: as well in claim settlement as in claim reporting.

- Outstanding Claims Reserves (delay in settlement)
  - lengths of delays vary according to the class of business (short / long tail)
  - regulation in general demands the use of individual estimates with respect to all known outstanding claims at the accounting date and hardly tolerates the use of over-all statistical methods
  - a ‘case reserve’ reflects the expected ultimate settlement value of a claim as established by the claims handling staff
- IBNR (delay in reporting)
  - requires a statistical treatment based on past experience and expected trends

IBNR reserves

For an insurance company, the ability to estimate its loss reserves correctly is of great importance:

- a correct view of the liabilities on the balance sheet
- premium calculation
- solvency
- ...

$\Longrightarrow$ Actuarial loss reserving methods (also ‘IBNR techniques’): to estimate the loss reserves statistically on aggregated data

IBNR reserves

- Traditionally: claims are aggregated and displayed in a run-off triangle
- Using a triangle simply avoids us having to introduce complicated notation to cope with all possible situations
- We assume that we have the following set of incremental claims data $\{Y_{ij} : i = 1, \ldots, t; j = 1, \ldots, s - i + 1\}$
- Most claims reserving methods usually assume that $t = s$
- We consider annual development and assume that the time it takes for the claims to be completely paid is fixed and known

Run-off triangle

Accident year	*Development year* 1	2	$\cdots$	j	$\cdots$	$t-1$	t
1	Y_{11}	Y_{12}	$\cdots$	Y_{1j}	$\cdots$	$Y_{1,t-1}$	Y_{1t}
2	Y_{21}	Y_{22}	$\cdots$	Y_{2j}	$\cdots$	$Y_{2,t-1}$	
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		
i	Y_{i1}	$\cdots$	$\cdots$	Y_{ij}			
$\vdots$	$\cdots$	$\cdots$	$\cdots$				
t	Y_{t1}						

Three directions

Fundamental influences (exogenous factors) in the direction of:

- Accident Year
  - changes in underwriting conditions (premium / coverage)
  - changes in the size of the portfolio
- Development Year
  - development pattern characteristics for short tail / long tail business
  - changes in the claim handling procedures changes in the finalization of the claims
- Calendar Year
  - monetary inflation
  - changes in jurisprudence

Remark: Accident years and development years mostly assumed to be independent; calendar year trends operate on both development years and accident years

Lognormal models

$$\begin{aligned} Z_{ij} &= \log Y_{ij} = \eta_{ij} + \epsilon_{ij} \qquad \eta_{ij} = (\mathbf{X}\vec{\beta})_{ij} \\ \epsilon_{ij} &\sim i.i.d\; N(0, \sigma^2) \end{aligned}$$

1. Transform the incremental claims by taking their logarithm
2. Fit a model to the transformed values using ordinary LS-regression analysis
3. Obtain estimates for the parameters in the linear predictor and the process variance
4. Fitted values (on a log scale) are obtained by forming the appropriate sum of estimates
5. Fitted values (on an untransformed scale) are **NOT** given by ~~$\hat{Y}_{ij} = \exp(\hat{\eta_{ij}})$~~
   $\hookrightarrow$ This gives an estimate of the median!

Lognormal models

$$\begin{array}{rcl} Z_{ij} & = & \log Y_{ij} = \eta_{ij} + \epsilon_{ij} \qquad \eta_{ij} = (\mathbf{X}\vec{\beta})_{ij} \\ \epsilon_{ij} & \sim & i.i.d\; N(0, \sigma^2) \end{array}$$

1. Transform the incremental claims by taking their logarithm
2. Fit a model to the transformed values using ordinary LS-regression analysis
3. Obtain estimates for the parameters in the linear predictor and the process variance
4. Fitted values (on a log scale) are obtained by forming the appropriate sum of estimates
5. Fitted values (on an untransformed scale) are given by $\hat{Y}_{ij} = \exp(\hat{\eta_{ij}} + \frac{1}{2}\hat{\sigma}^2_{ij})$ with $\hat{\sigma}^2_{ij} = \hat{\sigma}^2 \left(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right)_{ij} + \hat{\sigma}^2$
   $\mathbf{X}/\mathbf{R}$: design matrix corresponding to the upper triangle/square

Linear predictors

Examples

- Chain-ladder model

$$\eta_{ij} = \alpha_i + \beta_j, \qquad i + j \leq t + 1$$

- PTF

$$\eta_{ij} = \alpha_i + \sum_{k=1}^{j-1} \beta_k + \sum_{t=1}^{i+j-2} \gamma_t, \qquad i + j \leq t + 1$$

- Hoerl curve

$$\eta_{ij} = \alpha_i + \beta_i \text{log}(j) + \gamma_i j, \quad (j > 0) \qquad i + j \leq t + 1$$

Statistical analysis

- Check the model assumptions!
  - Gauss-Markov conditions of a regression model
  - Normality for inference
- Goodness-of-Fit
  - (Adjusted) coefficient of determination and AIC/BIC
  - Residual plots
  - Plot of the observed values vs. the fitted values
- Estimation of the parameters by maximum likelihood methods
  - $\hat{\sigma}^2 = \frac{1}{n}(\vec{Z} - \mathbf{X}\hat{\vec{\beta}})'(\vec{Z} - \mathbf{X}\hat{\vec{\beta}})$
  - $\hat{\vec{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\vec{Z}$

  Remark: $\tilde{\sigma}^2 = \frac{1}{n-p}(\vec{Z} - \mathbf{X}\hat{\vec{\beta}})'(\vec{Z} - \mathbf{X}\hat{\vec{\beta}}) \rightarrow$ unbiased estimator of σ^2

Lognormal models

- The mean of the IBNR reserve equals

$$W = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\vec{\beta})_{ij} + \frac{1}{2}\sigma^2\left(1+(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')_{ij}\right)}$$

- The unique UMVUE of the mean of the IBNR reserve is given by

$$\hat{W}_U = {}_0F_1\left(\frac{n-p}{2}; \frac{SS_z}{4}\right) \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\hat{\vec{\beta}})_{ij}},$$

  where ${}_0F_1(\alpha; z)$ denotes the hypergeometric function.

- The MLE of the mean of the IBNR reserve:

$$\hat{W}_M = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\hat{\vec{\beta}})_{ij} + \frac{1}{2}\hat{\sigma}^2\left(1+(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')_{ij}\right)}$$

Lognormal models

- Verrall (1991) has considered an estimator similar to $\hat{W}_M$, but with $\hat{\sigma}^2$ replaced with $\tilde{\sigma}^2$:

$$\hat{W}_V = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\hat{\vec{\beta}})_{ij} + \frac{1}{2}\tilde{\sigma}^2\left(1+(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')_{ij}\right)}$$

- Doray (1996) has considered the following simple estimator estimator

$$\hat{W}_D = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\hat{\vec{\beta}})_{ij} + \frac{1}{2}\tilde{\sigma}^2}$$

$\Rightarrow$ Now we have the order relation

$$\hat{W}_U < \hat{W}_D < \hat{W}_V,$$

which implies that $W = \mathrm{E}[\hat{W}_U] < \mathrm{E}[\hat{W}_D] < \mathrm{E}[\hat{W}_V]$

Generalized Linear Models

1. **Random component**

$$f(y_{ij};\theta_{ij},\phi) = \exp\left\{\left[y_{ij}\theta_{ij} - b(\theta_{ij})\right]/a(\phi) + c(y_{ij},\phi)\right\}$$

   - $f(.)$ belongs to the exponential family
   - $a(.)$, $b(.)$ en $c(.,.)$ are known functions: $a(\phi) = \phi/w_{ij}$
   - $\mathrm{E}[Y_{ij}] = \mu_{ij} = b'(\theta_{ij})$ and $\mathrm{Var}[Y_{ij}] = b''(\theta_{ij})a(\phi) = V(\mu_{ij})a(\phi)$

2. **Systematic Component**

$$\eta_{ij} = (\mathbf{X}\vec{\beta})_{ij} = \beta_1 R_{ij,1} + \cdots + \beta_p R_{ij,p}, \qquad i,j = 1,\ldots,t$$

3. **Link function**

$$\eta_{ij} = g(\mu_{ij})$$

g is a monotone, differentiable function

Generalized Linear Models: link function

- **Canonical link** $\rightarrow$ when $g(\mu_{ij}) = \theta_{ij}$
  $\hookrightarrow$ sufficient statistic in $\vec{\eta}$ (when $\vec{\eta} = \vec{\theta}$) given by $\mathbf{R}'\vec{Y}$
- Logarithmic link $\rightarrow$ multiplicative parametric structure + positive fitted values

Distribution	Density	ϕ	Canonical link $\theta(\mu)$	Mean function $\mu(\theta)$	Variance function $V(\mu)$
N(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}}\text{exp}\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$	σ^2	μ	θ	1
Poisson(μ)	$e^{-\mu}\frac{\mu^y}{y!}$	1	log(μ)	e^θ	μ
Gamma(μ, ν)	$\frac{1}{\Gamma(\nu)}\left(\frac{\nu y}{\mu}\right)^\nu \text{exp}\left(-\frac{\nu y}{\mu}\right)\frac{1}{y}$	$\frac{1}{\nu}$	$1/\mu$	$-1/\theta$	μ^2
IG(μ, σ^2)	$\frac{y^{-3/2}}{\sqrt{2\pi\sigma^2}}\text{exp}\left(\frac{-(y-\mu)^2}{2y\sigma^2\mu^2}\right)$	σ^2	$1/\mu^2$	$(-2\theta)^{-1/2}$	μ^3

Generalized Linear Models

- Estimation of the parameters by maximum likelihood methods (using iteratively reweighted least squares)
- Suppose:

  response is always positive
  data are invariably skew to the right
  variance increases with mean

  $\left.\begin{array}{l}\text{response is always positive}\\ \text{data are invariably skew to the right}\\ \text{variance increases with mean}\end{array}\right\} \Rightarrow$ no particular distr.

- **Quasi-likelihood** (Wedderburn, 1974) estimation allows us to model the response variable in a regression context without specifying its distribution. We need only to specify the link and variance functions to estimate the regression coefficients.
- If all the data are positive (greater than 0), identical parameter estimates are obtained using full or quasi-likelihood.

Generalized Linear Models

1. *Over-dispersed Poisson model:*
   The incremental claims Y_{ij} are distributed as independent over-dispersed Poisson random variables, with

$$\text{Var}[Y_{ij}] = \phi \mathrm{E}[Y_{ij}]$$

$\hookrightarrow$ not only suitable for data consisting exclusively of positive integers

$$\Downarrow$$

quasi-likelihood approach

2. *Gamma model:*

$$\text{Var}[Y_{ij}] = \phi \left(\mathrm{E}[Y_{ij}]\right)^2$$

Generalized Linear Models

Log-normal model:

$$Z_{ij} = \text{log}(Y_{ij}) \sim N(\mu_{ij}, \sigma^2)$$

$$\Downarrow$$

$$\eta_{ij} = \mu_{ij} \text{ and } \phi = \sigma^2$$

$\hookrightarrow$ limitation: incremental claim amounts must be positive

$$\boxed{\hat{Y}_{ij} = \exp(\hat{\eta_{ij}} + \frac{1}{2}\hat{\sigma}^2_{ij}) \overset{\eta_{ij}=g(\mu_{ij})}{\longleftrightarrow} \hat{Y}_{ij} = \hat{\mu}_{ij} = g^{-1}(\hat{\eta}_{ij})}$$

Quasi-likelihood equations

When using a logarithmic link function, the quasi-likelihood equations are given by

$$\sum_{j=1}^{t+1-i} e^{\eta_{ij}} = \sum_{j=1}^{t+1-i} Y_{ij} \quad 1 \le i \le t;$$

$$\sum_{i=1}^{t+1-j} e^{\eta_{ij}} = \sum_{i=1}^{t+1-j} Y_{ij} \quad 1 \le j \le t.$$

$\hookrightarrow$ The sum of the incremental claims in every row and column has to be non-negative $\Rightarrow$ problems when modelling incurred data with a large number of negative incremental claims in the later stages of development, which is the result of overestimates of case reserves in the first development years.

$\Rightarrow$ Work without GLM-software and without the log-link

Distribution of $\hat{\vec{\mu}}$

- $\mathbf{R}\hat{\vec{\beta}} \sim MN(\mathbf{R}\vec{\beta}, \mathbf{\Sigma}(\mathbf{R}\hat{\vec{\beta}}))$ (asymptotically) with
  - $\mathbf{\Sigma}(\mathbf{R}\hat{\vec{\beta}}) = \mathbf{\Sigma}^a = \{\sigma_{ij}^a\} = \mathbf{R}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{R}'$
  - $\mathbf{W} = \text{diag}\{w_{11}, \cdots, w_{t1}\}$ with $w_{ij} = \text{Var}[Y_{ij}]^{-1}(d\mu_{ij}/d\eta_{ij})^2$
- The function $g^{-1}(\eta_{11}, \cdots, \eta_{tt})$ has a nonzero differential $\vec{\psi} = (\psi_{11}, \cdots, \psi_{tt})'$ at $(\mathbf{R}\vec{\beta})$, where $\psi_{ij} = d\mu_{ij}/d\eta_{ij}$
- Delta method:

$$\left[\hat{\vec{\mu}} - \vec{\mu}\right] \xrightarrow{d} N\left(0, \mathbf{\Sigma}(\hat{\vec{\mu}})\right)$$

where $\mathbf{\Sigma}(\hat{\vec{\mu}}) = \vec{\psi}'\mathbf{\Sigma}^a\vec{\psi}$

Distribution of $\hat{\vec{\mu}}$

The n^{-1} bias of $\hat{\vec{\mu}}$: Cordeiro and McCullagh (1991)

$$\boxed{\mathbf{B}(\hat{\vec{\beta}}) = -\tfrac{1}{2}\mathbf{\Sigma}^b\mathbf{X}'\mathbf{\Sigma}_d^c\mathbf{F}_d\bar{\mathbf{1}},}$$

- $\mathbf{\Sigma}^b = \mathbf{\Sigma}(\hat{\vec{\beta}}) = \{\sigma_{ij}^b\} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}$
- $\mathbf{\Sigma}^c = \mathbf{\Sigma}(\mathbf{U}\hat{\vec{\beta}}) = \{\sigma_{ij}^c\} = \mathbf{X}\mathbf{\Sigma}^b\mathbf{X}'$ ($\mathbf{\Sigma}_d^c$ = diag$\{\sigma_{11}^c, \cdots, \sigma_{t1}^c\}$)
- $\mathbf{F}_d =$ diag$\{f_{11}, \cdots, f_{t1}\}$ with $f_{ij} =$ Var$[Y_{ij}]^{-1}\frac{d\mu_{ij}}{d\eta_{ij}}\frac{d^2\mu_{ij}}{d\eta_{ij}^2}$
- $\bar{\mathbf{1}}$: $t(t+1)/2 \times 1$ vector of ones

$$\Downarrow$$

$$\boxed{\mathbf{B}(\mathbf{R}\hat{\vec{\beta}}) = -\tfrac{1}{2}\mathbf{R}\mathbf{\Sigma}^b\mathbf{X}'\mathbf{\Sigma}_d^c\mathbf{F}_d\bar{\mathbf{1}}}$$

Distribution of $\hat{\vec{\mu}}$

Because $\hat{\mu}_{ij} = g^{-1}(\hat{\eta}_{ij}) = g^{-1}((\mathbf{R}\vec{\beta})_{ij})$ and the link function is monotone and twice differentiable, we can apply a Taylor series expansion of $\hat{\mu}_{ij}$ around η_{ij}:

$$
\begin{aligned}
\hat{\mu}_{ij} &\cong \mu_{ij} + \frac{d\mu_{ij}}{d\eta_{ij}}(\hat{\eta}_{ij} - \eta_{ij}) + \frac{1}{2}\frac{d^2\mu_{ij}}{d\eta_{ij}^2}(\hat{\eta}_{ij} - \eta_{ij})^2 \\
\hat{\mu}_{ij} - \mu_{ij} &\cong \frac{d\mu_{ij}}{d\eta_{ij}}(\hat{\eta}_{ij} - \eta_{ij}) + \frac{1}{2}\frac{d^2\mu_{ij}}{d\eta_{ij}^2}(\hat{\eta}_{ij} - \eta_{ij})^2 \\
\mathrm{E}[\hat{\mu}_{ij} - \mu_{ij}] &\cong \frac{d\mu_{ij}}{d\eta_{ij}}E[(\hat{\eta}_{ij} - \eta_{ij})] + \frac{1}{2}\frac{d^2\mu_{ij}}{d\eta_{ij}^2}\text{Var}(\hat{\eta}_{ij})
\end{aligned}
$$

Distribution of $\hat{\vec{\mu}}$

In matrix notation

$$\begin{aligned} \mathrm{E}[\hat{\vec{\mu}} - \vec{\mu}] &\cong \mathbf{G}_1 \mathrm{E}[(\hat{\vec{\eta}}_{-}\vec{\eta})] + \frac{1}{2}\mathbf{G}_2[\mathrm{Var}(\hat{\vec{\eta}})] \\ &\cong -\frac{1}{2}\mathbf{G}_1\mathbf{R}\boldsymbol{\Sigma}^b\mathbf{X}'\boldsymbol{\Sigma}_d^c\mathbf{F}_d\bar{\mathbf{1}} + \frac{1}{2}\mathbf{G}_2\boldsymbol{\Sigma}_d^a\tilde{\mathbf{1}} \end{aligned}$$

- $\mathbf{G}_1 = \mathrm{diag}\{\psi_{11}, \cdots, \psi_{tt}\}$ and $\psi_{ij} = d\mu_{ij}/d\eta_{ij}$
- $\mathbf{G}_2 = \mathrm{diag}\{\varphi_{11}, \cdots, \varphi_{tt}\}$ and $\varphi_{ij} = d^2\mu_{ij}/d\eta_{ij}^2$
- $\boldsymbol{\Sigma}_d^a$ = diag$\{\sigma_{11}^a, \cdots, \sigma_{tt}^a\}$
- $\tilde{\mathbf{1}}$: $t^2 \times 1$ vector of ones

$$\boxed{\mathbf{B}(\hat{\vec{\mu}}) = \tfrac{1}{2}\left\{\mathbf{G}_2\boldsymbol{\Sigma}_d^a\tilde{\mathbf{1}} - \mathbf{G}_1\mathbf{R}\boldsymbol{\Sigma}^b\mathbf{X}'\boldsymbol{\Sigma}_d^c\mathbf{F}_d\bar{\mathbf{1}}\right\}}$$

$\longrightarrow$ the corrected adjusted values are $\hat{\vec{\mu}}_c = \hat{\vec{\mu}} - \hat{\mathbf{B}}(\hat{\vec{\mu}})$

($\hat{\mathbf{B}}(.)$ = the value of $\mathbf{B}(.)$ at $(\hat{\phi}, \hat{\vec{\mu}})$)

Discounted IBNR reserve: lognormal framework

IBNR reserve

$$R \stackrel{def}{=} \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\hat{\vec{\beta}})_{ij}+\epsilon_{ij}}$$

$$\epsilon_{ij} \sim \text{i.i.d. } N(0,\sigma^2)$$

$$(\mathbf{R}\hat{\vec{\beta}})_{ij} \sim N\Big((\mathbf{R}\vec{\beta})_{ij}, \sigma^2 \left(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right)_{ij}\Big)$$

$\Downarrow$

Discounted IBNR reserve

$$S \stackrel{def}{=} \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} e^{(\mathbf{R}\hat{\vec{\beta}})_{ij}+\epsilon_{ij}-Y(i+j-t-1)}$$

$$Y(k) \sim N((\mu + \frac{\delta^2}{2})k, \delta^2 k)$$

Convex bounds discounted IBNR reserve (lognormal)

1. **Upper bound**

$$S^c = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} F^{-1}_{\text{exp}(W_{ij})}(U) F^{-1}_{\text{exp}(\epsilon_{ij})}(V)$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \text{exp}\left(\mathrm{E}[W_{ij}] + \sigma_{W_{ij}} \Phi^{-1}(U) + \sigma_{\epsilon_{ij}} \Phi^{-1}(V)\right)$$

with $W_{ij} = (\mathbf{R}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1)$

2. **Lower bound**

$$S^l = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \mathrm{E}[\text{exp}(W_{ij})|Z]\mathrm{E}[\text{exp}(\epsilon_{ij})] \qquad (Z \text{ normal distributed})$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \text{exp}\left(\mathrm{E}[W_{ij}] + \rho_{ij}\sigma_{W_{ij}} \Phi^{-1}(U) + \frac{1}{2}(1-\rho_{ij}^2)\sigma^2_{W_{ij}} + \frac{1}{2}\sigma^2_{\epsilon_{ij}}\right)$$

with $\rho_{ij} = \text{Corr}(Z, W_{ij})$

Convex bounds discounted IBNR reserve (lognormal)

- Choice of normal random variable Z?

$$Z = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \nu_{ij} Y(i+j-t-1)$$

with

$$\nu_{ij} = \text{exp}\left((\mathbf{R}\vec{\beta})_{ij} - (i+j-t-1)\mu \right)$$

- To compute the cdf's one can use the following result

$$F_{XY}(z) \quad = \quad \int_{-\infty}^{\infty} F_Y\left(\frac{z}{x}\right)\, dF_X(x) = \int_0^1 F_Y\left(\frac{z}{F_X^{-1}(u)}\right) du$$

Convex bounds discounted IBNR reserve (lognormal)

Upper bound

$$F_{S^c}(z) = \int_0^1 F_N\left(\log(z) - \log(F^{-1}_{\tilde{S}^c}(u))\right) du$$

with $F_N(x)$ the cdf of $N(0, \sigma^2)$ and

$$\tilde{S}^c = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \text{exp}\left(F^{-1}_{(\mathbf{R}\hat{\vec{\beta}})_{ij} - Y(i+j-t-1)}(U)\right)$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \text{exp}\left(\mathrm{E}[W_{ij}] + \sigma_{W_{ij}} \Phi^{-1}(U)\right)$$

Convex bounds discounted IBNR reserve (lognormal)

Lower bound

$$F_{s^l}^{-1}(p) = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} F^{-1}_{\mathrm{E}[V_{ij}|Z]\mathrm{E}[e^{\epsilon_{ij}}]}(p), \qquad p \in (0,1)$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \mathrm{E}[V_{ij}|Z = F_Z^{-1}(1-p)]\mathrm{E}[e^{\epsilon_{ij}}]$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \exp\left(\mathrm{E}[W_{ij}] - \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(p) + \frac{1}{2}(1-\rho_{ij}^2)\sigma_{W_{ij}}^2 + \frac{1}{2}\sigma_{\epsilon_{ij}}^2\right)$$

($\mathrm{E}[e^{W_{ij}}|Z]$: non-increasing function in Z since $\rho_{ij} \leq 0$)

$F_{S^l}(x) \rightarrow$ solving the equation:

$$\sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \exp\left(\mathrm{E}[W_{ij}] - \rho_{ij}\sigma_{W_{ij}}\Phi^{-1}(F_{S_l}(x)) + \frac{1}{2}(1-\rho_{ij}^2)\sigma_{W_{ij}}^2 + \frac{1}{2}\sigma_{\epsilon_{ij}}^2\right) = x$$

Discounted IBNR reserve: GLM framework

IBNR reserve

$$R \stackrel{def}{=} \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \hat{\mu}_{ij}$$

$$\left[\hat{\vec{\mu}} - \vec{\mu}\right] \xrightarrow{d} N(0, \vec{\psi}' \mathbf{\Sigma}^a \vec{\psi})$$

$$\hat{\vec{\mu}}_c = \hat{\vec{\mu}} - \hat{\mathbf{B}}(\hat{\vec{\mu}})$$

$\Downarrow$

Discounted IBNR reserve

$$S \stackrel{def}{=} \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \hat{\mu}_{ij} e^{-Y(i+j-t-1)}$$

$$Y(k) \sim N((\mu + \frac{\delta^2}{2})k, \delta^2 k)$$

Convex bounds discounted IBNR reserve (GLM)

1. **Upper bound**

$$S^c = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} F^{-1}_{\hat{\mu}_{ij}}(U) F^{-1}_{\text{exp}(V_{ij})}(V)$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \left(\mu_{ij} + \mathbf{B}(\hat{\vec{\mu}})_{ij} + \sqrt{\mathbf{\Sigma}(\hat{\vec{\mu}})_{ij}} \Phi^{-1}(V) \right) \text{exp}(\mathrm{E}[W_{ij}] + \sigma_{W_{ij}} \Phi^{-1}(U))$$

with $V_{ij} = -Y(i + j - t - 1)$

2. **Lower bound**

$$S^l = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \mathrm{E}[\hat{\mu}_{ij}] \mathrm{E}[\text{exp}(V_{ij}) | Z] \qquad (Z \text{ normal distributed})$$

$$= \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \left(\mu_{ij} + \mathbf{B}(\hat{\vec{\mu}})_{ij} \right) \text{exp} \left(E[W_{ij}] + \rho_{ij} \sigma_{W_{ij}} \Phi^{-1}(U) + \frac{1}{2}(1 - \rho^2_{ij}) \sigma^2_{W_{ij}} \right)$$

with $\rho_{ij} = \text{Corr}(Z, V_{ij})$

Convex bounds discounted IBNR reserve (GLM)

- Choice of normal random variable Z?

$$Z = \sum_{i=2}^{t} \sum_{j=t+2-i}^{t} \nu_{ij} Y(i+j-t-1)$$

with

$$\nu_{ij} = \left(\mu_{ij} + \mathbf{B}(\hat{\vec{\mu}})_{ij}\right) \text{exp}\left(-(i+j-t-1)\delta\right)$$

- The computation of the cdf's is analogous to the lognormal case

A numerical illustration: dataset

292686	683476	701376	747034	504265	312468	284954	170814	249348	69752
423113	991584	1032142	945156	500205	413863	434622	206319	342383	
344386	936335	971651	1104206	575666	416179	359195	246463		
308603	830615	864751	981609	504837	372329	353145			
338073	884174	895252	927435	647289	391208				
322270	927791	980275	952298	577483					
387598	1084439	1126376	1035701						
385603	1143038	1209301							
388795	951100								
308586									

A numerical illustration

- Statistical model:

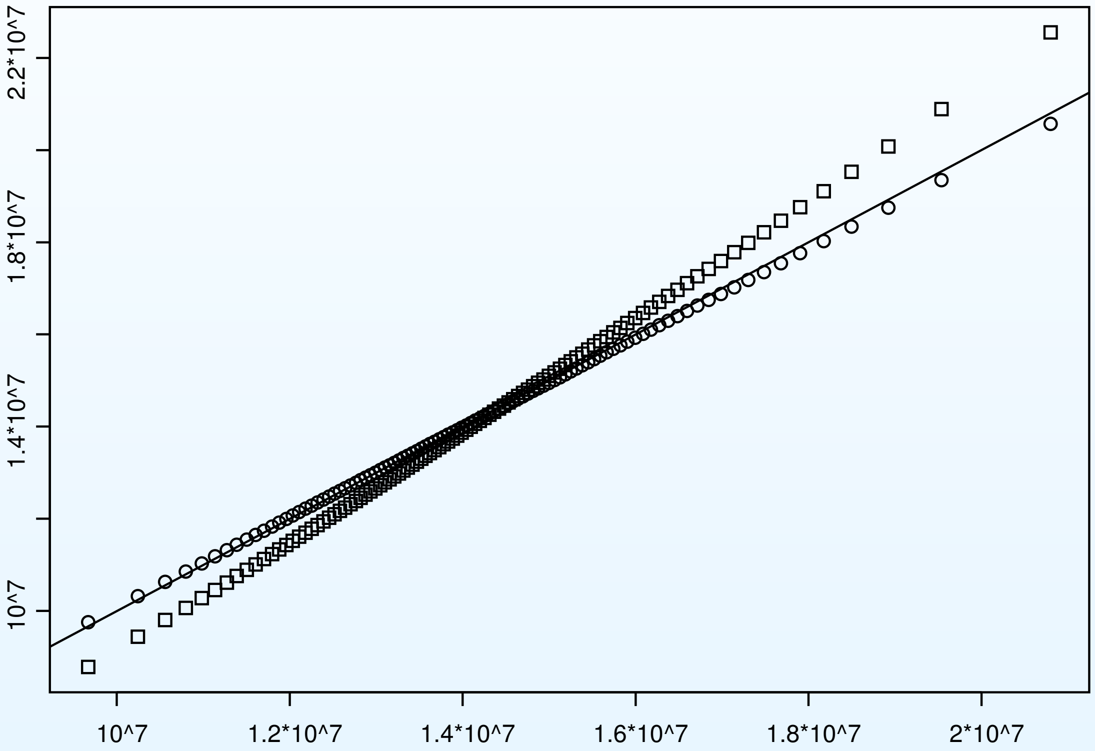
$$\begin{aligned} \mathrm{E}[Y_{ij}] &= \mu_{ij}, \\ \mathsf{Var}[Y_{ij}] &= \phi\mu_{ij}^2, \\ \mathsf{log}(\mu_{ij}) &= \eta_{ij}, \\ \eta_{ij} &= \alpha_i + \beta_j. \end{aligned}$$

- Return process: Black & Scholes model $\mu = 0.08, \sigma = 0.11$
- Simulation: 100 000 generated paths

A numerical illustration: Quantiles

p	$F_{S^l}^{-1}(p)$	$F_S^{-1}(p)$	$F_{S^c}^{-1}(p)$
0.95	17888702	18033971	18926155
0.975	18749885	18923975	20077389
0.99	19809569	19986346	21511663
0.995	20569107	20799492	22551353
0.999	22239104	22410022	24870374

A numerical illustration: QQ-plot



QQ-plot of the quantiles of S^l ($\circ$) and S^c ($\square$) versus those of S

A numerical illustration

year	F_{S^l}			F_S		
	95%	mean	st. dev.	95%	mean	st. dev.
2	102356	85934	9481	103187	85934	9747
3	462847	387251	43602	466609	387251	44775
4	619090	503187	66173	624112	503187	68014
5	1042181	842092	113871	1050345	842092	117188
6	1432744	1142369	164543	1444486	1142369	169224
7	2286615	1815836	266221	2305985	1815836	273721
8	3590200	2864235	410836	3619252	2864235	422643
9	4197088	3312169	499465	4231171	3312169	513473
10	4197710	3264577	524580	4231798	3264577	539321
total	17888702	14217631	2076583	18033971	14217631	2135185

A numerical illustration

- **Estimation error** $\longrightarrow$ from the estimation of the vector parameters $\hat{\beta}$ from the data
- **Statistical error** $\longrightarrow$ from the stochastic nature of the underlying model

$\Longrightarrow$ Use bootstrapping to construct statistical confidence intervals for the bounds incorporating the estimation error !

$$\Downarrow$$

1. Bootstrap an upper triangle: this involves resampling, with replacement, from the original residuals and then creating a new triangle of past claims payments using the resampled residuals together with the fitted values
2. Calculate for each bootstrap sample the desired percentile of the distribution of S^l

A numerical illustration

$Q_{S^l}(0.95)$-distribution based on 5000 bootstrapped run-off triangles

	Distribution of bootstrapped 95th percentiles of S^l	Simulated distribution of $F_S^{-1}(0.95)$
1 st percentile	16661827	16333152
2.5 th percentile	16861353	16576586
5 th percentile	17048933	16759301
10 th percentile	17233865	17101271
25 th percentile	17551891	17450048
50 th percentile	17913169	17904390
75 th percentile	18284619	18380651
90 th percentile	18641949	18832716
95 th percentile	18850593	19117307
97.5 th percentile	18999178	19264184
99 th percentile	19187288	19481477

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