Some New Classes of Consistent Risk Measures

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Abstract

Many types of insurance premium principles and/or risk measures can be characterized by means of a set of axioms, which in many cases are rather arbitrarily chosen and not always in accordance with economic reality. In the present paper we generalize Yaari's risk measure by relaxing his axioms. In addition, we derive translation invariant minimal Orlicz risk measures, which we call Haezendonck risk measures, and obtain sufficient conditions on the risk measure of Bernoulli risks to fulfill additivity and superadditivity properties for Orlicz premium principles.

Keywords: Consistent risk measures, Haezendonck risk measure, Monotone convergence theorem, Yaari's dual theory of choice under risks

1 Introduction

Recently, in Goovaerts *et al.* (2003a) it was argued that risk measures should be selected in an appropriate way in order to reflect the basic economic underlying reality. Indeed several examples can be given, which are relevant to real life insurance problems where evidently the properties that the risk measures should have are determined by the realities of the actuarial applications.[1]

Example 1.1 (Insurance – reinsurance). Suppose that a risk X is split into two parts as

$$X = [X - (X-d)_+] + (X-d)_+.$$

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[1]The importance of choosing the right set of desirable properties that actuarial risk measures should have for actuarial practice is discussed in the report of the Economic Capital Calculation and Allocation Subgroup of the Risk Management Task Force of the Society of Actuaries, 'Specialty Guide on Economic Capital' (version 1.4, dated June 2003, available from the SOA).

Clearly, both parts are comonotonic (see Dhaene et al. (2002a,b)) because they are increasing functions of X. A principle, say $\pi[\cdot]$, that is additive for comonotonic risks has

$$\pi\left[X\right] = \pi\left[X - (X-d)_+\right] + \pi\left[(X-d)_+\right].$$

This has some advantages for allocating the premium between the two parts involved. It provides for instance a tool to compare the part of the premiums charged for the risk $(X-d)_+$ with the reinsurance premium $\pi_R\left[(X-d)_+\right]$ that is actually charged by the reinsurer. The same also holds for the case

$$X = aX + (1-a)X, \qquad 0 \leq a \leq 1,$$

which leads to

$$\pi\left[X\right] = \pi\left[aX\right] + \pi\left[(1-a)X\right]$$

since the principle $\pi[\cdot]$ is additive for comonotonic risks. In both situations comonotonic additive risk measures are consistent with the practical problems at hand. Related discussions can be found for instance in Wang (1996).

Example 1.2 (Premium calculation). For practical reasons, in the case of premium calculation, splitting of the risk into two parts aX and $(1-a)X$ for some $0 < a < 1$ should not lead to a decrease in premiums. Hence,

$$\pi\left[X\right] \leq \pi\left[aX\right] + \pi\left[(1-a)X\right].$$

In this case subadditive risk measures for comonotonic risks are consistent with this particular situation. This property has been called the subdecomposability of a risk in Goovaerts *et al.* (1984).

Example 1.3 (Premium calculation from top-down). As in Bühlmann (1970) (see also Gerber (1979, 1985) and Kaas *et al.* (2001)), suppose that one approaches a premium calculation from top-down, for instance by considering a ruin probability model for the determination of the portfolio premium on the top level. In case of automobile insurance the risks constituting the portfolio can generally be considered to be independent. Then, in order to distribute the premium income at the top level among the risks at the down level, the use of an additive risk measure for independent risks,

$$\pi\left[X_1 + X_2\right] = \pi\left[X_1\right] + \pi\left[X_2\right] \qquad \text{for independent } X_1 \text{ and } X_2,$$

is consistent with the situation at hand.

Example 1.4 (Capital allocation). In order to keep the residual risk of the conglomerate, after the capital has been allocated, under control in the sense that the risk of the conglomerate benefits from the diversification, a superadditive risk measure is consistent. This is because

$$\pi\left[X_1 + X_2\right] \geq \pi\left[X_1\right] + \pi\left[X_2\right]$$

results in the residual capital to be asked to the shareholders satisfying

$$(X_1 + X_2 - \pi[X_1 + X_2])_+ \leq_{st} (X_1 - \pi[X_1])_+ + (X_2 - \pi[X_2])_+ ,$$

where $\leq_{st}$ denotes "stochastically not greater than".

Example 1.5 (Solvency margin). Consider the Bernoulli risk B_q with $q \in [0,1]$. For any $a > 0$ the risk measure $\pi[aB_q]$ should be increasing in $q \in [0,1]$ if this risk measure $\pi[\cdot]$ is used as a premium rule. However, when one aims at calculating a "solvency margin" (a provision for an adverse outcome that is much larger than the expected one) $\pi[aB_q]$ for this Bernoulli risk, it is clear that $\pi[aB_0] = \pi[aB_1] = 0$ because in both situations there is no uncertainty involved. One could think about $\pi[aB_q] = \pi[aB_{1-q}]$ to express the equality of uncertainty between the two risky situations aB_q and aB_{1-q}. One could also consider the financial picture in the following sense by means of an actuarial safety loading $\lambda > 0$:

$$\mathrm{E}[B_q](1+\lambda) = q(1+\alpha(1-q)) = q + \alpha q(1-q),$$

where α is a proportional part of the excess amount $1-q$ above the expected claim size q. For another Bernoulli risk $B_{q'}$ with $q' = 1-q$, one gets $(1-q)(1+\alpha q) = (1-q) + \alpha(1-q)q$. Consequently, $\pi[B_q] = \pi[B_{1-q}] = \alpha q(1-q)$ is an example of a consistent risk measure for calculating solvency margins.

The previous examples indicate that each realistic situation needs a specific set $\mathbb{S}$ of axioms. We introduce the following definition:

Definition 1.1. *Let $\mathbb{S}$ be a set of axioms for risk measures and α, $0 < \alpha < 1$, be a level. A risk measure $\pi[\cdot] = \pi_{(\mathbb{S},\alpha)}[\cdot] = \pi_\alpha[\cdot]$ is called $(\mathbb{S},\alpha)$–consistent if $\pi[\cdot]$ is a rule that assigns a value to each risk X satisfying the axioms $\mathbb{S}$ and such that $\pi[X] \geq F_X^{-1}(\alpha)$, where $F_X^{-1}(\alpha)$ is the αth quantile of the risk X and is defined, as usual, by $F_X^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\}$.*

In the present paper we first generalize some of the consistent risk measures for the following choice $\mathbb{S}_1$ of axioms:

A1. Monotonicity: $X \leq_{st} Y \Longrightarrow \pi[X] \leq \pi[Y]$;

A2. Exchangeability: $\pi[X^c + Y^c] = \pi[X^{*c} + Y^c]$ provided that $\pi[X] = \pi[X^*]$;

A3. Continuity: $\pi[X_n]$ converges to $\pi[X]$ if X_n is nondecreasing and converges weakly to X.

Here and throughout, for a random vector $(Y_1, \ldots, Y_n)$ we write by $(Y_1^c, \ldots, Y_n^c)$ a comonotonic random vector such that Y_i^c and Y_i have the same distribution for $i = 1, 2, \ldots, n$. Clearly, $\mathbb{S}_1$ is less restrictive than the axioms of Yaari's (1987) dual theory of choice under risks because, according to Yaari's axioms,

$$\pi[X^c + Y^c] = \pi[X^c] + \pi[Y^c] = \pi[X^{*c} + Y^c]$$

must hold and hence A2 follows immediately. We notice that the first axiom is not always valid in case one uses a risk measure as a solvency measure as was explained in Example 1.5.

Then, we consider another choice $\mathbb{S}_2$ of axioms as follows:

B1. Monotonicity: $X \leq_{st} Y \Longrightarrow \pi[X] \leq \pi[Y]$;

B2. Positive homogeneity: $\pi[aX] = a\pi[X]$ for $a > 0$;

B3. Subadditivity: $\pi[X_1 + X_2] \leq \pi[X_1] + \pi[X_2]$.

This is the set of axioms satisfied by an Orlicz premium principle; see Goovaerts *et al.* (1984, 2003b). By considering Orlicz distances for the risk $(X-u)_+$ and determining u optimally, one obtains in addition a translation invariance property for this risk measure; see Section 3, where we will introduce the Haezendonck risk measure.

2 Generalized Yaari risk measure

For simplicity, in this section when we mention a risk we mean that it is a nonnegative random variable. The result below can easily be extended to the more general case where the risk takes values on the whole real line.

Let $\pi[\cdot]$ be a risk measure such that for a risk X, the value $\pi[X]$ is uniquely determined by its distribution function F_X. We write $X \sim Y$ if $\pi[X] = \pi[Y]$ and denote by B_q with $0 \leq q \leq 1$ a Bernoulli variable satisfying

$$B_q = \begin{cases} 1 & \text{with probability } q, \\ 0 & \text{with probability } 1-q. \end{cases} \tag{2.1}$$

In terms of the risk measure $\pi[\cdot]$ and the Bernoulli variable B_q, we introduce two functions $v(\cdot,\cdot)$ and $f(\cdot)$ by

$$v(x,q) = \pi[xB_q], \qquad f(x) = v(x,1), \tag{2.2}$$

where $x \geq 0$ and $0 \leq q \leq 1$.

Theorem 2.1. *Let the risk measure $\pi[\cdot]$, the functions $f(\cdot)$ and $v(\cdot,\cdot)$ given above be such that*

$$w(q) = \frac{\partial}{\partial x} v(x,q)|_{x=0}$$

exists and is nondecreasing in $q \in [0,1]$ with $w(0) = 0$ and $w(1) = f'(0) > 0$. Then $\pi[\cdot]$ satisfies the monotonicity axiom A1, the exchangeability axiom A2 and the continuity axiom A3 if and only if the function $f(\cdot)$ is continuous and nondecreasing on $[0,+\infty)$ and

$$\pi[X] = f\left(\frac{1}{w(1)} \int_0^{+\infty} w\left(1 - F_X(x)\right) \mathrm{d}x\right). \tag{2.3}$$

Proof. **1. First we prove the "if" part.**

Assume that (2.3) holds with the function $f(\cdot)$ continuous and nondecreasing on $[0,+\infty)$. The proof of the monotonicity axiom A1 is trivial, and the continuity axiom A3 follows directly from the monotone convergence theorem. So we only need to prove the exchangeability axiom A2. For any risk X, in case $\int_0^{+\infty} w\left(1-F_X(x)\right)\mathrm{d}x=+\infty$, by (2.3) it holds that $\pi\left[X\right]=\pi\left[X^*\right]=f\left(+\infty\right)$. Hence by axiom A1, we can conclude that $\pi\left[X^c+Y^c\right]=\pi\left[X^{*c}+Y^c\right]=f\left(+\infty\right)$, which indicates A2. Symmetrically, the same discussion can be given for the case where $\int_0^{+\infty} w\left(1-F_{X^*}(x)\right)\mathrm{d}x=+\infty$. Thus it remains to prove A2 for the case where

$$\int_0^{+\infty} w\left(1-F_X(x)\right)\mathrm{d}x<+\infty \quad \text{and} \quad \int_0^{+\infty} w\left(1-F_{X^*}(x)\right)\mathrm{d}x<+\infty. \tag{2.4}$$

We apply an approximation device to prove the result. Write $Y_n=\min\{Y,n\}$, $n=1,2,\ldots$. The first relation in (2.4) indicates that

$$\begin{aligned}\int_0^{+\infty} w\left(1-F_{X^c+Y_n^c}(x)\right)\mathrm{d}x &\le \int_0^{+\infty} w\left(1-F_X(x-n)\right)\mathrm{d}x\\ &= nw(1)+\int_0^{+\infty} w\left(1-F_X(x)\right)\mathrm{d}x\\ &< +\infty.\end{aligned}$$

Thus, for each $n=1,2,\ldots$, applying integration by parts we have

$$\begin{aligned}\pi\left[X^c+Y_n^c\right] &= f\left(\frac{1}{w(1)}\int_0^1 F^{-1}_{X^c+Y_n^c}(1-y)\mathrm{d}w\left(y\right)\right)\\ &= f\left(\frac{1}{w(1)}\int_0^1 \left(F_X^{-1}(1-y)+F_{Y_n}^{-1}(1-y)\right)\mathrm{d}w\left(y\right)\right).\end{aligned}$$

Define the inverse function of $f(\cdot)$, as usual, by $f^{-1}(y)=\inf\{x: f(x)\ge y\}$. It follows immediately from (2.3) that

$$\pi\left[X^c+Y_n^c\right]=f\left(f^{-1}\left(\pi\left[X\right]\right)+\frac{1}{w(1)}\int_0^1 F_{Y_n}^{-1}(1-y)\mathrm{d}w\left(y\right)\right).$$

Symmetrically, it holds that

$$\pi\left[X^{*c}+Y_n^c\right]=f\left(f^{-1}\left(\pi\left[X^*\right]\right)+\frac{1}{w(1)}\int_0^1 F_{Y_n}^{-1}(1-y)\mathrm{d}w\left(y\right)\right).$$

This proves that the relation

$$\pi\left[X^c+Y_n^c\right]=\pi\left[X^{*c}+Y_n^c\right] \tag{2.5}$$

holds for each $n=1,2,\ldots$. Finally, by axiom A3, letting $n\to+\infty$ on both sides of (2.5) yields that $\pi\left[X^c+Y^c\right]=\pi\left[X^{*c}+Y^c\right]$.

2. Next we prove the "only if" part.

Clearly, by the monotonicity axiom A1 and the continuity axiom A3, the function $f(x)$ is nondecreasing and continuous in $x \in [0, +\infty)$ and the bivariate function $v(x, q)$ is also nondecreasing and continuous both in $x \in [0, +\infty)$ and in $q \in [0, 1]$. It follows immediately from (2.2) that

$$xB_q \sim f^{-1}\left(v(x, q)\right). \tag{2.6}$$

Now we formulate the remaining proof into three steps.

2.1. First we only consider a special case where the risk X has a discrete distribution function with finitely many supporting points $x_0 < x_1 < \ldots < x_n$, satisfying

$$\begin{cases} \Pr\left[X = x_i\right] = p_i \geq 0, & i = 0, 1, \ldots, n, \\ \sum\limits_{i=0}^{n} p_i = 1. \end{cases} \tag{2.7}$$

Without loss of generality we assume $x_0 = 0$. We define

$$\Delta_i = \begin{cases} x_i - x_{i-1} & \text{with probability } p_i + \ldots + p_n, \\ 0 & \text{with probability } p_0 + \ldots + p_{i-1}, \end{cases} \qquad i = 1, \ldots, n.$$

So Δ_i satisfies

$$\Delta_i =^d (x_i - x_{i-1}) B_{p_i + \ldots + p_n}, \qquad i = 1, \ldots, n,$$

where $=^d$ denotes "has the same distribution as". It is easy to see that

$$X =^d F_X^{-1}(U) = \sum_{i=1}^{n} F_{\Delta_i}^{-1}(U) =^d \sum_{i=1}^{n} (x_i - x_{i-1}) B^c_{p_i + \ldots + p_n}. \tag{2.8}$$

Hence,

$$\pi\left[X\right] = \pi\left[\sum_{i=1}^{n} (x_i - x_{i-1}) B^c_{p_i + \ldots + p_n}\right]. \tag{2.9}$$

Recall (2.6) and the exchangeability axiom A2. For any integer $m \geq 1$ we have that

$$\begin{aligned} \pi\left[X\right] &= \pi\left[\sum_{i=1}^{n} m\left(\frac{x_i - x_{i-1}}{m} B^c_{p_i + \ldots + p_n}\right)\right] \\ &= f\left(\sum_{i=1}^{n} m f^{-1}\left(v\left(\frac{x_i - x_{i-1}}{m},\ 1 - F(x_{i-1})\right)\right)\right). \end{aligned}$$

By the assumption on the function $v(\cdot, \cdot)$, one easily sees that

$$\lim_{x \searrow 0} \frac{f^{-1}\left(v(x, q)\right)}{x} = \lim_{v \searrow 0} \frac{f^{-1}(v)}{v} \cdot \lim_{x \searrow 0} \frac{v(x, q)}{x} = \frac{w(q)}{f'(0)} = \frac{w(q)}{w(1)}. \tag{2.10}$$

Therefore, by the continuity of the function $f(\cdot)$ and (2.10) we derive

$$
\begin{aligned}
\pi[X] &= \lim_{m\to+\infty} f\left(\sum_{i=1}^{n} m f^{-1}\left(v\left(\frac{x_i - x_{i-1}}{m},\ 1-F(x_{i-1})\right)\right)\right) \\
&= f\left(\sum_{i=1}^{n}(x_i - x_{i-1}) \lim_{m\to+\infty} \frac{f^{-1}\left(v\left((x_i - x_{i-1})/m,\ 1-F(x_{i-1})\right)\right)}{(x_i - x_{i-1})/m}\right) \\
&= f\left(\sum_{i=1}^{n}(x_i - x_{i-1}) \frac{w(1-F(x_{i-1}))}{w(1)}\right) \\
&= f\left(\frac{1}{w(1)} \int_0^{+\infty} w\left(1-F_X(x)\right) \mathrm{d}x\right).
\end{aligned}
$$

This indicates that formula (2.3) holds for the case where X has only finite supporting points.

2.2. Now we consider the general case where the risk X has a distribution function F_X that is supported on the half line $[0,+\infty)$. It is standard in measure theory that the distribution function F_X can be approximated by a sequence of nonincreasing distribution functions F_{X_n} of the discrete type given in (2.7). Then, applying the continuity axiom A3, the result obtained in step 2.1 and the continuity of the function $f(\cdot)$, respectively, we have

$$
\begin{aligned}
\pi[X] &= \lim_{n\to+\infty} \pi[X_n] \\
&= \lim_{n\to+\infty} f\left(\frac{1}{w(1)} \int_0^{+\infty} w\left(1-F_{X_n}(x)\right) \mathrm{d}x\right) \\
&= f\left(\frac{1}{w(1)} \lim_{n\to+\infty} \int_0^{+\infty} w\left(1-F_{X_n}(x)\right) \mathrm{d}x\right) \\
&= f\left(\frac{1}{w(1)} \int_0^{+\infty} w\left(1-F_X(x)\right) \mathrm{d}x\right), \qquad (2.11)
\end{aligned}
$$

where at the last step we applied the monotone convergence theorem. Thus, we obtain the announced result (2.3) for the general case.

This ends the proof of Theorem 2.1. □

3 The Haezendonck risk measure

Let X be a random variable with range $-\infty \le \min[X] \le \max[X] \le +\infty$. In this section we aim at deriving a new risk measure. For this purpose we introduce a nonnegative, strictly increasing and continuous function $\phi(\cdot)$ on $[0,+\infty)$ with $\phi(0)=0$, $\phi(1)=1$ and $\phi(+\infty)=+\infty$. Then, for any $x \in (-\infty,+\infty)$ and $\pi > x$, applying the method of Goovaerts *et al.* (2003b) we obtain that

$$
\Pr[X > \pi] = \Pr[X - x > \pi - x] \le \mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right]. \qquad (3.1)
$$

For inequality (3.1) to make sense, the function $\phi(\cdot)$ and the random variable X have to satisfy

$$\mathrm{E}\left[\phi(X/c)\right] < +\infty \qquad \text{for any } c > 0. \tag{3.2}$$

Hence, by assuming (3.2) the random variables considered are restricted to the class

$$\mathbb{X}_\phi = \left\{X : \mathrm{E}\left[\phi(X/c)\right] < +\infty \text{ for any } c > 0\right\}. \tag{3.3}$$

This will be assumed tacitly in this section. For any given value $0 < \alpha < 1$, consider the equation

$$\mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right] = 1 - \alpha. \tag{3.4}$$

It has no solution if $x \geq \max[X]$ and in this case we simply assume by convention that the solution is $+\infty$. So we only consider the case $-\infty < x < \max[X]$. Recall the restrictions made on the function $\phi(\cdot)$. By the monotone convergence theorem we easily see that

$$\lim_{\pi \searrow x} \mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right] = \phi\left(+\infty\right) \Pr\left[X > x\right] = +\infty$$

and that

$$\lim_{\pi \nearrow +\infty} \mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right] = \phi\left(0\right) \Pr\left[X > x\right] = 0.$$

Hence for any $-\infty < x < \max[X]$ and $0 < \alpha < 1$, equation (3.4) has a unique solution, say $\pi_\alpha\left[X, x\right]$, which lies in the interval $(x, +\infty)$. By (3.1), we also see that the solution $\pi_\alpha\left[X, x\right]$ gives an upper bound for the quantile $F_X^{-1}(\alpha)$.

We summarize this into a lemma as follows:

Lemma 3.1. *Let X be a risk and let $\phi(\cdot)$ be a nonnegative, strictly increasing and continuous function on $[0, +\infty)$ with $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(+\infty) = +\infty$. Then for any $-\infty < x < \max[X]$ and $0 < \alpha < 1$, equation (3.4) has a unique solution $\pi_\alpha\left[X, x\right]$ satisfying*

$$\pi_\alpha\left[X, x\right] \geq F_X^{-1}(\alpha) \qquad \text{and} \qquad \pi_\alpha\left[X, x\right] > x. \tag{3.5}$$

For the reader who wants some mathematical sophistication, Lemma 3.1 can be expressed in terms of Young functions and Orlicz norms. That is the reason why we called it Orlicz risk measure in Haezendonck and Goovaerts (1982). The idea originates from the Swiss premium calculation principle due to Bühlmann *et al.* (1977).

We prove another property of the solution of equation (3.4) below:

Lemma 3.2. *Let X and $\phi(\cdot)$ be as in Lemma 3.1 and let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of random variables. If $X_n \leq Y$ for some $Y \in \mathbb{X}_\phi$ and all $n = 1, 2, \ldots$ and X_n converges weakly to X, then $X \in \mathbb{X}_\phi$ and for any $-\infty < x < \max[X]$ and $0 < \alpha < 1$,*

$$\lim_{n \to \infty} \pi_\alpha\left[X_n, x\right] = \pi_\alpha\left[X, x\right]. \tag{3.6}$$

Proof. The assertion $X \in \mathbb{X}_\phi$ is an immediate consequence of the dominated convergence theorem guaranteed by $X_n \leq Y$ for all $n = 1, 2, \ldots$. In order to verify (3.6), it suffices to prove that any limit point of the sequence $\{\pi_\alpha [X_n, x], n = 1, 2, \ldots\}$, say

$$l = \lim_{n' \to \infty} \pi_\alpha [X_{n'}, x] \tag{3.7}$$

for a subsequence $\{X_{n'}, n' = 1, 2, \ldots\}$, should satisfy

$$l = \pi_\alpha [X, x]. \tag{3.8}$$

From Lemma 3.1 it is obvious that the limit l of (3.7) lies in the region $[x, +\infty]$. But in case $l = x$, applying the dominated convergence theorem it holds for any $\varepsilon > 0$ that

$$\begin{aligned} 1 - \alpha &= \liminf_{n' \to \infty} \mathrm{E}\left[\phi\left(\frac{(X_{n'} - x)_+}{\pi_\alpha [X_{n'}, x] - x}\right)\right] \\ &\geq \liminf_{n' \to \infty} \mathrm{E}\left[\phi\left(\frac{(X_{n'} - x)_+}{\varepsilon}\right)\right] \\ &= \mathrm{E}\left[\phi\left(\frac{(X - x)_+}{\varepsilon}\right)\right] \\ &\to +\infty \qquad \text{as } \varepsilon \to 0, \end{aligned}$$

which is a self-contradiction. Hence $l > x$. We apply the dominated convergence theorem once again and obtain that

$$1 - \alpha = \lim_{n' \to \infty} \mathrm{E}\left[\phi\left(\frac{(X_{n'} - x)_+}{\pi_\alpha [X_{n'}, x] - x}\right)\right] = \mathrm{E}\left[\phi\left(\frac{(X - x)_+}{l - x}\right)\right].$$

Hence,

$$\mathrm{E}\left[\phi\left(\frac{(X - x)_+}{l - x}\right)\right] = \mathrm{E}\left[\phi\left(\frac{(X - x)_+}{\pi_\alpha [X, x] - x}\right)\right],$$

which indicates (3.8) since $\phi(\cdot)$ is nonnegative and strictly increasing and $\max[X] > x$. This ends the proof of Lemma 3.2. ☐

Now we introduce an important notion in this section:

Definition 3.1. *Let $\phi(\cdot)$ be as in Lemma 3.1 and let $0 < \alpha < 1$ be arbitrarily fixed. We consider*

$$\pi_\alpha [X] = \inf_{-\infty < x < \max[X]} \pi_\alpha [X, x] \tag{3.9}$$

as the risk measure of a risk X, where $\pi_\alpha [X, x]$ is the unique solution of equation (3.4). In honor of the late J. Haezendonck we call it the Haezendonck risk measure*, which is a minimal Orlicz norm risk measure.*

Recall our convention that $\pi_\alpha [X, x] = \infty$ for $x \geq \max[X]$. Definition (3.9) can also be rewritten as

$$\pi_\alpha [X] = \inf_{-\infty < x < \infty} \pi_\alpha [X, x].$$

Theorem 3.1. *Let $\phi(\cdot)$ be as in Lemma 3.1. The Haezendonck risk measure $\pi_\alpha[X]$ satisfies*

$$F_X^{-1}(\alpha) \leq \pi_\alpha[X] \leq \max[X]. \tag{3.10}$$

Proof. The left-hand side of (3.10) is a direct consequence of Lemma 3.1. To prove the right-hand side of (3.10), we choose $x = F_X^{-1}(\alpha')$ for some $\alpha < \alpha' < 1$ and observe that

$$\mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\max[X]-x}\right)\right] \leq \Pr[X > x] \leq 1-\alpha' < 1-\alpha.$$

Comparing this with equation (3.4) gives $\pi_\alpha[X,x] \leq \max[X]$. Hence $\pi_\alpha[X] \leq \max[X]$. This ends the proof of Theorem 3.1. □

Example 3.1. Now we specify the risk in Definition 3.1 as B_q, a Bernoulli variable with

$$\Pr[B_q = 1] = 1 - \Pr[B_q = 0] = q \in [0,1].$$

Let $\phi(y) = y$ for $y \geq 0$ and let $-\infty < x < 1$ and $0 < \alpha < 1$ be arbitrarily given. In case $-\infty < x < 0$ equation (3.4) leads to

$$(1-q)\frac{-x}{\pi - x} + q\frac{1-x}{\pi - x} = 1-\alpha, \tag{3.11}$$

whereas in case $0 \leq x < 1$ it leads to

$$q\frac{1-x}{\pi - x} = 1-\alpha. \tag{3.12}$$

Clearly, equation (3.11) has a unique solution

$$\pi_\alpha[B_q, x] = \frac{q - \alpha x}{1-\alpha}, \qquad -\infty < x < 0,$$

whereas equation (3.12) has a unique solution

$$\pi_\alpha[B_q, x] = \frac{q}{1-\alpha} + \frac{1-\alpha-q}{1-\alpha}x, \qquad 0 \leq x < 1.$$

Simple analysis gives that

$$\pi_\alpha[B_q] = \inf_{-\infty < x < \max[X]} \pi_\alpha[B_q, x] = \min\left\{\frac{q}{1-\alpha}, 1\right\}.$$

Theorem 3.2. *Let $\pi_\alpha[\cdot]$ be the Haezendonck risk measure with $\phi(\cdot)$ given in Lemma 3.1 and let $0 < \alpha < 1$ be arbitrarily given. Then we have*

B1. Monotonicity: If $X \leq_{st} Y$ then $\pi_\alpha[X] \leq \pi_\alpha[Y]$;

B2. Positive homogeneity: $\pi_\alpha[cX] = c\pi_\alpha[X]$ for any $c > 0$;

B3. Subadditivity: If $\phi(\cdot)$ is convex, then $\pi_\alpha[X+Y] \le \pi_\alpha[X] + \pi_\alpha[Y]$ holds for any (X,Y) such that

$$\max[X+Y] = \max[X] + \max[Y]; \tag{3.13}$$

B4. Translation invariance: $\pi_\alpha[X+a] = \pi_\alpha[X] + a$ for any a;

B5. Preservation of convex ordering: If $\phi(\cdot)$ is convex, then $X \le_{cx} Y \Longrightarrow \pi_\alpha(X) \le \pi_\alpha(Y)$, where $X \le_{cx} Y$ means that $\mathrm{E}\varphi(X) \le \mathrm{E}\varphi(Y)$ holds for all convex functions $\varphi(\cdot)$ for which the expectations involved exist.

Proof. B1. Trivially, $X \le_{st} Y$ indicates that $\max[X] \le \max[Y]$ and that the inequality $\pi_\alpha[X,x] \le \pi_\alpha[Y,x]$ holds for any $-\infty < x < \max[X] \le \max[Y]$. Hence by (3.5) and (3.10),

$$\begin{aligned}
\pi_\alpha[Y] &= \min\left\{ \inf_{-\infty<x<\max[X]} \pi_\alpha[Y,x], \ \inf_{\max[X]\le x<\max[Y]} \pi_\alpha[Y,x] \right\} \\
&\ge \min\left\{ \inf_{-\infty<x<\max[X]} \pi_\alpha[X,x], \ \max[X] \right\} \\
&= \min\left\{ \pi_\alpha[X], \ \max[X] \right\} \\
&= \pi_\alpha[X].
\end{aligned}$$

B2. For any $c > 0$ and any $-\infty < x < \max[cX]$, by equation (3.4) we easily see that $\pi_\alpha[cX,x] = c\pi_\alpha[X,x/c]$. Hence,

$$\begin{aligned}
\pi_\alpha[cX] &= \inf_{-\infty<x<\max[cX]} \pi_\alpha[cX,x] \\
&= c \inf_{-\infty<x<\max[cX]} \pi_\alpha[X,x/c] \\
&= c \inf_{-\infty<x<\max[X]} \pi_\alpha[X,x] \\
&= c\pi_\alpha[X].
\end{aligned}$$

B3. Let (X,Y) be any pair of random variables such that (3.13) holds and let $-\infty < x < \max[X]$ and $-\infty < y < \max[Y]$, hence $-\infty < x+y < \max[X+Y]$. We derive

$$\begin{aligned}
&\mathrm{E}\left[\phi\left(\frac{(X+Y-x-y)_+}{\pi_\alpha[X,x]+\pi_\alpha[Y,y]-x-y}\right)\right] \\
&\le \mathrm{E}\left[\phi\left(\frac{(X-x)_+ + (Y-y)_+}{(\pi_\alpha[X,x]-x)+(\pi_\alpha[Y,y]-y)}\right)\right] \\
&\le \frac{\pi_\alpha[X,x]-x}{(\pi_\alpha[X,x]-x)+(\pi_\alpha[Y,y]-y)}\mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi_\alpha[X,x]-x}\right)\right] \\
&\qquad + \frac{\pi_\alpha[Y,y]-y}{(\pi_\alpha[X,x]-x)+(\pi_\alpha[Y,y]-y)}\mathrm{E}\left[\phi\left(\frac{(Y-y)_+}{\pi_\alpha[Y,y]-y}\right)\right] \\
&= 1-\alpha,
\end{aligned}$$

where we have used the convexity of the function ϕ and inequality (3.5). This proves that for any $-\infty < x < \max[X]$ and $-\infty < y < \max[Y]$,

$$\pi_\alpha [X+Y, x+y] \leq \pi_\alpha [X, x] + \pi_\alpha [Y, y]. \tag{3.14}$$

Recall our convention that $\pi_\alpha [X, x] = +\infty$ for $x \geq \max[X]$. Hence, inequality (3.14) holds for any $-\infty < x < +\infty$ and $-\infty < y < +\infty$. Therefore by (3.14) and the definition in (3.9),

$$\begin{aligned}
\pi_\alpha [X+Y] &= \inf_{-\infty < x+y < \infty} \pi_\alpha [X+Y, x+y] \\
&\leq \inf_{-\infty < x+y < \infty} \left(\pi_\alpha [X, x] + \pi_\alpha [Y, y]\right) \\
&= \inf_{-\infty < x < \infty} \pi_\alpha [X, x] + \inf_{-\infty < y < \infty} \pi_\alpha [Y, y] \\
&= \pi_\alpha [X] + \pi_\alpha [Y].
\end{aligned}$$

B4. Analogously to the proof of B2, for any $-\infty < a < +\infty$ and $-\infty < x < \max[X+a]$, by equation (3.4) we easily see that $\pi_\alpha [X+a, x] = \pi_\alpha [X, x-a] + a$. It follows that

$$\begin{aligned}
\pi_\alpha [X+a] &= \inf_{-\infty < x < \max[X+a]} \pi_\alpha [X+a, x] \\
&= \inf_{-\infty < x < \max[X+a]} \pi_\alpha [X, x-a] + a \\
&= \inf_{-\infty < y < \max[X]} \pi_\alpha [X, y] + a \\
&= \pi_\alpha [X] + a.
\end{aligned}$$

B5. For any $\pi > x$ we write

$$\varphi(t) = \varphi_{x,\pi}(t) = \phi\left(\frac{(t-x)_+}{\pi - x}\right) \qquad \text{for } -\infty < t < +\infty.$$

Since the function $\phi(\cdot)$ is assumed to be convex, the function $\varphi(t)$ is also convex in t. It follows that $\mathrm{E}\varphi(X) \leq \mathrm{E}\varphi(Y)$, that is

$$\mathrm{E}\phi\left(\frac{(X-x)_+}{\pi - x}\right) \leq \mathrm{E}\phi\left(\frac{(Y-x)_+}{\pi - x}\right) \qquad \text{for any } \pi > x.$$

This indicates that $\pi_\alpha [X, x] \leq \pi_\alpha [Y, x]$ holds for all $x \in (-\infty, +\infty)$. Hence, $\pi_\alpha(X) \leq \pi_\alpha(Y)$.

This ends the proof of Theorem 3.2. □

Remark 3.1. *Clearly, for any pair (X, Y) of random variables it holds that* $\max[X+Y] \leq \max[X] + \max[Y]$. *But in the proof of Property B3 we crucially applied assumption (3.13) because in case* $\max[X+Y] < \max[X] + \max[Y]$ *inequality (3.14) doesn't hold for those pairs (x, y) from a nonempty region*

$$A = \{(x, y) : x < \max[X],\ y < \max[Y],\ x+y > \max[X+Y]\}.$$

However, (3.13) is a very mild restriction in view that it holds for each of the following cases:

1. X and Y are independent;

2. X and Y are comonotonic;

3. X and Y are such that $\max[X+Y]=\infty$*;*

4. X and Y are weakly associated in the sense that the relation

$$\Pr\left[X+Y>x+y \mid X>x, Y>y\right]>0$$

holds for any choice of (x,y) *such that* $\Pr\left[X>x, Y>y\right]>0$.

The Orlicz and Haezendonck insurance premium principles and/or risk measures have some interesting ordering consequences. Now we consider the following definition.

Definition 3.2. *Let* $\phi_1(\cdot)$ *and* $\phi_2(\cdot)$ *be two real functions on* $(0,+\infty)$*. We say* $\phi_2(\cdot)$ *is convex (concave) in* $\phi_1(\cdot)$ *if and only if* $\phi_2\phi_1^{-1}(\cdot)$ *is convex (concave).*

We have the following result:

Theorem 3.3. *Let* $\phi_i(\cdot)$*,* $i=1,2$*, be two continuous and strictly increasing functions with* $\phi_i(x)=x$ *for* $x\in[0,1]$ *and* $\phi_i(+\infty)=+\infty$*, let* $\pi_\alpha^{(i)}\left[X,x\right]$*,* $i=1,2$*, be the solutions of (3.4) with* $\phi_i(\cdot)$ *and let the corresponding Haezendonck risk measures be*

$$\pi_\alpha^{(i)}\left[X\right]=\inf_{-\infty<x<\max[X]}\pi_\alpha^{(i)}\left[X,x\right], \qquad i=1,2.$$

1). If $\phi_2(\cdot)$ *is convex in* $\phi_1(\cdot)$ *then* $\pi_\alpha^{(1)}\left[X,x\right]\le\pi_\alpha^{(2)}\left[X,x\right]$*, hence* $\pi_\alpha^{(1)}\left[X\right]\le\pi_\alpha^{(2)}\left[X\right]$*;*
2). If $\phi_2(\cdot)$ *is concave in* $\phi_1(\cdot)$ *then* $\pi_\alpha^{(1)}\left[X,x\right]\ge\pi_\alpha^{(2)}\left[X,x\right]$*, hence* $\pi_\alpha^{(1)}\left[X\right]\ge\pi_\alpha^{(2)}\left[X\right]$*.*

Proof. We only give the proof of the first result since the proof for the second one can be given similarly. By the definition of $\pi_\alpha^{(2)}\left[X,x\right]$, we have

$$\mathrm{E}\left[\phi_2\left(\frac{(X-x)_+}{\pi_\alpha^{(2)}\left[X,x\right]-x}\right)\right]=1-\alpha. \tag{3.15}$$

Since the compound function $\phi_2\phi_1^{-1}(\cdot)$ is convex on $(0,+\infty)$, by Jensen's inequality we obtain

$$\begin{aligned}\mathrm{E}\left[\phi_2\left(\frac{(X-x)_+}{\pi_\alpha^{(1)}\left[X,x\right]-x}\right)\right]&=\mathrm{E}\left[\phi_2\phi_1^{-1}\phi_1\left(\frac{(X-x)_+}{\pi_\alpha^{(1)}\left[X,x\right]-x}\right)\right]\\&\ge\phi_2\phi_1^{-1}\left(\mathrm{E}\left[\phi_1\left(\frac{(X-x)_+}{\pi_\alpha^{(1)}\left[X,x\right]-x}\right)\right]\right)\\&=\phi_2\phi_1^{-1}\left(1-\alpha\right)\\&=1-\alpha.\end{aligned} \tag{3.16}$$

Comparing (3.16) with (3.15) yields that $\pi_\alpha^{(2)}\left[X,x\right]\ge\pi_\alpha^{(1)}\left[X,x\right]$. This ends the proof of Theorem 3.3. □

The following is an immediate consequence of Theorem 3.3, indicating that the Tail-VaR characterizes the intermediate case among the Haezendonck risk measures based on convex or concave functions $\phi(\cdot)$ on $(0, +\infty)$:

Corollary 3.1. *The Tail-VaR, which is defined by*

$$TVaR_{\alpha}\left[X\right] = F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\mathrm{E}\left[\left(X - F_X^{-1}(\alpha)\right)_+\right], \qquad \alpha \in (0,1),$$

is the smallest one among those Haezendonck risk measures $\pi_{\alpha}[X]$ that correspond to strictly increasing and convex *functions $\phi(\cdot)$ satisfying $\phi(x) = x$ for $0 < x < 1$, and is the largest one among those Haezendonck risk measures $\pi_{\alpha}[X]$ that correspond to strictly increasing and* concave *functions $\phi(\cdot)$ satisfying $\phi(x) = x$ for $0 < x < 1$ and $\phi(+\infty) = +\infty$.*

Proof. In the notation of Theorem 3.3, choose $\phi_1(x) = x$ for $x \in (0, \infty)$, then by equation (3.4) we derive that

$$\pi_{\alpha}^{(1)}\left[X, x\right] = x + \frac{1}{1-\alpha}\mathrm{E}\left[(X - x)_+\right] \qquad \text{for all } -\infty < x < \max[X].$$

Taking infimum over the range $x \in (-\infty, \max[X])$ yields that $\pi_{\alpha}^{(1)}[X] = TVaR_{\alpha}[X]$. Hence, the corollary can be proved by Theorem 3.3. □

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References

[1] Bühlmann, H. Mathematical methods in risk theory. Springer-Verlag, New York-Berlin, 1970.

[2] Bühlmann, H.; Gagliardi, B.; Gerber, H. U.; Straub, E. Some inequalities for stop-loss premiums. Astin Bulletin 9 (1977), no. 1, 75–83.

[3] Dhaene, J.; Denuit, M.; Goovaerts, M. J.; Kaas, R.; Vyncke, D. The concept of comonotonicity in actuarial science and finance: theory. Insurance: Mathematics & Economics 31 (2002a), no. 1, 3–33.

[4] Dhaene, J.; Denuit, M.; Goovaerts, M. J.; Kaas, R.; Vyncke, D. The concept of comonotonicity in actuarial science and finance: applications. Insurance: Mathematics & Economics 31 (2002b), no. 2, 133–161.

[5] Gerber, H. U. An introduction to mathematical risk theory. University of Pennsylvania, Philadelphia, 1979.

[6] Gerber, H. U. On additive principles of zero utility. Insurance: Mathematics & Economics 4 (1985), no. 4, 249–251.

[7] Goovaerts, M. J.; De Vijlder, F.; Haezendonck, J. Insurance premiums. Theory and applications. North-Holland Publishing Co., Amsterdam, 1984.

[8] Goovaerts, M. J.; Kaas, R.; Dhaene, J. Economic capital allocation derived from risk measures, North American Actuarial Journal 7 (2003a), no. 2. 44–59.

[9] Goovaerts, M. J.; Kaas, R.; Dhaene, J.; Tang, Q. A unified approach to generate risk measures. Astin Bulletin 33 (2003b), no. 2, 173–191.

[10] Haezendonck, J.; Goovaerts, M. A new premium calculation principle based on Orlicz norms. Insurance: Mathematics & Economics 1 (1982), no. 1, 41–53.

[11] Kaas, R.; Goovaerts, M. J.; Dhaene, J.; Denuit, M. Modern actuarial risk theory. Dordrecht: Kluwer Acad. Publ., 2001.

[12] Wang, S. S. Premium calculation by transforming the layer premium density. Astin Bulletin 26 (1996), no. 1, 71–92.

[13] Wang, S. S.; Young, V. R. Ordering risks: expected utility theory versus Yaari's dual theory of risk. Insurance: Mathematics & Economics 22 (1998), no. 2, 145–161.

[14] Yaari, M. E. The dual theory of choice under risk. Econometrica 55 (1987), no. 1, 95–115.