An accurate analytical approximation for the price of a European-style arithmetic Asian option

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Abstract

For discrete arithmetic Asian options the payoff depends on the price average of the underlying asset. Due to the dependence structure between the prices of the underlying asset, no simple exact pricing formula exists, not even in a Black-Scholes setting. In the recent literature, several approximations and bounds for the price of this type of option are proposed. One of these approximations consists of replacing the distribution of the stochastic price average by an ad hoc distribution (e.g. Lognormal or Inverse Gaussian) with the same first and second moment. In this paper we use a different approach and combine a lower and upper bound into a new analytical approximation. This approximation can be calculated efficiently, turns out to be very accurate and moreover, it has the correct first and second moment. Since the approximation is analytical, we can also calculate the corresponding hedging Greeks and construct a replicating strategy.

1 Introduction

Consider a risky asset (a non-dividend paying stock) with prices described by the stochastic process $\{A(t),\ t \geq 0\}$ and a risk-free continuously compounded rate δ that is constant through time. In this section all probabilities and expectations have to be considered as conditional on the information available

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at time 0, i.e. the prices of the risky asset up to time 0, unless otherwise stated. Note that in general, the conditional expectation (with respect to the physical probability measure) of $e^{-\delta t}A(t)$, given the information available at time 0, will differ from the current price $A(0)$. However, we will assume that we can find a unique equivalent probability measure Q such that the discounted price process $\{e^{-\delta t}\ A(t),\ t \geq 0\}$ is a martingale under this equivalent probability measure. This implies that for any $t \geq 0$, the conditional expectation (with respect to the equivalent martingale measure) of $e^{-\delta t}A(t)$, given the information $\mathcal{F}_0$ available at time 0, will be equal to the current price $A(0)$. Denoting this conditional expectation under the equivalent martingale measure by $\mathrm{E}^Q\left[e^{-\delta t}A(t)\right]$, we have that

$$\mathrm{E}^Q\left[e^{-\delta t}A(t)|\mathcal{F}_0\right] = A(0), \qquad t \geq 0. \tag{1}$$

The notation $F_{A_0(t)}(x)$ will be used for the conditional probability that $A(t)$ is smaller than or equal to x, under the equivalent martingale measure Q and given the information $\mathcal{F}_0$ available at time 0. Its inverse will be denoted by $F^{-1}_{A_0(t)}(p)$. The existence of an equivalent martingale measure is related to the absence of arbitrage in the securities market, while uniqueness of the equivalent martingale measure is related to market completeness. Two models incorporating such a unique equivalent martingale measure are the binomial tree model of Cox, Ross and Rubinstein (1979) and the geometric Brownian motion model of Black and Scholes (1973). The existence of the equivalent martingale measure allows one to reduce the pricing of options on the risky asset to calculating expected values of the discounted pay-offs, not with respect to the physical probability measure, but with respect to the equivalent martingale measure, see e.g. Harrison and Kreps (1979) or Harrison and Pliska (1981). A reference in the actuarial literature is Gerber and Shiu (1996).

A European call option on the risky asset, with exercise price K and exercise date T generates a pay-off $(A(T) - K)_+$ at time T, that is, if the price of the risky asset at time T exceeds the exercise price, the pay-off equals the difference; if not, the pay-off is zero. Note the similarity between such a pay-off and the payment on a stop-loss reinsurance contract. At time t this call option will trade against a price given by

$$\mathrm{EC}(K, T, t) = e^{-\delta(T-t)}\,\mathrm{E}^Q\left[(A_t(T) - K)_+\right] \tag{2}$$

A European-style arithmetic Asian call option with exercise date T, n averaging dates and exercise price K generates a pay-off $\left(\frac{1}{n}\sum_{i=0}^{n-1} A_T(T-i) - K\right)_+$

at T, that is, if the average of the prices of the risky asset at the latest n dates before T is more than K, the pay-off equals the difference; if not, the pay-off is zero. Such options protect the holder against manipulations of the asset price near the expiration date. The price of the Asian call option at time t is given by

$$\mathrm{AC}(n,K,T,t) = e^{-\delta(T-t)}\,\mathrm{E}^Q\left[\left(\frac{1}{n}\sum_{i=0}^{n-1} A_t(T-i) - K\right)_+\right] \tag{3}$$

and the price of the Asian put option at time t equals

$$\mathrm{AP}(n,K,T,t) = e^{-\delta(T-t)}\,\mathrm{E}^Q\left[\left(K - \frac{1}{n}\sum_{i=0}^{n-1} A_t(T-i)\right)_+\right].$$

Note that, due to the put-call parity, the price of an Asian put option can be easily derived from the price of an Asian call option:

$$\mathrm{AP}(n,K,T,t) = \mathrm{AC}(n,K,T,t) + \frac{e^{-\delta(T-t)}}{n}\left(K - \sum_{i=0}^{n-1}\mathrm{E}[A_t(T-i)]\right)$$

Hence, in the remainder we will only consider call options.

We can also assume that $T-n+1>t$. Indeed, if at time t the averaging has already started, i.e. $T-n+1\le t$, then we know the prices $A(T-n+1),\dots,A(\bar{t})$, where $\bar{t}$ denotes the integer part of t. Since at time t the prices $A(\bar{t}+1),\dots,A(T)$ are still random, we can write

$$\begin{aligned}\mathrm{AC}(n,K,T,t) &= e^{-\delta(T-t)}\frac{T-\bar{t}}{n}\,\mathrm{E}^Q\left[\left(\frac{1}{T-\bar{t}}\sum_{i=0}^{T-\bar{t}-1} A_t(T-i) - K'\right)_+\right] \\ &= \frac{n'}{n}\,\mathrm{AC}(n',K',T,t) \end{aligned}\tag{4}$$

where

$$n' = \min(T-\bar{t},n) \quad \text{and} \quad K' = \frac{nK - \sum_{i=T-\bar{t}}^{n-1} A_t(T-i)}{n'}.$$

With these rescaled parameters the averaging has not yet started since $T-n'+1>t$.

Determining the price of an Asian option is not a trivial task, because in general we do not have an explicit analytical expression for the distribution of the average $S_t = \sum_{i=0}^{n-1} A_t(T-i)$. One can use Monte-Carlo simulation techniques to obtain a numerical estimate of the price, see Kemna and Vorst (1990) and Vazquez-Abad and Dufresne (1998), or one can numerically solve a parabolic partial differential equation, see Rogers and Shi (1995). But as both approaches are rather time consuming, it would be helpful to have an accurate, easily computable analytical approximation of this price. In Jacques (1996) an approximation is obtained by replacing the distribution of the sum $\sum_{i=0}^{n-1} A(T-i)$ by a Lognormal or an Inverse Gaussian distribution.

From the expression for the price of an arithmetic Asian call option, we see that the problem of pricing such options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent random variables. This means that we can apply the results of Dhaene et al. (2002b,a) in order to find analytical lower and upper bounds for the price of Asian options. By combining these bounds a new approximation arises.

2 Bounds and approximations

Assume that at time t the averaging has not yet started and thus $A_t(T-n+1), \ldots, A_t(T)$ are random. The price $\mathrm{AC}(n, K, T, t)$ then essentially consists of a stop-loss premium of a sum of n dependent random variables. In Dhaene et al. (2002b,a) it is shown how to construct upper and lower bounds for such stop-loss premiums by using the theory on comonotonic risks.

In the actuarial literature it is common practice to replace a random variable by a "less attractive" random variable which has a simpler structure, making it easier to determine its distribution function, see e.g. Goovaerts et al. (1990), Kaas et al. (1994) or Denuit et al. (1999). Performing the computations (of premiums, reserves and so on) with the less attractive random variable is a prudent strategy. On the other, considering more attractive random variables could help to give an idea about the degree of overestimation involved in replacing the original variable by the less attractive random variable. Of course, we have to clarify what we mean by a less attractive random variable.

Definition 1. *Consider two random variables X and Y. Then X is said to precede Y in the stop-loss order sense, notation $X \leq_{s\ell} Y$, if and only if X*

has lower stop-loss premiums than Y:

$$\mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+], \qquad -\infty < d < +\infty.$$

Hence, $X \leq_{s\ell} Y$ means that X has uniformly smaller upper tails than Y, which in turn means that a payment Y is indeed less attractive than a payment X. Stop-loss order has a natural economic interpretation in terms of expected utility. Indeed, it can be shown that $X \leq_{s\ell} Y$ if and only if $\mathrm{E}[u(-X)] \geq \mathrm{E}[u(-Y)]$ holds for all non-decreasing concave real functions u for which the expectations exist. This means that any risk-averse decision maker would prefer to pay X instead of Y, which implies that acting as if the obligations X are replaced by Y indeed leads to conservative or prudent decisions. The characterization of stop-loss order in terms of utility functions is equivalent to $\mathrm{E}[v(X)] \leq \mathrm{E}[v(Y)]$ holding for all non-decreasing convex functions v for which the expectations exist. Therefore, stop-loss order is often called "increasing convex order" and denoted by $\leq_{icx}$. For more details and properties of stop-loss order in a general context, see Shaked and Shanthikumar (1994) or Kaas et al. (1994), where stochastic orders are considered in an actuarial context.

If $X \leq_{s\ell} Y$, then also $\mathrm{E}[X] \leq \mathrm{E}[Y]$, and it is intuitively clear that the best approximations arise in the borderline case where $\mathrm{E}[X] = \mathrm{E}[Y]$. This leads to the so-called convex order.

Definition 2. *Consider two random variables X and Y. Then X is said to precede Y in the convex order sense, notation $X \leq_{cx} Y$, if and only if $\mathrm{E}[X] = \mathrm{E}[Y]$ and*

$$\mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+], \qquad -\infty < d < +\infty$$

From $\mathrm{E}[(X-d)_+] - \mathrm{E}[(d-X)_+] = \mathrm{E}[X] - d$, we find

$$X \leq_{cx} Y \Leftrightarrow \begin{cases} \mathrm{E}[X] = \mathrm{E}[Y], \\ \mathrm{E}[(d-X)_+] \leq \mathrm{E}[(d-Y)_+], \qquad -\infty < d < +\infty. \end{cases}$$

Note that partial integration leads to

$$\mathrm{E}[(d-X)_+] = \int_{-\infty}^{d} F_X(x)\, dx,$$

which means that $\mathrm{E}[(d-X)_+]$ can be interpreted as the weight of a lower tail of X. We have seen that stop-loss order entails uniformly heavier upper tails. The additional condition of equal means implies that convex order also leads to uniformly heavier lower tails.

It can be proven that $X \leq_{cx} Y$ if and only $\mathrm{E}[v(X)] \leq \mathrm{E}[v(Y)]$ for all convex functions v, provided the expectations exist. This explains the name "convex" order. Note that when characterizing stop-loss order, the convex functions v are additionally required to be non-decreasing. Hence, stop-loss order is weaker: more pairs of random variables are ordered. We also find that $X \leq_{cx} Y$ if and only $\mathrm{E}[X] = \mathrm{E}[Y]$ and $E[u(-X)] \geq E[u(-Y)]$ for all non-decreasing concave functions u, provided the expectations exist. Hence, in a utility context, convex order represents the common preferences of all risk-averse decision makers between random variables with equal mean.

In case $X \leq_{cx} Y$, the upper tails as well as the lower tails of Y eclipse the corresponding tails of X, which means that extreme values are more likely to occur for Y than for X. This observation also implies that $X \leq_{cx} Y$ is equivalent to $-X \leq_{cx} -Y$. Hence, the interpretation of the random variables as payments or as incomes is irrelevant for the convex order.

As the function v defined by $v(x) = x^2$ is a convex function, it follows immediately that $X \leq_{cx} Y$ implies $\mathrm{Var}[X] \leq \mathrm{Var}[Y]$. The reverse implication does not hold in general.

For the problem at hand, we have the following result.

Theorem 1. *Consider a sum S of random variables $X_1, X_2, \ldots, X_n$ and define*

$$S^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U) \tag{5}$$

$$S^u = F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \ldots + F_{X_n|\Lambda}^{-1}(U) \tag{6}$$

$$S^\ell = \mathrm{E}[X_1 \mid \Lambda] + \mathrm{E}[X_2 \mid \Lambda] + \ldots + \mathrm{E}[X_n \mid \Lambda] \tag{7}$$

with $U \sim$ Uniform(0,1) and with Z an arbitrary random variable, independent of U. The following relations then hold:

$$\mathrm{E}[(S^\ell - t)_+] \leq \mathrm{E}[(S - t)_+] \leq \mathrm{E}[(S^u - t)_+] \leq \mathrm{E}[(S^c - t)_+], \tag{8}$$

for all real t, and $\mathrm{E}[S^\ell] = \mathrm{E}[S] = \mathrm{E}[S^u] = \mathrm{E}[S^c]$.

These lower and upper bounds can be considered as approximations for the distribution of a sum S of random variables. On the other hand, any convex combination of the stop-loss premiums of the lower bound S^ℓ and the upper bounds S^c or S^u too could serve as an approximation for the stop-loss premium of S. Since the bounds S^ℓ and S^c have the same mean as S, any random variable S^m defined by its stop-loss premiums

$$\mathrm{E}[(S^m - t)_+] = z\,\mathrm{E}[(S^\ell - t)_+] + (1-z)\,\mathrm{E}[(S^c - t)_+], \qquad 0 \le z \le 1,$$

will also have the same mean as S. By taking the (right-hand) derivative we find

$$F_{S^m}(x) = zF_{S^\ell}(x) + (1-z)F_{S^c}(x), \qquad 0 \le z \le 1,$$

so the distribution function of the approximation can be calculated fairly easily. By choosing the optimal weight z, we want S^m to be as *close* as possible to S. A natural optimality criterion could be to choose z such that

$$\int_{-\infty}^{\infty} \mathrm{E}[(S^m - t)_+] - \mathrm{E}[(S - t)_+] dt = 0.$$

Since S, S^c, S^ℓ and S^m all have the same mean, the relation (see Kaas et al. (2001))

$$\mathrm{Var}[X] = 2\int_{-\infty}^{\infty} (\mathrm{E}[(X - t)_+] - (\mathrm{E}[X] - t)_+) dt$$

implies that the optimal weight in this case equals

$$z = \frac{\mathrm{Var}[S^c] - \mathrm{Var}[S]}{\mathrm{Var}[S^c] - \mathrm{Var}[S^\ell]}. \tag{9}$$

Note that this choice doesn't depend on the retention t. Choosing z as in (9), we have that

$$\mathrm{Var}[S^m] = \mathrm{Var}[S]$$

so the optimal approximation S^m can also be seen to be a moments based approximation. As an alternative one could consider the improved upper bound S^u and define a second approximation as follows

$$\mathrm{E}[(S^{m2} - t)_+] = z\,\mathrm{E}[(S^\ell - t)_+] + (1-z)\,\mathrm{E}[(S^u - t)_+],$$

with $z = (\mathrm{Var}[S^u] - \mathrm{Var}[S])/(\mathrm{Var}[S^u] - \mathrm{Var}[S^\ell])$. For a comparison of these approximations we refer to Section 4 where it is shown that S^m and S^{m2} yield almost the same results.

This technique could also be used in other actuarial problems concerning sums of (dependent) random variables. Such a sum appears for instance when considering discounted payments related to a single policy or a portfolio at different future points in time, i.e. when combining the (actuarial) technical risk with the (financial) investment risk, see Dhaene et al. (2002b,a).

3 Application in a Black & Scholes setting

In the model of Black and Scholes (1973), the price of the risky asset is described by a stochastic process $\{A(t), t \geq 0\}$ following a geometric Brownian motion with constant drift μ and constant volatility σ:

$$\frac{dA(t)}{A(t)} = \mu dt + \sigma d\overline{B}(t), \qquad t \geq 0, \tag{10}$$

with initial value $A(0) > 0$ and where $\{\overline{B}(t),\ t \geq 0\}$ is a standard Brownian motion.

Under the equivalent martingale measure Q, the price process $\{A(t),\ t \geq 0\}$ still follows a geometric Brownian motion with the same volatility but with drift equal to the continuously compounded risk-free interest rate δ:

$$\frac{dA(t)}{A(t)} = \delta dt + \sigma dB(t), \qquad t \geq 0, \tag{11}$$

with initial value $A(0)$ and where $\{B(t),\ t \geq 0\}$ is a standard Brownian motion in the Q-dynamics. Hence, under the equivalent martingale measure, we have that

$$A_0(t) = A(0)\ e^{\left(\delta - \frac{\sigma^2}{2}\right)t + \sigma B(t)}, \qquad t \geq 0. \tag{12}$$

This implies that under the equivalent martingale measure, the random variables $\frac{A_0(t)}{A(0)}$ are lognormally distributed with parameters $\left(\delta - \frac{\sigma^2}{2}\right) t$ and $t\sigma^2$ respectively:

$$F_{A_0(t)}(x) = \text{Prob}\left[A(0)e^{(\delta - \frac{\sigma^2}{2})t + \sqrt{t}\sigma\Phi^{-1}(U)} \leq x\right], \tag{13}$$

where U is uniformly distributed on the interval $(0, 1)$.

Using the famous Black and Scholes (1973) pricing formula for European call options, we find

$$\begin{aligned} \text{EC}(K, T, t) &= e^{-\delta(T-t)}\, \text{E}^Q\left[(A_t(T) - K)_+\right] \\ &= A(t)\Phi(d_1) - Ke^{-\delta(T-t)}\Phi(d_2), \end{aligned} \tag{14}$$

where d_1 and d_2 are given by

$$d_1 = \frac{\ln(A(t)/K) + (\delta + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \tag{15}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{16}$$

Within the Black & Scholes model, no closed form expression is available for the price of an arithmetic Asian call option. Therefore, we will derive bounds and approximations for the price of such options. Because of (4) we will only consider the case that the averaging has not yet started. To avoid lengthy formulas, we will use T_i as a shorthand for $T-t-i$.

Comonotonic upper bound

From (5) we find the following comonotonic upper bound for the price of an Asian call option:

$$\begin{aligned} \mathrm{AC}(n,K,T,t) &\leq \frac{e^{-\delta(T-t)}}{n} \mathrm{E}\left[(S_t^c - nK)_+\right] \\ &= \frac{A(t)}{n}\sum_{i=0}^{n-1} e^{-\delta i}\Phi\left[\sigma\sqrt{T_i} - \Phi^{-1}\left(F_{S_t^c}(nK)\right)\right] \\ &\quad -e^{-\delta(T-t)}\ K\ \left(1 - F_{S_t^c}(nK)\right), \end{aligned} \tag{17}$$

which holds for any $K > 0$. Note that this upper bound corresponds to the optimal linear combination of the prices of European call options as mentioned in Dhaene et al. (2002a).

The remaining problem is how to calculate $F_{S_t^c}(nK)$. This quantity follows from

$$\sum_{i=0}^{n-1} F_{A_t(T-i)}^{-1}\left(F_{S_t^c}(nK)\right) = nK,$$

or, equivalently, from (12) we find that $F_{S_t^c}(nK)$ follows from

$$A(t)\sum_{i=0}^{n-1} \exp\left[\left(\delta - \frac{\sigma^2}{2}\right)T_i + \sigma\sqrt{T_i}\ \Phi^{-1}\left(F_{S_t^c}(nK)\right)\right] = nK.$$

Lower bound

For the lower bounds for $\mathrm{AC}(n,K,T,t)$ we consider the conditioning random variable Λ defined by

$$\Lambda = \sum_{j=0}^{n-1} e^{\left(\delta - \frac{\sigma^2}{2}\right)T_j} B_t(T-j). \tag{18}$$

From (12) we find that, in the Q-dynamics,

$$\sum_{i=0}^{n-1} A_t(T-i) = A(t)\sum_{i=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)T_i+\sigma B_t(T-i)}. \tag{19}$$

So, Λ is a linear transformation of a first order approximation to $\sum_{i=0}^{n-1} A_t(T-i)$. The variance of Λ is given by

$$\sigma_\Lambda^2 = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)(T_j+T_k)} \min(T_j, T_k). \tag{20}$$

We have that $(B_t(T-n+1), B_t(T-n+2), \ldots, B_t(T))$ has a multivariate normal distribution. This implies that given $\Lambda = \lambda$, the random variable $B_t(T-i)$ is normally distributed with mean $r_i \frac{\sqrt{T_i}}{\sigma_\Lambda}\lambda$ and variance $T_i\left(1-r_i^2\right)$ where

$$r_i = \frac{\mathrm{Cov}(B_t(T-i),\Lambda)}{\sigma_\Lambda\sqrt{T_i}} = \frac{\sum_{j=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)T_j}\min(T_i,T_j)}{\sigma_\Lambda\sqrt{T_i}}. \tag{21}$$

We find

$$\begin{aligned} S_t^\ell &\overset{d}{=} \mathrm{E}^Q\left[\sum_{i=0}^{n-1} A_t(T-i) \mid \Lambda\right] \\ &\overset{d}{=} A(t)\sum_{i=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}r_i^2\right)T_i+\sigma\ r_i\ \sqrt{T_i}\Phi^{-1}(U)} \end{aligned} \tag{22}$$

where U is uniformly distributed on the unit interval. From this expression, we see that S_t^ℓ is a comonotonic sum of lognormal random variables. Hence, we find the following lower bound for the price of the Asian call option:

$$\begin{aligned} \mathrm{AC}(n,K,T,t) &\geq \frac{e^{-\delta(T-t)}}{n}\,\mathrm{E}\left[\left(S_t^\ell - nK\right)_+\right] \\ &= \frac{A(t)}{n}\sum_{i=0}^{n-1} e^{-\delta i}\ \Phi\left[\sigma r_i\sqrt{T_i} - \Phi^{-1}\left(F_{S_t^\ell}(nK)\right)\right] \\ &\quad -e^{-\delta(T-t)}\ K\ \left(1-F_{S_t^\ell}(nK)\right) \end{aligned} \tag{23}$$

which holds for any $K>0$. In this case, $F_{S_t^\ell}(nK)$ follows from

$$A(t)\sum_{i=0}^{n-1}\exp\left[\left(\delta-\frac{\sigma^2}{2}r_i^2\right)T_i + \sigma r_i\sqrt{T_i}\Phi^{-1}\left(F_{S_t^\ell}(nK)\right)\right] = nK.$$

When the number of averaging dates n equals 1, the Asian call option reduces to a European call option. It is straightforward to prove that in this case the upper and the lower bounds (17) and (23) for the price of the Asian option both reduce to the Black & Scholes formula for the price of the European call option.

Improved upper bound

By Theorem 1 we can also construct a smaller upper bound for $\mathrm{AC}(n, K, T, t)$. We choose $\Lambda = B(T)$ since then the dependence structure of the terms in S^u is almost comonotonic, see Vanmaele et al. (2002). This yields

$$\mathrm{AC}(n, K, T, t) \leq \frac{e^{-\delta(T-t)}}{n} \mathrm{E}\left[(S_t^u - nK)_+\right]$$

with

$$\begin{aligned} &\mathrm{E}\left[(S_t^u - nK)_+\right] \\ &= -nK(1 - F_{S_t^u}(nK)) + \sum_{i=0}^{n-1} A(t) \exp\left[\left(\delta - \frac{\sigma^2}{2} r_i^2\right) T_i\right] \times \\ &\quad \int_0^1 e^{\sigma r_i \sqrt{T_i} \Phi^{-1}(v)} \Phi\left(\sigma\sqrt{1 - r_i^2}\sqrt{T_i} - \Phi^{-1}(F_{S_t^u|V=v}(nK))\right) dv. \end{aligned}$$

where the correlation coefficients r_i are given by

$$r_i = \frac{\mathrm{Cov}(B_t(T-i), \Lambda)}{\sigma_\Lambda \sqrt{T_i}} = \frac{\sqrt{T-t-i}}{\sqrt{T-t}}.$$

The conditional distribution $F_{S_t^u|V=v}(nK)$ follows from

$$A(t) \sum_{i=0}^{n-1} e^{(\delta - \frac{\sigma^2}{2})T_i + \sigma r_i \sqrt{T_i} \Phi^{-1}(v) + \sigma\sqrt{1-r_i^2}\sqrt{T_i}\Phi^{-1}(F_{S_t^u|V=v}(nK)} = nK.$$

Moments based approximations

Several authors propose to replace the unknown distribution of S by an ad hoc distribution with the correct first two moments. The question remains which distribution one should use. For 'reasonable' values of the parameters, Levy (1992) substantiates the lognormal distribution as an approximation for the distribution of a sum of lognormal random variables. Jacques (1996)

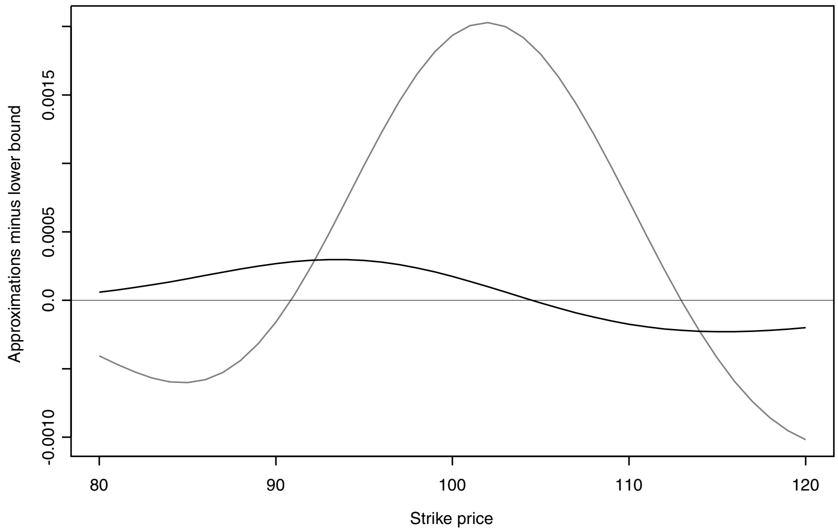

Figure 1: Price of an Asian option with $T = 120$, $n = 30$ and $\sigma = 0.2$ according to the lognormal approximation (black line) and the inverse gaussian approximation (grey line) minus the lower bound (23). Negative values show that the approximations perform worse than the lower bound.

concludes that an Inverse Gaussian approximation gives prices comparable to those given by the lognormal approximation when the parameters are chosen in the same range as in Levy (1992).

Although these approximations appear to be very accurate, they have two structural disadvantages. First, for some values of the parameters, the approximations turn out to be smaller than our theoretical lower bound, see Figure 1. Moreover, if a different process is used to model the stock prices, the approximations will not be valid anymore. By using the moments based approximation from Section 2 these drawbacks can be avoided.

A first approximation can be obtained by combining the lower bound S^ℓ and the comonotonic upper bound S^c. For the variance of S^ℓ and S^c we find

$$\mathrm{Var}[S^\ell] = A^2(t) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{\delta(T_i+T_j)} \left(e^{\sigma^2 r_i r_j \sqrt{T_i T_j}} - 1 \right)$$

and

$$\mathrm{Var}[S^c] = A^2(t) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{\delta(T_i+T_j)} \left(e^{\sigma^2 \sqrt{T_i T_j}} - 1 \right).$$

As shown in Section 2, the random variable $S^m = zS^\ell + (1-z)S^c$ will have the correct variance

$$\mathrm{Var}[S] = A^2(t) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{\delta(T_i+T_j)} \left(e^{\sigma^2 \min(T_i,T_j)} - 1 \right)$$

if we choose

$$z = \frac{\mathrm{Var}[S^c] - \mathrm{Var}[S]}{\mathrm{Var}[S^c] - \mathrm{Var}[S^\ell]}. \tag{24}$$

Replacing the comonotonic upper bound S^c by the improved upper bound S^u yields a second approximation $S^{m_2} = z_u S^\ell + (1-z_u)S^u$. In this case, the weight z_u is given by

$$z = \frac{\mathrm{Var}[S^u] - \mathrm{Var}[S]}{\mathrm{Var}[S^u] - \mathrm{Var}[S^\ell]}$$

where

$$\mathrm{Var}[S^u] = A^2(t) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{\delta(T_i+T_j)} \left(e^{\sigma^2 \left(r_i r_j + \sqrt{(1-r_i^2)(1-r_j^2)} \right) \sqrt{T_i T_j}} - 1 \right).$$

In the following section, we will show that the approximations S^m and S^{m_2} give almost equal prices. Hence, we propose to use S^m, since the computation of S^{m_2} involves much more calculations.

4 Numerical illustration

In this section we numerically illustrate the bounds and approximations for the price of an Asian option in a Black & Scholes setting, as obtained in the previous section. We consider a time unit of one day and set $t = 0$. The parameters that were used to generate the results given in Tables 1, 2 and 3 are the same as in Jacques (1996): an initial stock price $A(0) = 100$, a risk-free interest rate of 9% per year, three values (0.2, 0.3 and 0.4) for the yearly volatility, and five values (80, 90, 100, 110 and 120) for the exercise price K. Note that the risk-free force of interest per day is given by $\delta = \ln(1.09)/365$, while the daily volatility σ is obtained by dividing the yearly volatility by $\sqrt{365}$.

In Table 1 we compare the bounds and approximations with Monte Carlo estimates (based on 10000 paths each) in case $T = 120$ and $n = 30$. Note

σ	K	LB	MB	MB2	LN	IG	IUB	UB	MC (s.e.$\times 10^4$)
0.2	80	21.9212	21.9212	21.9212	21.9213	21.9208	21.9246	21.9269	21.9213 (2.51)
	90	12.6768	12.6768	12.6768	12.6771	12.6767	12.7038	12.7204	12.6764 (2.36)
	100	5.4609	5.4609	5.4609	5.4611	5.4629	5.5200	5.5557	5.4616 (2.31)
	110	1.6252	1.6252	1.6252	1.6250	1.6259	1.6762	1.7072	1.6250 (1.64)
	120	0.3317	0.3317	0.3317	0.3315	0.3307	0.3536	0.3673	0.3319 (1.15)
0.3	80	22.2332	22.2332	22.2332	22.2336	22.2313	22.2571	22.2720	22.2340 (5.69)
	90	13.8521	13.8521	13.8521	13.8528	13.8544	13.9137	13.9512	13.8519 (5.48)
	100	7.4787	7.4788	7.4788	7.4791	7.4855	7.5686	7.6229	7.4800 (5.36)
	110	3.4826	3.4827	3.4827	3.4825	3.4875	3.5690	3.6214	3.4829 (4.60)
	120	1.4125	1.4126	1.4126	1.4121	1.4123	1.4733	1.5105	1.4124 (3.52)
0.4	80	22.9646	22.9646	22.9646	22.9658	22.9638	23.0190	23.0525	22.9665 (10.05)
	90	15.3589	15.3589	15.3589	15.3602	15.3682	15.4539	15.5115	15.3600 (9.82)
	100	9.5113	9.5114	9.5114	9.5121	9.5277	9.6315	9.7041	9.5118 (9.05)
	110	5.4794	5.4795	5.4795	5.4795	5.4936	5.5994	5.6720	5.4794 (8.33)
	120	2.9608	2.9609	2.9609	2.9603	2.9666	3.0611	3.1222	2.9614 (7.36)

Table 1: Upper (UB, IUB) and lower bounds (LB) for the price of an Asian option at $t = 0$ with $T = 120$ and $n = 30$, compared to the Monte Carlo estimates (MC) with their standard error times 10000 (s.e.$\times 10^4$) and the moments based approximations (MB, MB2, LN, IG)

that the random paths are based on antithetic variables and that we use the geometric average as a control variate in order to reduce the variance of the Monte Carlo estimate. Also note that we generated different paths for each value of σ and K. For each estimate we computed the standard error. As is well-known, the (asymptotic) 95% confidence interval is given by the estimate plus or minus 1.96 times the standard error. On the other hand, the range between the lower bound and the (improved) upper bound contains the exact price with certainty.

Despite the quite large number of paths (and consequently a long computing time) and the variance reduction techniques used, the Monte Carlo estimate (MC) violates the lower bound (LB) 4 times out of 15. This might indicate that the lower bound is very close to the real price. The moments based approximations all give similar prices, but the lognormal approximation (LN) appears to violate the lower bound for options that are far out-of-the-money. Also the inverse gaussian approximation (IG) violates the lower bound, not only for out-of-the-money options but for in-the-money options too. Although the comonotonic upper bound (UB) and the improved upper bound (IUB) give quite different prices, the corresponding moments based approximations (MB, MB2) are almost equal.

In Table 2 we use the same parameters as in Table 1 but we change the expiration time to $T = 60$. In this case, the Monte Carlo estimate

σ	K	LB	MB	MB2	LN	IG	IUB	UB	MC (s.e.$\times 10^4$)
0.2	80	20.7841	20.7841	20.7841	20.7841	20.7841	20.7843	20.7845	20.7841 (2.43)
	90	11.0273	11.0273	11.0273	11.0277	11.0275	11.0470	11.0599	11.0276 (2.43)
	100	3.2013	3.2013	3.2013	3.2016	3.2021	3.2903	3.3443	3.2013 (2.15)
	110	0.3373	0.3373	0.3373	0.3367	0.3366	0.3805	0.4080	0.3372 (1.33)
	120	0.0116	0.0116	0.0116	0.0115	0.0114	0.0156	0.0185	0.0117 (0.55)
0.3	80	20.8122	20.8123	20.8123	20.8126	20.8122	20.8208	20.8268	20.8115 (5.44)
	90	11.4929	11.4929	11.4929	11.4944	11.4942	11.5599	11.6017	11.4931 (5.40)
	100	4.5063	4.5063	4.5063	4.5070	4.5086	4.6406	4.7221	4.5051 (4.55)
	110	1.1516	1.1517	1.1517	1.1505	1.1508	1.2515	1.3134	1.1522 (3.81)
	120	0.1915	0.1915	0.1915	0.1906	0.1898	0.2269	0.2503	0.1912 (1.97)
0.4	80	20.9708	20.9708	20.9708	20.9724	20.9709	21.0072	21.0309	20.9716 (9.57)
	90	12.2468	12.2469	12.2469	12.2498	12.2505	12.3655	12.4384	12.2482 (9.47)
	100	5.8157	5.8159	5.8159	5.8171	5.8210	5.9952	6.1038	5.8155 (8.49)
	110	2.2082	2.2083	2.2083	2.2067	2.2088	2.3630	2.4582	2.2091 (7.63)
	120	0.6783	0.6783	0.6783	0.6761	0.6750	0.7663	0.8223	0.6777 (4.96)

Table 2: Upper (UB, IUB) and lower bounds (LB) for the price of an Asian option at $t = 0$ with $T = 60$ and $n = 30$, compared to the Monte Carlo estimates (MC) with their standard error times 10000 (s.e.$\times 10^4$) and the moments based approximations (MB, MB2, LN, IG)

violates the lower bound 8 times out of 15. So again, the lower bound must be very close to the real price. For the lognormal and the inverse gaussian approximation, we see a similar pattern as in the previous case. The moments based approximations are again almost equal and very close to the lower bound.

In Table 3 we change the expiration time back to $T = 120$ but we reduce the number of averaging days to $n = 10$. With these parameters, the Monte Carlo estimate violates the lower bound 13 times out of 15. The first 4 columns (LB, MB, MB2, LN) are almost equal while the inverse gaussian approximation appears to underestimate the price of in-the-money options and out-of-the-money options.

Comparing the results in Tables 1 and 3, we see that the comonotonic upper bound performs better for the option with $n = 10$ than for the options with $n = 30$. This illustrates the fact that the dependency structure of the $A(T - i)$ is more "comonotonic-like" if all points in time $T - i$ are close to each other.

To assess the overall accuracy of the bounds and approximations, we assume that the Monte Carlo estimate gives the exact price and calculate the total absolute difference of all 45 cases. As can be seen from Table 4, the moments based approximations MB and MB2 both have the smallest error, closely followed by the lower bound. The lognormal approximation

σ	K	LB	MB	MB2	LN	IG	IUB	UB	MC (s.e.$\times 10^4$)
0.2	80	22.1712	22.1712	22.1712	22.1712	22.1706	22.1724	22.1735	22.1712 (0.85)
	90	13.0085	13.0085	13.0085	13.0085	13.0081	13.0162	13.0232	13.0083 (0.81)
	100	5.8630	5.8630	5.8630	5.8630	5.8651	5.8791	5.8934	5.8629 (0.75)
	110	1.9169	1.9169	1.9169	1.9168	1.9181	1.9313	1.9442	1.9168 (0.59)
	120	0.4534	0.4534	0.4534	0.4533	0.4525	0.4603	0.4665	0.4533 (0.33)
0.3	80	22.5656	22.5657	22.5657	22.5657	22.5631	22.5729	22.5795	22.5656 (1.89)
	90	14.3149	14.3149	14.3149	14.3150	14.3172	14.3321	14.3475	14.3148 (1.84)
	100	8.0101	8.0101	8.0101	8.0101	8.0178	8.0346	8.0563	8.0099 (1.72)
	110	3.9475	3.9475	3.9475	3.9475	3.9540	3.9715	3.9928	3.9474 (1.37)
	120	1.7297	1.7297	1.7297	1.7297	1.7307	1.7474	1.7633	1.7297 (1.14)
0.4	80	23.4194	23.4194	23.4194	23.4195	23.4181	23.4351	23.4493	23.4194 (3.43)
	90	15.9549	15.9549	15.9549	15.9550	15.9654	15.9811	16.0045	15.9554 (3.33)
	100	10.1735	10.1735	10.1735	10.1736	10.1925	10.2062	10.2354	10.1733 (3.02)
	110	6.1019	6.1019	6.1019	6.1019	6.1196	6.1349	6.1643	6.1018 (2.80)
	120	3.4683	3.4683	3.4683	3.4682	3.4775	3.4966	3.5220	3.4682 (2.41)

Table 3: Upper (UB, IUB) and lower bounds (LB) for the price of an Asian option at $t = 0$ with $T = 120$ and $n = 10$, compared to the Monte Carlo estimates (MC) with their standard error times 10000 (s.e.$\times 10^4$) and the moments based approximations (MB, MB2, LN, IG)

also performs quite well, while the inverse gaussian approximation yields a total absolute error which is 10 times bigger. In comparison to the lower bound, the upper bounds IUB and UB perform really bad, so we suggest to use them only to construct the moments based approximations.

5 Replicating portfolio

Since the bounds and approximations of Section 3 have an analytical form, we can explicitly calculate the so-called *Greeks* for these approximations. The Greeks are quantities representing the market sensitivities of the options as each Greek measures a different aspect of the risk in an option position. Through understanding and managing these Greeks, market traders can manage their risks appropriately. In this section, we will focus on the *delta* of the option since this quantity allows us to construct a dynamical replicating portfolio. The delta of an option is defined as the rate of change of the option price with respect to the price of the underlying asset, i.e.

$$\Delta(n, K, T, t) = \frac{\partial\, \mathrm{AC}(n, K, T, t)}{\partial A(t)}. \tag{25}$$

Approximation	Total absolute error
Moments based MB	0.0174745
Moments based MB2	0.0174745
Lower bound LB	0.0176689
Lognormal LN	0.0245574
Inverse Gaussian IG	0.1702414
Improved upper bound IUB	2.3196765
Comonotonic upper bound UB	3.8251189

Table 4: Total absolute error of the bounds and approximations in comparison to the Monte Carlo estimate.

The dynamical replicating portfolio consisting of $\Delta(n,K,T,t)$ shares of stock and

$$\Omega(n,K,T,t) = \frac{\mathrm{AC}(n,K,T,t) - \Delta(n,K,T,t)A(t)}{A(0)e^{\delta t}}.$$

shares of the bond, will reproduce the value of the option at maturity. This can be seen as follows: If the stock price drops with a unit amount, then we would lose $\Delta(n,K,T,t)$ on our portfolio. On the other hand, if we invest all our money in options and the underlying stock drops with a unit amount, then we would also lose $\Delta(n,K,T,t)$.

Since we can calculate the delta of the approximations of Section 3 explicitly, we can also use it to assess the quality of the approximations. In order to do that, we will construct the corresponding hedging portfolio and check how well the portfolio replicates the value of the option along a random trajectory of the stock price. For the comonotonic upper bound (17) we find

$$\Delta^c(n,K,T,t) = \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\Phi\left(\sigma\sqrt{T_i} - \Phi^{-1}(F_{S_t^c}(nK))\right)$$

if $T-n+1 > t$. The case $T-n+1 \leq t$ is essentially the same because of (4) but one has to be careful when $\bar{t} = t$. Then also K' is a function of $A(t)$ and we pick up an extra term in the differentiation. Analogously, for the lower bound (23) we find

$$\Delta^\ell(n,K,T,t) = \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\Phi\left(\sigma r_i\sqrt{T_i} - \Phi^{-1}(F_{S_t^\ell}(nK))\right).$$

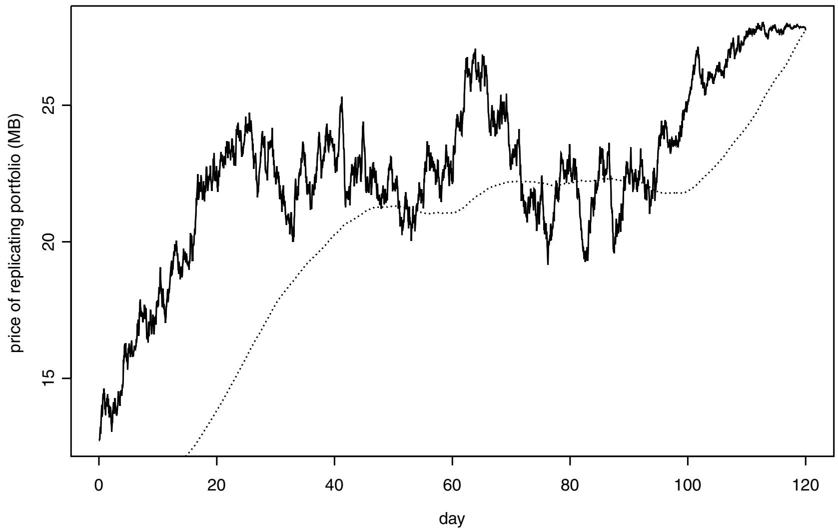

Figure 2: Moments based approximation for the price of an Asian option (black line) along a randomly generated path of the stock price, almost completely eclipsing the price of its replicating portfolio (grey line). The intrinsic value of the option is indicated by a dotted line.

Since z in (24) is independent of $A(t)$, the delta of the moments based approximation S^m equals

$$\Delta^m(n,K,T,t) = z\Delta^\ell(n,K,T,t) + (1-z)\Delta^c(n,K,T,t).$$

We will use this value as an approximation for the real delta (25) and construct the corresponding replicating portfolio. Note that in theory the replicating portfolio ought to be updated continuously, but in practice it will only be updated at discrete times. In our numerical example we will update the portfolio 24 times a day. Using a higher updating frequency doesn't seem to have a significant influence on the results.

Figure 2 shows the hedging portfolio along a randomly generated path of the stock price. The parameters are chosen as in Jacques (1996): an initial stock price $A(0) = 100$, a strike price $K = 90$, a risk-free interest rate $\delta = 9\%$ per year and a yearly volatility $\sigma = 20\%$. The number of days until maturity is set to $T = 120$ and the number of averaging days equals $n = 30$. Recall that for pricing purposes we have to replace the drift parameter μ with the risk-free rate of return δ. In the present setting however, we consider physical paths of the stock price process and hence we have to use the real rate of return μ. We choose this parameter to be 0.15, significantly larger than the risk-free rate of return. The price of the option is also plotted in Figure 2, but it is almost indistinguishable from the hedging portfolio. So, the hedging portfolio replicates the price of the option along the path with very high precision. We also calculate the intrinsic value of the option based on the n-periods moving average. This number indicates how much the option is worth assuming that maturity is attained at that time. For $t < n$, we need values of the stock price at negative times to compute the moving average and we assume that $A(t) = A(0)$ if $t < 0$. The smooth curve in Figure 2 is the intrinsic value at the current time. It can be seen that the hedging portfolio reproduces the intrinsic value of the option at maturity very well.

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