# Comparing Approximations for Risk Measures of Sums of Non-independent Lognormal Random Variables

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**Abstract**

In this paper we consider different approximations for computing the distribution function or risk measures related to a discrete sum of non-independent lognormal random variables. Comonotonic upper bound and lower bound approximations for such sums have been proposed in Dhaene et al. (2002a,b). We introduce the comonotonic "maximal variance" lower bound approximation. We also compare the comonotonic approximations with two well-known moment matching approximations: the lognormal and the reciprocal Gamma approximation. We find that for a wide range of parameter values the comonotonic "maximal variance" lower bound approximation outperforms the other approximations.
**Keywords:** comonotonicity, simulation, lognormal, reciprocal Gamma.

## 1 Introduction

In this paper we will consider and compare the performance of approximations for the distribution function (d.f.) and risk measures related to a random variable (r.v.) $S$ given by

$$S = \sum_{i=1}^{n} \alpha_i \, e^{Z_i}. \tag{1}$$

Here, the $\alpha_i$ are non-negative real numbers and $(Z_1, Z_2, ..., Z_n)$ is a multivariate normal random vector.

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The accumulated value at time $n$ of a series of future deterministic saving amounts $\alpha_i$ can be written in the form (1), where $Z_i$ denotes the random accumulation factor over the period $[i, n]$. Also the present value of a series of future deterministic payments $\alpha_i$ can be written in the form (1), where now $Z_i$ denotes the random discount factor over the period $[0, i]$. We refer to Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004) for more details.

The valuation of Asian or basket options in a Black & Scholes model and the setting of provisions and required capitals in an insurance context boils down to the evaluation of risk measures related to the distribution function of a random variable $S$ as defined in (1).

We will investigate how to (approximately) compute risk measures such as quantiles (Q) and conditional tail expectations (CTE) of the r.v. $S$ defined in (1). These risk measures are defined by

$$\mathrm{Q}_p[S] = \inf\{s \in R | F_S(s) \geq p\}, \qquad p \in (0,1) \tag{2}$$

and

$$\mathrm{CTE}_p[S] = E[S | S > Q_p[S]], \qquad p \in (0,1), \tag{3}$$

where $F_S(s) = \Pr[S \leq s]$ and by convention, $\inf\{\phi\} = +\infty$. Notice that the quantile risk measure is often called the Value-at-Risk, whereas the conditional tail expectation coincides with the Tail-Value-at-Risk. The latter holds true because $S$ is a continuous r.v.; see for instance Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004).

The r.v. $S$ defined in (1) will in general be a sum of non-independent lognormal r.v.'s. Its d.f. cannot be determined analytically and is too cumbersome to work with. In the literature, several techniques for approximating this d.f. have been proposed.

Moment matching methods approximate the unknown d.f. by a given d.f. in such a way that the first moments coincide; see for instance Klugman, Panjer and Willmot (1998) for an inventory of distributions that can be used for this purpose. Within this respect, practitioners often use a moment matching *lognormal approximation* for the distribution of $S$. The lognormal approximation is chosen such that its first two moments are equal to the corresponding moments of $S$.

The present value of a constant continuous perpetuity with lognormal return process has a *reciprocal (or inverse) Gamma distribution*; see for instance Dufresne (1990) or Milevsky (1997). Notice that this present value can be considered as the limiting case of the random variable $S$ as defined above. Motivated by the latter observation, Milevsky & Posner (1998) and Milevsky & Robinson (2000) propose a moment matching *reciprocal Gamma approximation* for the d.f. of $S$ such that the first two moments match. They use this technique for deriving closed form approximations for the price of Asian and basket options. Although these moment matching approximations find their origin in the continuous setting of the discussed problem, it makes sense to to test their accuracy also in the real-life discrete setting that we investigate in this paper.

Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b) derive *comonotonic upper bound and lower bound approximations* (in the convex order sense) for the d.f. of $S$. Especially the lower bound approximation, which is given by $\mathrm{E}[S \mid \Lambda]$ for an appropriate choice of the conditioning r.v. $\Lambda$, is extremely accurate; see for instance Vanduffel, Dhaene, Goovaerts & Kaas (2003). In view of its importance for practical applications we will focus here on deterministic sums of type (1). We point out that in this paper presented approximations can be also generalized to the case of stochastic sums. This is first done by Huang, Milevsky and Wang (2004) for the moment matching approximations and by Hoedemakers, Darkiewicz and Goovaerts (2005) for the comonotonic approximations. Huang, Milevsky & Wang (2004) compare the performance of different approximations for the probability that a person outlives his money in case of a lifelong continuous consumption pattern. As a special case, they also consider approximations for such a probability for the deterministic case, i.e. when the consumption period is fixed. On the other hand, Hoedemakers, Darkiewicz and Goovaerts (2005) use the concept of comonotonicity to derive approximations for the present value of life annuities.

Our paper is most related to Huang, Milevsky and Wang (2004), but instead of comparing the approximations of the ruin probabilities, we will evaluate the performance of the above mentioned techniques by comparing the approximated values of quantiles and conditional tail expectations of r.v.'s $S$ as defined in (1) and in this case $Z_i$ represents the random discount factor over the period $[0, i]$. The results of these paper are then threefold. First, in case of the comonotonic lower bound approximation, we propose a new choice for the conditioning r.v. $\Lambda$ and we provide theoretical evidence that supports this choice. Throughout this paper we will call this the "maximal variance" lower bound approximation. Secondly, we show that for a wide range of reasonable parameter values for the lognormal return process and time horizon, the "maximal variance" lower bound approximation often outperforms the comonotonic upper bound and both moment-matching methods. Finally, we show that even in the limiting case of a constant continuous perpetuity, when $S$ is guaranteed to be reciprocal Gamma distributed, the "maximal variance" lower bound approximation still performs very well. Overall, we believe it is the best all-round candidate to accurately approximate the risk measures of $S$.

The paper is organized as follows. In Section 2, we present the comonotonic approximations and we also focus on the optimal choice of the conditioning r.v. $\Lambda$ in case of the comonotonic lower bound $\mathrm{E}[S \mid \Lambda]$. We also propose in this section a new conditioning r.v. which is likely to make the variance of the approximation 'as close as possible' to the exact variance. In Section 3 we will briefly recall the mathematical techniques behind the reciprocal Gamma and lognormal moment matching techniques. Finally, in Section 4 we compare the comonotonic approximations with the moment matching techniques, using an extensive Monte Carlo simulation as the benchmark.

# 2 Comonotonic approximations

## 2.1 General results

In this section, we briefly repeat some results related to the comonotonic lower and upper bounds for the d.f. of the r.v. $S$ defined in (1). For proofs and more details, we refer to Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b).

A central concept in the theory on comonotonic r.v.'s is the concept of convex order. A r.v. $X$ is said to precede a r.v. $Y$ in the convex order sense, denoted $X \leq_{cx} Y$, if their means are equal and if their corresponding stop-loss premia are ordered uniformly for all retentions $d$, i.e., $E[(X-d)_+] \leq E[(Y-d)_+]$ for all $d$.

Replacing the copula describing the dependency structure of the terms in the sum (1) by the comonotonic copula yields an convex order upper bound for $S$. On the other hand, applying Jensen's inequality to $S$ provides us with a lower bound. These results are formalized in the following theorem, which is taken from Kaas, Dhaene & Goovaerts (2000).

**Theorem 1** *Let the r.v. $S$ be given by (1), where the $\alpha_i$ are non-negative real numbers and the random vector $(Z_1, Z_2, ..., Z_n)$ has a multivariate normal distribution. Consider the conditioning r.v. $\Lambda$, given by*

$$\Lambda = \sum_{i=1}^{n} \gamma_i Z_i. \tag{4}$$

*Also consider r.v.'s $S^l$ and $S^c$ defined by*

$$S^l = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2+r_i\sigma_{Z_i}\Phi^{-1}(U)} \tag{5}$$

*and*

$$S^c = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\sigma_{Z_i}\Phi^{-1}(U)}, \tag{6}$$

*respectively. Here $U$ is a Uniform$(0,1)$ r.v. and $\Phi$ is the cumulative d.f. of the $N(0,1)$ distribution. Further, the coefficients $r_i$ are defined by*

$$r_i = \frac{cov\left[Z_i, \Lambda\right]}{\sigma_{Z_i} \; \sigma_\Lambda}. \tag{7}$$

*For the r.v.'s $S, S^l$ and $S^c$, the following convex order relations hold:*

$$S^l \leq_{cx} S \leq_{cx} S^c. \tag{8}$$

A random vector is said to be comonotonic if all its components are non-decreasing functions of the same r.v.. This means that $S^c$ is a comonotonic sum. It implies that the quantiles and conditional tail expectations of $S^c$ are given by the sum of the corresponding risk measures for the marginals involved; see for instance Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004):

$$Q_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\sigma_{Z_i}\Phi^{-1}(p)}, \tag{9}$$

$$CTE_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1)\,. \tag{10}$$

Provided that all coefficients $r_i$ are positive, the terms in $S^l$ are also non-decreasing functions of the same r.v. $U$. Hence, $S^l$ will also be a comonotonic sum in this case. This implies that the quantiles and conditional tail expectations related to $S^l$ can be computed by summing the corresponding risk measures for the marginals involved. Hence, assuming that all $r_i$ are positive, we find the following expressions for quantiles and conditional tail expectations of $S^l$:

$$Q_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\left(1-r_i^2\right)\sigma^2_{Z_i}+r_i\sigma_{Z_i}\Phi^{-1}(p)}, \qquad p \in (0,1)\,, \tag{11}$$

$$CTE_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} \frac{\Phi\left(r_i \; \sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1)\,. \tag{12}$$

Finally, notice that the expected values of the r.v.'s $S$, $S^c$and $S^l$ are all equal:

$$E(S) = E(S^l) = E(S^c) = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}}, \tag{13}$$

whereas their variances are given by

$$Var(S) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Zj})}(e^{cov(Z_i,Z_j)}-1), \tag{14}$$

$$Var(S^l) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Zj})}(e^{r_ir_j\sigma_{Z_i}\sigma_{Z_j}}-1) \tag{15}$$

and

$$Var(S^c) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Zj})}(e^{\sigma_{Z_i}\sigma_{Z_j}}-1), \tag{16}$$

respectively.

## 2.2 The "maximal variance" lower bound approach

In the series of papers about the concept of comonotonicity and its applications, one almost always takes $\Lambda$ such that it can be seen as a kind of first order approximation of $S$; see for instance Dhaene, Denuit, Goovaerts, Kaas & Vyncke

(2002b). In this subsection we propose a new choice for the conditioning random variable $\Lambda$. In order to do so, recall that if $X \leq_{cx} Y$ and $X$ and $Y$ are not equal in distribution, then $Var[X] < Var[Y]$ must hold. An equality in variance would imply that $X \stackrel{d}{=} Y$. This indicates that if we replace $S$ by the less convex $S^l$, the best approximation arises when the variance of $S^l$ is 'as close as possible' to the variance of $S$. Indeed, in this case the d.f. of $S^l$ is also 'close' to the unknown d.f. of $S$. This means that we have to choose the coefficients $\gamma_i$ of the conditioning variable $\Lambda$ defined in (4) such that the variance of $S^l$ is maximized.

We will now prove that the first order approximation of the variance of $S^l$ will be maximized for the following choice of the parameters $\gamma_i$:

$$\gamma_i = \alpha_i e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}}, \quad i = 1, \ldots, n. \tag{17}$$

Indeed, from (15) we find that

$$\begin{aligned} Var(S^l) &\approx \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \ e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})}(r_i r_j \sigma_{Z_i}\sigma_{Z_j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \ e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})} \left(\frac{Cov[Z_i,\Lambda]Cov[Z_j,\Lambda]}{Var(\Lambda)}\right) \\ &= \frac{(Cov(\sum_{i=1}^{n} \alpha_i \ e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i, \Lambda))^2}{Var(\Lambda)} \\ &= (Corr(\sum_{i=1}^{n} \alpha_i \ e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i, \Lambda))^2 \ Var(\sum_{i=1}^{n} \alpha_i \ e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i). \end{aligned} \tag{18}$$

Hence, the first order approximation of $Var(S^l)$ is maximized when $\Lambda$ is given by

$$\Lambda = \sum_{i=1}^{n} \alpha_i \ e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i. \tag{19}$$

In the remainder of this paper, we will always assume that the conditioning r.v. $\Lambda$ is given by (19). Notice that this optimal choice for $\Lambda$ is slightly different from the choice that was made for this r.v. in Dhaene, Denuit, Kaas, Goovaerts & Vyncke (2002b). Numerical comparisons reveal that the choice proposed here in general leads to more accurate approximations.

One can easily prove that the first order approximation for $Var(S^l)$ with $\Lambda$ given by (19) is equal to the first order approximation of $Var(S)$. This observation gives an additional indication that this particular choice for $\Lambda$ will provide a good fit.

We emphasize that the conditioning r.v. $\Lambda$ as defined in (19) does not necessarily maximize the variance of $S^l$, but has to be understood as an approximation for the optimal $\Lambda$. Theoretically, one could use numerical procedures to determine the optimal $\Lambda$, but this would outweigh one of the main features of the convex bounds, namely that the quantiles and conditional tail expectations (and also other actuarial quantities such as stop-loss premiums) can easily be determined analytically. Having a ready-to-use approximation that can be implemented easily is important from a practical point of view.

# 3 Two well-known moment matching approximations

In this section we will briefly describe the reciprocal Gamma and the lognormal moment matching approximations. These two methods are frequently used to approximate the d.f. of the r.v. $S$ defined by (1).

## 3.1 The Reciprocal Gamma approximation

The r.v. $X$ is said to be Gamma distributed when its probability density function (p.d.f.) is given by

$$f_X(x;\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \qquad x > 0, \tag{20}$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma(.)$ denotes the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du \quad (\alpha > 0). \tag{21}$$

Consider now the r.v. $Y = 1/X$. This r.v. is said to be reciprocal (or inverse) Gamma distributed. Its p.d.f. is given by

$$f_Y(y;\alpha,\beta) = f_X(1/y;\alpha,\beta)/y^2, \qquad y > 0. \tag{22}$$

It is straightforward to prove that the quantiles and conditional tail expectations of $Y$ are given by

$$Q_p[Y] = \frac{1}{F_X^{-1}(1-p;\alpha,\beta)}, \qquad p \in (0,1) \tag{23}$$

and

$$CTE_p[Y] = \frac{F_X(F_X^{-1}(1-p;\alpha,\beta);\alpha-1,\beta)}{(1-p)(\alpha-1)\beta}, \qquad p \in (0,1), \tag{24}$$

where $F_X(.;\alpha,\beta)$ is the cumulative d.f. of the Gamma distribution with parameters $\alpha$ and $\beta$. Since the Gamma distribution is readily available in many statistical software packages, these risk measures can easily be determined.

The first two moments of the reciprocal Gamma distributed r.v. $Y$ are given by

$$E[Y] = \frac{1}{\beta(\alpha-1)} \tag{25}$$

and

$$E[Y^2] = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}. \tag{26}$$

Expressing the parameters $\alpha$ and $\beta$ in terms of $E[Y]$ and $E[Y^2]$ gives

$$\alpha = \frac{2E[Y^2] - E[Y]^2}{E[Y^2] - E[Y]^2} \tag{27}$$

and

$$\beta = \frac{E[Y^2] - E[Y]^2}{E[Y]E[Y^2]}. \tag{28}$$

The d.f. of the r.v. defined in (1) is now approximated by a reciprocal Gamma distribution with first two moments (13) and (14), respectively. The coefficients $\alpha$ and $\beta$ of the reciprocal Gamma approximation follow from (27) and (28). The reciprocal Gamma approximations for the quantiles and the conditional tail expectations are then given by (23) and (24).

The reciprocal Gamma moment matching method appears naturally in case one wants to approximate the d.f. of stochastic present values. Indeed, for the limiting case of the constant continuous perpetuity:

$$S_\infty = \int_0^\infty \exp\left[-(\mu - \frac{\sigma^2}{2})\tau - \sigma B(\tau)\right] \, d\tau, \tag{29}$$

where $B(\tau)$ represents a standard Brownian motion and $\mu > \frac{\sigma^2}{2}$, the risk measures can be calculated very easily since Dufresne (1990) proved that $S_\infty^{-1}$ is Gamma distributed with parameters $\frac{2\mu}{\sigma^2} - 1$ and $\frac{\sigma^2}{2}$. An elegant proof for this result can also be found in Milevsky (1997).

Expression (29) can be seen as a continuous counterpart of a discounted sum such as in (23) and one intuitively expects that the present value of a finite discrete annuity with a normal log return process with independent periodic returns, can be approximated by a reciprocal Gamma distribution, provided that the time period involved is long enough. This idea was set forward and explored in Milevsky & Posner (1998), Milevsky & Robinson (2000) and Huang, Milevsky & Wang (2004).

## 3.2 The lognormal approximation

The r.v. $X$ is said to be lognormal distributed if its p.d.f. is given by

$$f_X(x; \mu, \sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(\log x - \mu)^2 / 2\sigma^2}, \qquad x > 0, \tag{30}$$

where $\sigma > 0$.

The quantiles and conditional tail expectations of $X$ are given by

$$Q_p[X] = e^{\mu + \sigma \Phi^{-1}(p)}, \qquad p \in (0,1) \tag{31}$$

and

$$CTE_p[X] = e^{\mu + \frac{1}{2}\sigma^2} \frac{\Phi\left(\sigma - \Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1). \tag{32}$$

The first two moments of $X$ are given by

$$E[X] = e^{\mu + \frac{1}{2}\sigma^2} \tag{33}$$

and

$$E[X^2] = e^{2\mu + 2\sigma^2}. \tag{34}$$

Expressing the parameters $\mu$ and $\sigma^2$ of the lognormal distribution in terms of $E[X]$ and $E[X^2]$ leads to

$$\mu = \log\left(\frac{E[X]^2}{\sqrt{E[X^2]}}\right) \tag{35}$$

and

$$\sigma^2 = \log\left(\frac{E[X^2]}{E[X]^2}\right). \tag{36}$$

The same procedure as the one explained in the previous subsection can be followed in order to obtain a lognormal approximation for $S$, with the first two moments matched. Dufresne (2004) obtains a lognormal limit distribution for $S$ as the volatility $\sigma$ tends to zero and this provides a theoretical justification for the use of the lognormal approximation.

# 4 Comparing the approximations

In order to compare the performance of the different approximations presented above, we consider the r.v. $S_n$ which is defined as the random present value of a series of $n$ deterministic unit payment obligations due at times 1, 2, ..., $n$, respectively:

$$S_n = \sum_{i=1}^{n} e^{-Y_1 - Y_2 - \cdots - Y_i} \overset{def}{=} \sum_{i=1}^{n} e^{Z_i}. \tag{37}$$

where the r.v. $Y_i$ is the random return over the period $[i-1, i]$ and $e^{-(Y_1+Y_2+\cdots+Y_i)}$ $= e^{Z_i}$ is the random discount factor over the period $[0, i]$. We will assume that the periodical returns $Y_i$ are i.i.d. r.v.'s with mean $\left(\mu - \frac{\sigma^2}{2}\right)$ and variance $\sigma^2$. Notice that $S_n$ is a r.v. of the general type defined in (1).

The provision or reserve to set up at time 0 for these future unit payment obligations can be determined as $Q_p[S_n]$ or $CTE_p[S_n]$, with $p$ sufficiently large. A provision equal to $\mathrm{Q}_{0.95}[S_n]$ for instance, will guarantee that all payments can be made with a probability of 0.95; see for instance Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004).

As the time unit that we consider is long (1 year), assuming a Gaussian model for the returns seems to be appropriate, at least approximately, by the Central Limit Theorem. Empirical studies that confirm our theoretical setup can be found in Cesari & Cremonini (2003) and Levy (2004).

In order to compute the comonotonic approximations for quantiles and conditional tail expectations, notice that $E[Z_i]$, $\sigma^2_{Z_i}$ and $r_i$ are given by

$$E[Z_i] = -i(\mu - \frac{\sigma^2}{2}), \tag{38}$$
$$\sigma^2_{Z_i} = i\,\sigma^2 \tag{39}$$

and

$$r_i = \frac{\sum_{j=1}^{i}\sum_{k=j}^{n}\gamma_k}{\sqrt{i\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\gamma_k\right)^2}}, \tag{40}$$

with $\gamma_k$ given by

$$\gamma_k = e^{E[Z_k]+\frac{1}{2}\sigma^2_{Z_k}}, \quad k = 1,\ldots,n.$$

Notice that the correlation coefficients $r_i$ are positive, so that the formulae (11) and (12) can be applied.

Now we will compare the performance of the different approximation methods that were presented in Sections 2 and 3: the comonotonic upper bound method ($UB$), the comonotonic "maximal variance" lower bound method ($LB$), the reciprocal Gamma method ($RG$) and the lognormal method ($LN$).

We will compare the different approximations for quantiles and conditional tail expectations with the values obtained by Monte-Carlo simulation. The simulation results are based on generating 500,000 random paths. The estimates obtained from this time-consuming simulation will serve as benchmark. The random paths are based on antithetic variables in order to reduce the variance of the Monte-Carlo estimate.

The tables that we will present display the results obtained by Monte Carlo simulation ($MC$) for the risk measure at hand, as well as the deviations of the different approximation methods, relative to the Monte Carlo based result. For the quantiles and conditional tail expectations, these deviations are defined as follows:

$$\frac{Q_p[S_n^{approx}] - Q_p[S_n^{MC}]}{Q_p[S_n^{MC}]} \times 100\%$$

and

$$\frac{CTE_p[S_n^{approx}] - CTE_p[S_n^{MC}]}{CTE_p[S_n^{MC}]} \times 100\%,$$

where $S_n^{approx}$ corresponds to one of the approximation methods and $S_n^{MC}$ denotes the Monte Carlo simulation result. The figures displayed in bold in the tables correspond to the best approximations, this means the ones with the smallest deviation, relative to the Monte-Carlo result. In the tables, we also present the standard error (s.e.) of the Monte Carlo estimates. The standard error is formally defined as the square root of the estimated error variance and is here expressed as a percentage of the Monte Carlo estimate.

Table 1 summarizes the results for the 0.95-quantiles for different yearly volatilities $\sigma$ and for a yearly time horizon of $n = 20$ and $n = 40$, respectively.

| $n$ | Method | $\sigma = 0.05$ | $\sigma = 0.15$ | $\sigma = 0.25$ | $\sigma = 0.35$ |
|---|---|---|---|---|---|
| 20 | $UB$ | +3.24% | +8.02% | +9.36% | +7.50% |
| | $LB$ | **-0.01%** | **+0.02%** | **+0.00%** | **+0.35%** |
| | $RECG$ | +0.07% | -0.15% | -4.28% | -14.27% |
| | $LN$ | -0.16% | -0.06% | +2.99% | +9.04% |
| | $MC$ ($\pm s.e.$) | 12.1957 (0.04%) | 20.4592 (0.10%) | 41.5854 (0.25%) | 106.1389 (0.30%) |
| 40 | $UB$ | +4.39% | +10.26% | +9.42% | +1.47% |
| | $LB$ | **+0.00%** | **-0.06%** | **+0.06%** | **-0.83%** |
| | $RECG$ | +0.06% | -0.55% | -8.52% | -19.70% |
| | $LN$ | -0.23% | +0.58% | +9.73% | +9.96% |
| | $MC$ ($\pm s.e.$) | 15.4733 (0.04%) | 30.4033 (0.16%) | 87.7482 (0.32%) | 427.0793 (0.49%) |

**Table 1:** Approximations for the 0.95-quantile of $S_n$ for different horizons and volatilities ($\mu$=0.075; yearly payments of 1).

| Method | $p = 0.95$ | $p = 0.90$ | $p = 0.75$ | $p = 0.50$ | $p = 0.25$ |
|---|---|---|---|---|---|
| $UB$ | +8.02% | 5.62% | +1.93% | -2.25% | -6.33% |
| $LB$ | **+0.02%** | **-0.06%** | **+0.03%** | **-0.01%** | **+0.00%** |
| $RECG$ | -0.15% | -0.74% | -0.86% | -0.42% | +0.57% |
| $LN$ | -0.06% | +0.65% | +1.36% | +0.92% | -0.65% |
| $MC$ ($\pm s.e.$) | 20.4592 (0.10%) | 17.8221 (0.06%) | 14.2191 (0.05%) | 11.1986 (0.01%) | 8.9199 (0.04%) |

**Table 2:** Approximations for some selected quantiles of $S_{20}$ ($\mu$=0.075; $\sigma$=0.15; yearly payments of 1).

The yearly expected return $\mu$ has been set equal to 7.5%. The "maximal variance" lower bound approach ($LB$) turns out to fit the quantiles the best for all values of the parameters. Its quantiles fall almost always in the 95% confidence interval around $Q_p[S_n^{MC}]$. It appears that the performances of the lognormal and reciprocal Gamma approximations are worse for higher levels of volatility and for longer time horizons. As far as the reciprocal Gamma approximation is concerned, this result seems surprising given the convergence of the d.f. of $S_n$ to a reciprocal Gamma distribution. Note however that the variance of $S_\infty$ only exists provided that $\mu > 3\frac{\sigma^2}{2}$; see (26). This explains why for $\sigma = 0.25$ and $\sigma = 0.35$ both moment matching methods are less accurate.

Table 2 compares the different approximations for some selected quantiles of $S_{20}$, with a fixed yearly volatility $\sigma = 0.15$ and a yearly expected return $\mu = 0.075$. The results are in line with the previous ones. The lower bound approach outperforms all the others, for high as well as for low values of $p$.

Table 3 displays the approximated and simulated 95% conditional tail expectations for the same set of parameters as in Table 1. Again the "maximal

| $n$ | Method | $\sigma = 0.05$ | $\sigma = 0.15$ | $\sigma = 0.25$ | $\sigma = 0.35$ |
|---|---|---|---|---|---|
| 20 | $UB$ | +4.19% | +10.98% | +14.17% | +12.98% |
| | $LB$ | **-0.02%** | **-0.14%** | **-0.36%** | **-0.59%** |
| | $RECG$ | +0.21% | +1.18% | -0.98% | -15.41% |
| | $LN$ | -0.38% | -1.88% | -0.94% | +4.56% |
| | $MC$ ($\pm s.e.$) | 12.8231 (1.04%) | 24.4591 (2.16%) | 59.6646 (2.90%) | 198.0164 (3.27%) |
| 40 | $UB$ | +5.86% | +15.11% | +16.87% | +10.45% |
| | $LB$ | **+0.09%** | **-0.25%** | **-0.59%** | **-0.84%** |
| | $RECG$ | +0.28% | +0.87% | -7.49% | -40.77% |
| | $LN$ | -0.48% | -2.38% | +4.18% | +12.77% |
| | $MC$ ($\pm s.e.$) | 16.3994 (1.55%) | 38.2515 (2.61%) | 149.8569 (3.25%) | 1206.0858 (3.59%) |

**Table 3:** Approximations for the 0.95-conditional tail expectation of $S_n$ for different horizons and volatilities ($\mu$=0.075; yearly payments of 1).

| Method | $\mu = 0.05$ | $\mu = 0.075$ | $\mu = 0.10$ |
|---|---|---|---|
| $UB$ | +9.93% | +10.26% | +10.49% |
| $LB$ | **+0.15%** | **-0.06%** | **+0.02%** |
| $RECG$ | -1.05% | -0.55% | -0.13% |
| $LN$ | +1.24% | +0.58% | +0.29% |
| $MC$ ($\pm s.e.$) | 47.6988 (0.16%) | 30.4033 (0.16%) | 20.8469 (0.13%) |

**Table 4:** Approximations for $Q_{0.95}[S_{40}]$ for different expected returns (n=40; $\sigma$=0.15; yearly payments of 1).

variance" lower bound approach performs the best as an approximation for the conditional tail expectations.

In Table 4 we test the sensitivity of the results with respect to the yearly expected return $\mu$. This table reports the approximations for $Q_{0.95}[S_{40}]$ for different yearly expected returns $\mu$ and for a fixed yearly volatility $\sigma = 0.15$. The results show that the higher the yearly expected return $\mu$, the better any of the proposed approximations. Also in this case, the "maximal variance" lower bound approximation seems to provide the best fit.

At some point however, for increasing horizon and periodicity of the cash flows, the (limiting) reciprocal Gamma approximation must outperform the lower bound approximation. In Table 5, we compare the performances of the different approximations when increasing the yearly horizon, whereas in Table 6 we also increase the periodicity of the payments. More precisely, in Table 5 we consider yearly unit payments whereas in Table 6 we assume quarterly payments of 0.25 each. These tables confirm that increasing the periodicity and the total horizon positively impacts the performance of the reciprocal Gamma

| Method | $n = 40$ | $n = 100$ | $n = 250$ |
|---|---|---|---|
| $UB$ | +10.26% | +12.45% | +12.67% |
| $LB$ | **-0.06%** | **+0.09%** | +0.16% |
| $RECG$ | -0.55% | -0.19% | **-0.05%** |
| $LN$ | +0.58% | +2.01% | +2.23% |
| $MC$ ($\pm s.e$) | 30.4033 (0.16%) | 36.2960 (0.14%) | 36.5572 (0.17%) |

**Table 5:** Approximations for $Q_{0.95}[S_n]$ for different horizons ($\mu$=0.075; $\sigma$=0.15; yearly payments of 1).

| Method | $n = 40$ | $n = 100$ |
|---|---|---|
| $UB$ | +10.21% | +12.30% |
| $LB$ | **-0.10%** | **-0.03%** |
| $RECG$ | -0.49% | -0.16% |
| $LN$ | +0.49% | +1.86% |
| $MC$ ($\pm s.e.$) | 30.6718 (0.16%) | 36.7083 (0.17%) |

**Table 6:** Approximations for $Q_{0.95}[S_n]$ for different horizons ($\mu$=0.075; $\sigma$=0.15; quarterly payments of 0.25).

approximation.

Finally, in Table 7 we consider continuous perpetuities and confront the lower bound approximation with the exact reciprocal Gamma distribution. We consider different values of the volatility parameter $\sigma$ while $\mu$ is chosen equal to $\mu = 0.075$.and we compare the exact results for the 0.95-quantile of a constant continuous perpetuity (distributed like reciprocal Gamma) with the "maximal variance" lower bound approximation. The lower bound approximation has been obtained by a numerical evaluation of the quantile function of the "maximal variance" lower bound approximation for $S_\infty$ which is given by

$$Q_p[S_t^l] = \int_0^\infty e^{-(\mu - \frac{\sigma^2}{2})\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(p)} d\tau,$$

with $\Lambda = \int_0^\infty \exp\left[-(\mu - \sigma^2)\tau\right] \, B(\tau)d\tau$ and $r(\tau) = corr[Y(\tau), \Lambda]$. We note that also in this case the "maximal variance" lower bound approximation provides rather accurate approximate results.

From the 7 tables, one can observe that, generally speaking, the moment matching techniques perform poorly for high levels of $p$ and/or $\sigma$ whilst the "maximal variance" comonotonic lower bound continues to produce accurate approximations, also in these extreme cases.

| $n$ | Method | $\sigma = 0.05$ | $\sigma = 0.15$ | $\sigma = 0.25$ |
|---|---|---|---|---|
| $\infty$ | $LB$ | -0.02% | +0.01% | -0.96% |
| | $RECG$ | 27.2786 | 37.1132 | 219.2890 |

**Table 7:** Maximal variance lowerbound approximations for the 0.95-quantile of the constant continuous perpetuity for different volatilities ($\mu$=0.075).

# 5 Conclusion

In this paper, we compared in a discrete setting some approximation methods for a standard actuarial and financial problem: the determination of quantiles and conditional tail expectations of the present value of a series of cash-flows, when discounting is performed by a Brownian motion process. We introduce the comonotonic "maximal variance" lower bound approximation and we tested the accuracy of the comonotonic approximations and two moment matching approximations by comparing these approximations with the estimates obtained from extensive Monte Carlo simulations.

Overall, the comonotonic "maximal variance" lower bound approach provides the best fit and leads to accurate approximations under varying parameter assumptions, which are in line with realistic market values.

The comonotonic approach has the additional advantage that it gives rise to easily computable approximations for any risk measure that is additive for comonotonic risks. Examples of such risk measures are the distortion risk measures which were introduced in the actuarial literature by Wang (2000).

Also notice that the comonotonic approximations that we presented here can easily be transformed to the case when accumulating saving amounts to a final value, and also to the case where the cash flow payments vary from period to period; see Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004). Furthermore, Vanduffel, Dhaene & Goovaerts (2004) show that also in case of positive cash flow payments followed by negative withdrawals the comonotonic approximations can often be used to approximate the relevant risk measures. For all these mentioned cases the moment matching methods are less appropriate.

On the other hand, notice that Huang, Milevsky and Wang (2004) show in a continuous setting how the moment matching approximations can also be used in case the variable $n$ in equation (1) is itself stochastic. Also the comonotonic approximations can be used in this case. Indeed, Hoedemakers, Darkiewicz and Goovaerts (2005) use the theory on comonotonicity to obtain in a discrete setting approximations for the present value of a life annuity and a portfolio of life annuities.

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