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Consistent Assumptions for Modeling Credit Loss Correlations

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Abstract**

We consider a single period portfolio of n dependent credit risks that are subject to default during the period. We show that using stochastic loss given default random variables in conjunction with default correlations can give rise to an inconsistent set of assumptions for estimating the variance of the portfolio loss. Two sets of consistent assumptions are provided, which it turns out, also provide bounds on the variance of the portfolio's loss. An example of an inconsistent set of assumptions is given.

Key words and phrases: *default correlation, loss correlation, comonotonicity, economic capital*

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1 Introduction

Advanced credit portfolio models such as J.P. Morgan's CreditMetrics® (<http://www.creditmetrics.com>), Credit Suisse Financial Products' CreditRisk+® (<http://www.csfb.com/creditrisk>), McKinsey & Company's CreditPortfolioView® (Wilson 1997a and b), and KMV's PortfolioManager® (Kealhofer 1995) are widely used by banks to assess the credit default risk of their diverse loan portfolios.[1] Knowledge of this risk allows banks to set aside capital buffers to protect them against default. The implementation of these models is often the bank's first step toward developing what is now called an enterprise risk framework, i.e., a which can support consistent risk and reward management of the whole enterprise by integrating all risk components. Indeed, the capital used by different business units within a financial enterprise may adversely affect investment decisions and the performance of other business units.

Despite the commercial success of the above mentioned models, Deloitte & Touche's 2004 global risk management survey[2] has shown that many financial institutions have yet to set up such an integrated framework. Instead, some financial institutions have maintained the traditional variance-covariance portfolio model for the sake of transparency and practicality. In contrast to the credit risk models that compute the distribution of the portfolio loss, the variance-covariance approach focuses on the computation of the mean and the variance of this loss. The mean and variance are then linked to the required capital through a calibration on a known two-parameter distribution such as, for example, the beta distribution.

Using the variance-covariance framework requires information on the probability of default, exposure at default, the mean and variance of the loss given default, and the default correlation matrix among the various debtors. These parameters can also be found in the quantitative groundings of the 2004 Basel Accord.[3] Before setting up that

stage the loss given default is assumed to be constant, while in the second stage it was assumed to be stochastic.

[1]For a comparison of these models see, for example, Crouhy, Galai and Mark (2000). Gordy (2000) compares CreditMetrics® and CreditRisk+®.

[2]Deloitte & Touche's Global Risk Management Survey is available online at <http://www.deloitte.com>

[3]See "International Convergence of Capital Measurement and Capital Standards, a Revised Framework." Basel Committee for Banking Supervision, 2004.

[4]For example, when introducing the variance-covariance framework, a well known Belgian financial enterprise considered in inconsistent two-stage procedure. In the first

variance-covariance framework, however, we must specify assumptions and ensure that these assumptions are mutually consistent.[4]

We propose two consistent variance-covariance models. Both methods use a stochastic loss given default but but differ in their correlation assumptions. The first assumes independence among the stochastic loss given default they are comonotonic, meaning that they are all monotonic functions of a common random variable. We show that these two models are extremal in the sense that they provide bounds for the portfolio variance.

2 Description of the Problem

Consider a single period portfolio of n dependent credit risks at the start of the period. These risks, labeled $1, 2, \ldots, n$, can default during the period. For $i = 1, 2, \ldots, n$, let

I_i = Indicator random variable for the i^{th} risk's default during the period, i.e., $I_i = 1$ if default occurs and 0 otherwise;

q_i = $\mathbb{P}[I_i = 1]$ is the probability of default for the i^{th} risk;

M_i = Portfolio's exposure at default due to the i^{th} risk, i.e., the maximum amount of loss on risk i given that it defaults. M_i is assumed to be a finite deterministic quantity;

Θ_i = The loss given default random variable, which is the fraction of M_i that actually is lost given the i^{th} risk defaults;

L_i = $I_i M_i \Theta_i$ is the actual (unconditional) loss from the i^{th} risk's default during the period; and

L = $\sum_{i=1}^{n} L_i$ is the aggregate portfolio loss from defaults.

For any pair of random variables (X, Y) with finite variance, the notation $\rho(X, Y)$ is used to denote its Pearson's correlation coefficient where

$$\rho(X, Y) = \frac{\mathbb{C}\mathrm{ov}(X, Y)}{\sigma(X)\,\sigma(Y)}.$$

The default correlation of risk pair (i, j) is denoted by $\rho_{i,j}^{D}$ where

$$\rho_{i,j}^{D} = \rho\left(I_i, I_j\right), \tag{1}$$

where $\sigma^2(I_i) = q_i(1 - q_i)$ for $i = 1, 2, \ldots, n$. The loss given default correlation of the risk pair (i, j) is denoted by $\rho^{\Theta}_{i,j}$ where

$$\rho^{\Theta}_{i,j} = \rho\left(\Theta_i, \Theta_j\right). \tag{2}$$

Finally, the loss correlation of risk pair (i, j) is denoted by $\rho^{L}_{i,j}$ where

$$\rho^{L}_{i,j} = \rho\left(L_i, L_j\right). \tag{3}$$

We will discuss how to construct a consistent model of correlations $\rho^{D}_{i,j}, \rho^{\Theta}_{i,j}$ and $\rho^{L}_{i,j}$. In addition, we will show that while it is of course correct to consider Θ as a random variable, the consequences of this assumption should be carefully considered. For example, even though loss and default correlations are the same when the Θ_i's are deterministic, one cannot continue to assume that $\rho^{L}_{i,j} = \rho^{D}_{i,j}$ for all risk pairs (i, j) when the Θ_i's are random variables.

Though a number of authors have considered methods of estimating default correlations, e.g., the theoretical models of Hull and White (2001) and Zhou (2001), the estimates from real data that are used in Stevenson et al (1995) and Gollinger and Morgan (1993), it appears that much less work has been done on the more general concept of loss correlations. We hope this paper makes a contribution to the further understanding of loss correlations.

3 Some General Results

3.1 The Basic Assumption

Our first and most basic assumption is:

A1 The default indicator random variables I_i and the loss given default random variables Θ_j are mutually independent for any pair i and j, $i, j = 1, 2, \ldots, n$.

We emphasize that the mutual independence of I_i and Θ_i is just a technical assumption because only the variable $\Theta_i \mid I_i = 1$ is relevant. So we can choose any distribution function for $\Theta_i \mid I_i = 0$. A convenient choice is to assume that $\Theta_i \mid I_i = 0 \overset{d}{=} \Theta_i \mid I_i = 1$, where $\overset{d}{=}$ stands for equality in distribution. This is indeed a good choice, because it makes the random variables Θ_i and I_i mutually independent which is convenient from a mathematical point of view. The assumption of mutually independence between I_i and Θ_j for $i \neq j$ cannot be considered as a technical assumption, rather it is a simplifying assumption. As the Θ_i's are fractions of the M_i's, we can, without loss of generality, set $M_i = 1$. Results and conclusions can easily be generalized to the case where the M_i's are arbitrary.

Two well known results from probability are: for any triplet of random variables X, Y, and Z

$$\begin{aligned}\mathbb{C}\mathrm{ov}\,(X,Y) &= \mathbb{E}\,[\mathbb{C}\mathrm{ov}[(X,Y) \mid Z]] + \mathbb{C}\mathrm{ov}\,[\mathbb{E}\,(X \mid Z)\,,\mathbb{E}\,(Y \mid Z)]\\ \mathbb{V}\mathrm{ar}(L_i) &= \mathbb{V}\mathrm{ar}\,[\mathbb{E}\,(X \mid Z)] + \mathbb{E}\,[\mathbb{V}\mathrm{ar}\,(X \mid Z)]\end{aligned}$$

From assumption A1 we find that

$$\begin{aligned}\mathbb{C}\mathrm{ov}\left(L_i,L_j\right) &= \mathbb{E}\left(I_iI_j\right)\mathbb{C}\mathrm{ov}\left(\Theta_i,\Theta_j\right) + \mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j)\mathbb{C}\mathrm{ov}(I_i,I_j)\\ &= \left(\mathbb{C}\mathrm{ov}(I_i,I_j) + q_iq_j\right)\mathbb{C}\mathrm{ov}(\Theta_i,\Theta_j) + \mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j)\mathbb{C}\mathrm{ov}(I_i,I_j).\end{aligned} \tag{4}$$

Hence,

$$\begin{aligned}\rho^L_{i,j}\sigma(L_i)\sigma(L_j) = &\left[\rho^D_{i,j}\sigma(I_i)\sigma(I_j) + q_iq_j\right]\rho^\Theta_{i,j}\sigma(\Theta_i)\sigma(\Theta_j)\\ &+ \rho^D_{i,j}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\ \mathbb{E}(\Theta_j).\end{aligned} \tag{5}$$

and

$$\mathbb{V}\mathrm{ar}(L_i) = (\mathbb{E}(\Theta_i))^2\,q_i(1-q_i) + q_i\mathbb{V}\mathrm{ar}\,(\Theta_i)\,. \tag{6}$$

From the derivations above, we find that a general expression for $\mathbb{V}\mathrm{ar}(L)$ is given by

$$\begin{aligned}\mathbb{V}\mathrm{ar}(L) &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mathbb{C}\mathrm{ov}(L_i,L_j) + \sum_{i=1}^{n}\mathbb{V}\mathrm{ar}(L_i)\\ &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\rho^D_{i,j}\sigma(I_i)\sigma(I_j) + q_iq_j\right]\rho^\Theta_{i,j}\sigma(\Theta_i)\sigma(\Theta_j)\\ &\quad + \sum_{i\neq j}^{n}\rho^D_{i,j}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j)\\ &\quad + \sum_{i=1}^{n}q_i\left((\mathbb{E}(\Theta_i))^2\,(1-q_i) + \mathbb{V}\mathrm{ar}\,(\Theta_i)\right).\end{aligned} \tag{7}$$

3.2 First Model with Consistent Correlations

The simplest additional assumption that is consistent with assumption A1 is to assume that the Θ_i's are mutually independent, i.e.,

A2(a): Θ_i and Θ_j are mutually independent for $i, j = 1, 2, \ldots, n$ and $i \neq j$.

This assumption implies that $\rho^{\Theta}_{i,j} = 0$ for all $i \neq j$. In this case, we find from equation (5) that, for $i \neq j$,

$$\mathbb{C}\text{ov}(L_i, L_j) = \rho^D_{i,j}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j))$$

or equivalently,

$$\rho^L_{i,j} = \frac{\rho^D_{i,j}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j))}{\sigma(L_i)\sigma(L_j)} \tag{8}$$

From equation (7) we find now the following expression for the variance of the portfolio loss is:

$$\begin{aligned}\mathbb{V}\text{ar}(L) = & \sum_{i \neq j}^{n} \rho^D_{i,j}\sqrt{q_i(1-q_i)q_j(1-q_j)}\,\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j) \\ & + \sum_{i=1}^{n} q_i\left(\mathbb{E}^2(\Theta_i)(1-q_i) + \mathbb{V}\text{ar}\,(\Theta_i)\right).\end{aligned} \tag{9}$$

3.3 Second Model with Consistent Correlations

An alternative to assumption A2(a) is to assume that:

A2(b): The vector $(\Theta_1, \ldots, \Theta_n)$ is a comonotonic vector, i.e., the vector $(\Theta_1, \cdots, \Theta_n)$ has the same distribution as $\left(F^{-1}_{\Theta_1}(U), \cdots, F^{-1}_{\Theta_n}(U)\right)$, where U is uniformly distributed on the unit interval $(0, 1)$, and $F^{-1}_{\Theta_i}$ is the inverse distribution function of the random variable Θ_i.

[5] For more on the theory of comonotonicity see Dhaene and Goovaerts (1996), Kaas et al. (2000), and Dhaene et al. (2000a and b). The theory has been applied to a number of important financial and actuarial problems such as pricing Asian and basket options in a Black-Scholes model, setting provisions and required capitals in an insurance context, and determining optimal portfolio strategies; see, for example, Albrecher et al. (2005), Dhaene et al. (2002b), Dhaene et al. (2004), Vanduffel et al. (2002), and Vanduffel et al. (2005).

The assumption of comonotonicity implies that the different Θ_i are monotonic functions of a common random variable, U.[5]

One implication of comonotonicity is that

$$\mathbb{C}\text{ov}\left(\Theta_i, \Theta_j\right) = \mathbb{C}\text{ov}\left(F_{\Theta_i}^{-1}(U), F_{\Theta_j}^{-1}(U)\right) \quad \text{for all } (i, j). \tag{10}$$

Note that the vectors $(\Theta_1, \cdots, \Theta_n)$ and $\left(F_{\Theta_1}^{-1}(U), \cdots, F_{\Theta_n}^{-1}(U)\right)$ have the same marginal distributions, so that the Θ-correlations are given by

$$\rho_{i,j}^{\Theta} = \frac{\mathbb{C}\text{ov}\left(F_{\Theta_i}^{-1}(U), F_{\Theta_j}^{-1}(U)\right)}{\sqrt{\mathbb{V}\text{ar}\,(\Theta_i)\,\mathbb{V}\text{ar}\left(\Theta_j\right)}}. \tag{11}$$

It is straightforward to show that $\rho_{i,j}^{\Theta} = 1$ for all $i \neq j$ implies that the vector $(\Theta_1, \cdots, \Theta_n)$ is comonotonic; the reverse implication is only true if there exists a random variable Y, and real constants $a_i > 0$ and $-\infty < b_i < \infty$ such that the relation $\Theta_i \overset{d}{=} a_i Y + b_i$ for $i = 1, 2, \ldots, n$. In addition, Dhaene et al. (2000a) have proved that the comonotonicity of $(\Theta_1, , \cdots, \Theta_n)$ is equivalent with the maximization of the $\rho_{i,j}^{\Theta}$ for all pairs $\left(\Theta_i, \Theta_j\right)$ with $i \neq j$.

From equation (5) we find

$$\begin{aligned}\mathbb{C}\text{ov}(L_i, L_j) = {} & \left[\rho_{i,j}^{D}\sigma(I_i)\sigma(I_j) + q_i q_j\right]\mathbb{C}\text{ov}\left(F_{\Theta_i}^{-1}(U), F_{\Theta_j}^{-1}(U)\right) \\ & + \rho_{i,j}^{D}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j),\end{aligned}$$

or equivalently

$$\begin{aligned}\rho_{i,j}^{L}\sigma(L_i)\sigma(L_j) = {} & \left[\rho_{i,j}^{D}\sigma(I_i)\sigma(I_j) + q_i q_j\right]\mathbb{C}\text{ov}\left(F_{\Theta_i}^{-1}(U), F_{\Theta_j}^{-1}(U)\right) \\ & + \rho_{i,j}^{D}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j).\end{aligned} \tag{12}$$

The variance of the portfolio loss follows from equation (7):

$$\begin{aligned}\mathbb{V}\text{ar}(L) = {} & 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\rho_{i,j}^{D}\sigma(I_i)\sigma(I_j) + q_i q_j\right]\mathbb{C}\text{ov}\left(F_{\Theta_i}^{-1}(U), F_{\Theta_j}^{-1}(U)\right) \\ & + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\rho_{i,j}^{D}\sigma(I_i)\sigma(I_j)\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j) \\ & + \sum_{i=1}^{n} q_i\left(\mathbb{E}^2(\Theta_i)(1 - q_i) + \mathbb{V}\text{ar}\,(\Theta_i)\right).\end{aligned} \tag{13}$$

Assuming that $\rho^D_{i,j} \geq 0$ and $\rho^\Theta_{i,j} \geq 0$ for all (i,j), we find by comparing equations (5), (8) and (12), that:

$$\rho^L_{i,j}[\text{equation (8)}] \leq \rho^L_{i,j}[\text{equation (5)}] \leq \rho^L_{i,j}[\text{equation (12)}] \tag{14}$$

and also that

$$\mathbb{V}\text{ar}(L)[\text{equation (8)}] \leq \mathbb{V}\text{ar}(L)[\text{equation (5)}] \leq \mathbb{V}\text{ar}(L)[\text{equation (12)}]. \tag{15}$$

3.4 An Inconsistent Correlations Model

When the Θ_i are deterministic, it is straightforward to prove that for any risk pair (i,j) the loss correlation is equal to the default correlation. Suppose we make the following assumption:

A2(c): $\rho^L_{i,j} = \rho^D_{i,j}$ for all (i,j).

This assumption A2(c), however, leads to inconsistencies. Suppose the Θ_i and Θ_j are random variables, consider this numerical example: let $q_i = 0.001$, $q_j = 0.01$, $\mathbb{E}(\Theta_i) = 0.8$, $\mathbb{E}\left(\Theta_j\right) = 0.2$, $\mathbb{V}\text{ar}(\Theta_i) = 0.04$, $\mathbb{V}\text{ar}\left(\Theta_j\right) = 0.04$, and $\rho^D_{i,j} = \rho^L_{i,j} = 0.03$. We find from equation (6) that $\mathbb{V}\text{ar}(L_i) = 0.00068$ and $\mathbb{V}\text{ar}(L_j) = 0.00080$, while from equation (5) we find now that $\rho^\Theta_{i,j} = 1.669$, which is in contradiction with $\rho^\Theta_{i,j} \leq 1$. Hence assumptions A1 and A2(c) may lead to inconsistencies.

If we apply this example using assumption A2(a) instead, we find from equation (8) that $\rho^L_{i,j} = 0.021$ and not $\rho^L_{i,j} = 0.03$, as it was the case with assumption A2(c).

4 Final Remarks

The results of this paper continue to hold if we relax the assumption that the M_i's are all equal to one. For instance, assuming that $\rho^D_{i,j}$ and $\rho^\Theta_{i,j}$ are both non-negative for all (i,j) we find that the most general expression for the lower bound on the portfolio variance is given by

$$\mathbb{V}\mathrm{ar}(L) = \sum_{i \neq j}^{n} M_i M_j \rho_{i,j}^{D} \sqrt{q_i(1-q_i)q_j(1-q_j)}\,\mathbb{E}(\Theta_i)\mathbb{E}(\Theta_j) + \sum_{i=1}^{n} M_i^2 q_i \left(\mathbb{E}^2(\Theta_i)(1-q_i) + \mathbb{V}\mathrm{ar}\,(\Theta_i)\right). \tag{16}$$

Finally, we remark that all the results in this paper continue to hold if we generalize the model to the case that the defaults $(I_1, \cdots, I_n)$ depend on some conditioning random vector $(Q_1, \cdots, Q_n)$ such that $Q_i = \Pr[I_i = 1 \mid Q_i]$, which leads to

$$\Pr[I_i = 1] = \mathbb{E}(Q_i) = q_i. \tag{17}$$

Hence, the probability of default of risk i can be interpreted as the expectation of the conditioning random variable Q_i in this case.

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