COMONOTONIC BOUNDS ON THE SURVIVAL PROBABILITIES IN THE LEE-CARTER MODEL FOR MORTALITY PROJECTION

Michel Denuit
Institut des Sciences Actuarielles
Université Catholique de Louvain
Louvain-la-Neuve, Belgium
denuit@stat.ucl.ac.be

Jan Dhaene
Catholic University of Leuven
Center for Risk and Insurance Studies
B-3000 Leuven, Belgium
jan.dhaene@econ.kuleuven.ac.be

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Abstract

In the Lee-Carter framework, future survival probabilities are random variables with an intricate distribution function. In large homogeneous portfolios of life annuities, Value-at-Risk or Conditional Tail Expectation of the total yearly payout of the company are approximately equal to the corresponding quantities involving random survival probabilities. This paper aims to derive some bounds in the increasing convex (or stop-loss) sense on these random survival probabilities. These bounds are obtained with the help of comonotonic upper and lower bounds on sums of correlated random variables.

Key words and phrases: mortality projection, comonotonicity, risk measure, stop-loss order.

1 Introduction and Motivation

During the 20th century, the human mortality globally declined. These mortality improvements pose a challenge for the planning of public retirement systems as well as for the private life annuities business. When long-term living benefits are concerned, the calculation of expected present values (for pricing or reserving) requires an appropriate mortality projection in order to avoid underestimation of future costs. Actuaries have therefore to resort to life tables including a forecast of the future trends of mortality (the so-called projected life tables).

Different approaches for building projected life tables have been developed so far; see e.g., Pitacco (2004) and Wong-Fupuy & Haberman (2004) for a review. Lee & Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. The main statistical tool of Lee & Carter (1992) is least-squares estimation via singular value decomposition of the matrix of the log age-specific observed forces of mortality together with Box-Jenkins modelling for time series. For a review of recent applications of the Lee-Carter methodology, we refer the interested readers to Lee (2000).

The future lifetimes are all influenced by the same time index in the Lee-Carter framework. Since the future path of this index is unknown and modelled as a stochastic process, the policyholders' lifetimes become dependent on each other. When the Lee-Carter model applies, life annuity present values are correlated random variables, contrarily to the standard actuarial assumptions. Consequently, the risk does not disappear as the size of the portfolio increases: there always remains some systematic risk, that cannot be diversified, whatever the number of policies. This unexpected feature of the Lee-Carter model has been studied in Denuit & Frostig (2005).

This paper aims to apply the concept of comonotonicity (reviewed in details by Dhaene et al. (2002a,b)) to obtain approximations for stochastic survival probabilities in the Lee-Carter framework. The main contribution of this paper is that it provides a new application of the concept of comonotonicity, which deviates away from the natural financial applications of this theory (see, e.g., Dhaene et al. (2005) and Vanduffel et al. (2005) for illustrations). As such, it further extends the scope and the applicability of this theory.

The paper is organized as follows. Section 2 briefly reviews the basic features of the Lee-Carter model for mortality projections. Section 3 describes the survival probabilities in the Lee-Carter framework. Because they depend on the future trajectory of the time index, these probabilities are random variables. It is shown that computing their distribution function amounts to determine the distribution function of a sum of correlated LogNormal random variables. This allows us to derive stop-loss upper and lower bounds on the survival probabilities. It is shown in Denuit & Frostig (2005) that risk measures (like Value-at-Risk or Conditional Tail Expectation, for instance) of the insurer's annual payout for an homogeneous portfolio of life annuities are related to corresponding quantities involving survival probabilities. It is therefore important to be able to evaluate quantiles or conditional expectations of survival probabilities in the Lee-Carter model. This is the purpose of Section 4 where approximations for quantiles and bounds on conditional expectations are derived. Moreover, stop-loss lower and upper bounds on the number of survivors are proposed. The final Section 5 concludes.

2 Lee-Carter Stochastic Modelling for Dynamic Mortality

2.1 Notation and assumption

We analyze the changes in mortality as a function of both age x and time t. Henceforth, $T_x(t)$ is the remaining lifetime of an individual aged x on January the first of year t; this individual will die at age $x + T_x(t)$ in year $t + T_x(t)$. The mortality force at age x during calendar year t, denoted as $\mu_x(t)$, is defined as

$$\mu_x(t) = \lim_{\Delta \to 0} \frac{\Pr[x < T_0(t-x) \le x + \Delta | T_0(t-x) > x]}{\Delta}.$$

As pointed out by Dahl (2004), actuaries have traditionally been calculating premiums and reserves using a deterministic mortality intensity. Here, as in the paper by Dahl (2004), $\mu_x(t)$ will be described by a stochastic process.

In this paper, we assume that the age-specific mortality rates are constant within bands of age and time, but allowed to vary from one band to the next. Specifically, given any integer age x and calendar year t, it is supposed that

$$\mu_{x+\xi}(t + \tau) = \mu_x(t) \text{ for } 0 \le \xi, \tau < 1. \tag{2.1}$$

Under (2.1), we have for integer age x and calendar year t that

$$p_x(t) = \Pr[T_x(t) > 1] = \exp(-\mu_x(t)). \tag{2.2}$$

2.2 Lee-Carter model

Let us recall the basic features of the classical Lee-Carter approach. The model proposed by Lee & Carter (1992) is in essence a relational model assuming that

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t. \tag{2.3}$$

Interpretation of the parameters involved in model (2.3) is quite simple. The value of α_x is an average of $\ln \mu_x(t)$ over time t so that $\exp \alpha_x$ is the general shape of the mortality schedule. The actual forces of mortality change according to an overall mortality index κ_t modulated by an age response β_x. The shape of the β_x profile tells which rates decline rapidly and which slowly over time in response of change in κ_t. The time factor κ_t is intrinsically viewed as a stochastic process and Box-Jenkins techniques are then used to model and forecast κ_t.

2.3 Stochastic modelling of the time index

To forecast, Lee & Carter (1992) assume that the α_x's and β_x's remain constant over time and forecast future values of κ_t using a standard univariate time series model. After testing several specifications, they found that a random walk with drift was the most appropriate model for their data. They made clear that other ARIMA models might be preferable for different data sets, but in practice the random walk with drift model for κ_t is used almost exclusively. According to this model, the κ_t's obey to

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t \text{ with iid } \xi_t \sim \mathcal{N}or(0, \sigma^2), \tag{2.4}$$

where θ is known as the drift parameter and $\mathcal{N}or(0, \sigma^2)$ stands for the Normal distribution with mean 0 and variance σ^2. We will retain the model (2.4) throughout this paper. Note that since the κ_t's obey to the dynamics (2.4), the $\mu_x(t)$'s given in (2.3) are not constant but develop over time following a stochatic process.

We will assume in the remainder of this paper that the values $\kappa_1, \ldots, \kappa_{t_0}$ are known but that the κ_{t_0+k}'s, $k = 1, 2, \ldots$, are unknown and have to be projected from (2.4). To forecast the time index at time $t_0 + k$ with all data available up to t_0, we use the representation

$$\kappa_{t_0+k} = \kappa_{t_0} + k\theta + \sum_{j=1}^{k} \xi_{t_0+j}.$$

The point estimate of the stochastic forecast is thus

$$\mathbb{E}[\kappa_{t_0+k} | \kappa_1, \ldots, \kappa_{t_0}] = \kappa_{t_0} + k\theta$$

which follows a straight line as a function of the forecast horizon k, with slope θ. The conditional variance of the forecast is

$$\mathbb{V}[\kappa_{t_0+k} | \kappa_1, \ldots, \kappa_{t_0}] = k\sigma^2.$$

Therefore, the conditional standard errors for the forecast increase with the square root of the distance to the forecast horizon k.

3 Comonotonic Bounds

3.1 The d-year survival probability

For any non-negative integer d, let ${}_dP_{x_0}$ be the d-year survival probability for an individual aged x_0 in year t_0 given the trajectory of the time index $\boldsymbol{\kappa}$. More specifically, ${}_dP_{x_0} = \Pr[T_{x_0}(t_0) > d | \boldsymbol{\kappa}]$, where $\boldsymbol{\kappa}$ stands for the random vector $(\kappa_{t_0}, \ldots, \kappa_{t_0+\omega-x_0})$, where ω is the ultimate age of the life table. In the Lee-Carter framework and for integer d, this probability writes

$$\begin{aligned} {}_dP_{x_0} &= \prod_{j=0}^{d-1} p_{x_0+j}(t_0 + j) \\ &= \exp\left(- \sum_{j=0}^{d-1} \mu_{x_0+j}(t_0 + j) \right) \\ &= \exp\left(- \sum_{j=0}^{d-1} \exp\left(\alpha_{x_0+j} + \beta_{x_0+j} \kappa_{t_0+j} \right) \right). \end{aligned}$$

Note that ${}_dP_{x_0}$ is a random variable since it involves the κ_{t_0+j}'s obeying to (2.4).

As shown by Denuit & Frostig (2005), standard risk measures in large portfolios are functions of the survival probabilities. It is therefore interesting to derive the distribution of ${}_dP_{x_0}$.

3.2 Distribution of the d-year survival probability

Clearly,

$$ {}_dP_{x_0} = \exp(-S_d) \tag{3.1} $$

with

$$ S_d = \sum_{j=0}^{d-1} \exp\Big(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j}\Big) = \sum_{j=0}^{d-1} \delta_j \exp(X_j), $$

where $\delta_j = \exp(\alpha_{x_0+j}) > 0$ and $X_j = \beta_{x_0+j}\kappa_{t_0+j}$. Conditional upon κ_{t_0}, we have that $X_j \sim \mathcal{N}or(\mu_j, \sigma_j^2)$ with

$$ \mu_j = \beta_{x_0+j}(\kappa_{t_0} + j\theta) \text{ and } \sigma_j^2 = (\beta_{x_0+j})^2 j\sigma^2, \tag{3.2} $$

with the convention that a normally distributed random variable with zero variance is constantly equal to the mean.

The distribution function F_d of ${}_dP_{x_0}$ is given by

$$ F_d(p) = \Pr[{}_dP_{x_0} \leq p] = \Pr\left[S_d \geq -\ln p\right], \ \ 0 \leq p \leq 1, $$

where S_d is a linear combination of correlated LogNormal random variables. The analytical computation of F_d is difficult and numerical alternatives must be contemplated. A convenient procedure consists in simulating the X_j's (from the dynamics (2.4) for the κ_t's) to approximate the distribution function of the S_d's. In Section 5, we derive several accurate approximations for F_d and related quantities.

3.3 Some stochastic order relations

In the next section, we derive bounds on ${}_dP_{x_0}$ in the stop-loss (or increasing convex) sense. Here, we recall some definitions. For more details, the readers are referred, e.g., to Denuit et al. (2005). Consider two random variables X and Y. Then, X is said to be smaller than Y in the stop-loss order, henceforth denoted by $X \preceq_{\text{sl}} Y$, if $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \in \mathbb{R}^+$, that is, if their corresponding stop-loss premiums are ordered for all possible levels d of the deductible. In probability theory, the stop-loss order is usually referred to as the increasing convex order, since $X \preceq_{\text{sl}} Y \Leftrightarrow \mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for all the non-decreasing and convex functions g for which the expectations exist. A usual strengthening of the stop-loss order is obtained by requiring in addition that the means of the random variables to be compared are equal. More precisely, X is said to be smaller than Y in the convex order, henceforth denoted by $X \preceq_{\text{cx}} Y$ (or sometimes by $X \preceq_{\text{sl},=} Y$ in the actuarial literature), if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $X \preceq_{\text{sl}} Y$. The term "convex" is used since $X \preceq_{\text{cx}} Y \Leftrightarrow \mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for all convex functions g for which the expectations exist.

The supermodular order is based on the comparison of expectations of supermodular functions. A real-valued function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is called supermodular if the inequality

$$\Psi(\max\{x_1, y_1\}, \max\{x_2, y_2\}, \dots, \max\{x_n, y_n\}) + \Psi(\min\{x_1, y_1\}, \min\{x_2, y_2\}, \dots, \min\{x_n, y_n\})$$
$$\geq \Psi(x_1, x_2, \dots, x_n) + \Psi(y_1, y_2, \dots, y_n), \tag{3.3}$$

holds for all $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$. If Ψ has second partial derivatives then it is supermodular if, and only if, $\frac{\partial^2 \Psi}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$. Then, given two random vectors $(X_1, X_2, \dots, X_n)$ and $(Y_1, Y_2, \dots, Y_n)$, $(X_1, X_2, \dots, X_n)$ is said to precede $(Y_1, Y_2, \dots, Y_n)$ in the supermodular order, denoted as $(X_1, X_2, \dots, X_n) \preceq_{\text{sm}} (Y_1, Y_2, \dots, Y_n)$, if the inequality

$$\mathbb{E}[\Psi(X_1, X_2, \dots, X_n)] \leq \mathbb{E}[\Psi(Y_1, Y_2, \dots, Y_n)] \tag{3.4}$$

holds for all the supermodular functions $\Psi : \mathbb{R}^n \to \mathbb{R}$ for which the expectations in (3.4) exist. It is worth to mention that two random vectors ordered in the supermodular sense necessarily have the same univariate marginals, that is, X_i and Y_i are identically distributed for $i = 1, 2, \dots, n$.

3.4 Upper bound on ${}_dP_{x_0}$

We are now in a position to derive an upper bound on ${}_dP_{x_0}$ in the $\preceq_{\text{sl}}$-sense, based on a comonotonic version of the X_j's. To this end, let us first establish the next property, which is similar to Proposition 6.3.9 in Denuit et al. (2005).

Property 3.1. *Let* $(X_1, X_2, \dots, X_n)$ *and* $(Y_1, Y_2, \dots, Y_n)$ *be two random vectors, and consider a non-increasing, twice-differentiable supermodular function* $\Psi : \mathbb{R}^n \to \mathbb{R}$. *Then,*

$$(X_1, X_2, \dots, X_n) \preceq_{sm} (Y_1, Y_2, \dots, Y_n) \Rightarrow \Psi(X_1, X_2, \dots, X_n) \preceq_{sl} \Psi(Y_1, Y_2, \dots, Y_n).$$

Proof. To establish this statement, we need to show that, given any non-decreasing and convex function g, $g \circ \Psi$ is supermodular. From Denuit & Müller (2002), we know that it is enough to consider a twice differentiable function g. Then, a straightforward computation of the second mixed derivative of $g \circ \Psi$ gives for $i \neq j$

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} g \circ \Psi &= \frac{\partial}{\partial x_i}\left(g' \circ \Psi \times \frac{\partial}{\partial x_j}\Psi\right) \\ &= g'' \circ \Psi \times \frac{\partial}{\partial x_i}\Psi \times \frac{\partial}{\partial x_j}\Psi + g' \circ \Psi \times \frac{\partial^2}{\partial x_i \partial x_j}\Psi \geq 0, \end{aligned}$$

which ends the proof. □

From (3.1), we see that ${}_dP_{x_0}$ can be expressed as $\Psi(X_1, \dots, X_{d-1})$, with $\Psi(x_1, \dots, x_{d-1}) = \exp\left(-\sum_{j=0}^{d-1} \delta_j \exp(x_j)\right)$ fulfills the assumptions of Property 3.1. From Dhaene et al. (2002a), we know that

$$(X_1, \dots, X_n) \preceq_{\text{sm}} (\mu_1 + \sigma_1 Z, \dots, \mu_{d-1} + \sigma_{d-1} Z) \text{ where } Z \sim \mathcal{N}or(0, 1).$$

Let us define S_d^u as

$$S_d^u = \sum_{j=0}^{d-1} \delta_j \ \exp\Big(\mu_j + \sigma_j \ Z\Big), \text{ with } Z \sim \mathcal{N}or(0,1),$$

where μ_j and σ_j are given in (3.2). We then have that $S_d \preceq_{\text{cx}} S_d^u$ and ${}_dP_{x_0} \preceq_{\text{sl}} \exp(-S_d^u)$. The distribution function of S_d^u can be determined from the following algorithm: denoting as Φ the distribution function of Z, $F_{S_d^u}(x) = \Phi(\nu_x)$, with ν_x determined by

$$\sum_{j=0}^{d-1} \delta_j \ \exp\big(\mu_j + \sigma_j \ \nu_x\big) = x.$$

Note that the derivation of a convex upper bound on ${}_dP_{x_0}$ (which would be "closer" to ${}_dP_{x_0}$, sharing the same mean) requires the computation of $\mathbb{E}[{}_dP_{x_0}]$, which amounts to perform a $(d-1)$-dimensional integration. As pointed out by Vanduffel, Hoedemakers & Dhaene (2005) in a different context, this would outweigh one of the main features of the comonotonicity-based bounds, namely that the actuarial quantities of interest can easily be determined analytically.

Property 3.1 can also be used to determine a lower bound on ${}_dP_{x_0}$ as follows. The κ_t's obeying to (2.4) are associated, i.e. they satisfy $\mathbb{C}\text{ov}[\Psi_1(\boldsymbol{\kappa}), \Psi_2(\boldsymbol{\kappa})] \geq 0$ for all the non-decreasing functions Ψ_1 and Ψ_2 such that the covariance exists. The X_j's are therefore also associated. We know from Christophides & Vaggelatou (2004) that $(X_1^\perp, \ldots, X_{d-1}^\perp) \preceq_{\text{sm}} (X_1, \ldots, X_{d-1})$, where $(X_1^\perp, \ldots, X_{d-1}^\perp)$ is a vector made of independent components, with the same univariate marginals as the original $(X_1, \ldots, X_{d-1})$. Defining $S_d^\perp = \sum_{j=0}^{d-1} \delta_j \exp(X_j^\perp)$, we get $\exp(-S_d^\perp) \preceq_{\text{sl}} {}_dP_{x_0}$. Unfortunately, computing quantities involving $\exp(-S_d^\perp)$ requires approximately the same effort than computing them directly with ${}_dP_{x_0}$. In the next section, we derive a lower bound on ${}_dP_{x_0}$, as simple as the upper one derived above.

3.5 Lower bound on ${}_dP_{x_0}$

From Theorem 1 in Dhaene et al. (2002), we know that there exists a lower bound S_d^l in the convex sense on S_d that is obtained by conditioning S_d on some random variable Λ (since we know from Strassen's theorem that $S_d^l = \mathbb{E}[S_d|\Lambda] \preceq_{\text{cx}} S_d$). Following Kaas et al. (2000), we take

$$\Lambda = \sum_{j=0}^{d-1} \delta_j \exp(\mu_j) X_j.$$

The lower bound $\mathbb{E}[S_d|\Lambda]$ is then given by

$$S_d^l = \sum_{j=0}^{d-1} \delta_j \exp\Big(\mu_j + r_j\sigma_j Z + \frac{1}{2}(1 - r_j^2)\sigma_j^2\Big) \preceq_{\text{cx}} S_d$$

where r_i, $i = 0, \ldots, d-1$, is the correlation coefficient between Λ and X_i, that is,

$$r_i = \frac{\mathbb{C}\text{ov}[X_i, \Lambda]}{\sigma_i \sigma_\Lambda} = \frac{\sum_{j=0}^{d-1} \delta_j \exp(\mu_j) \mathbb{C}\text{ov}[X_i, X_j]}{\sigma_i \sqrt{\sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \delta_j \delta_k \exp(\mu_j + \mu_k) \beta_{x_0+j} \beta_{x_0+k} \min\{j,k\} \sigma^2}}$$

where

$$\mathbb{C}\text{ov}[X_i, X_j] = \beta_{x_0+i}\beta_{x_0+j}\min\{i,j\}\sigma^2.$$

Then, we have $\exp(-S_d^l) \preceq_{\text{sl}} {}_dP_{x_0}$.

In the application we have in mind, β_{x_0+i} and β_{x_0+j} typically have the same sign so that all the r_i's are non-negative. Then, S_d^l is the sum of d comonotonic random variables, which makes the derivation of its distribution function easy: this distribution function is given by $F_{S_d^l}(x) = \Phi(\nu_x)$ where ν_x is the root of the equation

$$\sum_{j=0}^{d-1} \delta_j \exp\left(\mu_j + r_j\sigma_j\nu_x + \frac{1}{2}(1-r_j^2)\sigma_j^2\right) = x.$$

4 Some Approximations and Bounds

As mentioned in the introduction, Denuit & Frostig (2005) proved that standard risk measures for the aggregate claim amounts are well approximated in large portfolios by the corresponding quantity for the random survival probabilities. This is why approximations for VaR's or CTE's of ${}_dP_{x_0}$ are of interest. The derivation of such approximations is the topic of this section.

4.1 Approximation for the VaR

It is clear that the quantiles of ${}_dP_{x_0}$ are related to the quantiles of S_d through the formula

$$F_d^{-1}(\epsilon) = \exp\Big(-F_{S_d}^{-1}(\epsilon)\Big).$$

We do not have bounds on $F_{S_d}^{-1}(\epsilon)$, but the quantile functions of S_d^l and S_d^u can be used as an approximation of it. It turns out that the lower bound outperforms the upper one as far as approximation of $F_{S_d}^{-1}(\epsilon)$ is concerned. Therefore, we can use the approximation

$$F_{S_d}^{-1}(\epsilon) \approx F_{S_d^l}^{-1}(\epsilon) = \sum_{j=0}^{d-1} \delta_j \left(\mu_j + \frac{1}{2}(1-r_j)^2\sigma_j^2 + r_j\sigma_j\Phi^{-1}(\epsilon)\right).$$

4.2 Bounds on CTE

Recall that the Conditional Tail Expectation of X at probability level ϵ is defined as $\text{CTE}[X;\epsilon] = \mathbb{E}[X|X > F_X^{-1}(\epsilon)]$. As ${}_dP_{x_0}$ possesses as continuous distribution function F_d, the Conditional Tail Expectation of ${}_dP_{x_0}$ agrees with a $\preceq_{\text{sl}}$-ranking and we have

$$B_l(\epsilon) \leq \text{CTE}[{}_dP_{x_0};\epsilon] \leq B_u(\epsilon)$$

where

$$B_l(\epsilon) = \text{CTE}\Big[\exp\big(-S_d^l\big);\epsilon\Big],$$

and

$$B_u(\epsilon) = \text{CTE}\Big[\exp\big(-S_d^u\big);\epsilon\Big].$$

4.3 Stop-loss bounds on the number of survivors

Let N_d be the number of survivors at time $t_0 + d$ from an initial group of n policyholders aged x_0 at time t_0. Given $\boldsymbol{\kappa}$, N_d obeys to the Binomial distribution with exponent n and parameter ${}_dP_{x_0}$.

Considering the extremal numbers of survivors N_d^+ and N_d^- with respective probability distributions

$$\Pr[N_d^- = k] = \binom{n}{k} \int_0^1 p^k(1-p)^{n-k} d\Pr[S_d^l \geq -\ln p],\ \ k = 0, 1, \ldots,$$

and

$$\Pr[N_d^+ = k] = \binom{n}{k} \int_0^1 p^k(1-p)^{n-k} d\Pr[S_d^u \geq -\ln p],\ \ k = 0, 1, \ldots,$$

we have

$$N_d^- \preceq_{\rm sl} N_d \preceq_{\rm sl} N_d^+.$$

Natural candidates for defining the benefits of longevity bonds or reinsurance treaties covering portfolios of life annuities involve the excess of the actual number of survivors to that expected from a reference life table. If the expected number of survivors at time $t_0 + d$ is $\overline{\nu}_d$ then the payoff could be related to $(N_d - \overline{\nu}_d)_+$. Bounds on $\mathbb{E}[(N_d - \overline{\nu}_d)_+]$ are then obtained by substituting N_d^- and N_d^+ for N_d.

5 Discussion

In this paper, we have derived some bounds (in the $\preceq_{\rm sl}$-sense) on the random survival probabilities in the Lee-Carter framework. We considered that the trajectory of the time index was adequately described by a random walk with drift. The approach remains nevertheless valid for any Gaussian process.

Of course, one could imagine many other stochastic processes for modelling the behavior of the time index κ_t. For instance, good candidates could be processes allowing for jumps. The idea is to modify (2.4) into

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t + I_t Z_t \text{ with } \xi_t \text{ iid } \mathcal{N}or(0, \sigma^2),$$

where I_t is a sequence of independent Bernoulli random variables with mean p indicating whether a jump occured in period t, and Z_t is a real valued random variable representing the size of the jump. In this setting, a positive jump corresponds to a catastrophe, like an epidemic, increasing suddenly the mortality in year t, whereas a negative jump means that a considerable improvement has been achieved, for instance thanks to the availability of a new medical treatment. The approach developed in this paper could be adapted to this case.

The determination of an optimal choice for the conditioning variable Λ used to define the lower bound on the random survival probability remains an open question. Recently, Vanduffel, Hoedemakers & Dhaene (2005) took for Λ the linear combination of the X_j's maximizing a first order approximation to the variance of the lower bound (making it intuitively speaking as close as possible to S_d). This maximal variance lower bound proposed could also be used to derive approximations for the survival probabilities in the Lee-Carter framework.

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