Some results on the CTE based capital allocation rule

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Abstract

Tasche (1999) introduces a capital allocation principle where the capital allocated to each risk unit can be expressed in terms of its contribution to the conditional tail expectation (CTE) of the aggregate risk. Panjer (2002) derives a closed-form expression for this allocation rule in the multivariate normal case. Landsman & Valdez (2003) generalise Panjer's result to the class of multivariate elliptical distributions.

In this paper we provide an alternative and simpler proof for the CTE based allocation formula in the elliptical case. Furthermore, we derive accurate and easy computable closed-form approximations for this allocation formula for sums that involve normal and lognormal risks.

1 Introduction

Evaluating the total capital requirement of a financial conglomerate as well as the allocation of this capital to its various business units is an important risk management issue. Recently, several authors have proposed the Conditional Tail Expectations as an appropriate risk measure for setting aggregate capital requirements of a financial institution, see for instance Wang (2002). For a given probability level p, the Conditional Tail Expectation $\text{CTE}_p[X]$ of the random variable (rv) X is defined by

$$\text{CTE}_p[X] = \text{E}\left[X \mid X > \text{Q}_p[X]\right], \qquad 0 < p < 1, \tag{1}$$

where Q_p stands for the quantile function:

$$\text{Q}_p[X] = \inf\{x \mid F_X(x) \geq p\}, \qquad 0 < p < 1 \tag{2}$$

Note that in this paper, expectations of rv's are assumed to exist when required. For a discussion on the suitability of $\text{CTE}_p[X]$ to set capital requirements in a one-period framework, see e.g. Dhaene *et al.* (2004).

Various capital allocation techniques have been proposed in the literature. Dhaene *et al.* (2005a) introduce a general capital allocation rule which is the solution of a distance

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minimisation problem. Several well-known allocation principles turn out to be special cases of this general allocation rule, and hence can be seen as solutions of a particular optimisation problem.

By the additivity property of the expectation operator, the CTE allows for a natural allocation of the aggregate capital attributed to $S = X_1 + \ldots + X_n$ among its various constituents X_k, $k = 1, 2, \ldots, n$. Indeed, based on the observation that

$$\mathrm{CTE}_p\left[S\right] = \sum_{k=1}^{n} \mathrm{E}[X_k | S > \mathrm{Q}_p\left[S\right]], \tag{3}$$

it appears 'natural' to consider the CTE based allocation rule where the amount $E[X_k|S > Q_p\left[S\right]]$ is attributed to the k-th risk.

Tasche (1999, 2004) obtains the CTE based allocation rule by a marginal cost argument. Denault (2001) finds this allocation rule within a game theoretical framework. Panjer (2002) provides a closed-form expression for this allocation rule when the risks are multivariate normally distributed, and Landsman & Valdez (2002) extend Panjer's result to the case where the risks are multivariate elliptically distributed. The proof of their result is rather technical and in this paper we give an elegant and shorter proof. Furthermore, we derive closed-form approximations for the CTE based allocation rule for sums that involve normal and lognormal risks. In the final section we provide a numerical example to illustrate the accuracy of the approximations.

2 The CTE based allocation rule for elliptical distributions

2.1 Elliptical distributions

In this section we recall some definitions and results concerning multivariate elliptical distributions. An extended reference to this class of distributions is Fang *et al.* (1990).

Definition 2.1 (Multivariate elliptical distributions). *Consider the vector* $\underline{\mu} = (\mu_1, ..., \mu_n)^T$ *and the positive semidefinite matrix* $\underline{\Sigma}$ *with elements* σ_{kl} $(k, l = 1, 2, ..., n)$*. The random vector* $\underline{X} = (X_1, \ldots, X_n)^T$ *is said to have an elliptical distribution with parameters* $\underline{\mu}$ *and* $\underline{\Sigma}$ *if its characteristic function* $E\left[\exp\left(i\underline{t}^T\underline{X}\right)\right]$ *is expressed as*

$$E\left[\exp\left(i\underline{t}^T\underline{X}\right)\right] = \exp\left(i\underline{t}^T\underline{\mu}\right)\ \phi\left(\frac{1}{2}\underline{t}^T\underline{\Sigma}\underline{t}\right), \qquad \underline{t}^T = \left(t_1, t_2, \ldots, t_n\right), \tag{4}$$

for some scalar function ϕ*.*

The function ϕ is called the *characteristic generator* of $\underline{X}$. If $\underline{X}$ is elliptically distributed as defined above we write $\underline{X} \sim E_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$. Note that the moments of $\underline{X} \sim E_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$ do not necessarily exist. However, in case the means exist, they are given by

$$\mathrm{E}\left[X_k\right] = \mu_k. \tag{5}$$

The existence of the covariances is equivalent to the existence of $\phi'(0)$ and in this case we find that

$$\text{Cov}\left[X_k, X_l\right] = -\phi'(0)\,\sigma_{kl}. \tag{6}$$

From (4) it follows that each component X_k of $\underline{X}\sim E_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$ is also elliptically distributed with the same characteristic generator:

$$X_k \sim E_1\left(\mu_k, \sigma_k^2, \phi\right), \qquad k = 1, \ldots, n, \tag{7}$$

and with $\sigma_k^2 = \sigma_{kk}$. Furthermore, the rv S defined by

$$S = \sum_{j=1}^{n} X_j \tag{8}$$

is elliptically distributed with the same characteristic generator:

$$S \sim E_1\left(\mu_S, \sigma_S^2, \phi\right), \tag{9}$$

and with parameters μ_S and σ_S^2 given by

$$\mu_S = \sum_{j=1}^{n} \mu_j \text{ and } \sigma_S^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jk}, \tag{10}$$

respectively. For the rv's X_k and S one can prove the following regression result:

$$E\left[X_k | S = s\right] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S^2}\left(s - \mu_S\right), \tag{11}$$

provided $\sigma_S^2 > 0$ and with $\sigma_{k,S}$ given by

$$\sigma_{k,S} = \sum_{j=1}^{n} \sigma_{kj}. \tag{12}$$

Not every multivariate elliptical distribution has a probability density function (pdf). The following well-known theorem gives necessary and sufficient condition for the existence of an elliptical density.

Theorem 2.1 (Elliptical densities). *Consider the random vector $\underline{Y}$. Then $\underline{Y}\sim E_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$ and has a density if and only if the pdf of $\underline{Y}$ is given by*

$$f_{\underline{Y}}(\underline{y}) = \frac{c_n}{\sqrt{|\underline{\Sigma}|}} g_n\left[\frac{1}{2}\left(\underline{y}-\underline{\mu}\right)^T \underline{\Sigma}^{-1}\left(\underline{y}-\underline{\mu}\right)\right] \tag{13}$$

for some non-negative function g_n satisfying the condition

$$0 < \int_0^\infty z^{n/2-1} g_n(z)dz < \infty, \tag{14}$$

a normalising constant c_n given by

$$c_n = \frac{\Gamma\left(n/2\right)}{(2\pi)^{n/2}}\left[\int_0^\infty z^{n/2-1} g_n(z)dz\right]^{-1}, \tag{15}$$

and with $\underline{\Sigma}$ positive definite.

The function g_n is called the *density generator*. When $n = 1$ we will often use the notations g and c instead of g_1 and c_1, respectively. One sometimes write $Y \sim E_n\left(\underline{\mu}, \underline{\Sigma}, g_n\right)$ to indicate n-dimensional elliptical distributions generated from the function g_n. A detailed proof of these results, using spherical transformations of rectangular coordinates, can be found in Landsman & Valdez (2002).

In the following example we consider multivariate normal distributions which are an important subclass of elliptical distributions.

Example 2.1 (Multivariate normal distribution). *Consider the vector* $\underline{\mu} = (\mu_1, ..., \mu_n)^T$ *and the positive semidefinite matrix* $\underline{\Sigma}$. *The n-dimensional random vector* $\underline{X}$ *has the multivariate normal distribution with parameters* $\underline{\mu}$ *and* $\underline{\Sigma}$, *notation* $\underline{X} \sim N_n\left(\underline{\mu}, \underline{\Sigma}\right)$, *if its characteristic function is given by*

$$E\left[\exp\left(i\underline{t}^T\underline{X}\right)\right] = \exp\left(i\underline{t}^T\underline{\mu}\right)\ \exp\left(-\tfrac{1}{2}\underline{t}^T\underline{\Sigma}\underline{t}\right). \tag{16}$$

From (4) we see that $N_n\left(\underline{\mu}, \underline{\Sigma}\right)$ *is an elliptical distribution with characteristic generator* ψ *given by*

$$\phi(t) = e^{-t}. \tag{17}$$

Since $\phi'(0) = -\frac{1}{2}$ *the matrix* $\underline{\Sigma}$ *in (16) is the covariance matrix of* $\underline{X}$.
In case $\underline{\Sigma}$ *is positive definite, the random vector* $\underline{X} \sim N_n\left(\underline{\mu}, \underline{\Sigma}\right)$ *has a pdf which is given by*

$$f_{\underline{X}}\left(\underline{x}\right) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|\underline{\Sigma}|}} \exp\left[-\tfrac{1}{2}\left(\underline{x}-\underline{\mu}\right)^T \underline{\Sigma}^{-1}\left(\underline{x}-\underline{\mu}\right)\right]. \tag{18}$$

Comparing (13) and (18) we find that the density generator g_n *and the normalising constant* c_n *of* $N_n\left(\underline{\mu}, \underline{\Sigma}\right)$ *are given by*

$$g_n(u) = e^{-u} \tag{19}$$

and

$$c_n = \frac{1}{(2\pi)^{\frac{n}{2}}}, \tag{20}$$

respectively.

In the following example we consider multivariate Laplace distributions which provide another subclass of elliptical distributions.

Example 2.2 (Multivariate Laplace distribution). *Following Andersen (1992), the n-dimensional random vector* $\underline{X}$ *is said to have a Multivariate Laplace pdf with mean vector* $\underline{\mu}$ *and positive definite variance-covariance matrix* $\underline{\Sigma}$ *if the pdf has the form*

$$f_{\underline{X}}\left(\underline{x}\right) = \frac{2}{(2\pi)^{\frac{n}{2}}\sqrt{|\underline{\Sigma}|}}\left[\tfrac{1}{2}\left(\underline{x}-\underline{\mu}\right)^T \underline{\Sigma}^{-1}\left(\underline{x}-\underline{\mu}\right)\right]^{\upsilon/2} K_\upsilon\left(2\sqrt{\frac{1}{2}\left(\underline{x}-\underline{\mu}\right)^T \underline{\Sigma}^{-1}\left(\underline{x}-\underline{\mu}\right)}\right). \tag{21}$$

Here, $\upsilon = (2-n)/2$, *while* $K_\upsilon(u)$ *is the modified Bessel function of the* 3^{rd} *kind, see Abramovich & Stegun (1965, p. 376). We write* $\underline{X} \sim La_n\left(\underline{\mu}, \underline{\Sigma}\right)$. *Comparing (13) and (21)* *we find that $\underline{X}$ is elliptically distributed with density generator g_n and normalising constant c_n given by*

$$g_n(u) = 2u^{v/2}K_v(2\sqrt{u}), \qquad u > 0 \tag{22}$$

and

$$c_n = \frac{1}{(2\pi)^{\frac{n}{2}}}. \tag{23}$$

When $n = 1$ we have that

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\exp(-x), \qquad x > 0 \tag{24}$$

and we obtain the Laplace (or double exponential) pdf:

$$f_X(x) = \frac{1}{\sqrt{2}\sigma}\exp(-\sqrt{2}\frac{|x-\mu|}{\sigma}). \tag{25}$$

The characteristic function of $\underline{X} \sim La_n\left(\underline{\mu}, \underline{\Sigma}\right)$ is given by

$$E\left[\exp\left(i\underline{t}^T\underline{X}\right)\right] = \exp\left(i\underline{t}^T\underline{\mu}\right)(1+\frac{1}{2}\underline{t}^T\underline{\Sigma}\underline{t})^{-1}, \tag{26}$$

which implies that the characteristic generator ϕ is given by

$$\phi(t) = \frac{1}{1+t}. \tag{27}$$

Note that since $\phi'(0) = -1$ the matrix $\underline{\Sigma}$ in (21) is indeed a covariance matrix.

We refer to Fang *et al.* (1990) for an extended list of examples of multivariate elliptical distributions. Actuarial applications of elliptical distributions are considered in Landsman & Valdez (2003) and Valdez & Dhaene (2004), amongst others. In the remainder of the paper we will only consider rv's with a finite mean.

2.2 CTE's and the CTE based allocation rule for elliptical distributions

Let $X \sim E_1(\mu, \sigma^2, g)$ with $\sigma^2 > 0$. Landsman & Valdez (2003) prove that its Conditional Tail Expectations are given by

$$\text{CTE}_p\left[X\right] = \mu + \sigma\frac{c}{(1-p)}\int_{\frac{1}{2}Q_p^2\left[\frac{X-\mu}{\sigma}\right]}^{\infty} g(x)\ dx, \qquad 0 < p < 1, \tag{28}$$

with c being the appropriate normalising constant as defined in (15).

In the following two examples we derive expressions for the conditional tail expectations of normally and Laplace distributed rv's.

Example 2.3 (CTE's of a normal random variable). *Assume that* $X \sim N(\mu, \sigma^2)$ *with* $\sigma^2 > 0$. *From (28) we find the well-known expressions for the CTE's of* X :

$$CTE_p[X] = \mu + \sigma \frac{\Phi'(\Phi^{-1}(p))}{1-p}, \qquad 0 < p < 1, \tag{29}$$

where Φ *denotes the cumulative distribution function (cdf) and* Φ' *the related pdf of the standard normally distributed rv* $Z \sim N(0,1)$. *Furthermore,* Φ^{-1} *is the quantile function of the standard normal cdf.*

Example 2.4 (CTE's of a Laplace random variable). *Assume that* $X \sim La(\mu, \sigma^2)$ *with* $\sigma^2 > 0$. *Using (24) and (23), we find from (28) that*

$$CTE_p[X] = \mu + \frac{\sigma}{(1-p)\sqrt{2}} \int_{\frac{1}{2}Q_p^2\left[\frac{X-\mu}{\sigma}\right]}^{\infty} \exp(-2\sqrt{u})du.$$

After some straightforward calculations this expression transforms into

$$CTE_p[X] = \mu + \sigma \frac{e^{-\sqrt{2}\,\Lambda^{-1}(p)}}{2(1-p)}\left(\frac{\sqrt{2}}{2} + \Lambda^{-1}(p)\right), \qquad 0 < p < 1, \tag{30}$$

where Λ *and* Λ^{-1} *denote the cdf and the related quantile function* $Q_p[X]$ *of the standard Laplace distributed rv* $Z \sim La(0,1)$. *One can prove that*

$$\Lambda^{-1}(p) = \begin{cases} \frac{1}{\sqrt{2}}\ln(2p) & : 0 < p \le \frac{1}{2}, \\ -\frac{1}{\sqrt{2}}\ln(2(1-p)) & : \frac{1}{2} < p < 1. \end{cases} \tag{31}$$

Landsman & Valdez (2003) derived a closed-form expression of the CTE based allocation rule for elliptical rv's. In the following theorem we restate their result and we give an elegant and short proof.

Theorem 2.2 (The CTE based allocation rule for elliptical random variables). *Let* $\underline{X} \sim E_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$ *and let* $S = X_1 + \ldots + X_n$ *with* $\sigma_S > 0$. *Then we have that the contribution* $E[X_k | S > Q_p[S]]$ *of the k-th risk,* $k = 1, 2, \ldots, n$ *to* $CTE_p[S]$ *is given by*

$$E[X_k | S > Q_p[S]] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S^2}\left(CTE_p[S] - \mu_S\right), \qquad 0 < p < 1, \tag{32}$$

with μ_S, σ_S *and* $\sigma_{k,S}$ *given by (10) and (12).*

Proof. From the Law of Total Probability we find that

$$E[X_k | S > Q_p[S]] = \int_{Q_p[S]}^{\infty} E[X_k | S = s]\; dF_S\left(s \mid S > Q_p[S]\right). \tag{33}$$

Substituting the expression (11) for $\mathrm{E}[X_k | S = s]$ in (33) leads to (32). ☐

From the proof of Theorem 2.2 we find that relation (32) can be rewritten as

$$E[X_k | S > Q_p[S]] = E[E[X_k | S] | S > Q_p[S]], \qquad 0 < p < 1. \tag{34}$$

In the following examples we apply Theorem 2.2 to the classes $N_n\left(\underline{\mu}, \underline{\Sigma}\right)$ and $La_n\left(\underline{\mu}, \underline{\Sigma}\right)$.

Example 2.5 (CTE based allocation for normal random variables). *In case* $\underline{X} \sim N_n\left(\underline{\mu}, \underline{\Sigma}\right)$ *with* $\underline{\Sigma}$ *positive definite we have that* $S \sim N_1\left(\mu_S, \sigma_S^2\right)$ *with* $\sigma_S^2 > 0$*, see (9). From (29) and (32) we find that* $E[X_k|S > Q_p\left[S\right]]$ *is given by*

$$E[X_k|S > Q_p\left[S\right]] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S} \times \frac{\Phi'\left(\Phi^{-1}\left(p\right)\right)}{1-p}, \qquad 0 < p < 1. \tag{35}$$

This expression can be found in Panjer (2002).

Example 2.6 (CTE based allocation for Laplace random variables). *In case* $\underline{X} \sim La_n\left(\underline{\mu}, \underline{\Sigma}\right)$ *with* $\underline{\Sigma}$ *positive definite, we have that* $S \sim La_1\left(\mu_S, \sigma_S^2\right)$ *with* $\sigma_S^2 > 0$*, see (9). From (30) and (32) we find that* $E[X_k|S > Q_p\left[S\right]]$ *is given by*

$$E[X_k|S > Q_p\left[S\right]] = \mu_k + \frac{\sigma_{k,S}}{\sigma_S} \times \frac{e^{-\sqrt{2}\,\Lambda^{-1}(p)}}{2(1-p)}\left(\frac{\sqrt{2}}{2} + \Lambda^{-1}(p)\right), \qquad 0 < p < 1. \tag{36}$$

From (32) we see that CTE based allocation rule for elliptical distributions is embedded in a mean- (co-)variance framework. This is due to the properties of elliptical distributions, see e.g. Embrechts *et al.* (2002), Landsman (2006), Landsman & Tsanakas (2006), Tsanakas (2007) and Landsman & Nešlehová (2007).

3 Approximations for the CTE based allocation rule

3.1 The CTE based allocation rule for comonotonic risks

In the previous section we derived an explicit expression (32) for the contributions $\mathrm{E}[X_k|S > \mathrm{Q}_p\left[S\right]]$ in the multivariate elliptical case. Unfortunately such an explicit formula for the $\mathrm{E}[X_k|S > \mathrm{Q}_p\left[S\right]]$ is often not available for general types of distributions, and in this case we suggest to consider approximations for these conditional expectations based on the theory of comonotonicity. In order to derive the approximations we first have to consider the case where $\underline{X}$ is a comonotonic random vector.

Comonotonicity of the random vector $\underline{X}$ means that there exist non-decreasing functions $f_1, f_2, \ldots, f_n$ and a rv Z such that

$$\underline{X} \stackrel{d}{=} \left(f_1(Z), f_2(Z), \ldots, f_n(Z)\right), \tag{37}$$

where '$\stackrel{d}{=}$' stands for 'equality in distribution'. Equivalently, comonotonicity can be characterised as

$$\underline{X} \stackrel{d}{=} \left(\mathrm{Q}_U\left[X_1\right], \mathrm{Q}_U\left[X_2\right], \ldots, \mathrm{Q}_U\left[X_n\right]\right), \tag{38}$$

where U is an uniformly distributed rv over the unit interval $(0, 1)$. For more details on the notion of comonotonicity and some of its applications in insurance and finance we refer to Dhaene *et al.* (2002a,b). Hereafter, we will restrict to comonotonic random vectors with continuous marginal cumulative distribution functions.

Theorem 3.1 (The CTE based allocation rule for comonotonic random variables). *Let $\underline{X}$ be a comonotonic random vector with continuous marginal cdf's $F_{X_k}(x) = \Pr[X_k \leq x]$. The contribution $E[X_k|S > Q_p[S]]$ of the k-th risk, $k = 1, \ldots, n$, to the Conditional Tail Expectation $CTE_p[S]$ of $S = X_1 + \ldots + X_n$ is given by*

$$E[X_k|S > Q_p[S]] = CTE_p[X_k], \qquad 0 < p < 1. \tag{39}$$

Proof. In case $\underline{X}$ is comonotonic we have that $Q_p[S] = \sum_{k=1}^{n} \mathrm{Q}_p[X_k]$. Furthermore, the continuity of the marginal cdf's implies that each $\mathrm{Q}_p[X_k]$ is a strictly increasing function in p, $0 < p < 1$. Combining these results the following equivalence relations hold for each k:

$$\sum_{j=1}^{n} \mathrm{Q}_U[X_j] > \sum_{j=1}^{n} \mathrm{Q}_p[X_j] \Leftrightarrow U > p \Leftrightarrow \mathrm{Q}_U[X_k] > \mathrm{Q}_p[X_k].$$

Hence,

$$\begin{aligned} & E[X_k \ |S > Q_p[S]] \\ & \quad = E\left[\mathrm{Q}_U[X_k] \ |\sum_{j=1}^{n} \mathrm{Q}_U[X_j] > \sum_{j=1}^{n} \mathrm{Q}_p[X_j]\right] \\ & \quad = E\left[\mathrm{Q}_U[X_k] \ |\mathrm{Q}_U[X_k] > \mathrm{Q}_p[X_k]\right] \\ & \quad = CTE_p[X_k]. \end{aligned}$$

□

From Theorem 3.1 we can conclude that for a comonotonic random vector $\underline{X}$ with continuous marginals, The CTE based rule for allocating $CTE_p[S]$ comes down to allocating to each component X_k its Conditional Tail Expectation $CTE_p[X_k]$.

For a general random vector $\underline{X}$ and its sum $S = X_1 + \ldots + X_n$, it may be difficult to determine $E[X_k|S > Q_p[S]]$. This problem can sometimes be solved by considering approximations for the contribution of the k-th risk to $\mathrm{CTE}_p[S]$. As in Kaas *et al.* (2000) we propose to approximate the rv S by the rv S^l defined by

$$S^l = \mathrm{E}[S \mid \Lambda] = \sum_{k=1}^{n} \mathrm{E}[X_k \mid \Lambda]. \tag{40}$$

Here Λ is some appropriately chosen conditioning rv in the sense that the rv $\mathrm{E}[S \mid \Lambda]$ is sufficiently 'close' to the rv S and explicit expression can be obtained for it. Note that $\mathrm{E}[S \mid \Lambda] \equiv S$ when taking $\Lambda \equiv S$ but this ideal choice for Λ is not feasible because it does not allow explicit calculations. Since $\mathrm{E}[S \mid \Lambda]$ essentially predicts S based on Λ, hereby eliminating the randomness of S that cannot be explained by Λ, one intuitively expects the conditional expectation $\mathrm{E}[S \mid \Lambda]$ to be 'less variable' than S and this idea will be used to derive an optimal choice for Λ. In fact it can be proven that

$$\mathrm{Var}[\mathrm{E}[S|\Lambda]] \leq \mathrm{Var}[S], \tag{41}$$

and also that

$$\mathrm{CTE}_p[\mathrm{E}[S|\Lambda]] \leq \mathrm{CTE}_p[S], \qquad \text{for any } p \in (0,1), \tag{42}$$

see e.g. Denuit *et al.* (2005) or Dhaene *et al.* (2006). In equations (41) and (42) an equality will occur when $\Lambda \equiv S$ in which case $\mathrm{E}[S \mid \Lambda] \equiv S$. Hence we find that a good conditioning rv Λ could be defined as one that allows explicit calculations for $\mathrm{E}[S \mid \Lambda]$ whilst obtaining as large as possible values for $\mathrm{Var}[\mathrm{E}[S \mid \Lambda]]$ or, alternatively, for $\mathrm{CTE}_p\left[\mathrm{E}[S|\Lambda]\right]$ .Note that the latter approach is more focusing on finding optimal approximations for S in case S takes large values whereas the former intends to provide a global optimal fit to S. We refer to Vanduffel *et al.* (2006) for a more detailed discussion on the topic of appropriately choosing the conditioning rv Λ.

Next, we propose to approximate $\mathrm{E}[X_k \mid S > Q_p[S]]$ by $\mathrm{E}[X_k \mid S^l > Q_p\left[S^l\right]]$:

$$E[X_k \mid S > Q_p[S]] \approx E[X_k \mid S^l > Q_p\left[S^l\right]], \qquad 0 < p < 1. \tag{43}$$

From the above reasoning it becomes clear that this approximation will perform well provided $\mathrm{E}[S \mid \Lambda]$ is a good approximation for S, especially for large values of S (i.e. when S exceeds $Q_p[S]$). In the final section a numerical example will further demonstrate the accuracy of the approximations. Note that the approximation (43) can also be written as

$$E[X_k \mid S > Q_p[S]] \approx E\left[E[X_k \mid \Lambda] | S^l > Q_p\left[S^l\right]\right], \qquad 0 < p < 1. \tag{44}$$

Let us now assume that the conditioning rv Λ is such that $(E[X_1 \mid \Lambda], E[X_2 \mid \Lambda], \ldots, E[X_n \mid \Lambda])$ is a comonotonic random vector with continuous marginal cdf's. Combining (43) and Theorem 3.1 we find the following approximation for the k-th contribution to $CTE_p[S]$:

$$E[X_k \mid S > Q_p[S]] \approx CTE_p\left[E[X_k \mid \Lambda]\right], \qquad 0 < p < 1. \tag{45}$$

This result will be used in Section 3.3 to derive approximations for the the CTE based allocation rule for sums that involve (log)normal risks.

3.2 Lognormal and logelliptical distributions

For any n-dimensional vector $\underline{x} = (x_1, \ldots, x_n)^T$ with positive components x_i, we define

$$\ln \underline{x} = (\ln x_1, \ln x_2, ..., \ln x_n)^T. \tag{46}$$

The random vector $\underline{X}$ is said to have a multivariate logelliptical distribution if $\ln \underline{X}$ has a multivariate elliptical distribution. We denote $\ln \underline{X} \sim E_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$ as $\underline{X} \sim LE_n\left(\underline{\mu}, \underline{\Sigma}, \phi\right)$. Similar notations hold for the class of lognormal and logLaplace distributions.

Let $\ln X \sim E_1\left(\mu,\sigma^2, \phi\right)$ with $\sigma^2 > 0$ and assume that there exists a $\delta > \frac{\sigma^2}{2}$ such that the characteristic generator $\phi(u)$, which is actually defined on $[0, \infty)$, can be positively extended to the interval $[-\delta, \infty)$. Then, the following expression can be derived for the Conditional Tail Expectations of X:

$$\mathrm{CTE}_p[X] = \frac{e^{\mu}}{1-p}\phi\left(-\frac{\sigma^2}{2}\right)\Pr\left[Z^* > Q_p[Z]\right], \qquad 0 < p < 1, \tag{47}$$

where $Z = \frac{\ln X - \mu}{\sigma}$ and Z^* is a rv with pdf given by

$$f_{Z^*}(x) = \frac{f_Z(x)e^{\sigma x}}{\phi\left(-\frac{\sigma^2}{2}\right)}, \tag{48}$$

see Valdez & Dhaene (2004)). In the next two examples we make use of (47) to derive expressions for the CTE's of lognormal and logLaplace distributions.

Example 3.1 (CTE's of a lognormal random variable). *When $X \sim LN_1(\mu, \sigma^2)$ with $\sigma^2 > 0$, we find from (17) that $\phi\left(-\frac{\sigma^2}{2}\right) = e^{\frac{\sigma^2}{2}}$ and hence that $Z^* \sim N(\sigma, 1)$. From (47), we find that the CTE's are given by*

$$CTE_p[X] = \frac{e^{\mu+\frac{\sigma^2}{2}}}{1-p}\Phi\left(\sigma - \Phi^{-1}(p)\right), \qquad 0 < p < 1. \tag{49}$$

where, as before, Φ and Φ^{-1}denote the standard normal cdf and its associated quantile function respectively.

Example 3.2 (CTE's of a log-Laplace random variable). *When $X \sim LLa_1(\mu, \sigma)$ with $\sigma^2 > 0$, we find from (27) that $\phi\left(-\frac{\sigma^2}{2}\right) = \frac{1}{1-\frac{\sigma^2}{2}}$, which is positive when $\sigma < \sqrt{2}$. From (25) and (48), we find that $f_{Z^*}(x)$ is given by*

$$f_{Z^*}(x) = \frac{1-\sigma^2/2}{\sqrt{2}}e^{\sigma x - \sqrt{2}|x|}, \qquad \sigma < \sqrt{2}. \tag{50}$$

From (31) and (47) we find the following expression for the CTE's of the logLaplace distribution if $\sigma < \sqrt{2}$:

$$\begin{aligned} CTE_p[X] &= \frac{e^{\mu}}{1-p}\phi\left(\frac{-\sigma^2}{2}\right)\frac{1-\sigma^2/2}{\sqrt{2}}\int_{Q_p[Z]}^{\infty}\exp((\sigma-\sqrt{2})x)dx, \\ &= \begin{cases} \frac{(2p)^{\sigma/\sqrt{2}}e^{\mu}}{2p(1-p)(2-\sqrt{2}\sigma)} & : 0 < p < \frac{1}{2}, \\ \frac{\sqrt{2}\, e^{\mu}}{[2(1-p)]^{\sigma/\sqrt{2}}(\sqrt{2}-\sigma)} & : \frac{1}{2} \le p < 1. \end{cases} \end{aligned} \tag{51}$$

3.3 The CTE based allocation rule for (log)normal sums

In this section we consider the random vector $\underline{X} = (X_1, \ldots, X_n)$ given by

$$(X_1, \ldots, X_n) = (Y_1, \ldots, Y_m, e^{Y_{m+1}}, \ldots, e^{Y_n}), \tag{52}$$

where $\underline{Y} = (Y_1, \ldots, Y_n) \sim N_n\left(\underline{\mu}, \underline{\Sigma}\right)$. As before, we denote the elements of the vector $\underline{\mu}$ by μ_k and the elements of the positive definite assumed matrix $\underline{\Sigma}$ by σ_{kl}, $k, l = 1, 2, ..., n$. The aggregate risk is denoted by S:

$$S = \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{Y_k}. \tag{53}$$

As it is not possible to derive an analytical expression for $\mathrm{E}[X_k \mid S > Q_p[S]]$ in this case, we propose to approximate the rv S by the rv S^l defined by

$$S^l = \mathrm{E}[S \mid \Lambda] = \sum_{k=1}^{m} E[Y_k \mid \Lambda] + \sum_{k=m+1}^{n} E[e^{Y_k} \mid \Lambda], \tag{54}$$

where the conditioning rv Λ is given by

$$\Lambda = \sum_{k=1}^{n} \beta_k \; Y_k \tag{55}$$

Let us denote the correlation between Y_k and Λ by r_k:

$$r_k = \text{corr}\left[Y_k, \Lambda\right] = \frac{1}{\sigma_k \; \sigma_\Lambda} \sum_{l=1}^{n} \beta_l \; \sigma_{kl}, \qquad k = 1, 2, ..., n, \tag{56}$$

with σ_Λ^2 given by

$$\sigma_\Lambda^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_k \; \beta_l \; \sigma_{kl}. \tag{57}$$

As far as the choice for Λ is concerned, we make the following general suggestion for the coefficients β_l :

$$\beta_l = \begin{cases} 1 & : k = 1, 2, ..., m, \\ e^{\mu_k} & : k = m + 1, , ..., n. \end{cases} \tag{58}$$

Indeed, we notice that this choice makes Λ a linear transformation of a first-order approximation to the sum S. This can be easily deduced from the following computation:

$$\begin{aligned} S &= \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{\mu_k + (Y_k - \mu_k)} \\ &\approx \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{\mu_k} [1 + Y_k - \mu_k] \\ &\approx C + \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{\mu_k} Y_k, \end{aligned} \tag{59}$$

with C an appropriate constant. Consequently, this choice for Λ ensures that E[Var[$S \mid \Lambda$]] will become 'small'. Since Var[S] =E[Var[$S \mid \Lambda$]] +Var[E[$S \mid \Lambda$]] this is equivalent to saying that Var[E[$S \mid \Lambda$]] becomes 'large', see e.g. Kaas *et al.* (2000) or Vanduffel *et al.* (2006).

Next, as in (43) we propose to approximate E[$X_k \mid S > Q_p[S]$] by E[$X_k \mid S^l > Q_p\left[S^l\right]$]. In the following theorem, we derive a closed-form expression for this approximation of the k-th contribution to CTE$_p\,[S]$ in case all correlations r_k are positive.

Theorem 3.2 (CTE based allocation for sums of (log)normal random variables). *Using the notations and assumptions introduced in Subsection 3.3 and assuming that all correlations r_k are positive, we have that the approximation $E[X_k \mid S^l > Q_p\left[S^l\right]]$, $0 < p < 1$, for the k-th contribution to CTE[S] is given by*

$$E[X_k | S^l > Q_p[S^l]] = \mu_k + r_k \; \sigma_k \times \frac{\Phi'\left(\Phi^{-1}(p)\right)}{1-p} \tag{60}$$

when $k = 1, 2, ..., m,$ and by

$$E[X_k | S^l > Q_p[S^l]] = \frac{e^{\mu_k + \frac{\sigma_k^2}{2}}}{1-p} \Phi[r_k \; \sigma_k - \Phi^{-1}(p)] \tag{61}$$

when $k = m + 1, ..., n.$

Proof. Conditional on $\Lambda = \lambda$, we have that $Y_k \sim N_1\left(\mu_k + r_k\frac{\sigma_k}{\sigma_\Lambda}(\lambda - E[\Lambda]);(1-r_k^2)\,\sigma_k^2\right)$. For $k = 1, 2, ..., m$, this leads to

$$E[X_k|\Lambda] = E[Y_k \mid \Lambda] = \mu_k + r_k\ \sigma_k\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right) \sim N_1\left(\mu_k;\ r_k^2\ \sigma_k^2\right),$$

whilst for $k = m+1, ..., n$, we have that

$$E[X_k|\Lambda] = E[e^{Y_k} \mid \Lambda] = e^{\mu_k + (1-r_k^2)\frac{\sigma_k^2}{2} + r_k\ \sigma_k\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right)} \sim LN_1\left(\mu_k + (1-r_k^2)\frac{\sigma_k^2}{2};\ r_k^2\ \sigma_k^2\right).$$

From these observations and the assumption that all r_k are positive, we find that the random vector $(E[X_1 \mid \Lambda], E[X_2 \mid \Lambda], \ldots, E[X_n \mid \Lambda])$ is comonotonic with continuous marginal cdf's. Applying (45), we can conclude that

$$E[X_k \mid S^l > Q_p[S^l]] = CTE_p[E[X_k \mid \Lambda]], \qquad 0 < p < 1.$$

Taking into account expressions (29) and (49) for the CTE's of (log)normal rv's, we find (60) and (61).

□

From the previous results we also find an expression for the approximation $E[S\ |S^l > Q_p[S^l]]$ of $\text{CTE}_p[S]$:

$$E[S|S^l > Q_p[S^l]] = \sum_{k=1}^{m}\left[\mu_k + r_k\sigma_k \times \frac{\Phi'\left(\Phi^{-1}(p)\right)}{1-p}\right] + \sum_{k=m+1}^{n}\frac{e^{\mu_k + \frac{\sigma_k^2}{2}}}{1-p}\Phi[r_k\ \sigma_k - \Phi^{-1}(p), \qquad 0 < p < 1. \tag{62}$$

This approximation for $\text{CTE}_p[S]$ can also be found in Dhaene *et al.* (2005b) where it was derived by considering $E[S^l|S^l > Q_p[S^l]]$.

4 Numerical Illustration

Consider an insurance company with 4 business lines. The multivariate risk $\underline{X} = (X_1, \ldots, X_4)^T$ faced by this company is assumed to be multivariate lognormally distributed:

$$\underline{X} \sim LN_4\left(\underline{\mu}, \underline{\Sigma}\right),$$

with E$[X_k]$ and Var$[X_k]$, $k = 1, \ldots 4$, given by

$$(E[X_1], E[X_2], E[X_3], E[X_4]) = (20, 40, 10, 5)$$

and

$$(Var[X_1], Var[X_2], Var[X_3], Var[X_4]) = \left(5^2, 15^2, 2^2, 2^2\right),$$

respectively. Furthermore, the matrix $\underline{\Sigma}$ is given by

$$\underline{\Sigma} = \begin{pmatrix} \sigma_1^2 & \alpha\sigma_1\sigma_2 & \alpha\sigma_1\sigma_3 & \alpha\sigma_1\sigma_4 \\ \alpha\sigma_2\sigma_1 & \sigma_2^2 & \alpha\sigma_2\sigma_3 & \alpha\sigma_2\sigma_4 \\ \alpha\sigma_3\sigma_1 & \alpha\sigma_3\sigma_2 & \sigma_3^2 & \alpha\sigma_3\sigma_4 \\ \alpha\sigma_4\sigma_1 & \alpha\sigma_4\sigma_2 & \alpha\sigma_4\sigma_3 & \sigma_4^2 \end{pmatrix}$$

We will consider the cases $\alpha = 0.75$ and $\alpha = 0$, respectively.

For each business line k we determine the contribution $E[X_k|S > Q_{0.9995}[S]]$ in two ways. Firstly, Monte Carlo simulations are performed. Due to the high probability level $p = 0.9995$, a large sample of 10^6 realisations of S is used. These estimations are performed 10 times in order to obtain an estimate for the standard deviation of the sampling error. Secondly, the comonotonic approximation

$$E[X_k|S^l > Q_{0.9995}[S^l]] = \frac{e^{\mu_k + \frac{\sigma_k^2}{2}}}{1 - 0.9995} \Phi[r_k \ \sigma_k - \Phi^{-1}(0.9995)] \tag{63}$$

is calculated. As far as the choice of the conditioning rv Λ is concerned we could make use of the general expression (58) to determine the choice of the parameters β_k. However, since the sum S only involves lognormals we can also apply a slightly more involved approach hereby relying on inequality (42). Hence, we suggest to choose the parameters β_k for Λ such that a first order approximation for the $\mathrm{CTE}_{0.9995}\left[S^l\right]$ becomes maximised:

$$\beta_l = \begin{cases} \Phi'\left(\Phi^{-1}\left(0.9995\right)\right) & : l = 1, 2, ..., m, \\ e^{\mu_l + \frac{1}{2}\sigma_l^2} \times e^{-\frac{1}{2}(r_l^* \sigma_l - \Phi^{-1}(0.9995))^2} & : l = m+1, , ..., n. \end{cases} \tag{64}$$

Here, r_k^* is the correlation between $Y_k = \ln(X_k)$ and the rv $\Lambda^* = \sum_{k=1}^m e^{\mu_k + \frac{1}{2}\sigma_k^2} Y_k$. This choice is designed to make Λ such that S^l is 'close' to S for large values of S, which for our purposes is to be preferred above the general choice (58) for the parameters β_k, see also Vanduffel *et al.* (2006). Note that for $\alpha = 0.75$ and $\alpha = 0$, the choice (64) for the parameters β_l ensures that the correlations r_k defined in (56) are positive.

Let us first consider the case that $\alpha = 0.75$, which means that the different business lines are rather strongly positively dependent. In Table 1 we provide the Monte Carlo based estimates as well as the approximations (63) of the different contributions to $CTE_{0.9995}\left[S\right]$. From Table 1 we can conclude that the comonotonic approximations (63) closely match the values obtained by the extensive Monte Carlo simulation.

Table 1: Estimation of the contributions $C_{0.9995}[X_k] = E[X_k|S > Q_{0.9995}[S]]$, $\alpha = 0.75$.

	$C_{0.9995}[X_1]$	$C_{0.9995}[X_2]$	$C_{0.9995}[X_3]$	$C_{0.9995}[X_4]$	$CTE_{0.9995p}[S]$
Monte Carlo Estim.	34.89	94.67	15.68	10.68	155.65
Sample Stand. Dev.	0.19	0.53	0.10	0.10	0.69
Comon. Approx.	34.95	94.20	15.72	10.67	155.54
Relative Difference	-0.18 %	0.50 %	-0.27 %	0.12 %	0.07 %

Next in Table 2 we present the Monte Carlo based estimates as well as the approximations (63) of the different contributions to $CTE_{0.9995}\left[S\right]$ for $\alpha = 0$. Intuitively, one expects that the less correlated the business lines the worse the comonotonic approximation (63) will perform. This is because when S is a comonotonic sum we have that $S \equiv S^l$ so that in this case the contributions based on the rv S^l will coincide with those based on S. However, from Table 2 we can conclude that the approximations (63) based on the theory of comonotonicity continue to perform very well in case the business lines are assumed to be independent.

Table 2: Estimation of the contributions $C_{0.9995}[X_k| = E[X_k|S > Q_{0.995}[S]]$, $\alpha = 0$.

	$C_{0.9995}[X_1]$	$C_{0.9995}[X_2]$	$C_{0.9995}[X_3]$	$C_{0.9995}[X_4]$	$CTE_{0.9995}[S]$
Monte Carlo Estim.	22.62	94.11	10.39	5.32	132.45
Sample Stand. Dev.	0.18	0.36	0.08	0.06	0.33
Comon. Approx.	23.00	93.54	10.44	5.33	132.30
Relative Difference	-1.6 %	0.61 %	-0.44 %	-0.18 %	0.11 %

5 Final Remarks

The Enterprise Risk Management process of a financial institution usually contains a procedure to allocate, or subdivide, the total risk capital of the company into its different business units. Several capital allocation rules have been described in the literature. We refer to Dhaene *et al.* (2005a) for a general framework that incorporates many of these capital allocation rules.

The CTE based allocation rule as proposed by Tasche (1999) and Denault (2001) decomposes the CTE of the aggregate risk into its marginals' contributions. Panjer (2002) provides a closed-form expression for this allocation when the risks are all multivariate normally distributed, whereas Landsman & Valdez (2003) consider the multivariate elliptical case. The proof of their result is rather technical and in this paper we gave a shorter and straightforward proof of the Landsman & Valdez formula.

We also extend the field where analytical solutions for the CTE based allocation rule are available by deriving accurate and easy to compute closed-form approximations for this rule in the case that the risks of the different units have a multivariate (log)normal distribution.

Note that in recent literature other extensions have been investigated as well. Furman & Landsman (2005) derive analytical expressions for CTE's and the CTE based allocation for multivariate gamma distributions. Cai & Li (2005) and Chiragiev & Landsman (2006) consider the same problem for multivariate phase type distributions and Pareto distributions, respectively. Finally, Furman & Landsman (2007) derive analytical expressions for these quantities related with multivariate Poisson distributions.

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