Optimal Approximations for Risk Measures of Sums of Lognormals based on Conditional Expectations

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Abstract

In this paper we investigate approximations for the distribution function of a sum S of lognormal random variables. These approximations are obtained by considering the conditional expectation $\mathrm{E}[S \mid \Lambda]$ of S with respect to a conditioning random variable Λ.

The choice for Λ is crucial in order to obtain accurate approximations. The different alternatives for Λ that have been proposed in literature to date are 'global' in the sense that Λ is chosen such that the entire distribution of the approximation $\mathrm{E}[S \mid \Lambda]$ is 'close' to the corresponding distribution of the original sum S.

In an actuarial or a financial context one is often only interested in a particular tail of the distribution of S. Therefore in this paper we propose approximations $\mathrm{E}[S \mid \Lambda]$ which are only locally optimal, in the sense that the relevant tail of the distribution of $\mathrm{E}[S \mid \Lambda]$ is an accurate approximation for the corresponding tail of the distribution of S. Numerical illustrations reveal that local optimal choices for Λ can improve the quality of the approximations in the relevant tail significantly.

We also explore asymptotic properties of the approximations $\mathrm{E}[S \mid \Lambda]$ and investigate links with results from Asmussen & Royas-Nandayapa (2005). Finally, we briefly adress the sub-optimality of Asian options from the point of view of risk averse decision makers with a fixed investment horizon.

Keywords: comonotonicity, lognormal, maximal variance, conditional expectation. Jensen's inequality.

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1 Introduction

Many problems in actuarial science and finance involve the evaluation of the distribution function (d.f.) of a random variable (r.v.) S of the form

$$S = \sum_{i=1}^{n} \alpha_i \, e^{Z_i}, \tag{1}$$

where the α_i $(i = 1, 2, ...n)$ are real numbers and $(Z_1, Z_2, ..., Z_n)$ is a multivariate random vector with means and variances denoted by $\mathrm{E}[Z_i]$ and $\sigma^2_{Z_i}$ respectively. If Z_i $(i = 1, 2, ...n)$ denotes the stochastic logreturn of the period $[i, n]$, the r.v. S can be interpreted as the accumulated value at time n of a series of future deterministic saving amounts α_i. On the other hand, when $-Z_i$ denotes the stochastic logreturn over the period $[0, i]$, e^{Z_i} can be interpreted as the stochastic discount factor over the period $[0, i]$. In this case, the r.v. S can be interpreted as the stochastic present value of a series of future deterministic payments α_i. Examples of financial and actuarial problems that involve a sum S as defined in (1) include the valuation of exotic options such as Asian and Basket options, optimal portfolio selection problems and the calculation of provisions and required capital.

The classical work horse in finance for modelling asset returns is the Gaussian model. Both the celebrated Capital Asset Pricing Model and Black & Scholes' option pricing formulas have been derived in this setting. Apart from mathematical convenience such a Gaussian model for the returns often seems to be appropriate when the time unit is sufficiently long, because of a 'Central Limit Theorem' effect. Empirical studies that support this theoretical setup can be found in Cesari & Cremonini (2003), Levy (2004) and McNeil et al. (2005).

A sum of lognormals also appears as a crucial r.v. in other disciplines such as physics and engineering. For a reference to applications in physics, see Romeo et al. (2003). In engineering sums of lognormals appear when considering communication problems, computer network design problems and traffic flow problems. In literature on wireless systems it occurs in outage analysis and received signal power analysis, see e.g. Stüber (Ch2, 1996) and Fenton (1960). A sum of lognormals also arises when modelling the cost of a routed path in a computer network, see e.g. Rasmusson (2002). In the latter case, the lognormals are strongly correlated and the sums are highly dimensional. Arroyo and Kornhauser (2005) consider sums of lognormals to model travel time distributions on a road network.

The various applications in finance, insurance and engineering differ with respect to the dimensionality of the random vector $(Z_1, Z_2, ..., Z_n)$ involved, the assumed equality (or inequality) of the marginal distributions, the assumed independence (or dependence) of the marginals, the level of the volatilities and the relevant region of the distribution function.

Most applications deal with positive coefficients α_i. Therefore, in the remainder of this paper we will assume that all α_i are positive. Furthermore, we will assume that $(Z_1, Z_2, ..., Z_n)$ has a multivariate normal density given by

$$f_{Z_1,Z_2,...,Z_n}(z_1, z_2, ..., z_n) = \frac{1}{(2\pi)^{\frac{n}{2}} \, |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp(-\frac{1}{2}(\mathbf{z} - \boldsymbol{\mu})^{'} \boldsymbol{\Sigma}^{-1} (\mathbf{z} - \boldsymbol{\mu})), \tag{2}$$

where $\mathbf{z}^{'} = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$, $\boldsymbol{\mu}^{'} \in \mathbb{R}^n$ is the vector of the means and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is the covariance matrix. Note that $(\boldsymbol{\mu})_i = \mathrm{E}[Z_i]$ and $(\boldsymbol{\Sigma})_{ii} = \sigma^2_{Z_i}$. We also note that every covariance matrix $\mathbf{\Sigma}$ is necessarily symmetric and positive semidefinite, whereas the existence of $\mathbf{\Sigma}^{-1}$ is guaranteed by assuming that $\mathbf{\Sigma}$ is positive definite. From the assumed existence of a multivariate normal density for $(Z_1, Z_2, ..., Z_n)$ it follows that the Z_i as well as the key r.v. of interest S have a continuous and strictly increasing d.f.

Most applications mentioned above amount to the evaluation of risk measures of S. In this paper we will focus on some risk measures that are often used in practice. The p-quantile risk measure for a r.v. X, also called the Value-at-Risk at level p, is denoted by $Q_p[X]$. It is defined as

$$Q_p\left[X\right] = \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq p\right\}, \qquad p \in (0,1)\,, \tag{3}$$

where $F_X(x) = \Pr\left[X \leq x\right]$.
The Conditional Tail Expectation at level p, denoted by $\mathrm{CTE}_p\left[X\right]$, is defined as

$$\mathrm{CTE}_p\left[X\right] = \mathrm{E}\left[X \mid X > Q_p\left[X\right]\right], \qquad p \in (0,1)\,. \tag{4}$$

The Conditional Left Tail Expectation at level p, denoted by $\mathrm{CLTE}_p\left[X\right]$, is defined as

$$\mathrm{CLTE}_p\left[X\right] = \mathrm{E}\left[X \mid X < Q_p\left[X\right]\right], \qquad p \in (0,1)\,. \tag{5}$$

Finally, the stop-loss premium with retention d of the r.v. X is defined by $\mathrm{E}[(X-d)_+]$, where the notation $(x-d)_+$ stands for $\max\left(x-d, 0\right)$. By using partial integration, we obtain

$$\mathrm{E}[(X-d)_+] = \int_d^{\infty} \Pr\left[X > x\right]\, dx, \qquad -\infty < d < +\infty. \tag{6}$$

We refer to Denuit et al. (2005) or Dhaene et al. (2006) for a discussion on risk measures and the relations that hold between them.

In most cases it is impossible to obtain analytical expressions for risk measures of a sum S of lognormal random variables. Based on an idea of Rogers & Shiu (1995) in an Asian option context, Kaas et al. (2000) propose to approximate the d.f. of S by the d.f. of the r.v. S^l which is defined by

$$S^l \stackrel{\text{def}}{=} \mathrm{E}[S|\Lambda] = \sum_{i=1}^{n} \alpha_i\ \mathrm{E}[e^{Z_i} \mid \Lambda] \tag{7}$$

for an appropriate choice of the conditioning r.v. Λ. Loosely speaking, this approach allows one to transform the stochastic multi-dimensionality of the problem, caused by $(Z_1, Z_2, ..., Z_n)$, to a single dimension, caused by Λ. Moreover, an appropriate choice of Λ will lead to a comonotonic random vector $(\alpha_1\mathrm{E}[e^{Z_1} \mid \Lambda], \alpha_2\mathrm{E}[e^{Z_2} \mid \Lambda], ..., \alpha_n\mathrm{E}[e^{Z_n} \mid \Lambda])$, which means that all $\alpha_i\mathrm{E}[e^{Z_i} \mid \Lambda]$ are non-decreasing functions of the conditioning random variable Λ. Note that by taking $-\Lambda$ as the conditioning random variable we find that comonotonicity could also be characterised by requiring the different components of the random vector to be non-increasing in the conditioning random variable. In this paper we will always use the former characterisation.

Risk measures related to the d.f. of S are then approximated by the corresponding risk measures of S^l. These approximations are straightforward to calculate, taking into account the additivity properties of sums of comonotonic r.v.'s. For an extensive overview on the theory of comonotonicity and some of its applications, we refer to Dhaene et al. (2002a, 2002b). Various applications of this theory have been discussed in Deelstra et al. (2007), Denuit et al. (2006), Dhaene et al. (2004, 2006, 2007), Vanduffel et al. (2002) and Vanmaele et al., amongst others.

The technique of taking conditional expectations has proven to provide accurate and easy to compute approximations for several risk measures of sums of lognormal r.v.'s, see for example Huang et al. (2004) or Vanduffel et al. (2005b) for detailed numerical investigations. Intuitively, Λ should be chosen such that it is 'close' to the original r.v. S. In literature, various choices for Λ have been proposed that are in line with this approach. Kaas et al. (2000) propose to determine Λ as a normal r.v. which can be considered as a first-order approximation of the original sum S. Vanduffel et al. (2005a) propose to choose Λ as a normal r.v. such that a first-order approximation of the variance of $\mathrm{E}[S|\Lambda]$ is 'as close as possible' to the variance of S. Both choices for Λ are 'global' in the sense that the d.f. of $\mathrm{E}[S|\Lambda]$ can be considered as a good approximation for the entire d.f. of S. Note however that there are many financial and actuarial problems where one is only interested in a particular tail of the distribution of S, and as such the approximation is only required to perform well in that particular area of the distribution function. Therefore in this paper we will propose and investigate comonotonic approximations for the d.f. of S which are only 'locally' optimal in some sense.

The rest of this paper is organised as follows. In Section 2, we give an overview of general results concerning comonotonic approximations that will be used in later sections. In Section 3, we recall and discuss 'global' optimal choices for the conditioning r.v. Λ. In particular, we describe the 'Taylor-based' and the 'Maximal Variance' approximations. In Section 4, we propose new choices for Λ that are 'locally' optimal. We also discuss their asymptotic characteristics and relate these with results of Asmussen and Nandayapa (2005). In Section 5 we apply the locally optimal approximations to discounted or compounded sums and numerically investigate their accuracy. In Section 6 we apply the approximations to the pricing of Asian Options and we briefly discuss the optimality of these. Finally, Section 7 concludes the paper.

2 Comonotonic approximations

Let the r.v. S be given by (1), where the α_i are non-negative real numbers and the random vector $(Z_1, Z_2, ..., Z_n)$ has a multivariate normal density given by (2). Consider the conditioning r.v. Λ which is defined as the linear combination of $Z_1, Z_2, ..., Z_n$ determined by

$$\Lambda = \sum_{j=1}^{n} \lambda_j Z_j \tag{8}$$

for given real numbers λ_j, $j = 1, 2, ..., n$. We denote the mean and the variance of Λ by $\mathrm{E}[\Lambda]$ and σ_Λ^2, respectively. From (7) and (8), we find that S^l can be written as

$$S^l = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2 + r_i \sigma_{Z_i} \frac{\Lambda - \mathrm{E}[\Lambda]}{\sigma_\Lambda}}. \tag{9}$$

Here r_i is the correlation between Z_i and Λ:

$$r_i = \frac{\mathrm{cov}\left[Z_i, \Lambda\right]}{\sigma_{Z_i} \; \sigma_\Lambda} = \frac{1}{\sigma_{Z_i} \; \sigma_\Lambda} \sum_{j=1}^{n} \lambda_j \; \mathrm{cov}\left[Z_i, Z_j\right], \qquad i = 1, 2, ..., n. \tag{10}$$

Note that the expected values of the random variables S and S^l are equal:

$$\mathrm{E}[S] = \mathrm{E}[S^l] = \sum_{i=1}^{n} \alpha_i \; \mathrm{E}\left[e^{Z_i}\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\sigma_{Z_i}^2}, \tag{11}$$

whereas their variances are given by

$$\mathrm{Var}[S] = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \; \mathrm{E}\left[e^{Z_i}\right] \mathrm{E}\left[e^{Z_j}\right] \left(e^{\mathrm{Cov}(Z_i, Z_j)} - 1\right) \tag{12}$$

and

$$\mathrm{Var}[S^l] = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \; \mathrm{E}\left[e^{Z_i}\right] \mathrm{E}\left[e^{Z_j}\right] \left(e^{r_i r_j \sigma_{Z_i} \sigma_{Z_j}} - 1\right), \tag{13}$$

respectively.

If all the correlation coefficients r_i defined in (10) are non-negative, we find from (9) that S^l is a comonotonic sum. In this case, the quantiles and the conditional (left) tail expectations of S^l are given by the sum of the corresponding risk measures of the marginals involved. Hence, in case all $r_i \geq 0$, we have that

$$Q_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2 + r_i \sigma_{Z_i} \Phi^{-1}(p)}, \tag{14}$$

$$\mathrm{CTE}_p\left[S^l\right] = \frac{1}{1-p} \sum_{i=1}^{n} \alpha_i \; \mathrm{E}\left[e^{Z_i}\right] \; \Phi\left(r_i \; \sigma_{Z_i} - \Phi^{-1}(p)\right) \tag{15}$$

and

$$\mathrm{CLTE}_p\left[S^l\right] = \frac{1}{p} \sum_{i=1}^{n} \alpha_i \; \mathrm{E}\left[e^{Z_i}\right] \left(1 - \Phi\left(r_i \; \sigma_{Z_i} - \Phi^{-1}(p)\right)\right), \tag{16}$$

which holds for all $p \in (0, 1)$, and where Φ denotes the standard normal distribution. Furthermore, in case all $r_i \geq 0$ the stop-loss premium with retention d, $0 < d < \infty$, of S^l is given by

$$\mathrm{E}\left[\left(S^l - d\right)_+\right] = \sum_{i=1}^{n} \alpha_i \; \mathrm{E}\left[e^{Z_i}\right] \; \Phi\left(r_i \sigma_{Z_i} - \Phi^{-1}(p)\right) - d(1-p), \tag{17}$$

where p is the root of

$$Q_p\left[S^l\right] = d. \tag{18}$$

From (10) it is easy to see that a sufficient condition for all r_i to be non-negative is that all $\lambda_j \geq 0$ and also all $\mathrm{cov}[Z_i, Z_j] \geq 0$.

Since the definition (7) of S^l involves a conditional expectation, eliminating the randomness that cannot be explained by Λ, one may expect that S^l will be 'less risky' than S, and examining equations (12) and (13) reveals that at least the ordering of the variances supports this intuition. As a matter of fact a much stronger result holds. From Jensen's inequality one can prove that S^l is smaller in convex order than S:

$$S^l \leq_{cx} S, \tag{19}$$

which means that for any convex function $v(x)$ it holds that

$$\mathrm{E}[v(S^l)] \leq \mathrm{E}[v(S)], \tag{20}$$

provided the expectations exist. In this case we also say that S^l is a convex lower bound of S. The convex order relation (19) implies any of the following relations:

$$\mathrm{CTE}_p\left[S^l\right] \leq \mathrm{CTE}_p\left[S\right], \qquad \text{for any } p \in (0,1), \tag{21}$$

$$\mathrm{CLTE}_p\left[S\right] \leq \mathrm{CLTE}_p\left[S^l\right], \qquad \text{for any } p \in (0,1) \tag{22}$$

and also

$$\mathrm{E}\left[\left(S^l - d\right)_+\right] \leq \mathrm{E}\left[(S-d)_+\right], \qquad \text{for any } d \in \mathbb{R}. \tag{23}$$

In literature a comonotonic upper bound for the r.v. S has also been proposed and we denote this by S^c; see e.g. Dhaene et al. (2002b). In our lognormal context S^c can be defined by imposing the correlations in expression (9) to be equal to one. Formally:

$$S^c \stackrel{d}{=} \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \sigma_{Z_i}\Phi^{-1}(U)}, \tag{24}$$

with U a uniformly $(0,1)$ distributed r.v. Next, we find expressions for the different risk measures of S^c by setting the correlations r_i to be equal to 1 in the expressions (14), (15), (16) and (17). In particular we obtain that

$$Q_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{\mathrm{E}[Z_i] + \sigma_{Z_i}\Phi^{-1}(p)}. \tag{25}$$

and

$$\mathrm{CTE}_p\left[S^c\right] = \frac{1}{1-p} \sum_{i=1}^{n} \alpha_i \; \mathrm{E}\left[e^{Z_i}\right] \; \Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right) \tag{26}$$

It can be proven that S is convex smaller than S^c and hence inequalities similar to (21), (22) and (23) can be derived. In particular we obtain,

$$\mathrm{CTE}_p\left[S^l\right] \leq \mathrm{CTE}_p\left[S\right] \leq \mathrm{CTE}_p\left[S^c\right], \qquad p \in (0,1), \tag{27}$$

$$\text{CLTE}_p\left[S^l\right] \geq \text{CLTE}_p\left[S\right] \geq \text{CLTE}_p\left[S^c\right], \qquad p \in (0,1), \tag{28}$$

and

$$\text{E}\left[(S^l - d)_+\right] \leq \text{E}\left[(S-d)_+\right] \leq \text{E}\left[(S^c - d)_+\right], \qquad d \in \mathbb{R}. \tag{29}$$

For more details about the results summarised in this section, we refer to Dhaene et al. (2002b, 2006).

3 Globally optimal choices for Λ

3.1 The 'Taylor-based' approximation

From Kaas et al. (1994, p. 68) it follows that if $X \leq_{cx} Y$ we have that

$$\int_{-\infty}^{\infty} (\text{E}[(Y-t)_+] - \text{E}[(X-t)_+])\, dt = \frac{1}{2}\{\text{Var}[Y] - \text{Var}[X]\} \tag{30}$$

Hence, $\frac{1}{2}\{\text{Var}[Y]-\text{Var}[X]\}$ can be fairly interpreted as a measure for the total error made when approximating the stop-loss premiums of Y by those of the convex smaller X. Since the integrand in the left hand side of (30) is non-negative, we also find that if $X \leq_{cx} Y$ whilst $\text{Var}[X] = \text{Var}[Y]$, then this means that X and Y must have equal stop-loss premiums and hence the same d.f. This suggests that if we wish to replace S by the convex smaller S^l, the best approximations will occur when Λ is chosen such that $\text{Var}[S^l]$ is as large as possible or equivalently, since $\text{Var}[S] = \text{Var}[S^l] + \text{E}[\text{Var}[S|\Lambda]]$, when $\text{E}[\text{Var}[S|\Lambda]]$ is as small as possible. We notice that either of this criteria means that Λ and S should be 'as alike as' possible. Therefore, Kaas et al. (2000) propose to choose the conditioning r.v. Λ as the linear combination of the Z_j defined in (8), with the coefficients λ_j given by

$$\lambda_j^{TB} = \alpha_j e^{\text{E}[Z_j]}, \qquad j = 1, \cdots, n. \tag{31}$$

Indeed, this choice makes Λ a linear transformation of a first-order approximation of the sum S. This can easily be seen from the following derivation:

$$S = \sum_{j=1}^{n} \alpha_j e^{\text{E}[Z_j]}\, e^{(Z_j - \text{E}[Z_j])} \approx C + \sum_{j=1}^{n} \alpha_j e^{\text{E}[Z_j]} Z_j, \tag{32}$$

where C is some appropriate constant. We will call the approximation based on the λ_j defined in (31) the **Taylor-based approximation**. The conditioning r.v. Λ is denoted by Λ^{TB} in this case:

$$\Lambda^{TB} = \sum_{j=1}^{n} \alpha_j e^{\text{E}[Z_j]} Z_j. \tag{33}$$

Furthermore, the correlations $\text{corr}\left[Z_i, \Lambda^{TB}\right]$ are denoted by r_i^{TB}:

$$r_i^{TB} = \frac{1}{\sigma_{Z_i}\ \sigma_{\Lambda}^{TB}} \sum_{j=1}^{n} \alpha_j e^{\text{E}[Z_j]}\ \text{cov}\left[Z_i, Z_j\right], \qquad i = 1, 2, ..., n. \tag{34}$$

Here, σ_{Λ}^{TB} is given by

$$\sigma_{\Lambda}^{TB} = (\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j e^{\text{E}[Z_i]} e^{\text{E}[Z_j]} \text{cov}\left[Z_i, Z_j\right])^{\frac{1}{2}}. \tag{35}$$

3.2 The 'maximal variance' approximation

The best approximations for the d.f. of S based on S^l will be the ones where the variance of S^l is 'as large as possible'. The Taylor-based approach assumes a rather intuitive approach to derive a Λ that gives rise to a 'large' value for $\mathrm{Var}[S^l]$. Vanduffel et al. (2005a) use a more explicit approach to derive Λ. They derive the following first order approximation of the variance of S^l:

$$\begin{aligned}\mathrm{Var}\left[S^l\right] &\approx \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\ e^{\mathrm{E}[Z_i]+\mathrm{E}[Z_j]+\frac{1}{2}(\sigma_{Z_i}^2+\sigma_{Z_j}^2)}(r_i r_j \sigma_{Z_i}\sigma_{Z_j}) \\ &= \left(\mathrm{Corr}\left[\sum_{j=1}^{n}\alpha_j \mathrm{E}\left[e^{Z_j}\right], \Lambda)\right]\right)^2 \ \mathrm{Var}\left[\sum_{j=1}^{n}\alpha_j \mathrm{E}\left[e^{Z_j}\right] Z_j\right]. \end{aligned} \tag{36}$$

They then propose to choose the conditioning r.v. Λ as the linear combination of the Z_j defined in (8), with the coefficients λ_j such that the first order approximation (36) of $\mathrm{Var}\left[S^l\right]$ is maximised:

$$\lambda_j^{MV} = \alpha_j\ \mathrm{E}\left[e^{Z_j}\right], \quad j = 1, \ldots, n. \tag{37}$$

We call the approximation S^l based on the coefficients λ_j defined in (37) the **maximal variance approximation**. The condition r.v. Λ is denoted by Λ^{MV} in this case:

$$\Lambda^{MV} = \sum_{j=1}^{n}\alpha_j \mathrm{E}\left[e^{Z_j}\right] Z_j, \tag{38}$$

whereas the correlations $\mathrm{corr}\left[Z_i, \Lambda^{MV}\right]$ are denoted by r_i^{MV}:

$$r_i^{MV} = \frac{1}{\sigma_{Z_i}\ \sigma_\Lambda^{MV}}\sum_{j=1}^{n}\alpha_j \mathrm{E}\left[e^{Z_j}\right]\ \mathrm{cov}\left[Z_i, Z_j\right], \qquad i = 1, 2, ..., n. \tag{39}$$

Here, σ_Λ^{MV} is given by

$$\sigma_\Lambda^{MV} = (\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j \mathrm{E}\left[e^{Z_i}\right]\mathrm{E}\left[e^{Z_j}\right]\mathrm{cov}\left[Z_i, Z_j\right])^{\frac{1}{2}}. \tag{40}$$

Note that the conditioning r.v. Λ^{MV} does not necessarily maximise the variance of S^l, but has to be understood as an approximate solution. One could use numerical procedures to determine the 'real' Λ that maximises $\mathrm{Var}\left[S^l\right]$ but obviously this would be at the cost of losing one of the main features of the approximations, namely that quantiles, conditional (left) tail expectations and stop-loss premiums can be easily determined analytically.

4 Locally optimal choices for Λ

4.1 The 'CTE_p-based' approximation

For several practical applications one only needs to focus on a particular tail of the distribution for S. The provision to be established at time 0 for future payment obligations can be determined as either $Q_p[S]$ or $\mathrm{CTE}_p[S]$, with p sufficiently large. For example, a provision equal to $Q_{0.95}[S]$ guarantees a non-ruin probability of 0.95. Determining Asian or basket European type option prices only involves the calculation of upper or lower tails of the d.f. of S. As a result it makes sense to consider choices for the conditioning r.v. Λ that are only optimal in a particular upper or lower tail of the d.f. under consideration. The underlying intuitive idea is that when only requiring a good fit between the distributions of S and S^l in a particular region, we will be able to find better approximations, at least when constrained to that particular region.

In order to determine an optimal Λ for approximating the upper tail risk measure $\mathrm{CTE}_p[S]$, recall that (21) states that $\mathrm{CTE}_p[S^l] \leq \mathrm{CTE}_p[S]$ holds for all p in $(0,1)$. This observation suggests that determining Λ such that $\mathrm{CTE}_p[S^l]$ is as 'large as possible' is a feasible choice for that purpose.

In the case that all correlations $r_i = \mathrm{corr}[Z_i, \Lambda]$ are non-negative we have that $\mathrm{CTE}_p\left[S^l\right]$ is given by (15). We will show that the following choice of the parameters λ_j maximises a first-order approximation of the formula (15):

$$\lambda_j^{(p)} = \alpha_j \ \mathrm{E}\left[e^{Z_j}\right] \ \Phi' \left[r_j^{MV} \sigma_{Z_j} - \Phi^{-1}(p)\right], \quad j = 1, \ldots, n, \tag{41}$$

where r_j^{MV} is defined in (39). Notice that these optimal $\lambda_j^{(p)}$ depend on the probability level p, reflecting the fact that they are indeed constructed to be locally optimal in some sense.

In order to derive the coefficients (41), we start by expanding (15) around the correlations r_i^{MV}. Then we find:

$$\begin{aligned} \mathrm{CTE}_p\left[S^l\right] &\approx \frac{1}{1-p} \sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_j}\right] \ \Phi\left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \\ &\quad + \frac{1}{1-p} \sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_j}\right] \ \Phi' \left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \left(r_i - r_i^{MV}\right) \sigma_{Z_i} \end{aligned} \tag{42}$$

Hence, the first-order approximation (42) of the expression (15) of $\mathrm{CTE}_p\left[S^l\right]$ is maximised when

$$\sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_j}\right] \ \Phi' \left(r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right) \ r_i \ \sigma_{Z_i} \tag{43}$$

is maximised. As

$$\begin{aligned} &\sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_i}\right] \ \Phi' \left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \ r_i \ \sigma_{Z_i} \\ &= \frac{1}{\sigma_\Lambda} \mathrm{Cov}\left(\sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_i}\right] \ \Phi' \left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \ Z_i, \Lambda\right) \\ &= \mathrm{Corr}\left(\sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_i}\right] \ \Phi' \left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \ Z_i, \Lambda\right) \\ &\times \left(\mathrm{Var}\left[\sum_{i=1}^n \alpha_i \ \mathrm{E}\left[e^{Z_j}\right] \ \Phi' \left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \ Z_i)\right]\right)^{\frac{1}{2}}, \end{aligned} \tag{44}$$

it follows that the choice (41) for the parameters λ_j maximises the first-order approximation (42) of $\mathrm{CTE}_p\left[S^l\right]$.

We will call the approximation S^l based on the coefficients λ_j defined in (41) the **CTE$_p$-based approximation**. The conditioning r.v. Λ is denoted by $\Lambda^{(p)}$ in this case:

$$\Lambda^{(p)} = \sum_{j=1}^{n} \alpha_j \; \mathrm{E}\left[e^{Z_j}\right] \;\; \Phi' \left[r_j^{MV} \sigma_{Z_j} - \Phi^{-1}(p)\right] Z_j. \tag{45}$$

Furthermore, the correlations $r_i = \mathrm{corr}\left[Z_i, \Lambda^{(p)}\right]$ are denoted by $r_i^{(p)}$. Hence,

$$r_i^{(p)} = \frac{1}{\sigma_{Z_i} \; \sigma_\Lambda^{(p)}} \sum_{j=1}^{n} \alpha_j \mathrm{E}\left[e^{Z_j}\right] \;\; \Phi' \left[r_j^{MV} \sigma_{Z_j} - \Phi^{-1}(p)\right] \; \mathrm{cov}\left[Z_i, Z_j\right], \qquad i = 1, 2, ..., n. \tag{46}$$

Here, $\sigma_\Lambda^{(p)}$ is given by

$$\sigma_\Lambda^{(p)} = \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \mathrm{E}\left[e^{Z_i}\right] \mathrm{E}\left[e^{Z_j}\right] \;\; \Phi' \left[r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)\right] \;\; \Phi' \left[r_j^{MV} \sigma_{Z_j} - \Phi^{-1}(p)\right] \; \mathrm{cov}\left[Z_i, Z_j\right]\Big)^{\frac{1}{2}}. \tag{47}$$

Since expression (15) requires the correlations r_i to be non-negative we expect that the choice (41) for the parameters λ_j will only perform well in case the true but unknown optimal Λ, i.e. the one that maximises $\mathrm{CTE}_p\left[S^l\right]$, has non-negative correlations r_i. One can expect that this will hold true in case all $r_i^{(p)}$ in (46) are non-negative. The accuracy of the approximations based on $\mathrm{E}[S|\Lambda^{(p)}]$ will be addressed in Sections 5 and 6.

Note that in (42) the Taylor expansion of $\mathrm{CTE}_p\left[S^l\right]$ is performed around the correlations r_i^{MV} $(i = 1, 2, ...n)$. It can be easily verified that a naive expansion of $\mathrm{CTE}_p\left[S^l\right]$ around zero correlations would have provided Λ^{MV} as an optimal choice. This gives some more indication that the CTE$_p$-based approximation is likely to provide a better fit than the Taylor-based or maximal variance lower bound approximations.

Since the construction of $\Lambda^{(p)}$ involves a first order approximation of $\mathrm{CTE}_p\left[S^l\right]$ it remains an approximation to the 'optimal' Λ. Using numerical techniques to optimise $\mathrm{CTE}_p\left[S^l\right]$ instead of its first order approximation (42) would undoubtedly provide better choices for Λ, but this would be at the expense of losing a full analytical solution. Having a readily available approximation that can be implemented easily is important from a practical point of view.

4.2 An 'asymptotically optimal' approximation

In the previous section we argued that the best convex lower bound to measure the upper tail arises when Λ is such that $\mathrm{CTE}_p\left[S^l\right]$ is maximised. Unfortunately, it appears that it is not possible to find an analytical solution for this optimisation problem in general. Therefore, we considered the maximisation of a first order approximation to $\mathrm{CTE}_p\left[S^l\right]$ and this gave rise to the CTE$_p$-based approximation. However, in the asymptotic case, when p tends to 1, the solution for the maximisation of $\mathrm{CTE}_p\left[S^l\right]$ can be derived analytically.

In addition to the assumption of non-negative α_i made throughout this paper, in this section we will also assume that all Z_i in (1) are positively correlated:

$$\text{cov}\left[Z_i, Z_j\right] \geq 0, \qquad i,j = 1,2,...,n. \tag{48}$$

In many applications this assumption will hold true, see Sections 5 and 6. In order to prove the asymptotic results of this section, we will need the following lemma.

Lemma 1 *If c_1 and c_2 are real numbers such that $0 \leq c_1 < c_2$, then we have*

$$\lim_{p \to 1} \frac{\Phi\left(c_2 - \Phi^{-1}(p)\right)}{\Phi\left(c_1 - \Phi^{-1}(p)\right)} = \infty. \tag{49}$$

Proof. This follows by substituting z for $\Phi^{-1}(p)$ and then applying de L'Hôpital's rule. ■

Without loss of generality, in this section we assume that the Z_i are ranked such that

$$\sigma_{Z_1} \geq \sigma_{Z_2} \geq ... \geq \sigma_{Z_n}. \tag{50}$$

Furthermore, we assume that the ranking of the Z_i is such that

$$\sigma_{Z_i} = \sigma_{Z_{i+1}} \text{ for some } i \Rightarrow \alpha_i e^{\text{E}[Z_i]} \geq \alpha_{i+1} e^{\text{E}[Z_{i+1}]}. \tag{51}$$

The following theorem provides results regarding the lower bound approximations that are asymptotically optimal.

Theorem 1 *For any conditioning r.v. Λ of the form (8) with correlations r_i defined in (10) such that $r_i \geq 0$, $i = 1,2,...,n$, we have that*

$$(a)\ \lim_{p \to 1} \frac{CTE_p\left[E[S \mid Z_1]\right]}{CTE_p\left[E[S \mid \Lambda]\right]} \geq 1 \tag{52}$$

$$(b)\ \lim_{p \to 1} \frac{CTE_p\left[E[S \mid Z_1]\right]}{CTE_p[\alpha_1 e^{Z_1}]} = 1 \tag{53}$$

In case $\sigma_{Z_1} > \sigma_{Z_i}$ for all $i = 2,...,n$, we also have

$$(c)\ \lim_{p \to 1} \frac{CTE_p\left[S^c\right]}{CTE_p[\alpha_1 e^{Z_1}]} = 1 \tag{54}$$

$$(d)\ \lim_{p \to 1} \frac{CTE_p\left[S\right]}{CTE_p[\alpha_1 e^{Z_1}]} = 1 \tag{55}$$

Proof. We first prove (a).
From (15) we find that

$$\text{CTE}_p\left[\text{E}[S \mid Z_1]\right] = \frac{1}{1-p} \sum_{i=1}^{n} \alpha_i \,\text{E}\left[e^{Z_i}\right]\ \Phi\left(\text{corr}\left[Z_i, Z_1\right]\ \sigma_{Z_i} - \Phi^{-1}(p)\right), \qquad 0 < p < 1. \tag{56}$$

Next, from (56) and (15) we obtain

$$\frac{\mathrm{CTE}_p\left[\mathrm{E}[S \mid Z_1]\right]}{\mathrm{CTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]} = \frac{\alpha_1 \mathrm{E}[e^{Z_1}]\ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right)}{\alpha_j \mathrm{E}[e^{Z_j}]\ \Phi\left(r_j \sigma_{Z_j} - \Phi^{-1}(p)\right)} \frac{1 + \sum_{i=2}^n \frac{\alpha_i \mathrm{E}[e^{Z_i}]\ \Phi\left(\mathrm{corr}[Z_i, Z_1]\ \sigma_{Z_i} - \Phi^{-1}(p)\right)}{\alpha_1 \mathrm{E}[e^{Z_1}]\ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right)}}{1 + \sum_{i \neq j}^n \frac{\alpha_i \mathrm{E}[e^{Z_i}]\ \Phi\left(r_i\ \sigma_{Z_i} - \Phi^{-1}(p)\right)}{\alpha_j \mathrm{E}[e^{Z_j}]\ \Phi\left(r_j \sigma_{Z_j} - \Phi^{-1}(p)\right)}}, \tag{57}$$

where we have chosen $j \in \{1, 2, ...n\}$ such that $r_j\ \sigma_{Z_j} \geq r_i\ \sigma_{Z_i} \geq 0$ for all $i = 1, 2, ..., n$. From the positive definiteness of the variance-covariance matrix $\mathbf{\Sigma}$ it follows that $\mathrm{corr}[Z_i, Z_1] < 1$ for $i = 2, ..., n$. Indeed, suppose that $\mathrm{corr}[Z_i, Z_1] = 1$ for a particular $i > 2$, then $\mathrm{Var}[-\sigma_{Z_1} Z_i + \sigma_{Z_i} Z_1] = 0$, which contradicts the assumption of positive definiteness. Hence, $\mathrm{corr}[Z_i, Z_1] .\sigma_{Z_i} < \sigma_{Z_1}$, and from (57) and Lemma 1 it follows that

$$\lim_{p \to 1} \frac{\mathrm{CTE}_p\left[\mathrm{E}[S \mid Z_1]\right]}{\mathrm{CTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]} = \lim_{p \to 1} \frac{\alpha_1 \mathrm{E}[e^{Z_1}]\ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right)}{\alpha_j \mathrm{E}[e^{Z_j}]\ \Phi\left(r_j \sigma_{Z_j} - \Phi^{-1}(p)\right)} \frac{1}{1 + \frac{\sum_{i \neq j}^n \alpha_i \mathrm{E}[e^{Z_i}]\ \Phi\left(r_i\ \sigma_{Z_i} - \Phi^{-1}(p)\right)}{\alpha_j \mathrm{E}[e^{Z_j}]\ \Phi\left(r_j \sigma_{Z_j} - \Phi^{-1}(p)\right)}} \tag{58}$$

From our previously stated assumptions it follows that $r_i\ \sigma_{Z_i} \leq r_j\ \sigma_{Z_j} \leq \sigma_{Z_1}$ holds for all $i = 1, 2, ..., n$.
Let us first investigate the case where $r_j\ \sigma_{Z_j} < \sigma_{Z_1}$. In this case (a) follows as an application of Lemma 1 to expression (58).
On the other hand, when $r_j\ \sigma_{Z_j} = \sigma_{Z_1}$, it follows from (50) that $r_j = 1$ and $\sigma_{Z_1} = \sigma_{Z_j}$. In this case we find that

$$\lim_{p \to 1} \frac{\mathrm{CTE}_p\left[\mathrm{E}[S \mid Z_1]\right]}{\mathrm{CTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]} = \frac{\alpha_1 \mathrm{E}[e^{Z_1}]}{\alpha_j \mathrm{E}[e^{Z_j}]} \lim_{p \to 1} \frac{1}{1 + \sum_{i \neq j}^n \frac{\alpha_i \mathrm{E}[e^{Z_i}]\ \Phi\left(r_i\ \sigma_{Z_i} - \Phi^{-1}(p)\right)}{\alpha_j \mathrm{E}[e^{Z_j}]\ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right)}} \tag{59}$$

By analogous reasoning the positive definiteness of $\mathbf{\Sigma}$ will imply that for all $i \neq j$ we have that $r_i < r_j = 1$. Taking into account (50), this implies that $r_i\ \sigma_{Z_i} < \sigma_{Z_1}$ holds for all $i \neq j$. Inequality (a) will then follow from Lemma 1 and (51), which implies that $\alpha_1 \mathrm{E}[e^{Z_1}] \geq \alpha_j \mathrm{E}[e^{Z_j}]$.
Next, we prove (b).
As $\alpha_1 e^{Z_1} = \mathrm{E}[\alpha_1 e^{Z_1} | Z_1]$, we have that $\mathrm{CTE}_p[\alpha_1 e^{Z_1}]$ can be found as a special case of (56):

$$\mathrm{CTE}_p[\alpha_1 e^{Z_1}] = \frac{1}{1-p} \alpha_1\ \mathrm{E}\left[e^{Z_1}\right]\ \ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right), \qquad 0 < p < 1. \tag{60}$$

Combining (56) and (60), we find

$$\frac{\mathrm{CTE}_p\left[\mathrm{E}[S \mid Z_1]\right]}{\mathrm{CTE}_p[\alpha_1 e^{Z_1}]} = 1 + \sum_{i=2}^n \frac{\alpha_i\ \mathrm{E}[e^{Z_i}]\ \Phi\left(\mathrm{corr}\left[Z_i, Z_1\right]\ \sigma_{Z_i} - \Phi^{-1}(p)\right)}{\alpha_1 \mathrm{E}[e^{Z_1}]\ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right)}, \qquad 0 < p < 1. \tag{61}$$

As we have that $0 \leq \mathrm{corr}[Z_i, Z_1] < 1$, $i = 2, 3, ..., n$, statement (b) follows as a straightforward application of Lemma 1.

We will now prove (c). Using the expressions (26) and (60) for the relevant conditional tail expectations, we obtain

$$\frac{\mathrm{CTE}_p\left[S^c\right]}{\mathrm{CTE}_p[\alpha_1 e^{Z_1}]} = 1 + \sum_{i=2}^{n} \frac{\alpha_i \ \mathrm{E}[e^{Z_i}] \ \Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right)}{\alpha_1 \ \mathrm{E}[e^{Z_1}] \ \Phi\left(\sigma_{Z_1} - \Phi^{-1}(p)\right)}, \qquad 0 < p < 1. \tag{62}$$

Since $\sigma_{Z_1} > \sigma_{Z_i}$ for all $i = 2, ..., n$, statement (c) follows as a direct consequence of Lemma 1.
Finally, from (27) we have

$$\frac{\mathrm{CTE}_p\left[\mathrm{E}[S \mid Z_1]\right]}{\mathrm{CTE}_p[\alpha_1 e^{Z_1}]} \leq \frac{\mathrm{CTE}_p\left[S\right]}{\mathrm{CTE}_p[\alpha_1 e^{Z_1}]} \leq \frac{\mathrm{CTE}_p\left[S^c\right]}{\mathrm{CTE}_p[\alpha_1 e^{Z_1}]}, \qquad 0 < p < 1.$$

Taking into account (b) and (c), these inequalities imply (d). ■

In the remainder of this section we will assume that

$$\sigma_{Z_1} > \sigma_{Z_i} \text{ for } i = 2, .., n. \tag{63}$$

From (52) we can conclude that in an asymptotic sense, the largest, and hence the best approximation $\mathrm{CTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]$ for $\mathrm{CTE}_p\left[S\right]$ is obtained by choosing Λ equal to Z_1. Results (53), (54) and (55) state that the upper tail of the sum S, as well as the upper tail of its approximations $\mathrm{E}[S \mid Z_1]$ and S^c, all measured by their respective CTE's, will asymptotically behave in the same way as the upper tail of the first term $\alpha_1 e^{Z_1}$ of S.

From the inequalities (27) and the relation

$$\mathrm{CTE}_p\left[X\right] = \frac{1}{1-p} \int_p^1 Q_q\left[X\right] \ dq, \qquad p \in (0,1), \tag{64}$$

which holds for continuously distributed r.v.'s, it follows that

$$\lim_{p \to 1} \frac{Q_p\left[\mathrm{E}[S \mid Z_1]\right]}{Q_p\left[S^c\right]} \leq \lim_{p \to 1} \frac{Q_p\left[S\right]}{Q_p\left[S^c\right]} \leq 1. \tag{65}$$

provided these limits exist. The proof of these inequalities follows by showing that the opposite inequality leads to a contradiction with (27). Now, from (14) and (25) we immediately obtain that

$$\begin{aligned}
\lim_{p \to 1} \frac{Q_p\left[\mathrm{E}[S \mid Z_1]\right]}{Q_p\left[S^c\right]} &= \lim_{p \to 1} \frac{\sum_{i=1}^{n} \alpha_i \ e^{\mathrm{E}[Z_i] + \frac{1}{2}(1 - \ \mathrm{corr}[Z_i, Z_1])\sigma_{Z_i}^2 + \ \mathrm{corr}[Z_i, Z_1] \ \sigma_{Z_i} \Phi^{-1}(p)}}{\sum_{i=1}^{n} \alpha_i \ e^{\mathrm{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(p)}} \\
&= \lim_{p \to 1} \frac{1 + \sum_{i=2}^{n} \frac{\alpha_i \ e^{\mathrm{E}[Z_i] + \frac{1}{2}(1 - \ \mathrm{corr}[Z_i, Z_1])\sigma_{Z_i}^2 + \ \mathrm{corr}[Z_i, Z_1] \ \sigma_{Z_i} \Phi^{-1}(p)}}{\alpha_1 \ e^{\mathrm{E}[Z_1] + \sigma_{Z_1} \Phi^{-1}(p)}}}{1 + \sum_{i=2}^{n} \frac{\alpha_i \ e^{\mathrm{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(p)}}{\alpha_1 \ e^{\mathrm{E}[Z_1] + \sigma_{Z_1} \Phi^{-1}(p)}}} \\
&= 1.
\end{aligned} \tag{66}$$

Combining (65) and (66) it follows that

$$\lim_{p \to 1} \frac{Q_p\left[\mathrm{E}[S \mid Z_1]\right]}{Q_p\left[S^c\right]} = \lim_{p \to 1} \frac{Q_p\left[S\right]}{Q_p[S^c]} = 1. \tag{67}$$

Furthermore, we have that

$$\lim_{p\to 1} \frac{Q_p[S^c]}{Q_p[\alpha_1 e^{Z_1}]} = \lim_{p\to 1} \left(1 + \sum_{i=2}^{n} \frac{\alpha_i \ e^{\mathrm{E}[Z_i]+\sigma_{Z_i}\Phi^{-1}(p)}}{\alpha_1 \ e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(p)}} \right) = 1. \tag{68}$$

From (67) and (68) we can conclude that

$$\lim_{p\to 1} \frac{Q_p[S]}{Q_p[\alpha_1 e^{Z_1}]} = 1. \tag{69}$$

The results derived above mean that asymptotically the exact quantiles $Q_p[S]$ and their approximations $Q_p[\mathrm{E}[S \mid Z_1]]$ and $Q_p[S^c]$ all coincide with the quantiles of the first term of S.

Asmussen & Royas-Nandayapa (2005) investigate sums of lognormals in a more general setting. In particular, they have proven that under condition (63) similar asymptotic behaviour is found for the tail probabilities:

$$\lim_{x\to\infty} \frac{\Pr[S > x]}{\Pr[\alpha_1 e^{Z_1} > x]} = 1. \tag{70}$$

Hence, both in (69) and (70), the sum S asymptotically behaves like the component $\alpha_i e^{Z_i}$ with the largest value for σ_{Z_i}.

In the following theorem, we prove (70) by using the results on convex ordering.

Theorem 2 *Under the condition (63) we have that*

$$\lim_{x\to\infty} \frac{\Pr[S > x]}{\Pr[\alpha_1 e^{Z_1} > x]} = 1 \tag{71}$$

Proof. From (6) and the stop loss ordering relation (29) it follows that

$$\lim_{x\to\infty} \frac{\Pr[S > x]}{\Pr[\alpha_1 e^{Z_1} > x]} \le \lim_{x\to\infty} \frac{\Pr[S^c > x]}{\Pr[\alpha_1 e^{Z_1} > x]}, \tag{72}$$

provided these limits exist. The proof of this inequality follows by showing that the opposite inequality leads to a contradiction with (29).
Because of (50) we have that for any $i \ge 2$ it holds that

$$\alpha_i \ e^{\mathrm{E}[Z_i]+\sigma_{Z_i}\Phi^{-1}(U)} \le \alpha_2 \ e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)},$$

provided U is sufficiently large. This implies that for x sufficiently large, we have that

$$\Pr\left[S^c > x\right] \le \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} + (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} > x\right]. \tag{73}$$

From the Law of Total Probability it follows that the right hand side of this inequality can be written as

$$\begin{aligned} & \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} + (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} > x\right] \\ = & \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} + (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} > x,\ \alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} > x\right] \\ & + \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} + (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} > x,\ \alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} \le x\right]. \end{aligned} \tag{74}$$

On the one hand, we have that

$$\begin{aligned} & \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} + (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} > x,\ \alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} > x\right] \\ = \ & \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} > x\right]. \end{aligned} \tag{75}$$

On the other hand, since

$$\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} \leq x \Leftrightarrow (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} \leq \gamma x^{\beta} \tag{76}$$

where $\beta = \frac{\sigma_{Z_2}}{\sigma_{Z_1}}$ and $\gamma = (n-1)\alpha_2 e^{\mathrm{E}[Z_2]}(\frac{1}{\alpha_1}e^{-\mathrm{E}[Z_1]})^{\beta}$, we find that.

$$\begin{aligned} & \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} + (n-1)\alpha_2 e^{\mathrm{E}[Z_2]+\sigma_{Z_2}\Phi^{-1}(U)} > x,\ \alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} \leq x\right] \\ & \leq \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} > x - \gamma x^{\beta}, \alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} \leq x\right], \end{aligned} \tag{77}$$

Combining (73), (75) and (77) we find for x sufficiently large that

$$\Pr\left[S^c > x\right] \leq \Pr\left[\alpha_1 e^{\mathrm{E}[Z_1]+\sigma_{Z_1}\Phi^{-1}(U)} > x - \gamma x^{\beta}\right],$$

and therefore we obtain that

$$\begin{aligned} \lim_{x\to\infty} \frac{\Pr[S^c > x]}{\Pr[\alpha_1 e^{Z_1} > x]} & \leq \lim_{x\to\infty} \frac{\Pr[\alpha_1 e^{Z_1} > x - \gamma x^{\beta}]}{\Pr[\alpha_1 e^{Z_1} > x]} \\ & = 1, \end{aligned} \tag{78}$$

where the last equality can be be proven using de L'Hôpital's rule and the fact that $\beta < 1$.
Obviously we also have that

$$\lim_{x\to\infty} \frac{\Pr[S > x]}{\Pr[\alpha_1 e^{Z_1} > x]} \geq 1. \tag{79}$$

The stated result (71) follows then from (72), (78) and (79). ■

4.3 The 'CLTE$_p$-based' approximation

In practice, there are also applications where one focuses on the lower tails of the distribution function of a sum of lognormal random variables as defined in (1). An example is the determination of put option prices of arithmetic Asian options. In this case, a 'locally optimal' approximation $\mathrm{E}[S \mid \Lambda]$ can be defined as the one for which $\mathrm{CLTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]$ is 'as close as possible' to $\mathrm{CLTE}_p\left[S\right]$. From (22), it follows that Λ should be chosen such that $\mathrm{CLTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]$ is minimised in order to obtain the optimal approximation for the exact $\mathrm{CLTE}_p\left[S\right]$. As

$$\mathrm{E}\left[S\right] = \mathrm{E}[S \mid \Lambda] = p\ \mathrm{CLTE}_p\left[\mathrm{E}[S \mid \Lambda]\ \right] + (1-p)\ \mathrm{CTE}_p\left[\mathrm{E}[S \mid \Lambda]\ \right], \tag{80}$$

it follows that minimising $\mathrm{CLTE}_p\left[\mathrm{E}[S \mid \Lambda]\ \right]$ provides the same solution for Λ as maximising $\mathrm{CTE}_p\left[\mathrm{E}[S \mid \Lambda]\ \right]$. Therefore, the choice (41) for the parameters λ_i minimises a first order approximation for $\mathrm{CLTE}_p\left[\mathrm{E}[S \mid \Lambda]\right]$.

5 Application to Discounting and Compounding

5.1 Discounted sums

Let us consider the random variable S_n^d which represents the random present value of a series of n deterministic unit cash flows:

$$S_n^d = \sum_{i=1}^{n} e^{-Y_1 - Y_2 - \cdots - Y_i} \stackrel{\text{def}}{=} \sum_{i=1}^{n} e^{Z_i^d}. \tag{81}$$

Here the r.v.'s Y_i denote the random return over the period $[i-1, i]$, and $e^{-(Y_1+Y_2+\cdots+Y_i)} = e^{Z_i^d}$ is the random discount factor over the period $[0, i]$. We will assume that the periodic returns Y_i's are i.i.d. normally distributed random variables with mean $\mu - \frac{\sigma^2}{2}$ and variance σ^2.

Notice that S_n^d is a r.v. of the general type defined in (1) with $\mathrm{E}[Z_i^d]$, $\sigma^2_{Z_i^d}$ and $\mathrm{Cov}[Z_i^d, Z_j^d]$ given as

$$\begin{aligned} \mathrm{E}[Z_i^d] &= -i(\mu - \frac{\sigma^2}{2}), \\ \sigma^2_{Z_i^d} &= i\,\sigma^2, \\ \mathrm{Cov}[Z_i^d, Z_j^d] &= \mathrm{Min}(i,j)\sigma^2. \end{aligned} \tag{82}$$

In Table 1 we compare the different approximations for the 0.95-conditional tail expectation of S_n^d for different levels of the yearly volatility σ using the result of Monte Carlo simulations as the benchmark. We fixed the number of yearly payments to $n = 20$ and the yearly expected return μ has been set equal to 0.075. Note that we do not mention the results of the 'asymptotically optimal approximations' explicitly. The reason for this is that more detailed numerical investigations revealed that in a financial context these underperform the other approximations significantly for all reasonable values for σ and p, also indicating that the convergence speed is low in these instances.

Then the CTE_p-based approximation which corresponds to the use of the conditioning r.v. $\Lambda = \Lambda^{(p)}$ in the approximations based on $\mathrm{E}[S_n^d \mid \Lambda]$, turns out to provide the best for the conditional tail expectations for all values of the parameter σ whereas the maximal variance approximation (Λ^{MV}) outperforms the Taylor-based approximation (Λ^{TB}).

n	Method using	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
20	Λ^{TB}	24.39	59.02	193.69
	Λ^{MV}	24.42	59.45	196.85
	$\Lambda^{(p)}$	**24.46**	**59.64**	**197.28**
	MC ($\pm$s.e)	24.48 (0.029)	59.84 (0.126)	198.23 (0.833)

Table 1: Approximations for the 0.95-conditional tail expectation of the discounted sum S_n^d for different volatilities (μ=0.075; yearly payments of 1). The figures in brackets represent the standard error on the Monte Carlo results.

5.2 Compounded sums

We consider the random variable S_n^c defined as the random compounded value of a series of n deterministic unit cash flows:

$$S_n^c = \sum_{i=1}^{n} e^{Y_i + Y_{i+1} + \cdots + Y_n} \stackrel{\text{def}}{=} \sum_{i=1}^{n} e^{Z_i^c}, \tag{83}$$

where the Z_i^c $(i = 1, 2, ..., n)$ now represent cumulative log-returns over the period $[i-1, n]$ .Note that S_n^c is a r.v. of the general type defined in (1) with $\mathrm{E}[Z_i^c], \sigma^2_{Z_i^c}$ and $\mathrm{Cov}[Z_i^c, Z_j^c]$ given as

$$\begin{aligned} \mathrm{E}[Z_i^c] &= (n-i+1)(\mu - \frac{\sigma^2}{2}), &(84)\\ \sigma^2_{Z_i^c} &= (n-i+1)\ \sigma^2, &(85)\\ \mathrm{Cov}[Z_i^c, Z_j^c] &= (n - \mathrm{Max}(i-1, j-1))\sigma^2. &(86) \end{aligned}$$

Table 2 compares the different approximations for the 0.05-conditional left tail expectation of S_n^c again for different levels of the yearly volatility σ whilst taking $n = 20$ and $\mu = 0.075$. The results are also compared with Monte Carlo simulations. Keeping in mind (22) we find that also in this case the CTE_p-based approximation, which coincides with the CLTE_p-based approximation, provides the best results. Moreover, the relative increase in accuracy as compared to the maximal variance and Taylor-based approximation is significant. It is interesting to observe that as far as these global choices for Λ are concerned the maximal variance approximation appears to be less accurate than the Taylor-based approximation in this example. The reason for this is that the maximal variance approximation is more sensitive to the right, unbounded, tail of S_n^c, and this is at the expense of losing some accuracy in the left tail of S_n^c.

This suggests that when choosing between the Taylor-based and maximal variance approximation, the former one is often more appropriate in case of risk measures that focus on the left tail of the distribution such as the CLTE whereas the latter is better in case one focuses on the right tail of the distributions.

n	Method	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
	Λ^{TB}	17.80	9.35	5.22
20	Λ^{MV}	17.82	9.48	5.51
	$\Lambda^{(p)}$	**17.75**	**9.21**	**5.09**
	MC (±s.e)	17.73 (0.028)	9.16 (0.019)	4.94 (0.01)

Table 2: Approximations for the 0.05-conditional left tail expectation of the compounded sum S_n^c for different volatilities (μ=0.075; yearly saving of 1). The figures in brackets represent the standard error on the Monte Carlo results.

6 Application to the Pricing of Asian Options

In this section we will assess the accuracy of the different approximations for discrete arithmetic Asian option prices. We refer to Dhaene et al (2002b) or Vanmaele et al

(2006) for extensive reviews on how the prices of these instruments can be approximated using the theory on comonotonicity and convex ordering.

Consider a risky asset with a known price $P(0)$ at time $i = 0$ and unknown prices $P(i)$ at times $i = 1, 2, ..., n$. A discrete Asian option with maturity n, strike K and $n - j$ averaging dates is a financial instrument that generates at maturity n a pay-off that is equal to $(\frac{1}{n-j}\sum_{i=j+1}^{n} P(i) - K)_+$. When averaging is carried out during the entire period $[0, n]$ at equidistant intermediate times $i = 1, 2, ..., n$, we find that the pay-off can also be represented by $\left(\frac{1}{n}P(0)S_n^a - K\right)_+$ with S_n^a given by

$$S_n^a = \sum_{i=1}^{n} e^{Y_1+Y_2+\cdots+Y_i} \stackrel{\text{def}}{=} \sum_{i=1}^{n} e^{Z_i^a}, \tag{87}$$

where the Z_i^a $(i = 1, 2, ..., n)$, are cumulative log-returns over the period $[0, i]$. Note that S_n^a is a r.v. of the general type defined in (1). Furthermore, in the absence of arbitrage opportunities and assuming a Black & Scholes market, the cost for an Asian option with strike K will be denoted by C_K and is given as

$$C_K = e^{-rn}\mathrm{E}[(\frac{1}{n}P(0)S_n^a - K)_+], \tag{88}$$

with r the risk free rate. Here, $\mathrm{E}[Z_i^a]$, $\sigma^2_{Z_i^a}$ and $\mathrm{Cov}[Z_i^a, Z_j^a]$ are given as

$$\begin{aligned} \mathrm{E}[Z_i^a] &= i(r - \frac{\sigma^2}{2}), \\ \sigma^2_{Z_i^a} &= i\ \sigma^2. && (89) \\ \mathrm{Cov}[Z_i^a, Z_j^a] &= \mathrm{Min}(i, j)\sigma^2. && (90) \end{aligned}$$

In fact, for arbitrage-free pricing purposes the expectation in (88) will be taken with respect to the risk neutral measure, and in this case we will explicitly denote the expectations operator by E_r whereas the notation E_μ will be used when expectations are taken with respect to the initial (physical) probability measure; We refer to e.g. Harrison & Kreps (1979) or Harrison & Pliska (1981) for extensive theory on arbitrage-free pricing.

We will now assess the quality of the different lower bounds using the parameter setting from Vanmaele et al (2006, p.29); see also Brückner (2007). The time-unit is assumed to be one month, and averaging is done over the whole period taking into account the monthly end prices of the underlying stock. Furthermore, the monthly volatility σ is given by $\sigma = \frac{0.25}{\sqrt{12}}$ whereas for the monthly risk free rate r we have that $r = \frac{0.04}{12}$. In Table 3 we compare lower bound approximations for the prices of Asian call options for different strike prices K. The other parameters are fixed and are stated in the table. The last column, indicated by Λ^{GA}, corresponds to the case that the approximation is based on the conditioning r.v. Λ taken as the standardised logarithm of the geometric average $\sqrt[n]{\prod_{i=1}^{n} e^{Z_i}}$. The probability '$p$' in the CTE_p-based approximation is determined as the root of $Q_p\left[(S_n^a)^l\right] = n\frac{K}{P(0)}$. In line with the previous results we find that the newly proposed CTE_p-based approximation will always outperform the other approximations, and we also find that the relative increase in accuracy is quite significant for out-of-the money call options when the

σ	K	MC($\pm$s.e)	$\Lambda^{(p)}$	Λ^{MV}	Λ^{TB}	Λ^{GA}
0.25	50	50.0481 (0.0069)	**50.0475**	50.0472	50.0473	50.0473
	80	24.7507 (0.0099)	**24.7478**	24.7443	24.7457	24.7461
	90	17.9358 (0.0119)	**17.9319**	17.9298	17.9311	17.9314
	100	12.4802 (0.0132)	12.4758	12.4754	**12.4759**	**12.4759**
	110	8.3909 (0.0127)	**8.3864**	8.3864	8.3860	8.3857
	150	1.3797 (0.0062)	**1.3770**	1.3736	1.3717	1.3711
	180	0.3223 (0.0034)	**0.3212**	0.3182	0.3171	0.3168
	200	0.1214 (0.0022)	**0.1209**	0.1189	0.1183	0.1181

Table 3: Different approximations for Asian call option prices for different strikes K (σ=0.25; r=0.04; P(0)=100; T=3; n=36). The figures in brackets represent the standard error on the Monte Carlo results.

strike K is larger than the current stock price. Since we focus on the right tail, also the maximal variance approximation will outperform the Taylor-based approximation. We notice that the quality of the approximation that uses Λ^{GA} as the conditioning r.v. decreases as K increases.

We will now further investigate the case of an Asian option with strike K equal to zero. More specifically, we will compare the zero-strike pay-off $\frac{1}{n}P(0)S_n^a$ with its conditional expectation $\mathrm{E}_\mu[\frac{1}{n}P(0)S_n^a \mid \Lambda]$ where Λ is taken to be equal to Z_n^a. Note that the expectation is taken with respect to the initial probability measure. We find that $\mathrm{E}_\mu[\frac{1}{n}P(0)S_n^a \mid Z_n^a]$ is given by

$$\mathrm{E}_\mu[\frac{1}{n}P(0)S_n^a \mid Z_n^a] = \sum_{i=1}^{n} \frac{1}{n} e^{\frac{1}{2}\left(1-\frac{i}{n}\right)i\ \sigma^2 + \frac{i}{n}\Lambda}. \tag{91}$$

It is important to note that the r.v. $\mathrm{E}_\mu[\frac{1}{n}P(0)S_n^a \mid Z_n^a]$ does not depend on μ. Its arbitrage free price C is given by:

$$\begin{aligned} C &= e^{-rn}\mathrm{E}_r[\mathrm{E}_\mu[\frac{1}{n}P(0)S_n^a \mid Z_n^a]] \\ &= e^{-rn}\mathrm{E}_r[\mathrm{E}_r[\frac{1}{n}P(0)S_n^a \mid Z_n^a]] \qquad (92) \\ &= e^{-rn}\mathrm{E}_r[\frac{1}{n}P(0)S_n^a] \\ &= C_0 \qquad (93) \end{aligned}$$

Although the price of the zero-strike payoff $\frac{1}{n}P(0)S_n^a$ will coincide with its conditional expectation $\mathrm{E}_\mu[\frac{1}{n}P(0)S_n^a \mid Z_n^a]$ the latter is convex smaller, and will be preferred by all risk averse decision makers. Note that the latter pay-off, as it only depends on the final state value Z_n^a of the underlying return process, is path independent whereas the former pay-off depends on the intermediate states and is path-dependent.

The sub-optimality of path dependent structures was already discussed in Cox & Leland (1982); see also Dybvig (1988). However, in this paper we present a short and elegant proof regarding the sub-optimality for a particular choice of path dependent pay-offs. We believe that these results can be generalised to other path dependent structures and other asset return processes but this will be the topic of a subsequent paper.

7 Concluding remarks

The stochastically discounted or compounded value of a series of cash flows is often a key quantity in finance and actuarial science. Yet even for most realistic stochastic return models, it is often difficult to obtain analytic expressions for the risk measures involving these discounted sums. Following the works of Kaas et al. (2000), Dhaene et al. (2002a, 2002b) and Vanduffel et al. (2005a) we show in this paper how to improve the so-called convex lower bound approximations by suitably choosing the conditioning variable Λ. It has already been documented in literature that choosing this conditioning variable using either a Taylor-based or a maximal variance approximation provides in some sense an overall goodness of fit. However, we can further improve the approximations if we concentrate on a local neighborhood of the distribution function such as the lower or upper tails. In these instances, we find that the approximations for various risk measures can be improved significantly if we use conditioning variables on the basis of a first-order approximation of the conditional tail expectation, if upper tails are concerned, and on a first-order approximation of the conditional left tail expectation, if lower tails are concerned. We also present some asymptotic results regarding the optimality of the approximations which show that these do not perform arbitrarily bad in case p approaches 1 (or zero). We provide numerical illustrations that show that the newly proposed CTE_p-based approximation usually provides better fits in the tail, and we briefly address the sub-optimality of path dependent pay-offs in a restricted setting.

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