Correlation order, merging and diversification[1]

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Abstract

We investigate the influence of the dependence between random losses on the shortfall and on the diversification benefit that arises from merging these losses.
We prove that increasing the dependence between losses, expressed in terms of correlation order, has an increasing effect on the shortfall, expressed in terms of an appropriate integral stochastic order. Furthermore, increasing the dependence between losses decreases the diversification benefit.
We also consider merging comonotonic losses and show that even in this extreme case a non-negative diversification benefit will often arise.

Key words and phrases: Correlation order, supermodularity, shortfall risk, diversification, comonotonicity.

1 Introduction and motivation

Consider a market of insurance portfolios with respective random losses X_i, $i = 1, 2, \ldots, n$, defined on a common probability space $(\Omega, \mathcal{F}, \Pr)$. We assume that all X_i have a finite mean. In order to protect the policyholders and other debtholders against insolvency, the regulatory authority will require each of the portfolios i to operate above a minimal solvency capital requirement $\rho[X_i]$, which means that for each portfolio i, the available capital K_i has to be larger than or equal to the prescribed minimal level $\rho[X_i]$.

The minimal capital requirement $\rho : \Gamma \to \mathbb{R}$ is assumed to be a risk measure on some set Γ of real-valued random variables (r.v.'s) defined on $(\Omega, \mathcal{F}, P)$. The set Γ contains the random losses X_i, $i = 1, 2, \ldots, n$. We assume that $X_1, X_2 \in \Gamma$ implies that $X_1 + X_2 \in \Gamma$, and also $aX_1 \in \Gamma$ for any $a > 0$ and $X_1 + b \in \Gamma$ for any real b.

We will often impose properties such as law invariance, positive homogeneity, translation invariance and/or subadditivity to the capital requirement $\rho[.]$. These properties are defined hereafter. The notation $\stackrel{d}{=}$ is used to denote 'equality in distribution'.

- *Law invariance*: for any $X_1, X_2 \in \Gamma$ with $X_1 \stackrel{d}{=} X_2$, $\rho[X_1] = \rho[X_2]$.
- *Positive homogeneity*: for any $X \in \Gamma$ and $a > 0$, $\rho[aX] = a\rho[X]$.
- *Translation invariance*: for any $X \in \Gamma$ and $b \in \mathbb{R}$, $\rho[X + b] = \rho[X] + b$.
- *Subadditivity*: for any $X_1, X_2 \in \Gamma$, $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$.

The shortfall of portfolio i with loss X_i and available capital K_i is defined as

$$(X_i - K_i)_+ = \max(X_i - K_i, 0). \tag{1}$$

The insurer of portfolio i will be able to fulfill his obligations provided X_i is smaller than K_i, whereas in case X_i exceeds K_i, the policyholders will suffer a loss $X_i - K_i$. Hence, $(X_i - K_i)_+$ reflects the shortfall risk that is faced by policyholders and possible other debtholders of portfolio i.

The shortfall of the market is defined by

$$\sum_{i=1}^{n} (X_i - K_i)_+. \tag{2}$$

This r.v. represents the random amount that the group of all policyholders in the market may lose due to the insolvency of one or more portfolios.

Intuitively one expects that merging portfolios will decrease the market shortfall because within a merged portfolio the shortfall of one of the units may be compensated by the gains of the other(s). This decrease of the market shortfall due to merging leads to a so-called *diversification benefit*.

In practice, merging portfolios may change management, business strategy, cost structure, and so on, and may as such have an impact on the distribution of the losses under consideration. In this paper however, we will only focus on the 'pure' diversification effect caused by pooling losses. This means that we will assume that merging does not change the distribution function of (X_1, X_2).

We will investigate how the dependency structure between X_1 and X_2 influences the market shortfall, the diversification benefit as well as the preferences of the group of policyholders. Since the shortfall is random, we will rely on stochastic orders to compare market shortfalls and diversification benefits for different underlying dependency structures of the losses X_1 and X_2.

In Section 2 we introduce the stochastic orders that we will use throughout the paper. In Section 3 we define two kinds of diversification benefit and summarize some of their properties. In Section 4 we compare market shortfalls under different dependency structures in terms of appropriate integral stochastic orders. In Section 5, we investigate the effect of increasing the dependence between X_1 and X_2 on the diversification benefit. Finally in Section 6 we show that the expected diversification benefit is strictly positive under very general conditions, which hold even when the losses under consideration are comonotonic.

2 Integral stochastic orders

2.1 Univariate integral stochastic orders

Consider a given class $\mathcal{D}$ of functions $f : \mathbb{R} \to \mathbb{R}$. A r.v. X is said to be smaller than the r.v. Y in the $\preceq_{\mathcal{D}}$-sense if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ holds for all functions f in $\mathcal{D}$ for which the expectations exist.

Taking for $\mathcal{D}$ the class of non-decreasing functions yields the well-known stochastic dominance $\preceq_{\mathrm{ST}}$.

In case $\mathcal{D}$ is set equal to the class of non-decreasing convex functions yields the increasing convex order $\preceq_{\text{ICX}}$, also known as stop-loss order $\preceq_{\text{SL}}$ in the actuarial literature.

Choosing $\mathcal{D}$ equal to the class of the convex functions yields the convex order $\preceq_{\text{CX}}$. This order can be seen as a strengthening of the stop-loss order obtained by requiring in addition that the means of the r.v.'s to be compared are equal.

The importance of the above mentioned orders becomes apparent when relating them to theories for decision making under risk. For instance, it holds in both the classical utility theory from von Neuman & Morgenstern (1947) and Yaari's (1987) dual theory of choice under risk that $X \preceq_{\text{ICX}} Y$ reflects the preferences of all risk averse decision makers when choosing between different losses X and Y. We refer to Denuit et al. (1999, 2005) for more details and background on integral stochastic orders in an actuarial context.

2.2 Multivariate integral stochastic orders

The definition of multivariate integral stochastic orders is a direct extension of the univariate case by considering classes $\mathcal{D}$ of functions $\mathbb{R}^n \to \mathbb{R}$. In this paper we will mainly consider the order between random couples with equal marginals, generated by the class of bivariate supermodular functions.

Let $\underline{e}_i$ denote the i-th n-dimensional unit vector. For $\underline{x} = (x_1, \cdots, x_n)$ and an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$, we define $\Delta_i^{\varepsilon} f(\underline{x}) = f(\underline{x} + \epsilon\ \underline{e}_i) - f(\underline{x})$.

Definition 1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular if*

$$\Delta_i^{\delta} \Delta_j^{\epsilon}\ f(\underline{x}) \geq 0$$

holds for all $\underline{x} \in \mathbb{R}^n$, $1 \leq i < j \leq n$ and all $\delta, \epsilon > 0$.

In case $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable then f is supermodular if, and only if,

$$\frac{\partial^2}{\partial x_i \partial x_j} f\left(\underline{x}\right) \geq 0$$

holds for every $\underline{x} \in \mathbb{R}^n$ and $1 \leq i < j \leq n$.

The bivariate supermodular order coincides with the correlation order which was introduced in the actuarial literature by Dhaene et al. (1996, 1997).

Definition 2 (Correlation order) *Consider the random couples* (X_1, X_2) *and* (Y_1, Y_2) *with* $X_1 \stackrel{d}{=} Y_1$ *and* $X_2 \stackrel{d}{=} Y_2$*. Then* (X_1, X_2) *is said to be less correlated than* (Y_1, Y_2)*, notation* $(X_1, X_2) \preceq_{CORR} (Y_1, Y_2)$*, if any of the following equivalent conditions holds true:*

(i) $Cov\,[f(X_1),\, g(X_2)] \leq Cov\,[f(Y_1),\, g(Y_2)]$ *holds for all non-decreasing functions* $f : \mathbb{R} \rightarrow \mathbb{R}$ *and* $g : \mathbb{R} \rightarrow \mathbb{R}$ *for which the covariances exist;*

(ii) $\Pr[X_1 \leq x_1, X_2 \leq x_2] \leq \Pr[Y_1 \leq x_1, Y_2 \leq x_2]$ *holds for all* $x_1, x_2 \in \mathbb{R}$*;*

(iii) $\Pr[X_1 > x_1, X_2 > x_2] \leq \Pr[Y_1 > x_1, Y_2 > x_2]$ *holds for all* $x_1, x_2 \in \mathbb{R}$*;*

(iv) $\mathbb{E}[f(X_1, X_2)] \leq \mathbb{E}[f(Y_1, Y_2)]$ *holds for all supermodular functions* $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ *for which the expectations exist.*

(v) $\mathbb{E}[f(X_1, X_2)] \leq \mathbb{E}[f(Y_1, Y_2)]$ *holds for all twice differentiable functions* $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ *satisfying* $\frac{\partial^2}{\partial x_1 \partial x_2} f \geq 0$ *and such that the expectations exist.*

The intuitive meaning of a ranking with respect to $\preceq_{\text{CORR}}$ is clear from Definition 2. Indeed, $\Pr[X_1 \leq x_1, X_2 \leq x_2]$ reads as 'X_1 and X_2 are both small', meaning that X_1 is smaller than the threshold x_1, whereas X_2 is smaller than the threshold x_2. A similar interpretation can be given to $\Pr[Y_1 \leq x_1, Y_2 \leq x_2]$. Hence, $\underline{X} \preceq_{\text{CORR}} \underline{Y}$ means that the probability that Y_1 and Y_2 are both small is larger than the corresponding probability for X_1 and X_2. Similarly, $\underline{X} \preceq_{\text{CORR}} \underline{Y}$ ensures that the probability that X_1 and X_2 are both large is smaller than the corresponding probability for Y_1 and Y_2. Hence, correlation order corresponds to the intuitive meaning of '(Y_1, Y_2) being more positively dependent than (X_1, X_2)'.

In Dhaene & Goovaerts (1996) the following expression is considered for the stop-loss premiums of a sum of two random variables:

$$\mathbb{E}\left[(X_1 + X_2 - d)_+\right] = \mathbb{E}\left[X_1\right] + \mathbb{E}\left[X_2\right] - d + \int_{-\infty}^{+\infty} F_{X_1,X_2}(x, d - x)\, dx, \quad (3)$$

which holds for any real d. Combining (3) and Definition 2(ii) they find the following implication:

$$(X_1, X_2) \preceq_{\text{CORR}} (Y_1, Y_2) \Rightarrow X_1 + X_2 \preceq_{\text{CX}} Y_1 + Y_2, \quad (4)$$

which states that more correlated random couples lead to larger stop-loss premiums for the corresponding sums.

A straightforward generalization of correlation order to the n-dimensional case is the supermodular order.

Definition 3 (Supermodular order) *Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ and $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two n-dimensional random vectors. Then $\underline{X}$ is said to be smaller than $\underline{Y}$ in the supermodular order, notation $\underline{X} \preceq_{SM} \underline{Y}$, if any of the following equivalent conditions holds true:*

(i) $\mathbb{E}[f(\underline{X})] \leq \mathbb{E}[f(\underline{Y}))]$ *holds for all supermodular functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ for which the expectations exist.*

(ii) $\mathbb{E}[f(\underline{X})] \leq \mathbb{E}[f(\underline{Y}))]$ *holds for all twice differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying $\frac{\partial^2}{\partial x_i \partial x_j} f \geq 0$ for every $1 \leq i < j \leq n$ and such that the expectations exist.*

If $\underline{X} \preceq_{\mathrm{SM}} \underline{Y}$, then $X_i \overset{d}{=} Y_i$ for $i = 1, 2, \ldots, n$. Hence, only distributions with the same marginals can be compared in the supermodular sense.

The implication (4) can be generalized to the supermodular order in the following way: For all non-decreasing functions ϕ_i it holds that

$$\underline{X} \preceq_{\mathrm{SM}} \underline{Y} \Rightarrow \sum_{i=1}^{n} \phi_i(X_i) \preceq_{\mathrm{CX}} \sum_{i=1}^{n} \phi_i(Y_i), \tag{5}$$

see Müller (1997). Notice that from (5) one finds in particular that

$$\underline{X} \preceq_{\mathrm{SM}} \underline{Y} \Rightarrow \operatorname{var}\left(\sum_{i=1}^{n} \phi_i(X_i)\right) \leq \operatorname{var}\left(\sum_{i=1}^{n} \phi_i(Y_i)\right), \tag{6}$$

holds for all non-decreasing functions ϕ_i.

Taking into account that the indicator functions $\underline{y} \mapsto \mathbb{I}\left(\underline{y} \leq \underline{x}\right)$and $\underline{y} \mapsto \mathbb{I}\left(\underline{y} > \underline{x}\right)$are supermodular for each fixed $\underline{x}$, we immediately find that

$$\underline{X} \preceq_{\mathrm{SM}} \underline{Y} \Rightarrow \Pr\left[\underline{X} \leq \underline{x}\right] \leq \Pr\left[\underline{Y} \leq \underline{x}\right] \text{ for all } \underline{x} \in \mathbb{R}^n$$

and

$$\underline{X} \preceq_{\mathrm{SM}} \underline{Y} \Rightarrow \Pr\left[\underline{X} > \underline{x}\right] \leq \Pr\left[\underline{Y} > \underline{x}\right] \text{ for all } \underline{x} \in \mathbb{R}^n.$$

As a consequence of these implications, we find that

$$\underline{X} \preceq_{\mathrm{SM}} \underline{Y} \Rightarrow (X_i, X_j) \preceq_{\mathrm{CORR}} (Y_i, Y_j) \text{ for all } i \neq j.$$

Furthermore, if $\underline{X} \preceq_{\text{SM}} \underline{Y}$ then Pearson's, Kendall's and Spearman's correlation coefficients are smaller for any pair (X_i, X_j) compared to the corresponding pair (Y_i, Y_j) for any $i \neq j$, see e.g. Denuit et al. (2005). We can conclude that the intuitive meaning of $\underline{X} \preceq_{\text{SM}} \underline{Y}$ is that the components of $\underline{X}$ and $\underline{Y}$ have the same marginal behavior, whereas the components of $\underline{Y}$ are 'more positively dependent' than those of $\underline{X}$.

3 Diversification

We start this section by formalizing the notion of 'diversification benefit'.

Definition 4 (Diversification benefit $D(\underline{X}, \underline{K})$) *Consider the vector of random losses $\underline{X} = (X_1, X_2, \ldots, X_n)$ with respective capitals $\underline{K} = (K_1, K_2, \ldots, K_n)$. The diversification benefit $D(\underline{X}, \underline{K})$ is defined as*

$$D(\underline{X}, \underline{K}) = \sum_{i=1}^{n} (X_i - K_i)_+ - \left(\sum_{i=1}^{n} (X_i - K_i) \right)_+ . \tag{7}$$

Hence the random diversification benefit $D(\underline{X}, \underline{K})$ is equal to the decrease in the market shortfall caused by merging the n portfolios X_i and adding their respective available capitals K_i.

It is easy to prove that the following inequality holds:

$$\left(\sum_{i=1}^{n} (X_i - K_i) \right)_+ \leq \sum_{i=1}^{n} (X_i - K_i)_+ . \tag{8}$$

Here and in the remainder of the paper, a stochastic inequality $X \leq Y$ has to be understood as $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. Such inequality implies a (P-)almost sure inequality.

From (8) and Definition 4 we find that

$$D(\underline{X}, \underline{K}) \geq 0, \tag{9}$$

regardless of the dependency structure between X_1 and X_2. Inequality (9) expresses that from the viewpoint of decreasing the market shortfall, and hence of creating a diversification benefit, it is optimal to merge portfolios, at least when capitals are added.

Assume that the group of policyholders expresses its preferences in terms of a (non-decreasing) utility function u and an initial wealth w. In this case, inequality (8) implies

$$u\left(w-\sum_{i=1}^{n}\left(X_i-K_i\right)\right)\geq u\left(w-\sum_{i=1}^{n}\left(X_i-K_i\right)_+\right). \tag{10}$$

Inequality (10) expresses that from the viewpoint of maximizing utility, the group of policyholders always benefit from merging when adding the stand-alone capitals.

In the next definition, we consider the diversification benefit that arises when the required capitals $\rho\left[X_i\right]$ are considered instead of the available capitals K_i. In this case, after a merger between the X_i has taken place, the required capital for the merged portfolios is determined by $\rho\left[\sum_{i=1}^n X_i\right]$, which will in general be different from $\sum_{i=1}^n \rho\left[X_i\right]$. The capital requirement $\rho\left[.\right]$ may be imposed by the regulator or required by the rating agency or by internal policy.

Definition 5 (Diversification benefit $D_\rho\left(\underline{X}\right)$) *Consider the vector of random losses $\underline{X}=\left(X_1,X_2,\ldots,X_n\right)$ and a capital requirement ρ. The diversification benefit $D_\rho\left(\underline{X}\right)$ is defined as*

$$D_\rho\left(\underline{X}\right)=\sum_{i=1}^{n}\left(X_i-\rho\left[X_i\right]\right)_+-\left(\sum_{i=1}^{n}X_i-\rho\left[\sum_{i=1}^{n}X_i\right]\right)_+. \tag{11}$$

Hence the random diversification benefit $D_\rho\left(\underline{X}\right)$ is the decrease in shortfall caused by merging the portfolios X_i and replacing the required capitals $\rho\left[X_i\right]$ of the individual portfolios by the required capital $\rho\left[\sum_{i=1}^n X_i\right]$ of the merged portfolio.

It is straightforward to prove that

$$\rho\left[\sum_{i=1}^{n}X_i\right]\leq\sum_{i=1}^{n}\rho\left[X_i\right]\Rightarrow D_\rho\left(\underline{X}\right)\leq D\left(\underline{X},\left(\rho\left[X_1\right],\rho\left[X_2\right],\ldots,\rho\left[X_n\right]\right)\right), \tag{12}$$

Hence, for subadditive risk measures, the diversification benefit $D_\rho\left(\underline{X}\right)$ is never larger than the corresponding diversification benefit $D\left(\underline{X},\left(\rho\left[X_1\right],\rho\left[X_2\right],\ldots,\rho\left[X_n\right]\right)\right)$.

From (9), we know that the diversification benefit $D(\underline{X}, \underline{K})$ when adding the available capitals is non-negative. Let us now consider the diversification benefit $D_\rho(\underline{X})$ where the required capital after the merger is given by $\rho\left[\sum_{i=1}^n X_i\right]$. In order to see why $D_\rho(\underline{X}) \geq 0$ will not hold in general consider the situation where two companies are merged and the new required capital $\rho\left[X_1 + X_2\right] < \rho\left[X_1\right] + \rho\left[X_2\right]$. Now suppose that the event '$X_1 > \rho\left[X_1\right]$ and $X_2 > \rho\left[X_2\right]$' takes place, which means that on a stand-alone basis (before merging) both companies default. Then also the event '$X_1 + X_2 > \rho\left[X_1 + X_2\right]$' is taking place, which means that merging the stand-alone companies could not avoid bankruptcy. After the merger, the shortfall for the policyholders is even higher than the shortfall in the stand-alone situation, because upon merging some capital has been released, e.g. reimbursed to the shareholders, rather than being used to compensate policyholders. We can conclude that when the capital requirement is such that $\rho\left[X_1 + X_2\right] < \rho\left[X_1\right] + \rho\left[X_2\right]$, then $X_1 > \rho\left[X_1\right]$ and $X_2 > \rho\left[X_2\right]$ imply $D_\rho(\underline{X}) < 0$.

Even the weaker result $\mathbb{E}\left[D_\rho(\underline{X})\right] \geq 0$ will not hold in general. This fact is illustrated in the following example.

Example 6 (Dhaene et al. (2008a)) *Consider the random couple* (X_1, X_2) *where the* X_i *are both uniformly distributed on the unit interval* $(0,1)$. *The r.v.* X_2 *is defined by*

$$X_2 = \begin{cases} 0.9U & \text{if } 0 < X_1 \leq 0.9, \\ X_1 & \text{if } 0.9 < X_1 < 1, \end{cases}$$

where U *is uniformly distributed on* $(0,1)$ *and independent of* X_1.
One can prove that in this case

$$\mathbb{E}\left[D_{TVaR_{0.85}}(\underline{X})\right] < 0, \tag{13}$$

where $TVaR_{0.85}$ *stands for the Tail Value at Risk at level* 0.85*:*

$$TVaR_{0.85}\left[X_i\right] = \frac{1}{1-0.85}\int_{0.85}^{1} F_{X_i}^{-1}(q)dq,$$

and $F_X^{-1}(q)$ *is the quantile, or the Value at Risk at level* q*:*

$$F_X^{-1}(q) = \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq q\right\}, \qquad q \in [0,1] \tag{14}$$

with $\inf \emptyset = +\infty$ *and* $\sup \emptyset = -\infty$ *by convention. From (13) one can conclude that the subadditive* $TVaR_{0.85}$ *risk measure is 'too subadditive' in this particular example.*
For more details, we refer to DHAENE ET AL. *(2008a).* ∇

Recall that the distributions of the r.v.'s X_i, $i = 1, 2, \ldots, n$, are said to belong to the same location-scale family of distributions if there exists a r.v. Z, as well as positive real constants a_i and real constants b_i such that the relation

$$X_i \stackrel{d}{=} a_i \; Z + b_i, \tag{15}$$

holds for $i = 1, 2, \ldots, n$. Without loss of generality, we will always assume that $\mathbb{E}[Z] = 0$ and $\mathrm{Var}\,[Z] = 1$.

As an illustration of a location-scale family of distributions, consider the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ with characteristic function given by

$$\mathbb{E}\left[\exp\left(i\underline{t}\underline{X}^T\right)\right] = \exp\left(i\underline{t}\underline{\mu}^T\right)\phi\left(\underline{t}\underline{\Sigma}\underline{t}^T\right), \qquad \underline{t} = (t_1, t_2, \ldots, t_n), \tag{16}$$

for a given scalar function ϕ, an n-dimensional vector $\underline{\mu}$ and where $\underline{\Sigma}$ is of the form $\underline{\Sigma} = \underline{A}\underline{A}^T$ for some $2 \times m$ matrix $\underline{A}$. In this case, the random vector $\underline{X}$ is said to be elliptically distributed with characteristic generator ϕ. Choosing the characteristic generator equal to $\phi(u) = \exp(-u/2)$ gives rise to the multivariate normal distribution. The components X_i of the multivariate elliptically distributed random vector $\underline{X}$ belong to the same location-scale family of distributions. Also any linear combination of the X_i belongs to the same location scale family of distributions. A standard reference for the theory of elliptical distributions is FANG, KOTZ & NG (1987). For applications of elliptical distributions in insurance and finance, see LANDSMAN & VALDEZ (2003), DHAENE ET AL. (2008b) and VALDEZ ET AL (2008).

The following theorem, a proof of which can be found in DHAENE ET AL. (2008a), states that $\mathbb{E}[D_\rho(\underline{X})] \geq 0$ holds for a broad class of capital requirements ρ, in case X_1, X_2 and $X_1 + X_2$ belong to the same location-scale family of distributions.

Theorem 7 (Diversification and location-scale families) *For any law invariant, translation invariant and positively homogeneous risk measure* ρ *and any random vector* $\underline{X} = (X_1, X_2, \ldots, X_n)$ *such that* $X_1, X_2, \ldots,$ X_n *and*

$X_1 + X_2 + \ldots + X_n$ belong to the same location-scale family of distributions and have finite variances, one has that

$$\mathbb{E}\left[D_\rho(\underline{X})\right] \geq 0. \tag{17}$$

It immediately follows that the theorem holds in particular for multivariate elliptical random vectors and a Value-at-Risk capital requirement.

4 Correlation order and market shortfalls

In this section, we compare two markets (X_1, X_2) and (Y_1, Y_2) with equal marginal distributions which are ordered in the correlation order sense. We compare both markets before as well as after a merger has taken place. Intuitively we would expect that increasing the positive dependence between the losses, expressed in terms of correlation order, will decrease the expected utility of the risk averse group of policyholders. This result is expected to hold before as well as after merging.

Theorem 8 (Correlation order and shortfalls) *Consider two random couples $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ with $X_1 \overset{d}{=} Y_1$ and $X_2 \overset{d}{=} Y_2$. Furthermore, let $\underline{K} = (K_1, K_2)$.*
Then $\underline{X} \preceq_{CORR} \underline{Y}$ implies

$$\sum_{i=1}^{2} (X_i - K_i)_+ \preceq_{CX} \sum_{i=1}^{2} (Y_i - K_i)_+ \tag{18}$$

and

$$(X_1 + X_2 - K_1 - K_2)_+ \preceq_{SL} (Y_1 + Y_2 - K_1 - K_2)_+ . \tag{19}$$

Proof. (a) From $X_i \overset{d}{=} Y_i$ we find that $(X_i - K_i)_+ \overset{d}{=} (Y_i - K_i)_+$ for $i = 1, 2$. Taking into account Definition 2(i) one finds that $(X_1, X_2) \preceq_{\text{CORR}} (Y_1, Y_2)$ implies that

$$\left((X_1 - K_1)_+ ,\ (X_2 - K_2)_+\right) \preceq_{\text{CORR}} \left((Y_1 - K_1)_+ ,\ (Y_2 - K_2)_+\right). \tag{20}$$

The convex order relation (18) follows then from (4).
(b) The assumed correlation order between (X_1, X_2) and (Y_1, Y_2) implies that $X_1 + X_2 \preceq_{\text{CX}} Y_1 + Y_2$. This convex order relation in turn implies that

$(X_1 + X_2 - d)_+ \preceq_{\text{SL}} (Y_1 + Y_2 - d)_+$ for all real d. Choosing $d = K_1 + K_2$ leads to (19). ■

Equation (18) can be rewritten in terms of expected utilities as

$$\mathbb{E}\left[u\left(w - \sum_{i=1}^{2}(X_i - K_i)_+\right)\right] \geq \mathbb{E}\left[u\left(w - \sum_{i=1}^{2}(Y_i - K_i)_+\right)\right], \tag{21}$$

which has to hold for all concave functions u, see e.g. DENUIT ET AL. (2005). Hence, (18) can be interpreted as follows: In a market without merged risks, the risk averse group of policyholders wanting to maximize their expected utility will prefer the less correlated market (X_1, X_2).

Similarly, equation (19) can be rewritten as

$$\mathbb{E}\left[u\left(w - (X_1 + X_2 - K_1 - K_2)_+\right)\right] \geq \mathbb{E}\left[u\left(w - (Y_1 + Y_2 - K_1 - K_2)_+\right)\right], \tag{22}$$

which has to hold for all non-decreasing concave functions u. Consequently, also in merged markets where capital requirements are added, the group of policyholders will prefer the less correlated couple (X_1, X_2). Both (21) and (22) are in correspondence with intuition.

In (19) and (22), we considered the preferences of the group of policyholders after a merger when adding the available capitals K_i. Let us now consider their preferences after a merger, taking into account the required capitals. In general, it will not hold that $(X_1, X_2) \preceq_{\text{CORR}} (Y_1, Y_2)$ implies that $(X_1 + X_2 - \rho[X_1 + X_2])_+ \preceq_{\text{SL}} (Y_1 + Y_2 - \rho[Y_1 + Y_2])_+$. This means that there may exist situations where ρ is such that after the merger, the more correlated couple is preferred.

However, in the following theorem, we show that this situation cannot occur in case the r.v.'s involved belong to the same location-scale family of distributions.

Theorem 9 (Location scale families, correlation order and shorfalls) *Assume that the capital requirement ρ is law invariant, translation invariant and positively homogeneous. Furthermore assume that $X_1 \stackrel{d}{=} Y_1$ and $X_2 \stackrel{d}{=} Y_2$ and also that their sums $X_1 + X_2$ and $Y_1 + Y_2$ belong to the same location-scale family of distributions and have finite variances. Then one has that $(X_1, X_2) \preceq_{CORR} (Y_1, Y_2)$ implies*

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ \preceq_{ST} (Y_1 + Y_2 - \rho[Y_1 + Y_2])_+. \tag{23}$$

Proof. We further write $\text{var}[X_j] = \sigma_j^2$, $j = 1, 2$, $\text{var}[X_1 + X_2] = \sigma^2_{X_1+X_2}$ and $\text{var}[Y_1 + Y_2] = \sigma^2_{Y_1+Y_2}$. Then we immediately find that

$$(X_1 + X_2 - \rho\,[X_1 + X_2])_+ \stackrel{d}{=} \sigma_{X_1+X_2}\,(Z - \rho\,[Z])_+$$

and

$$(Y_1 + Y_2 - \rho\,[Y_1 + Y_2])_+ \stackrel{d}{=} \sigma_{Y_1+Y_2}\,(Z - \rho\,[Z])_+\,.$$

From (4) it follows immediately that

$$\sigma_{X_1+X_2} \leq \sigma_{Y_1+Y_2}.$$

Combining these results leads to (23). ■

The stochastic inequality (23) can be expressed as

$$\mathbb{E}\left[u\left(w - (X_1 + X_2 - \rho\,[X_1 + X_2])_+\right)\right] \geq \mathbb{E}\left[u\left(w - (Y_1 + Y_2 - \rho\,[Y_1 + Y_2])_+\right)\right] \quad (24)$$

which has to hold for all non-decreasing functions u. Hence, under the conditions of the theorem we find that in merged markets, where the merged capital requirements are given by $\rho\,[X_1 + X_2]$ and $\rho\,[Y_1 + Y_2]$, respectively, the group of policyholders always prefers the less correlated couple of portfolios. This result is independent of whether or not ρ is subadditive. In particular, it holds for a Value-at-Risk risk measure in a bivariate elliptical market.

In this section, we have interpreted the integral order relations (18), (19) and (23) that hold between shortfalls of random couples in terms of expected utility theory. Similarly, we can interpret these results in terms of Yaari's dual theory of choice under risk.

In Theorems 8 and 9 we investigated implications of the correlation order $(X_1, X_2) \preceq_{\text{CORR}} (Y_1, Y_2)$ on the market shortfall. These results can be generalized in a straightforward way to implications of the supermodular order $\underline{X} \preceq_{\text{SM}} \underline{Y}$. The proof of these generalizations is based essentially on (5) and (6).

5 Correlation order and diversification

In this section, we investigate the relation between correlation order as defined in Definition 2 and the diversification benefits as defined in Definitions 7 and 11, respectively. In the following theorem we prove that increasing the dependence between the components of a random couple with given marginals, in terms of correlation order, decreases the expected diversification benefit from merging.

Theorem 10 (Increasing dependence ⇒ decreasing expected benefit) *Consider two couples of random losses* $\underline{X} = (X_1, X_2)$ *and* $\underline{Y} = (Y_1, Y_2)$ *with* $X_1 \stackrel{d}{=} Y_1$ *and* $X_2 \stackrel{d}{=} Y_2$*. Furthermore, let* $\underline{K} = (K_1, K_2)$*. Then,*

$$\underline{X} \preceq_{CORR} \underline{Y} \Rightarrow \mathbb{E}[D(\underline{X}, \underline{K})] \geq \mathbb{E}[D(\underline{Y}, \underline{K})]. \tag{25}$$

Proof. From (4) we find that for all real d it holds that

$$\mathbb{E}[(X_1 + X_2 - d)_+] \leq \mathbb{E}[(Y_1 + Y_2 - d)_+].$$

The result now follows from Definition 4. ■

The result in Theorem 10 is intuitive. This result cannot be readily generalized to the diversification benefit for the minimal capital requirement as defined in (11). This fact is illustrated in the following example.

Example 11 *Consider the random couple* $\underline{X} = (X_1, X_2)$ *and the r.v.* U *as defined in Example 6. Further, consider the comonotonic random couple* $\underline{Y} = (U, U)$*.*
It is straightforward to prove that

$$\underline{X} \preceq_{CORR} \underline{Y},$$

and also that $\mathbb{E}\left[D_{TVaR_{0.85}}(\underline{Y})\right] = 0$*. Furthermore, from Example 6, we know that* $\mathbb{E}\left[D_{TVaR_{0.85}}(\underline{X})\right] < 0$*. Hence, we have found that*

$$\mathbb{E}\left[D_{TVaR_{0.85}}(\underline{X})\right] < \mathbb{E}\left[D_{TVaR_{0.85}}(\underline{Y})\right],$$

which might at first sight be counterintuitive because of the above-mentioned correlation order. ▽

In the following Theorem, we show that under the assumption of location-scale distributions, correlation order will imply the appropriate order of the expected diversification benefits.

Theorem 12 (Location scale families and diversification) *Assume that the capital requirement ρ is law invariant, translation invariant and positively homogeneous. Furthermore, consider two couples of random losses $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ with $X_1 \stackrel{d}{=} Y_1$ and $X_2 \stackrel{d}{=} Y_2$. Assume that their sums $X_1 + X_2$ and $Y_1 + Y_2$ belong to the same location-scale family of distributions and have finite variances. Then one has that*

$$\underline{X} \preceq_{CORR} \underline{Y} \Rightarrow \mathbb{E}[D_\rho(\underline{X})] \geq \mathbb{E}[D_\rho(\underline{Y})]. \tag{26}$$

Proof. From Theorem 9 we have that

$$\mathbb{E}\left[(X_1 + X_2 - \rho\left[X_1 + X_2\right])_+\right] \leq \mathbb{E}\left[(Y_1 + Y_2 - \rho\left[Y_1 + Y_2\right])_+\right].$$

The implication (26) follows then from $X_1 \stackrel{d}{=} Y_1$ and $X_2 \stackrel{d}{=} Y_2$ and Definition 11 . ■

Theorems 10 and 12 can be generalized in a straightforward way to the supermodular order case. The proofs for the generalized versions are based on (5) and on the generalization of Theorem 9 to the supermodular case.

6 Strictly positive expected diversification benefits

In this section, we investigate conditions for the expected diversification benefit to be strictly positive. Moreover, we investigate the validity of the belief that merging comonotonic losses cannot lead to a diversification benefit.

For simplicity reasons, throughout this section we consider a random couple $\underline{X} = (X_1, X_2)$ such that its marginal cdf's F_{X_i} are strictly increasing and continuous on $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$, where $F_{X_i}^{-1}(p)$ is defined in (14) and $F_{X_i}^{-1+}(p)$ is defined as

$$F_{X_i}^{-1+}(p) = \sup\left\{x \in \mathbb{R} \mid F_X(x) \leq p\right\}, \qquad p \in [0,1], \tag{27}$$

with $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$ by convention.

Let $\underline{K} = (K_1, K_2)$, where K_i is the available capital for portfolio i. We assume that

$$F_{X_i}^{-1+}(0) < K_i < F_{X_i}^{-1}(1), \qquad i = 1, 2. \tag{28}$$

The symbol U will be used to denote a r.v. which is uniformly distributed on the unit interval $(0,1)$.

The stochastic order inequalities

$$\left(F_{X_1}^{-1}(U),\ F_{X_2}^{-1}(1-U)\right) \preceq_{\text{CORR}} \underline{X} \preceq_{\text{CORR}} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \tag{29}$$

allow us to conclude that in the class of all random couples with given marginal distributions, the extremal members with respect to correlation order correspond to the famous Fréchet-Höffding bounds: the comonotonic couple $\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right.$ is largest in correlation order sense, whereas the countermonotonic couple $\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U)\right)$ is smallest in correlation order sense. For more details, we refer to Dhaene et al. (1996) and Wang et al. (1998).

Combining Theorem 10 and the stochastic order inequalities (29) leads to

$$\begin{aligned} 0 &\leq \mathbb{E}[D(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \underline{K})] \\ &\leq \mathbb{E}[D(\underline{X}, \underline{K})] \leq \mathbb{E}[D(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U), \underline{K})]. \end{aligned} \tag{30}$$

This means that the expected diversification benefit for a random couple of losses with given marginals is smallest when the copula connecting the marginals is the comonotonic copula and largest when it is the countermonotonic copula. Notice that under the assumptions stated in Theorem 12, we find a similar result for the expected diversification benefit $\mathbb{E}[D_\rho(\underline{X})]$.

In Kaas et al. (2002) it is proven that

$$\begin{aligned} \left(F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) - K_1 - K_2\right)_+ &= \sum_{i=1}^{2} \left(F_{X_i}^{-1}(U) - F_{X_i}^{-1}(p^*)\right)_+ \\ &\leq \sum_{i=1}^{2} \left(F_{X_i}^{-1}(U) - K_i\right)_+, \end{aligned} \tag{31}$$

where $p^* \in (0,1)$ is given by

$$p^* = F_{F_{X_1}^{-1}(U)+F_{X_2}^{-1}(U)}(K_1 + K_2). \tag{32}$$

From (31) we find that the diversification benefit from merging the comonotonic losses $F_{X_1}^{-1}(U)$ and $F_{X_2}^{-1}(U)$ is given by

$$D\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \underline{K}\right) = \sum_{i=1}^{2}\left(F_{X_i}^{-1}(U) - K_i\right)_+ - \sum_{i=1}^{2}\left(F_{X_i}^{-1}(U) - F_{X_i}^{-1}(p^*)\right)_+ \geq 0. \tag{33}$$

In the following theorem we consider a condition that ensures the expected diversification benefit to be strictly positive.

Theorem 13 (Strictly positive expected diversification benefit) *Consider a random couple $\underline{X} = (X_1, X_2)$ with strictly increasing marginal cdf's F_{X_i} which are continuous on $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$. Further, let $\underline{K} = (K_1, K_2)$ be such that $F_{X_i}^{-1+}(0) < K_i < F_{X_i}^{-1}(1)$ for $i = 1, 2$.*
In this case, the condition

$$F_{X_1}(K_1) \neq F_{X_2}(K_2) \tag{34}$$

implies

$$\mathbb{E}\left[D\left(\underline{X}, \underline{K}\right)\right] > 0. \tag{35}$$

Proof. (a) From (32) and the additivity property of quantiles of a comonotonic sum we find that

$$F_{X_1}^{-1}(p^*) + F_{X_2}^{-1}(p^*) = K_1 + K_2 \tag{36}$$

with p^* defined in (32).
Without loss of generality, let us assume that

$$F_{X_1}(K_1) < F_{X_2}(K_2). \tag{37}$$

Next, we define p_1 and $p_2 \in (0, 1)$ such that

$$K_i = F_{X_i}^{-1}(p_i), \qquad i = 1, 2. \tag{38}$$

The inequality (37) can then be rewritten as

$$p_1 < p_2. \tag{39}$$

From (36) and (38) we find that

$$F_{X_1}^{-1}(p^*) + F_{X_2}^{-1}(p^*) = F_{X_1}^{-1}(p_1) + F_{X_2}^{-1}(p_2). \tag{40}$$

From (39), (40) and the strictly increasingness of the quantile functions $F_{X_i}^{-1}(q)$ it follows that

$$p_1 < p^* < p_2.$$

In case U is such that

$$p_1 < U < p^* < p_2,$$

we find from (33) that

$$D\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \underline{K}) = F_{X_1}^{-1}(U) - F_{X_1}^{-1}(p_1) > 0.$$

Hence, $\Pr\left[D\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \underline{K}) > 0\right] \geq p^* - p_1 > 0$.
Combining this result with the fact that $D\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \underline{K}) \geq 0$ holds for all outcomes of U, we can conclude that (35) holds for the comonotonic couple $\underline{X} \equiv \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right)$.
(b) Let us now consider a general random couple $\underline{X}$ for which the condition (34) holds. As $X_i \stackrel{d}{=} F_{X_i}^{-1}(U)$, we can rewrite condition (34) as

$$F_{F_{X_1}^{-1}(U)}(K_1) \neq F_{F_{X_2}^{-1}(U)}(K_2).$$

From (a) we can conclude that $\mathbb{E}\left[D\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \underline{K})\right] > 0$. Combining this result with (30) proves (35). ■

Theorem 13 states that merging any random couple of losses leads to a strictly positive expected diversification benefit, in case the non-ruin probabilities $F_{X_1}(K_1)$ and $F_{X_2}(K_2)$ are different. This result holds in particular for comonotonic losses.

The condition (34) is equivalent to the condition that there exist not any p in $(0,1)$ such that $\underline{K} = \left(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p)\right)$. Hence, the condition (34) says that both capitals K_i do not have a VaR_p representation for some fixed p in $(0,1)$.

Notice that the condition (34) can never be satisfied in case $X_1 \stackrel{d}{=} X_2$ and $K_1 = K_2$.

In the following theorem we consider the case that $F_{X_1}(K_1) = F_{X_2}(K_2)$ for a comonotonic couple of losses $\underline{X}$.

Theorem 14 (Comonotonicity and zero-diversification) *Consider a comonotonic random couple* $\underline{X} \equiv \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right)$ *with strictly increasing marginal cdf's* F_{X_i} *which are continuous on* $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$. *Further, let* $\underline{K} = (K_1, K_2)$

such that $F_{X_i}^{-1+}(0) < K_i < F_{X_i}^{-1}(1)$ *for* $i = 1, 2$.
In this case, the condition

$$F_{X_1}(K_1) = F_{X_2}(K_2) \tag{41}$$

implies

$$D\left(\underline{X}, \underline{K}\right) = 0. \tag{42}$$

Proof. The condition (41) is equivalent to the condition that there exist a p in $(0, 1)$ such that

$$\underline{K} = \left(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p)\right) \tag{43}$$

From (36) and (43) we find that

$$F_{X_1}^{-1}(p^*) + F_{X_2}^{-1}(p^*) = F_{X_1}^{-1}(p) + F_{X_2}^{-1}(p).$$

The strictly increasingness of $F_{X_1}^{-1}(q) + F_{X_2}^{-1}(q)$ implies $p = p^*$. The result (42) follows then from (33). ■

The theorem states that, under the appropriate assumptions, in a market consisting of comonotonic losses where a quantile risk measure is used to set capitals, merging will never lead to a diversification benefit.

Theorem 15 (Location scale families and VaR) *In addition to law invariance, assume that the risk measure* ρ *is translation invariant and positively homogeneous. Further, consider a location-scale family of distributions generated by the (cdf of) the r.v.* Z. *Assume that* F_Z *is strictly increasing and continuous, and also that* $F_Z^{-1+}(0) < \rho[Z] < F_Z^{-1}(1)$.
For any r.v. X *belonging to this location-scale family, one has that*

$$\rho[X] = F_X^{-1}\left[F_Z\left(\rho[Z]\right)\right]. \tag{44}$$

Proof. Define $p \in (0, 1)$ such that

$$\rho[Z] = F_Z^{-1}(p), \tag{45}$$

or, equivalently,

$$p = F_Z\left(\rho[Z]\right). \tag{46}$$

For any r.v. X belonging to the location-scale family, there exist a positive real constant a and a real constant b such that

$$X \stackrel{d}{=} aZ + b.$$

Hence, it holds that

$$F_X^{-1}(p) = a\ F_Z^{-1}(p) + b \tag{47}$$

and also

$$\rho\,[X] = a\ \rho\,[Z] + b. \tag{48}$$

Combining (45), (46), (47) and (48) we find that $\rho\,[X]$ can be expressed as (44). ■

We can conclude that any risk measure ρ satisfying the requirements of Theorem 15 can be seen as a Value-at-Risk at a fixed level p, when restricted to a location-scale family as described in the theorem. This conclusion holds in particular for distortion risk measures applied to the components of a multivariate elliptical distributed random vector.

Theorem 16 (Location scale families, comonotonicity and diversification) *In addition to law invariance let us assume that the risk measure ρ is translation invariant and positively homogeneous. Further, consider a location-scale family of distributions generated by the (cdf of) the r.v. Z. Assume that F_Z is strictly increasing and continuous, and also that $F_Z^{-1+}(0) < \rho\,[Z] < F_Z^{-1}(1)$. For any comonotonic random couple $\underline{X} \equiv \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right)$ of which the X_i belong to the location-scale family generated by Z one has that*

$$D\left(\underline{X}, \left(\rho\,[X_1]\,, \rho\,[X_2]\right)\right) = 0. \tag{49}$$

Moreover, in case ρ is subadditive, one has that

$$D_\rho\left(\underline{X}\right) \leq 0. \tag{50}$$

Proof. From Theorem 15, we find that

$$\left(\rho\,[X_1]\,, \rho\,[X_2]\right) = \left(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p)\right), \tag{51}$$

with $p \in (0,1)$ given by (46). Furthermore, the assumptions imposed on F_Z and ρ imply that the cdf's F_{X_i} are strictly increasing and continuous on $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$ and also that $F_{X_i}^{-1+}(0) < \rho\,[X_i] < F_{X_i}^{-1}(1)$, $i = 1, 2$.
Applying Theorem 13 leads to (49).
The inequality (50) follows immediately from (12) and (49). ■

Theorem 16 holds in particular for distortion risk measures applied to the class of elliptical r.v.'s. with a given characteristic generator.

We conclude this section with an example of merging two lognormal losses.

Example 17 *Consider the lognormal losses X_1 and X_2 with parameters μ_i and σ_i^2, $i = 1, 2$, respectively. The capital risk measure ρ is given by the Tail Value-at-Risk at level $p \in (0, 1)$:*

$$\begin{aligned} \rho\left[X_i\right] &= TVaR_p\left[X_i\right] = \frac{1}{1-p}\int_p^1 F_{X_i}^{-1}(q)\ \mathrm{d}q \\ &= e^{\mu_i+\sigma_i^2/2}\frac{\Phi\left(\sigma_i-\Phi^{-1}(p)\right)}{1-p}, \qquad i = 1, 2. \end{aligned} \tag{52}$$

Let the p_i be defined by

$$F_{X_i}^{-1}(p_i) = TVaR_p\left[X_i\right].$$

As

$$F_{X_i}^{-1}(p_i) = e^{\mu_i+\sigma_i\Phi^{-1}(p_i)}, \tag{53}$$

we find that the p_i are given by

$$p_i = \Phi\left(\frac{\sigma_i}{2} + \frac{1}{\sigma_i}\ln\left(\frac{\Phi\left(\sigma_i-\Phi^{-1}(p)\right)}{1-p}\right)\right). \tag{54}$$

In general, $\sigma_1 \neq \sigma_2$ implies that $p_1 \neq p_2$ which means that the capitals $TVaR_p\left[X_i\right]$, $i = 1, 2$, do not correspond with a unique VaR_p. From Theorem 13 we can conclude that $\mathbb{E}[D(\underline{X}, (TVaR_p\left[X_1\right], TVaR_p\left[X_2\right]))] > 0$. in this case.

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