BUY-AND-HOLD STRATEGIES AND COMONOTONIC APPROXIMATIONS[1]

J. Marín-Solano[2], Universitat de Barcelona
O. Roch, Universitat de Barcelona
J. Dhaene, Katholieque Universiteit Leuven
C. Ribas, Universitat de Barcelona
M. Bosch-Príncep, Universitat de Barcelona
S. Vanduffel, Vrije Universiteit Brussel

[1]M. Bosch-Príncep, J. Marín-Solano, C. Ribas and O. Roch acknowledge partial financial support by MEC (Spain) Grants MTM2006-13468 and ECO2009-08274. Jan Dhaene and Steven Vanduffel acknowledge the financial support of the Onderzoeksfonds K.U. Leuven (GOA/07: Risk Modeling and Valuation of Insurance and Financial Cash Flows, with Applications to Pricing, Provisioning and Solvency). They also acknowledge the financial support of Fortis (K.U.Leuven Fortis Chair in Financial and Actuarial Risk Management).

[2]Corresponding author: Jesús Marín-Solano, Dept. Matemàtica econòmica, financera i actuarial, Universitat de Barcelona, Avda. Diagonal 690, E-08034 Barcelona, Spain. E-mail address: jmarin@ub.edu; Tel.: +34-93-402-1991; fax: +34-93-403-4892

BUY-AND-HOLD STRATEGIES AND COMONOTONIC APPROXIMATIONS

Abstract

We investigate optimal buy-and-hold strategies for terminal wealth problems in a multi-period framework. As terminal wealth is a sum of dependent random variables, the distribution function of final wealth cannot be determined analytically for any realistic model. By calculating lower bounds in the convex order sense, we consider approximations that reduce the multivariate randomness to univariate randomness. These approximations are used to determine buy-and-hold strategies that optimize, for a given probability level, the Value at Risk and the Conditional Left Tail Expectation of the distribution function of final wealth. Finally, the accurateness of the different approximations is investigated numerically.

Keywords: comonotonicity, lognormal variables, lower bounds, optimal portfolios, risk measures.

1 Introduction

Optimal portfolio selection can be defined as the problem that consists in identifying the best allocation of wealth among a basket of securities. The investor chooses an initial asset mix and a particular investment strategy within a given set of strategies, according to which he will buy and sell assets during the whole time period under consideration.

The simplest class of strategies are the so-called "buy-and-hold" strategies, where an initial asset mix is chosen and no rebalancing is performed during the investment period.

In this paper, we aim at finding optimal buy-and-hold strategies for final wealth problems. Note that the case of constant mix strategies was analyzed in Dhaene *et al.* (2005). Buy-and-hold strategies are an important and popular class of investment strategies. Firstly, they do not require a dynamic follow-up and are easy to implement. Secondly, since no intermediate trading is required, they do not involve transaction costs.

As the investment horizon that we consider is typically long, the Central Limit Theorem provides some justification for the use of a Gaussian model for the stochastic returns, see e.g. Cesari and Cremonini (2003) and McNeil *et al.* (2005).

We assume that the aim of the decision maker is to maximize the "benefit" he attracts from the final value of his investment. Hence, we maximize a quantity related to terminal wealth, thereby also reflecting the decision maker's risk aversion. In this paper we do not work within the framework of expected utility (Von Neumann & Morgenstern (1947)). Instead, we use distorted expectations within the framework of Yaari's dual theory of choice under risk (Yaari (1987)). We consider strategies that maximize the quantile (or Value-at-Risk) of the final wealth corresponding to a given probability level.

For any buy-and-hold strategy, terminal wealth is a sum of dependent random variables (rv's). In any realistic multiperiod asset model, the distribution function of final wealth cannot be determined analytically. Therefore, we look for accurate analytic approximations for the distribution function (df) or the risk measure at hand. The most direct approximation is given by the so-called "comonotonic upper bound", which is an upper bound for the exact df in the convex order sense, see Kaas *et al.* (2000). However, much better approximations can be obtained by using comonotonic lower bound approximations; see Dhaene *et al.* (2002a,b), Vanduffel *et al.* (2005) and Vanduffel *et al.* (2008). The advantages of working with these approximations are related to the fact that, for any given investment strategy, they enable accurate and easy-to-compute approximations to be obtained for risk measures that are additive for comonotonic risks, such as quantiles, conditional tail expectations and, more generally, distortion risk measures.

The paper is organized as follows. Section 2 gives a brief review of some important risk measures, such as Value-at-Risk and Conditional Left Tail Expectation, and also introduces the different comonotonic bounds for sums of rv's used throughout the paper. In Section 3, the basic variables of the problem, such as dynamic price equations and investment strategies are introduced, and buy-and-hold strategies are described. In Section 4, we derive explicit expressions for upper and lower comonotonic bounds for terminal wealth when following a buy-and-hold strategy. Section 5 is devoted to finding optimal buy-and-hold strategies in the case where one is focusing on maximizing a Value-at-Risk or a Conditional Left Tail Expectation. The results are investigated numerically to illustrate the level of accurateness of the different comonotonic approximations. Section 6 concludes the paper.

2 Preliminaries

2.1 Risk measures

All rv's considered in this paper are defined on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$.

In order to make decisions, we use risk measures. A risk measure is a mapping from a set of relevant rv's to the real line $\mathbb{R}$. Firstly, let us consider the Value-at-Risk at level p (also called the p-quantile) of a rv X. It is defined as

$$Q_p[X] = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\} \ , \ p \in (0,1) \ ,$$

where $F_X(x) = Pr(X \leq x)$ and by convention $\inf\{\emptyset\} = +\infty$.

We can also define the related risk measure

$$Q_p^+[X] = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\} \ , \ p \in (0,1) \ ,$$

where by convention $\sup\{\emptyset\} = -\infty$.

If F_X is strictly increasing, then $Q_p[X] = Q_p^+[X]$, for every $p \in (0,1)$.

In this paper, we also use the Conditional Left Tail Expectation at level p, which is denoted by $CLTE_p[X]$. It is defined as

$$CLTE_p[X] = E\left[X \mid X < Q_p^+[X]\right] \ , \ p \in (0,1) \ .$$

If $CTE_p[X] = E\left[X \mid X > Q_p[X]\right]$ denotes the Conditional Tail Expectation,

$$CLTE_{1-p}[X] = -CTE_p[-X] \ . \tag{2.1}$$

We refer to Dhaene *et al.* (2006) for an overview of the properties of distortion risk measures.

2.2 Comonotonic bounds for sums of random variables

A random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is said to be *comonotonic* if

$$(X_1, X_2, \ldots, X_n) \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)) \ ,$$

where U is a rv uniformly distributed on the unit interval. We refer to Dhaene *et al.* (2002a,b) for an extensive overview on comonotonicity and a discussion of some of its applications.

The risk measures Q_p and $CLTE_p$ have the convenient property that they are additive for sums of comonotonic risks, i.e., if $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a comonotonic random vector and $S = X_1 + X_2 + \cdots + X_n$, then we have that

$$Q_p[S] = \sum_{i=1}^{n} Q_p[X_i]$$

and

$$CLTE_p[S] = \sum_{i=1}^{n} CLTE_p[X_i] \ ,$$

provided all marginal distributions F_{X_i} are continuous.

Now, let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector of dependent rv's X_i, $i = 1, \ldots, n$, and let $S = X_1 + X_2 + \cdots + X_n$ be the corresponding sum. In some cases the df of S can be determined; for instance, when $\mathbf{X}$ is a multivariate normally or elliptically distributed rv, but in general this a difficult exercise. Kaas *et al.* (2000) and Dhaene *et al* (2002a,b) showed that there are situations where good and analytically tractable approximations for the df and the risk measures of S can be found. These approximations are bounds in convex order. A rv X is said to be convex smaller than another rv Y, denoted by $X \leq_{cx} Y$, if

$$\begin{aligned} E\left[X\right] &= \mathrm{E}\left[Y\right], \\ \mathrm{E}\left[(X-d)_+\right] &\leq \mathrm{E}\left[(Y-d)_+\right], \text{ for all } d \in \mathbb{R} \ . \end{aligned}$$

Let U be the uniform distribution on the unit interval. For any random vector $(X_1, X_2, \ldots, X_n)$ and any rv Λ, we define

$$S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U), \text{ and } S^l = \sum_{i=1}^{n} E\left[X_i \mid \Lambda\right].$$

It can be proven that $S^l \leq_{cx} S \leq_{cx} S^c$, see Kaas *et al.* (2000). The bound S^c is the so-called *comonotonic upper bound*, and whilst its risk measures are often readily available they do not provide us with good approximations for the risk measures of S in general. Essentially, this is because the comonotonic vector $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$ entails a maximal correlation between the rv's X_i and X_j, for every $i, j = 1, \ldots, n$. On the other hand, for the lower bound S^l to be of real use, we need more explicit expressions for the rv's $E\left[X_i \mid \Lambda\right]$. Fortunately, in the lognormal case such expressions are readily available, as we show below. The challenge consists in choosing the rv Λ in such a way that the convex lower bound $S^l = E\left[S \mid \Lambda\right]$ is 'close' to the rv S.

2.3 Sums of log-normal random variables

Consider the multivariate normal random vector $(Z_1, Z_2, \ldots, Z_n)$, and the non-negative real numbers α_i, $i = 1, \ldots, n$. In this case, the sum S defined by

$$S = \sum_{i=1}^{n} \alpha_i \, e^{Z_i}$$

is a sum of dependent lognormal rv's.

The comonotonic upper bound S^c for S is given by

$$S^c = \sum_{i=1}^{n} F_{\alpha_i e^{Z_i}}^{-1}(U) = \sum_{i=1}^{n} \alpha_i \, e^{E\left[Z_i\right] + \sigma_{Z_i} \Phi^{-1}(U)}. \tag{2.2}$$

In order to obtain a lower bound S^l for S, we consider a conditioning rv Λ which is a linear combination of the different Z_i,

$$\Lambda = \sum_{j=1}^{n} \gamma_j Z_j \, .$$

After some computations (see Dhaene *et al.* (2002b)), we find that the lower bound $S^l = \sum_{i=1}^{n} \alpha_i \, E\left[e^{Z_i} \mid \Lambda\right]$ is given by

$$S^l = \sum_{i=1}^{n} \alpha_i \, e^{E\left[Z_i | \Lambda\right] + \frac{1}{2} Var\left[Z_i | \Lambda\right]}, \tag{2.3}$$

with

$$\begin{aligned} E\left[Z_i \mid \Lambda\right] &= E\left[Z_i\right] + r_i \sigma_{Z_i} \frac{\Lambda - E[\Lambda]}{\sigma_\Lambda} \\ Var\left[Z_i \mid \Lambda\right] &= \left(1 - r_i^2\right) \sigma_{Z_i}^2, \end{aligned}$$

where r_i is the correlation coefficient between Z_i and Λ, σ_Λ is the standard deviation of the rv Λ and Φ denotes the standard normal df. If all r_i are positive, then S^l is a comonotonic sum.

In order to obtain accurate approximations for the df of S, we choose the coefficients γ_j in such a way that they minimize some "distance" between S and S^l. In this paper, we use four different approaches.

1. **The 'Taylor-based' lower bound approach**. In Kaas *et al.* (2000) and Dhaene *et al.* (2002b), the parameters γ_j are chosen such that Λ is a linear transformation of a first order approximation to S. After a straightforward derivation, the parameters γ_j turn out to be given by

$$\gamma_j = \alpha_j \, e^{E[Z_j]} \ . \tag{2.4}$$

2. **The 'Maximal Variance' lower bound approach**. As we have that $\text{Var}[S^l] \leq \text{Var}[S^l] + \text{E}[\text{Var}[S|\Lambda]] = \text{Var}[S]$, it seems reasonable to choose the coefficients γ_j such that the variance of S^l is maximized. This idea led Vanduffel *et al.* (2005) to maximise an approximate expression for $\text{Var}[S^l]$. They obtain

$$\gamma_j = \alpha_j \, e^{E[Z_j] + \frac{1}{2}\sigma^2_{Z_j}} = \alpha_j E\left[e^{Z_j}\right] \ . \tag{2.5}$$

3. **The 'MV-Minimal $CLTE_p$' lower bound approach**. The two lower bounds described above are constructed in such a way that they lead to an overall good approximation for the distribution function for the sum S. In Vanduffel *et al.* (2008) a 'locally' optimal Λ was introduced such that the df of the corresponding lower bound $E[S \mid \Lambda]$ is close to the df of S in a particular upper or lower tail of the distribution.

   The convex ordering that exists between the rv's S^l, S and S^c implies that $CLTE_p[S^c] \leq CLTE_p[S] \leq CLTE_p[S^l]$; see Dhaene *et al.* (2006). Then, Λ is optimal for measuring the lower tail for the df of S in case $CLTE_p[S^l]$ becomes 'as small as possible'. In particular, let r_i denote the correlation coefficients between Z_i and the rv Λ obtained from the 'Maximal Variance' approach. Then, the parameters γ_j minimizing a first-order approximation for the $CLTE_p[S^l]$ in a neighborhood of r_i are given by

$$\gamma_j = \alpha_j \, e^{E[Z_j] + \frac{1}{2}\sigma^2_{Z_j}} \cdot e^{-\frac{1}{2}(r_j \sigma_{Z_j} - \Phi^{-1}(p))^2} \tag{2.6}$$

   with

$$r_j = \frac{\sum_{k=1}^n \alpha_k E\left[e^{Z_k}\right] \text{Cov}\left[Z_j, Z_k\right]}{\sigma_{Z_j} \cdot \sqrt{\sum_{k=1}^n \sum_{l=1}^n \alpha_k \alpha_l E\left[e^{Z_k}\right] E\left[e^{Z_l}\right] \text{Cov}\left[Z_k, Z_l\right]}} \ .$$

   Note that from relation (2.1) it follows that minimizing $CLTE_p[S^l]$ is equivalent to maximizing $CTE_p[S^l]$. Therefore, the coefficients (2.6) also give rise to lower bound approximations that provide a good fit in the upper tail.

4. **The 'T-Minimal $CLTE_p$' lower bound approach**. In this paper we introduce this bound, which is similar to the previous one, but now the first order approximation is performed in a neighborhood of the correlation coefficient r_i which represents the correlation between the Z_i and the rv Λ obtained from the 'Taylor' approach. In this case, we find that the coefficients γ_j in Λ are given by (2.6) with

$$r_j = \frac{\sum_{k=1}^{n} \alpha_k e^{E[Z_k]} \text{Cov}\left[Z_j, Z_k\right]}{\sigma_{Z_j} \cdot \sqrt{\sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \alpha_l e^{E[Z_k]} e^{E[Z_l]} \text{Cov}\left[Z_k, Z_l\right]}} .$$

Indeed, Vanduffel *et al.* (2008) provided some evidence that the 'Taylor-based' lower bound approach is likely to be more appropriate in the approximation of the left tail of the distribution of S, whereas the 'Maximal Variance' lower bound approach is more accurate in the case where one focuses on the right tail of S. As we illustrate numerically, the same kind of observations also holds, as expected, for the related 'T-Minimal $CLTE_p$' and 'MV-Minimal $CLTE_p$' lower bound approaches. Note that from the investors' point of view, the risk of the final wealth rv is in the left tail of its distribution, which corresponds to small outcomes of final wealth.

3 General description of the problem

3.1 The Black & Scholes setting

We adopt the classical continuous-time framework pioneered by Merton (1971), and which is nowadays mostly referred to as the Black and Scholes setting. Let $t = 0$ be now and let the time unit be equal to 1 year. We assume that there $m + 1$ securities available in the financial market. One of them is a risk-free security (for instance, a cash account). Its unit price, denoted as $P^0(t)$, evolves according to the following ordinary differential equation:

$$\frac{\mathrm{d}P^0(t)}{P^0(t)} = r\mathrm{d}t ,$$

where $r > 0$ and $P^0(0) = p_0 > 0$. There are also m risky assets (stock funds, for instance). Let $P^i(t)$, $i = 1, \ldots, m$, denote the price for 1 unit of the risky asset i at time t. We assume that $P^i(t)$ evolves according to a geometric Brownian motion, described by the following system of differential equations:

$$\frac{\mathrm{d}P^i(t)}{P^i(t)} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}B^i(t) , \; i = 1, \ldots, m ,$$

where $P^i(0) = p_i > 0$, $(B^1(t), \ldots, B^m(t))$ is a m-dimensional Brownian motion process. The $B^i(t)$ are standard Brownian motions with $\text{Cov}\left(B^i(t), B^j(t+s)\right) = \frac{\sigma_{ij}}{\sigma_i \sigma_j} t$, for $t, s \geq 0$.

We assume that r and the drift vector of the risky assets $\mu = (\mu_1, \ldots, \mu_m)$ remain constant over time, and also that $\mu \neq (r, \ldots, r)$.

We define the matrix $\Sigma = (\sigma_{ij})$, $i, j = 1, \ldots, m$, with $\sigma_{ii} \equiv \sigma_i^2$. We assume that Σ is positive definite. In particular, this implies that all $\sigma_{ii} > 0$ (all m risky assets are indeed risky) and that Σ is nonsingular.

Finally, let us analyze the return in one year for an amount of 1 unit that is invested at time $k-1$ in asset i. If Y_k^i denotes the random yearly log-return of account i in year k, then $\mathrm{e}^{Y_k^i} = \dfrac{P^i(k)}{P^i(k-1)}$.

The random yearly returns Y_k^i, $i = 1, ..., m$, are independently and normally distributed with

$$\begin{aligned} E\left[Y_k^i\right] &= \mu_i - \frac{1}{2}\sigma_i^2 \, , \\ \mathrm{Var}\left[Y_k^i\right] &= \sigma_i^2 \, , \\ \mathrm{Cov}\left[Y_k^i, Y_l^j\right] &= \begin{cases} 0 & \text{if} \quad k \neq l \\ \sigma_{ij} & \text{if} \quad k = l \, . \end{cases} \end{aligned}$$

Hence, Σ is the Variance-Covariance Matrix of the one-period logarithms $(Y_k^1, \ldots, Y_k^n)$.

3.2 Buy-and-Hold strategies and terminal wealth

In this paper, we focus on buy-and-hold strategies. We consider the following terminal wealth problem:

- New investments are made once a year, with $\alpha_i \geq 0$ the investment at time i, $i = 0, 1, \ldots, n-1$.
- The α_i are invested in the $m+1$ assets according to a buy-and-hold strategy characterized by the vector of pre-determined proportions $\Pi(t) = (\pi_0(t), \ldots, \pi_m(t))$, for $t = 0, 1, \ldots, n-1$, with $\sum_{j=0}^m \pi_j(t) = 1$.
- The proportions according to which the new investments are made do not vary over time, i.e. $\Pi(t) = (\pi_0, \pi_1, \ldots, \pi_m)$.
- The investor does not perform any other trading activity during the investment period $[0, n]$.

Our aim is to evaluate the random terminal wealth $W_n(\Pi)$ for a given buy-and-hold strategy $\Pi = (\pi_0, \pi_1, \ldots, \pi_m)$ and a given (deterministic) vector of savings $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$.

Let Z_j^i be the total log-return, over the period $[j, n]$ of 1 unit of capital invested at time $t = j$ in asset i, $i = 0, 1, \ldots, m$:

$$Z_j^i = \sum_{k=j+1}^{n} Y_k^i \, . \tag{3.1}$$

Note that, for every asset i, $i = 1, \ldots, m$, the different Z_j^i are n dependent normally distributed rv's with

$$E\left[Z_j^i\right] = (n-j)\left[\mu_i - \frac{1}{2}\sigma_i^2\right] , \tag{3.2}$$

$$\sigma^2_{Z_j^i} = (n-j)\sigma_i^2 , \tag{3.3}$$

whereas for the risk-free component ($i = 0$) we find that Z_j^0 is given by $Z_j^0 = E\left[Z_j^i\right] = (n-j)r$. Hence, by denoting $\mu_0 = r$ and $\sigma_0^2 = 0$, we find that expressions (3.1) also cover the case $i = 0$.

Investing according to the buy-and-hold strategy $\Pi = (\pi_0, \pi_1, \ldots, \pi_m)$, we find that the terminal wealth of the investments in asset class i is given by

$$W_n^i(\Pi) = \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{Z_j^i} .$$

The total terminal wealth $W_n(\Pi)$ is then given by

$$W_n(\Pi) = \sum_{i=0}^{m} W_n^i(\Pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{Z_j^i} . \tag{3.4}$$

4 Upper and lower bounds for the terminal wealth

From (3.4) it becomes clear that $W_n(\Pi)$ is the sum of $m \cdot n$ dependent log-normal rv's and a constant term which represents the final wealth of the risk free investments. In general, it is not possible to determine the df of $W_n(\Pi)$ analytically. In order to obtain good analytical approximations for risk measures related to $W_n(\Pi)$, we determine the comonotonic bounds described in Section 2.3.

4.1 Comonotonic upper bound

The terminal wealth for the buy-and-hold strategy $\Pi = (\pi_0, \pi_1, \ldots, \pi_m)$ is given by (3.4). From (3.2), (3.3) and (2.2), we obtain

$$W_n^c(\Pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{(n-j)(\mu_i - \frac{1}{2}\sigma_i^2) + \sqrt{n-j}\sigma_i \Phi^{-1}(U)} . \tag{4.1}$$

Note that $W_n^c(\Pi)$ is linear in the investment proportions π_i, $i = 0, 1, \ldots, m$.

4.2 The 'Taylor-based' lower bound

For the sum of log-normal rv's and the constant term given by (3.4), we know from Section 2.2 that lower bounds can be obtained as $W_n^l(\Pi) = \mathrm{E}[W_n(\Pi) \mid \Lambda(\Pi)]$, where $\Lambda(\Pi)$ is a linear combination of Z_j^i. Following the results in Section 2.3, we choose

$$\Lambda(\Pi) = \sum_{i=1}^{m} \sum_{j=0}^{n-1} \gamma_{ij}(\Pi) \cdot Z_j^i \ . \tag{4.2}$$

From (2.4), it follows that the coefficients $\gamma_{ij}(\Pi)$ for the Taylor-based approach are given by

$$\gamma_{ij}(\Pi) = \pi_i \, \alpha_j \, e^{E\,[Z_j^i]} \ . \tag{4.3}$$

Therefore, from (3.2) we obtain

$$\Lambda(\Pi) = \sum_{i=1}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{(n-j)\left[\mu_i - \frac{1}{2}\sigma_i^2\right]} Z_j^i \ .$$

From (2.3) we know that

$$W_n^l(\Pi) \stackrel{d}{=} \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{A_j^i(\Pi)} \ , \tag{4.4}$$

where

$$\begin{aligned} A_j^i(\Pi) &= E[Z_j^i] + \frac{1}{2}(1 - r_{ij}^2(\Pi))\sigma_{Z_j^i}^2 + r_{ij}(\Pi)\sigma_{Z_j^i}\Phi^{-1}(U) \\ &= (n-j)\left(\mu_i - \frac{1}{2} r_{ij}^2(\Pi)\sigma_i^2\right) + r_{ij}(\Pi)\sqrt{n-j}\sigma_i\Phi^{-1}(U) \ . \end{aligned}$$

It remains to compute the correlation coefficients $r_{ij}(\Pi)$, for $i = 1, \ldots, m$, $j = 0, 1, \ldots, n-1$:

$$r_{ij}(\Pi) = \frac{\mathrm{Cov}[Z_j^i, \Lambda(\Pi)]}{\sqrt{\mathrm{Var}[Z_j^i]}\sqrt{\mathrm{Var}[\Lambda(\Pi)]}} \ .$$

First, note that

$$\sqrt{\mathrm{Var}[Z_j^i]} = \sqrt{n-j}\, \sigma_i \ .$$

Moreover, since $\Lambda(\Pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \gamma_{ij}(\Pi) \, Z_j^i$ we find that

$$\mathrm{Var}[\Lambda(\Pi)] = \sum_{i=1}^{m} \sum_{k=1}^{m} \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \gamma_{ij}(\Pi) \, \gamma_{kl}(\Pi) \, \mathrm{Cov}\left[Z_j^i, Z_l^k\right] \ . \tag{4.5}$$

Lemma 1 *For every $i, k = 1, \ldots, m$, and $j, l = 0, 1, \ldots n-1$, it holds that*

$$Cov\left[Z_j^i, Z_l^k\right] = (n - \max(j,l))\sigma_{ik} \ .$$

Proof: Straightforward.

□

From relation (4.5), we obtain by using Lemma 1 that

$$\mathrm{Var}[\Lambda(\Pi)] = \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \gamma_{ij}(\Pi)\,\gamma_{kl}(\Pi)\,(n-\max(j,l))\sigma_{ik}\ .$$

Finally note that, for $i = 1, \ldots, m$,

$$\begin{aligned}\mathrm{Cov}\left[Z_j^i, \Lambda(\Pi)\right] &= \mathrm{Cov}\left[Z_j^i, \sum_{k=0}^{m}\sum_{l=0}^{n-1}\gamma_{kl}(\Pi)\,Z_l^k\right] = \sum_{k=1}^{m}\sum_{l=0}^{n-1}\gamma_{kl}(\Pi)\,\mathrm{Cov}\left[Z_j^i, Z_l^k\right] \\ &= \sum_{k=1}^{m}\sum_{l=0}^{n-1}\gamma_{kl}(\Pi)(n-\max(j,l))\,\sigma_{ik}\ . \qquad (4.6)\end{aligned}$$

From (4.2)-(4.6) we arrive at the following result:

Proposition 1 *The Taylor-based lower bound is determined by*

$$W_n^l(\Pi) \overset{d}{=} \sum_{i=0}^{m}\sum_{j=0}^{n-1}\pi_i\,\alpha_j\,e^{(n-j)\left(\mu_i-\frac{1}{2}\bar{r}_{ij}^2(\Pi)\,\sigma_i^2\right)+\bar{r}_{ij}(\Pi)\sqrt{n-j}\,\sigma_i\Phi^{-1}(U)}\ , \qquad (4.7)$$

where the correlation coefficients $\bar{r}_{ij}(\Pi)$ are given by

$$\bar{r}_{ij}(\Pi) = \qquad (4.8)$$

$$\frac{\sum_{k=1}^{m}\sum_{l=0}^{n-1}\pi_k\,\alpha_l\,(n-\max(j,l))\,\sigma_{ik}\,e^{(n-l)\left[\mu_k-\frac{1}{2}\sigma_k^2\right]}}{\sigma_i\left[(n-j)\sum_{s,k=1}^{m}\sum_{t,l=0}^{n-1}\pi_s\pi_k\alpha_t\alpha_l(n-\max(t,l))\sigma_{sk}e^{(n-t)\left[\mu_s-\frac{1}{2}\sigma_s^2\right]+(n-l)\left[\mu_k-\frac{1}{2}\sigma_k^2\right]}\right]^{1/2}}$$

for $i = 1, \ldots, m$, $j = 0, \ldots, n-1$, and $\bar{r}_{0j}(\Pi) = 0$.

Note that, for $\alpha_i \geq 0$, $i = 0, 1, \ldots, n-1$, it holds that $\bar{r}_{ij}(\Pi) \geq 0$.

4.3 The 'Maximal Variance' lower bound

For the 'Maximal Variance' lower bound approach, the coefficients $\gamma_{ij}(\Pi)$ in (4.2) are chosen according to (2.5). Hence,

$$\gamma_{ij}(\Pi) = \pi_i\,\alpha_j\,e^{E\,[Z_j^i]+\frac{1}{2}\sigma^2_{Z_j^i}}\ . \qquad (4.9)$$

Since $\mathrm{E}[Z_j^i] + \frac{1}{2}\sigma^2_{Z_j^i} = (n-j)\,\mu_i$, see (3.2)-(3.3), we find

$$\Lambda(\Pi) = \sum_{i=1}^{m}\sum_{j=0}^{n-1}\pi_i\,\alpha_j \mathrm{e}^{(n-j)\mu_i}\,Z_j^i\ .$$

As before, from (4.4)-(4.9) we arrive at the following result:

Proposition 2 *The 'Maximal Variance' lower bound is determined by (4.7) with the correlation coefficients $\bar{r}_{ij}(\Pi)$ replaced by*

$$\tilde{r}_{ij}(\Pi) = \frac{\sum_{k=1}^{m}\sum_{l=0}^{n-1} \pi_k\, \alpha_l\, (n - \max(j,l))\, \sigma_{ik}\, e^{(n-l)\mu_k}}{\sigma_i \left[(n-j)\sum_{s,k=1}^{m}\sum_{t,l=0}^{n-1} \pi_s\pi_k\alpha_t\alpha_l(n-\max(t,l))\sigma_{sk}\, e^{(n-t)\mu_s+(n-l)\mu_k}\right]^{1/2}} \tag{4.10}$$

for $i = 1, \ldots, m$, $j = 0, \ldots, n-1$, and $\tilde{r}_{0j}(\Pi) = 0$.

For $\alpha_i \geq 0$, $i = 0, 1, \ldots, n-1$, it holds that $\tilde{r}_{ij}(\Pi) \geq 0$.

4.4 The 'MV-Minimal $CLTE_p$' lower bound

In a similar way to the previous section, applying (2.6), we find that the coefficients $\gamma_{ij}(\Pi)$ in (4.2) are given by

$$\gamma_{ij}(\Pi) = \pi_i\, \alpha_j\, e^{(n-j)\mu_i} \cdot e^{-\frac{1}{2}(\tilde{r}_{ij}(\Pi)\sqrt{n-j}\sigma_i - \Phi^{-1}(p))^2} \,. \tag{4.11}$$

Then we have:

Proposition 3 *The 'MV-Minimal $CLTE_p$' lower bound is determined by (4.7) with the correlation coefficients $\bar{r}_{ij}(\Pi)$ replaced by*

$$r_{ij}(\Pi) = \frac{\sum_{k=1}^{m}\sum_{l=0}^{n-1} \gamma_{kl}(\Pi)\, (n - \max(j,l))\, \sigma_{ik}}{\left(\sqrt{n-j}\,\sigma_i\right)\left[\sum_{s=1}^{m}\sum_{k=1}^{m}\sum_{t=0}^{n-1}\sum_{l=0}^{n-1} \gamma_{st}(\Pi)\, \gamma_{kl}(\Pi)\, (n-\max(t,l))\, \sigma_{sk}\right]^{1/2}}\,, \tag{4.12}$$

for $i = 1, \ldots, m$, $j = 0, \ldots, n-1$, where $\gamma_{ij}(\Pi)$ are given by (4.11); and $r_{0j}(\Pi) = 0$.

For $\alpha_i \geq 0$, $i = 0, 1, \ldots, n-1$, it holds that $r_{ij}(\Pi) \geq 0$.

4.5 The 'T-Minimal $CLTE_p$' lower bound

In this case, the coefficients $\gamma_{ij}(\Pi)$ in (4.2) are given by

$$\gamma_{ij}(\Pi) = \pi_i\, \alpha_j\, e^{(n-j)\mu_i} \cdot e^{-\frac{1}{2}(\bar{r}_{ij}(\Pi)\sqrt{n-j}\sigma_i - \Phi^{-1}(p))^2} \,. \tag{4.13}$$

In comparison with expression (4.11), note that the only difference is in the correlation coefficient. Then we find the following result:

Proposition 4 *The Taylor-based 'Minimal $CLTE_p$' lower bound is determined by (4.7) with the correlation coefficients $r_{ij}(\Pi)$ replaced by (4.12), where $\gamma_{ij}(\Pi)$ are given by (4.13); and $r_{0j}(\Pi) = 0$.*

For $\alpha_i \geq 0$, $i = 0, 1, \ldots, n-1$, it holds that $r_{ij}(\Pi) \geq 0$.

4.6 Numerical illustration

In this section we numerically illustrate the accuracy of the analytic bounds presented in the previous sections.

We consider a portfolio with two risky assets and one risk-free asset. Yearly drifts of the risky assets are $\mu_1 = 0.06$ and $\mu_2 = 0.1$, whereas volatilities are given by $\sigma_1 = 0.1$ and $\sigma_2 = 0.2$, respectively. Moreover, $\sigma_{12} = 0.01$, hence Pearson's correlation between these assets is $r(Y_k^1, Y_k^2) = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2} = 0.5$. The yearly return of the risk-free asset is considered to be 0.03. Every period i, $i = 0, ..., n-1$, an amount of one unit ($\alpha_i = 1$) is invested in the following proportions: 19% in the risk-free asset, 45% in the first risky asset, while the remaining 36% will be invested in the second risky asset. At time $i = n$ the invested amount $\alpha_n = 0$.

The following tables comprise the results of the comparison between the simulated and the corresponding approximated values obtained by means of the different comonotonic approximations of the terminal wealth. The simulated results were obtained using 500,000 random paths.

First we compare quantiles of terminal wealth. For our particular problem, we are interested in low quantiles, corresponding to relatively small outcomes of final wealth. For any $p \in (0,1)$, $Q_p[W_n(\Pi)]$ is the (smallest) wealth that will be reached with a probability of (at least) $1 - p$.

In order to compute the different quantiles, note that the correlation coefficients $r_{ij}(\Pi)$ are all non-negative for any approximation method. Hence, $W_n^l(\Pi)$ is a comonotonic sum for the 'Taylor based', 'Maximal Variance', 'MV-Minimal $CLTE_p$' and 'T-Minimal $CLTE_p$' lower bound approaches. This implies that

$$Q_p[W_n^l(\Pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{(n-j)\left(\mu_i - \frac{1}{2} r_{ij}^2(\Pi)\, \sigma_i^2\right) + r_{ij}(\Pi) \sqrt{n-j}\, \sigma_i \Phi^{-1}(p)} ,$$

where the $r_{ij}(\Pi)$ are chosen according to the appropriate method (Propositions 1-4).

For $n = 20$, the results for the tails of the distribution function of the terminal wealth obtained by the Monte Carlo simulation, as well as the procentual difference between the analytic and the simulated values, are given in Table 1. We make the following notational convention: **MC** denotes the result for the Monte Carlo simulation, and **T**, **MV**, $\mathbf{MCLTE_T}$ and $\mathbf{MCLTE_{MV}}$ denote the results for the 'Taylor-based', 'Maximal-Variance', 'T-Minimal $CLTE_p$' and 'MV-Minimal $CLTE_p$' lower bounds, respectively. We also include the results for the comonotonic upper bound approach (**CUB**) for the sake of comparison. The percentage is calculated as the difference between the approximated and the simulated values, divided by the simulated value.

TABLE 1

Comparing the results obtained with the Monte Carlo simulation, all the lower bound approximations seem to perform reasonably well; some of them are excellent, mainly for high quantiles, but also for low quantiles. In order to discuss the approximations for the left tail of the distribution (low quantiles), we calculate the tails for the case $n = 30$ (Table 2).

TABLE 2

When the number of years n increases, the approximations become worse. In particular, for the left tail, the approximation given by the 'Maximal Variance' lower bound approach becomes clearly worse. Except when p approaches 0, the Taylor-based approximation appears to work reasonably well. However, if we look for a better approximation, the best one is given by the T-Minimal $CLTE_p$ approach. A drawback of the Minimal $CLTE_p$ approaches is that they require an additional calculation as compared to the 'Taylor' or 'Maximal Variance' approaches. Hence, when the number of years is not too high, the approximations given by the 'Taylor-based' and 'Maximal Variance' approaches for the left and right tails, respectively, could be used. For the problem analyzed in this paper, this means that the Taylor lower bound can be a good choice (recall that we are mainly interested in the lower tails of the distribution function), unless p is very small. When the number of periods (years) become very high, the Minimal $CLTE_p$ approaches seem to be an appropriate choice.

To assess the performance of the approximations when the number of assets is high, we also consider a more realistic portfolio consisting of 30 risky assets plus one riskfree asset. In this example, all pairs of risky assets are affected by different degrees of positive correlation. The annualized expected returns range from 0.035 to 0.15, whereas the volatilities range from 0.12 to 0.40. In every period, one unit of capital is evenly distributed among the assets so that the proportions π_i, $i = 0, \ldots, 30$, are all equal. In the last period, nothing is invested ($\alpha_{20} = 0$).

TABLE 3

As can be seen in Table 3, increasing the number of assets does not affect significantly the performance of the approximations. Contrary to the case when the number of periods increases, raising the number of assets does not seem to deteriorate drastically the accuracy of the analytical bounds. Only the precision of the comonotonic upper bound is deeply affected despite the higher complexity of the model.

Finally, since in the following section we also work with an optimization criterion based on the Conditional Left Tail Expectation, we numerically illustrate the approximated values corresponding to the $CLTE_p$ in the three cases described above. Tables 4-6 summarize the results for $n = 20$, $n = 30$ and $m = 30$ $(n = 20)$ respectively.

TABLE 4

TABLE 5

TABLE 6

Clearly, the approximations are much better for the Minimal $CLTE_p$ criteria. In fact, for $n = 20$, the 'Maximal Variance' approximation is not accurate enough, and for $n = 30$ only the $MCLTE$ criteria seem to be adequate.

5 Optimal portfolio selection

In the remainder of the paper, we look for portfolios that maximize the risk measures $Q_{1-p}[W_n(\Pi)]$ and $CLTE_{1-p}[W_n(\Pi)]$, respectively. A natural justification of this choice is given by Yaari's (1987) dual theory of choice under risk. Within this framework, the investor chooses the optimal investment strategy as the one that maximizes the distorted expectation of the final wealth:

$$\Pi^* = \arg\max_{\Pi} \rho_f[W_n(\Pi)] = \arg\max_{\Pi} \int_0^{\infty} f(\Pr(W_n(\Pi) > x))\,\mathrm{d}x \ ,$$

where the distortion function f is a non-decreasing function on the interval $[0, 1]$, $f(0) = 0$ and $f(1) = 1$. It is easy to prove that the risk measures $Q_{1-p}[W_n(\Pi)]$ and $CLTE_{1-p}[W_n(\Pi)]$ correspond to distorted expectations $\rho_f[W_n(\pi)]$ for appropriate choices of the distortion funtion f. For more details, we refer to Dhaene *et al.*(2006).

5.1 Maximizing the Value at Risk

For a given probability level p and a given investment strategy Π, let the p-target capital be defined as the $(1-p)$-th order quantile of terminal wealth. The problem of the investor consists in looking for the optimal target capital K_p^* obtained as the maximizer of the quantile, whose maximization is performed over all buy-and-hold strategies Π:

$$K_p^* = \max_{\Pi} Q_{1-p}[W_n(\Pi)] \ .$$

As it is impossible to determine $Q_{1-p}[W_n(\Pi)]$ analytically, we first try to solve the optimization problem for the comonotonic approximations $W_n^c(\Pi)$ of $W_n(\Pi)$:

$$K_p^{c*} = \max_{\Pi} Q_{1-p}[W_n^c(\Pi)] \ .$$

Using the expression (4.1) for $W_n^c(\Pi)$, it is clear that

$$Q_{1-p}[W_n^c(\Pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{(n-j)(\mu_i - \frac{1}{2}\sigma_i^2) + \sqrt{n-j}\sigma_i \Phi^{-1}(1-p)} \ . \tag{5.1}$$

Use of the comonotonic upper bound approximations is not appropriate in our buy-and-hold context. Firstly, as we have illustrated numerically, the comonotonic upper bound does not give an accurate approximation to terminal wealth. Secondly, as shown in (5.1), $Q_p[W_n^c(\Pi)]$ is a linear combination of the proportions π_i, $i = 0, 1, \ldots, m$. Therefore, the solution to the optimization problem will be trivial: the investor invests all her/his capital in only one asset. It is obvious that such an investment strategy will be far from optimal in general.

Therefore, we address our attention to solving the approximate problem

$$K_p^{l*} = \max_{\Pi} Q_{1-p}[W_n^l(\Pi)] \ , \tag{5.2}$$

where

$$Q_{1-p}[W_n^l(\Pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{(n-j)\left(\mu_i - \frac{1}{2} r_{ij}^2(\Pi)\, \sigma_i^2\right) + r_{ij}(\Pi)\sqrt{n-j}\, \sigma_i \Phi^{-1}(1-p)} \ ,$$

and the $r_{ij}(\Pi)$ are chosen according to the appropriate method (Propositions 1-4).

Let us illustrate numerically the results for the approximated optimal values obtained from (5.2) using the examples given in Section 4.6. In order to avoid corner solutions (all the available money is allocated in the risk-free asset or in the risky assets), we impose a (reasonable) constraint consisting in a minimal expected return. In particular, we assume that the portfolio has an expected return not lower than 6%, and we look for the portfolio maximizing Q_{1-p} for $p = 0.95$ (and so $1 - p = 0.05$) satisfying this constraint and such that $\pi_i \geq 0$, for $i = 0, \ldots m$.

For $n = 20$ we obtain the results given in Table 7.

TABLE 7

For $p = 0.9$ (and so $1 - p = 0.1$), the results are given in Table 8.

TABLE 8

Note that the results are relatively close to each other for all the lower bound approximations.

For $n = 30$, we restrict our attention to the Taylor-based and the minimal $CLTE_p$ lower bound approaches (the approximation given for the 0.05 quantile by the 'Maximal Variance' approach was not accurate enough). For $p = 0.95$ ($1 - p = 0.05$), the results are given in Table 9.

TABLE 9

For $p = 0.9$ ($1 - p = 0.1$), the results are given in Table 10.

TABLE 10

5.2 Maximizing Conditional Left Tail Expectations

Now let us calculate the optimal investment strategy by maximizing the CLTE for a given probability level p,

$$\Pi^* = \arg\max_{\Pi} CLTE_{1-p}[W_n(\Pi)] \ . \tag{5.3}$$

This optimization problem describes decisions of risk-averse investors. Recall that the conditional left tail expectation has the following nice property:

$$CLTE_{1-p}[W_n^c(\Pi)] \leq CLTE_{1-p}[W_n(\Pi)] \leq CLTE_{1-p}[W_n^l(\Pi)] \ ,$$

for every $p \in (0, 1)$.

Once again, we solve the optimization problem for the lower bound approximations of $W_n(\Pi)$, since the upper comonotonic bound exhibits the same problems as those described in the previous subsection. Indeed, from (4.1), it is clear that

$$CLTE_p[W_n^c(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{\mu_i (n-j)} \, \frac{1 - \Phi(\sqrt{n-j}\sigma_i - \Phi^{-1}(p))}{p} \, .$$

Therefore, we solve numerically the approximate problem

$$\arg\max_{\Pi} CLTE_{1-p}[W_n^l(\Pi)] \, . \tag{5.4}$$

Since $W_n^l(\Pi)$ is a comonotonic sum for the 'Taylor-based', 'Maximal Variance', 'MV-Minimal $CLTE_p$' and 'Taylor-Minimal $CLTE_p$' lower bound approaches, we have

$$CLTE_p[W_n^l(\Pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \, \alpha_j \, e^{\mu_i (n-j)} \, \frac{1 - \Phi(\sqrt{n-j}\, r_{ij}(\Pi) \, \sigma_i - \Phi^{-1}(p))}{p}$$

with the appropriate $r_{ij}(\Pi)$ for each lower bound method.

Next, we numerically illustrate the approximated optimal portfolios obtained from (5.4) for the same problem discussed in the previous section.

For $n = 20$ and $1 - p = 0.05$, the optimal portfolios for the different bounds are given in Table 11.

TABLE 11

For $n = 20$ and $1 - p = 0.1$, the optimal portfolios are given in Table 12.

TABLE 12

For $n = 30$, the results for the Taylor-based and the minimal $CLTE_p$ lower bound approaches are given in Tables 13 (for $1 - p = 0.05$) and 14 (for $1 - p = 0.1$).

TABLE 13

TABLE 14

It is clear from the numerical results that in both cases ($n = 20$ and $n = 30$) the best approximation for the optimal target capital K_p^* is given by the T-Minimal $CLTE_p$ lower bound approximation.

6 Conclusions

In Dhaene *et al.* (2005), the 'Maximal Variance' lower bound to the sum of log-normal dependent variables was applied in the search for optimal portfolios within the class of constant mix strategies. In this paper, we use a similar approach for the analysis of buy-and-hold strategies, obtaining in this way analytic approximations of the df of terminal wealth. An advantage of buy-and-hold strategies compared with constant mix strategies is that much lower transactions costs are involved. However, the comonotonic bounds used in obtaining an analytic approximation of the df of terminal wealth seem to be more sensitive to the number of periods and assets in a buy-and-hold strategy than in a constant mix strategy. Therefore, in this paper we calculate not only the comonotonic lower bounds for uniform values of the conditioning variable Λ (the so-called 'Taylor-based' (Dhaene *et al.* (2002b)) and 'Maximal Variance' (Vanduffel *et al.* (2005)) lower bound approaches), but also the bounds obtained for specific choices of Λ approximating the tails of the sum of log-normal variables. These new approximations were introduced in Vanduffel *et al.* (2008) by using a nice property of the Conditional (Left) Tail Expectation. We call such an approximation the 'MV-Minimal $CLTE_p$ lower bound'. Since in our context the Taylor-based approach works better than the 'Maximal Variance' one, we introduce a different version of this comonotonic lower bound, which we call the 'T-Minimal $CLTE_p$ lower bound', and which has proved to be the best analytic approximation for our particular problem. Finally, we compare the performance of the different approximations in the problem of finding the buy-and-hold strategy that maximizes the target capital.

References

[1] R. Cesari and D. Cremonini, Benchmarking, Portfolio Insurance and Technical Analysis: A Monte Carlo Comparison of Dynamic Strategies of Asset Allocation, *Journal of Economic Dynamics and Control* **27** (2003) 987-1011.

[2] J. Dhaene, M. Denuit, M. Goovaerts, R. Kaas and D. Vyncke, The concept of comonotonicity in actuarial science and finance: Theory, *Insurance: Mathematics & Economics* **31** (2002a) 3-33.

[3] J. Dhaene, M. Denuit, M. Goovaerts, R. Kaas and D. Vyncke, The concept of comonotonicity in actuarial science and finance: Applications, *Insurance: Mathematics & Economics* **31** (2002b) 133-161.

[4] J. Dhaene, S. Vanduffel, M. Goovaerts, R. Kaas and D. Vyncke, Comonotonic approximations for optimal portfolio selection problems, *Journal of Risk and Insurance* **72** (2005) 253-301.

[5] J. Dhaene, S. Vanduffel, Q.H. Tang, M. Goovaerts, R. Kaas and D. Vyncke, Risk measures and comonotonicity: a review, *Stochastic Models* **22** (2006) 573-606.

[6] R. Kaas, J. Dhaene and M. Goovaerts, Upper and lower bounds for sums of random variables, *Insurance: Mathematics and Mathematics* **27** (2000) 151-168.

[7] A.J. McNeil, R. Frey and P. Embrechts, P., *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton Series in Finance, Princeton University Press, 2005.

[8] R. Merton, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *Journal of Economic Theory* **3** (1971) 373-413.

[9] S. Vanduffel, T. Hoedemakers and J. Dhaene, Comparing approximations for risk measures of sums of non-independent lognormal random variables, *North American Actuarial Journal* **9** (2005) 71-82.

[10] S. Vanduffel, X. Chen, X., J. Dhaene, M. Goovaerts, L. Henrard and R. Kaas. Optimal approximations for risk measures of sums of lognormals based on conditional expectations *Journal of Computational and Applied Mathematics* **221** (2008) 202-218.

[11] J. Von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton University Press, Princeton, 1947.

[12] M. Yaari, The Dual Theory of Choice Under Risk, *Econometrica* **55** (1987) 95-115.

p	**MC**	**T**	**MV**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	**CUB**
0.01	21.0088	1.51%	2.44%	0.63%	0.78%	-18.44%
0.025	23.0171	1.03%	1.73%	0.57%	0.68%	-16.90%
0.05	25.0385	0.64%	1.14%	0.46%	0.54%	-15.40%
0.1	27.7600	0.28%	0.57%	0.33%	0.38%	-13.41%
0.95	86.4381	-0.11%	0.04%	-0.07%	-0.09%	10.93%
0.975	101.7844	-0.55%	-0.17%	-0.05%	-0.07%	13.53%
0.99	124.4009	-1.25%	-0.56%	0.03%	0.02%	16.33%

Table 1: Procentual difference between simulated and approximated values of $Q_p[W_{20}(\Pi)]$.

p	**MC**	**T**	**MV**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	**CUB**
0.01	38.2135	3.10%	5.21%	1.64%	2.14%	-22.59%
0.025	42.9505	2.20%	3.82%	1.41%	1.76%	-20.99%
0.05	48.0219	1.22%	2.43%	0.92%	1.17%	-19.55%
0.1	55.0187	0.51%	1.27%	0.63%	0.63%	-17.32%
0.95	267.6211	-0.01%	0.15%	-0.06%	-0.08%	10.57%
0.975	337.2806	-0.48%	0.06%	0.09%	0.07%	13.12%
0.99	449.9011	-1.81%	-0.72%	-0.20%	-0.22%	15.09%

Table 2: Procentual difference between simulated and approximated values of $Q_p[W_{30}(\Pi)]$.

p	**MC**	**T**	**MV**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	**CUB**
0.01	14.2801	1.58%	3.18%	0.88%	0.97%	-52.33%
0.025	16.6095	1.31%	2.53%	0.98%	1.05%	-49.31%
0.05	19.0727	1.06%	1.95%	0.94%	1.00%	-46.30%
0.1	22.5801	0.81%	1.35%	0.84%	0.89%	-42.23%
0.95	136.2118	-0.26%	-0.17%	-0.36%	-0.37%	26.16%
0.975	174.3170	-0.61%	-0.16%	-0.28%	-0.30%	38.88%
0.99	237.0702	-1.87%	-0.89%	-0.75%	-0.77%	54.33%

Table 3: Procentual difference between simulated and approximated values of $Q_p[W_{20}(\Pi)]$ when $m = 30$.

p	MC	T	MV	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	CUB
0.01	19.4627	2.14%	3.27%	0.54%	0.66%	-19.39%
0.025	21.0590	1.55%	2.47%	0.41%	0.48%	-18.27%
0.05	22.5796	1.15%	1.90%	0.36%	0.41%	-17.11%
0.1	24.5304	0.76%	1.31%	0.31%	0.33%	-15.61%

Table 4: Procentual difference between simulated and approximated values for $CLTE_p[W_{20}(\Pi)]$.

p	MC	T	MV	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	CUB
0.01	34.6499	4.28%	6.80%	1.40%	1.82%	-23.33%
0.025	38.3641	3.19%	5.27%	1.05%	1.32%	-22.29%
0.05	42.0104	2.34%	4.05%	0.80%	0.97%	-21.17%
0.1	46.8531	1.50%	2.79%	0.58%	0.67%	-19.61%

Table 5: Procentual difference between simulated and approximated values for $CLTE_p[W_{30}(\Pi)]$.

p	MC	T	MV	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	CUB
0.01	12.5704	2.34%	4.28%	1.10%	1.17%	-54.12%
0.025	14.3601	1.75%	3.32%	0.95%	0.99%	-51.85%
0.05	16.1374	1.43%	2.71%	0.93%	0.96%	-49.53%
0.1	18.5524	1.16%	2.11%	0.90%	0.92%	-46.45%

Table 6: Procentual difference between simulated and approximated values for $CLTE_p[W_{20}(\Pi)]$ when $m = 30$.

$\geq 6\%$	**T**	**MV**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	**CUB**
π_0	12.48%	12.14%	11.97%	11.82%	40.00%
π_1	55.04%	55.72%	56.06%	56.36%	0.00%
π_2	32.48%	32.14%	31.97%	31.82%	60.00%
K^*	25.1802	25.3254	25.145	25.1703	21.3226

Table 7: Optimal portfolio weights in the case of maximizing $Q_{0.05}[W_{20}(\Pi)]$.

$\geq 6\%$	**T**	**MV**	**MCLTE$_{\mathbf{T}}$**	**MCLTE$_{\mathbf{MV}}$**	**CUB**
π_0	0.00%	0.00%	0.00%	0.00%	40.00%
π_1	66.25%	65.90%	66.28%	66.32%	0.00%
π_2	33.75%	34.10%	33.72%	33.68%	60.00%
K^*	27.9625	28.0683	27.9847	28.0072	24.0377

Table 8: Optimal portfolio weights in the case of maximizing $Q_{0.1}[W_{20}(\Pi)]$.

$\geq 6\%$	**T**	**MCLTE$_{\mathbf{T}}$**	**MCLTE$_{\mathbf{MV}}$**
π_0	11.13%	10.43%	9.92%
π_1	57.74%	59.14%	60.16%
π_2	31.13%	30.43%	29.92%
K^*	48.8106	48.7112	48.8998

Table 9: Optimal portfolio weights in the case of maximizing $Q_{0.05}[W_{30}(\Pi)]$.

$\geq 6\%$	**T**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$
π_0	0.00%	0.00%	0.00%
π_1	58.85%	59.40%	60.30%
π_2	41.15%	40.60%	39.70%
K^*	56.7152	56.806	56.9404

Table 10: Optimal portfolio weights in the case of maximizing $Q_{0.1}[W_{30}(\Pi)]$.

$\geq 6\%$	**T**	**MV**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$	**CUB**
π_0	15.98%	15.08%	15.23%	15.03%	40.00%
π_1	48.05%	49.85%	49.55%	49.94%	0.00%
π_2	35.97%	35.07%	35.22%	35.03%	60.00%
K^*	22.714	22.8947	22.5359	22.5485	19.1586

Table 11: Optimal portfolio weights in the case of maximizing $CLTE_{0.05}[W_{20}(\Pi)]$.

$\geq 6\%$	**T**	**MV**	**MCLTE$_\mathbf{T}$**	**MCLTE$_\mathbf{MV}$**	**CUB**
π_0	13.68%	13.17%	12.86%	12.76%	40.00%
π_1	52.64%	53.67%	54.28%	54.48%	0.00%
π_2	33.68%	33.16%	32.86%	32.76%	60.00%
K^*	24.6638	24.8168	24.5598	24.5679	20.9498

Table 12: Optimal portfolio weights in the case of maximizing $CLTE_{0.1}[W_{20}(\Pi)]$.

$\geq 6\%$	**T**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$
π_0	14.35%	13.19%	12.54%
π_1	51.30%	53.61%	54.91%
π_2	34.35%	33.19%	32.54%
K^*	42.8765	42.2428	42.3493

Table 13: Optimal portfolio weights in the case of maximizing $CLTE_{0.05}[W_{30}(\Pi)]$.

$\geq 6\%$	**T**	$\mathbf{MCLTE_T}$	$\mathbf{MCLTE_{MV}}$
π_0	12.24%	11.01%	10.61%
π_1	55.52%	57.98%	58.78%
π_2	32.24%	31.01%	30.61%
K^*	47.6574	47.2594	47.3327

Table 14: Optimal portfolio weights in the case of maximizing $CLTE_{0.1}[W_{30}(\Pi)]$.