A Recursive Approach to Mortality-linked Derivative Pricing

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Abstract

In this paper, we develop a recursive method to derive an exact numerical and nearly analytical representation of the Laplace transform of the transition density function with respect to the time variable for time-homogeneous diffusion processes. We further apply this recursion algorithm to the pricing of mortality-linked derivatives. Given an arbitrary stochastic future lifetime $\mathbb{T}$, the probability distribution function of the present value of a cash flow depending on $\mathbb{T}$ can be approximated by a mixture of exponentials, based on Jacobi polynomial expansions. In case of mortality-linked derivative pricing, the required Laplace inversion can be avoided by introducing this mixture of exponentials as an approximation of the distribution of the survival time $\mathbb{T}$ in the recursion scheme. This approximation significantly improves the efficiency of the algorithm.

Keywords: mortality-linked derivative, diffusion process, transition density function, Feynman-Kac integral.

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1 Introduction

Significant increase of the human life expectancy has been observed in most developed countries over the past few decades. Unexpected improvement of human mortality rates naturally leads to an underestimation of the prices and provisions determined for mortality related products for insurance companies and public pension system. Therefore, financial instruments, such as mortality swap, longevity bonds, or other related option-style derivatives, need to be developed to hedge the risk exposure of insurance companies.

Under the assumption of no-arbitrage and market completeness, a unique risk-neutral price can be found using an equivalent martingale measure (EMM). However, it is argued that the EMM is not unique once the derivatives are associated with mortality or longevity since the market is then incomplete, that is, there may exist several EMMs to determine the fair risk premium under the incomplete markets setting. Different methods were proposed in the literature to handle this problem, although there is still no consensus on how to price these products in a fair way. The Wang transform (Wang (2000)) was introduced with the parameter of market price of risk λ for the pricing of mortality-linked derivatives, see Lin and Cox (2005) and Denuit et al. (2007). The Wang transform was criticized by Pelsser (2008), who states that the Wang transform can not be treated as a universal financial measure for financial and insurance pricing, see also Goovaerts and Laeven (2008) and Lauschagne and Offwood (2010). Other methods for the pricing and hedging in incomplete markets are super-replication, Follmer-Schweizer-Sondermann approach, indifference pricing based on insurer's utility function, etc., see e.g. Embrechts (2000) and Møller (2000) for an overview. Given the difficulty to fairly price the mortality-linked derivatives in incomplete market, it would still be interesting to obtain physical and "risk-neutral" density functions as we can simply consider the terminal payoff function as series of future cash flows, which leads to an easy calculation of the mortality-linked insurance contracts.

One of the prerequisites for the pricing of mortality-linked derivatives is to capture the dynamics of future stochastic mortality. Different models were introduced in the literature for the stochastic modeling of mortality rates: the one-factor Lee-Carter model (Lee and Carter (1992, 2000)), the refined Lee-Carter model in Renshaw and Haberman (2003), the two-factor model by Cairns et al. (2006), etc. For a quantitative comparison of these models, we refer to Cairns et al. (2009). Diffusion processes were introduced to model the force of mortality in Milevsky and Promislow (2001) and Dahl (2004). It turns out that one can hardly obtain a closed-form valuation formula, and therefore simulation based methods are usually used for the calculations. Affine jump diffusion processes based on the results of Duffie et al. (2000) were used in Biffis (2005) to describe the dynamics of the force of mortality for the sake of its tractability. The underlying difficulties for the diffusion approach is that the long-term transition probabilities are usually not available in closed-form and too complicated to obtain accurate approximations, whereas the typical life insurance contracts are usually long-term. In this paper, we work under an assumed equivalent martingale measure $\mathbb{Q}$. Furthermore, we assume that the underlying financial assets follow time-homogeneous diffusion processes, whereas the uncertainties for the future mortality can be represented by the stochastic future life time which is assumed to be independent of the dynamics of the underlying financial assets. Goovaerts et al. (2010) presented a recursive approach for the nearly analytical approximation of Laplace transformed transition densities, using a Feynman-Kac integral formalism and the delta-perturbation theory. This recursive approach generates very accurate approximations for the transition densities, not only for the short-time horizon, but also for long-time perspective. Having accurate approximations for transition densities over the long-term available in nearly analytical form is particularly useful to price these products in a probabilistic approach.

Theoretically, we usually calculate the transition densities by numerically solving the forward Kolmogorov equation, such as in the finite difference method (FDM). Nevertheless, the major complication with this approach is the fact that the initial condition of this forward equation is a Dirac's delta function which could result in unreliable and unrobust numerical outcomes, whereas in the Feynman formalism, the delta function is already self-contained within the Feynman-Kac integral framework. For an overview of the Feynman-Kac integral approach to the approximation of the transition densities, we refer to Goovaerts et al. (2004), where closed-form approximations were obtained based on the concepts of convex ordering and comonotonicity which have been used extensively in the ordering of risk theory in actuarial science.

Apart from the Feynman-Kac integral approach, Jensen and Poulsen (2002) summarized some other methods used in the approximation of the transition densities in the literature: Euler, binomial, simulation, FDM (finite difference method), and Hermite expansion as proposed in Aït-Sahalia (1999). They concluded that the method based on the Hermite expansion is preferred to the other methods in terms of the trade-offs between speed and accuracy. However, compared with the Feynman-Kac integral approach, the Hermite expansion approximation may deteriorate with the increase of the time horizon, that is, it may only be applicable for modeling short term dynamics.

In this paper we apply the recursion algorithm presented in Goovaerts et al. (2010) to the pricing of mortality-linked derivatives. By incorporating the approximation of the probability distribution of the future lifetime $\mathbb{T}$ into the recursive scheme by using certain mixtures of exponentials, the risk-neutral pricing kernel can be derived without performing an additional real Laplace inversion. This approach significantly improves the efficiency of the recursive scheme.

The outline of the paper is as follows. Section 2 gives a brief introduction to transition densities and the Feynman-Kac integral formulation of the recursion algorithm. The idea of approximating probability distribution by certain mixtures of exponentials using Jacobian polynomials is presented in Section 3. Section 4 concentrates on mortality-linked derivative pricing using the recursion algorithm. Some numerical examples are studied in section 5.

2 The recursion algorithm for Laplace transformed transition densities

In this section, we construct a nearly analytical recursion algorithm for the calculation of the Laplace transform of transition density. For more details concerning this recursion scheme, its applications and extensions, we refer to Goovaerts et al. (2010). We start this section with some results on the connections between the Feynman-Kac integral and the diffusion processes.

2.1 Diffusion processes and transition density

A general diffusion process is conventionally defined by the following stochastic differential equation:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \qquad X_0 = x_0, \tag{1}$$

where X_t denotes the variable of interest, W_t is a standard Brownian motion and $\mu(X_t, t)$ and $\sigma(X_t, t)$ are the drift and diffusion terms satisfying certain conditions, namely, smoothness of the coefficients, nondegeneracy of the diffusion and boundary behavior (Aït-Sahalia (1999)). In this contribution, we consider the time-homogeneous diffusion process

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0 \tag{2}$$

where "time-homogeneous" refers to the fact that the diffusion process is independent of the time evolution. Here we assume that the stochastic differential equation is defined by Itô's left point discretization criteria.
The diffusion process can be best understood by its transition density function defined as

$$p(x_t, t|x_0, 0) = \frac{d}{dx_t} Prob[X_t \leq x|X_0 = x_0]. \tag{3}$$

The transition density function also satisfies the semigroup property, sometimes called Chapman-Kolmogorov property,

$$p(x_t, t|x_0, 0) = \int p(x_t, t|x_s, s)p(x_s, s|x_0, 0)dx_s.$$

Given a time-homogeneous diffusion process of the form (2), we can always transform (2) to its corresponding unit diffusion process using Lamperti transform

$$Y_t := \psi(X_t) = \int^{X_t} dr/\sigma(r),$$

where $\psi(\cdot)$ is non-decreasing and invertible, and the unit diffusion process reads

$$dY_t = \tilde{\mu}(Y_t)dt + dW_t \tag{4}$$

where

$$\tilde{\mu}(y) = \frac{\mu(\psi^{-1}(y))}{\sigma(\psi^{-1}(y))} - \frac{1}{2}\frac{\partial \sigma}{\partial x}(\psi^{-1}(y)).$$

The transition density function with underlying unit diffusion process (4) satisfies the Fokker-Planck equation

$$\frac{1}{2}\frac{\partial^2}{\partial y_t^2}(p(y_t,t|y_0,0)) - \frac{1}{2}\frac{\partial}{\partial y_t}(\tilde{\mu}(y_t)p(y_t,t|y_0,0)) = \frac{\partial}{\partial t}p(y_t,t|y_0,0) \tag{5}$$

with initial condition $p(y_t,0|y_0,0) = \delta(y_t - y_0)$.

One can prove that any unit diffusion process has the Feynman-Kac integral representation for its transition density function (see Goovaerts et al. (2004)).

Theorem 2.1 (Feynman-Kac integral representation of the transition density)**.** *A stochastic differential equation with unit diffusion*

$$dY_t = \tilde{\mu}(Y_t)dt + dW_t \tag{6}$$

can always be expressed by the Feynman-Kac integral formalism as

$$p(y_t,t|y_0,0) = e^{\int_{y_0}^{y_t}\tilde{\mu}(y)dy}\mathbb{E}^{y_0,y_t,t}\left[e^{-\int_0^t V(y)d\tau}\right]. \tag{7}$$

where $V(y) = \frac{1}{2}\left(\tilde{\mu}^2(y) + \frac{\partial\tilde{\mu}(y)}{\partial y}\right)$ *and* $\mathbb{E}^{y_0,y_t,t}[\cdot]$ *is a conditional expectation given the initial and end points states. This notation can be rewritten as* $\mathbb{E}_{(0,t)}^{(y_0,y_t)}$ *which is the notation related to Brownian bridge. It can also be written in Feynman notation as*

$$p(y_t,t|y_0,0) = e^{\int_{y_0}^{y_t}\tilde{\mu}(y)dy}\int_{(y_0,0)}^{(y_t,t)}\mathcal{D}y(\tau)e^{-\frac{1}{2}\int_0^t\left(\frac{dy}{d\tau}\right)^2 d\tau - \int_0^t V(y)d\tau} \tag{8}$$

where $\mathcal{D}y(\tau)$ *is the probability measure representing the limit sum between* $(y_0,0)$ *and* (y_t,t)*. A list of the function* $V(x)$ *for some popular diffusion models is provided in Appendix.*

The probability density function $p_X(x_t,t|x_0,0)$ can be easily found by applying the Jacobian formula

$$\begin{aligned} p_X(x_t,t|x_0,0) &= \frac{1}{\sigma(x_t)}p_Y(\psi(x_t),t|\psi(x_0)) \\ &= \frac{1}{\sigma(x_t)}e^{\int_{\psi(x_0)}^{\psi(x_t)}\tilde{\mu}(y)dy}\mathbb{E}^{\psi(x_0),\psi(x_t),t}\left[e^{-\int_0^t V(y)d\tau}\right]. \end{aligned}$$

2.2 Feynman-Kac integral decomposition and recursion algorithm

The first proof of the following recursive formula between a Feynman-Kac integral and the one with an additional δ-function perturbation was obtained in Goovaerts (1985), see also Goovaerts and Broeckx (1985) and Grosche (1990). The idea behind the recursion algorithm is that we decompose the piecewise continuous potential function $V(x)$ into two parts and derive an exact recursion scheme for calculation of the Laplace transform (with respect to t) in case the analytical expression of one of the decomposed potentials is known.

Consider the Feynman-Kac integral for a sum of two functions (i.e., potentials) $V_1(x)$ and $V_2(x)$,

$$k_{1,2}(x_t, t|x_0, 0) = \mathbb{E}^{x_0,x_t,t}\left[e^{-\int_0^t V_1(x)d\tau - \int_0^t V_2(x)d\tau}\right]. \tag{9}$$

Suppose that there exists a closed-form expression for

$$k_1(x_t, t|x_0, 0) = \mathbb{E}^{x_0,x_t,t}\left[e^{-\int_0^t V_1(x)d\tau}\right].$$

One then expands $\exp\left(-\int_0^t V_2(X_\tau)d\tau\right)$. This expansion gives a so-called *Born series*:

$$\begin{aligned}\exp\left(-\int_0^t V_2(X_\tau)d\tau\right) &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\left[\int_0^t V_2(X_\tau)d\tau\right]^n\\ &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\int_0^t d\tau_1\cdots\int_0^t d\tau_n V_2(X_{\tau_1})\cdots V_2(X_{\tau_n}).\end{aligned}$$

Due to symmetry, the right hand side of the expression above reduces to

$$\sum_{n=0}^{+\infty}(-1)^n\int_0^t d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots\int_0^{\tau_2}d\tau_1 V_2(X_{\tau_n})\cdots V_2(X_{\tau_1}).$$

Hence, substituting the above expression in (9) and using the Chapman-Kolmogorov property one obtains

$$\begin{aligned}k_{1,2}(x_t, t|x_0, 0) = &\sum_{n=0}^{+\infty}(-1)^n\int_0^t d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots\int_0^{\tau_2}d\tau_1\\ &\times\int_{-\infty}^{+\infty}dx_n\cdots\int_{-\infty}^{+\infty}dx_1 k_1(x_t, t|x_n, \tau_n)V_2(x_n)\cdots V_2(x_1)k_1(x_1, \tau_1|x_0, 0).\end{aligned}$$

Taking the Laplace transform of the transition probability with respect to s and using as a notation

$$\rho_{1,2}^s(x_t, x_0) := \int_0^{+\infty}e^{-st}k_{1,2}(x_t, t|x_0, 0)dt, \qquad \text{and} \qquad \rho_1^s(x_t, x_0) := \int_0^{+\infty}e^{-st}k_1(x_t, t|x_0, 0)dt,$$

one gets

$$\rho_{1,2}^s(x_t, x_0) = \sum_{n=0}^{+\infty}(-1)^n\int_{-\infty}^{+\infty}dx_n\cdots\int_{-\infty}^{+\infty}dx_1\rho_1^s(x_t, x_n)V_2(x_n)\cdots\rho_1^s(x_1, x_0)V_2(x_1). \tag{10}$$

For an arbitrary potential $V(x)$, we introduce the following integral representation:

$$\begin{aligned}V(x) &= \int_{-\infty}^{+\infty}V(a)\delta(x-a)da\\ &= \lim_{\max(a_{j+1}-a_j)\to 0}\sum_{j=0}^{+\infty}(a_{j+1}-a_j)V(a_j)\delta(x-a_j),\end{aligned}$$

here we define the partition $-\infty = a_0 < a_1 < \cdots < a_\infty = +\infty$ and denote by $\delta(x)$ Dirac's delta function. We start with a linear combination of $m+1$ delta function potentials:

$$V^{(m)}(x) = \sum_{j=0}^{m}(a_{j+1}-a_j)V(a_j)\delta(x-a_j).$$

As we remarked before, the Feynman formalism allows for the calculation of combinations of delta function potentials. In the sequel, we assume $V(x) \geq 0$ to guarantee that the recursion algorithm will be convergent. In case of negative potential $V(x)$, we could first make the potential for the recursion positive and then put an extra factor outside the Feynman-Kac integral. For details, we refer to Remark 4 in Goovaerts et al. (2010).

Suppose we know $\rho^{s,(m)}(x, x_0)$ for some $V_1(x) = V^{(m)}(x)$, adding an additional δ function $V_2(x) = (a_{m+2} - a_{m+1})V(a_{m+1})\delta(x - a_{m+1})$ and recalling (10), we obtain

$$\begin{aligned}
\rho^{s,(m+1)}(x, x_0) &= \sum_{n=0}^{+\infty}(-1)^n \int_{-\infty}^{+\infty} dx_n \cdots \int_{-\infty}^{+\infty} dx_1 \\
&\quad \times \rho^{s,(m)}(x_t, x_n)(a_{m+2} - a_{m+1})V(a_{m+1})\delta(x_n - a_{m+1}) \\
&\quad \cdots (a_{m+2} - a_{m+1})V(a_{m+1})\delta(x_1 - a_{m+1})\rho^{s,(m)}(x_1, x_0) \\
&= \sum_{n=0}^{+\infty}(-1)^n(a_{m+2} - a_{m+1})^n V^n(a_{m+1}) \\
&\quad \times \rho^{s,(m)}(x_t, a_{m+1}) \left(\rho^{s,(m)}(a_{m+1}, a_{m+1})\right)^{n-1} \rho^{s,(m)}(a_{m+1}, x_0) \\
&= \rho^{s,(m)}(x_t, x_0) - \frac{\rho^{s,(m)}(x_t, a_{m+1})\rho^{s,(m)}(a_{m+1}, x_0)V(a_{m+1})(a_{m+2} - a_{m+1})}{1 + \rho^{s,(m)}(a_{m+1}, a_{m+1})V(a_{m+1})(a_{m+2} - a_{m+1})},
\end{aligned} \tag{11}$$

where the last equality holds by virtue of the geometric series.

The recursion algorithm can be summarized as follows

The Recursive Scheme of the Laplace Transformed Transition Densities

Input: potential $V(x)$, constant a_0, Laplace transform parameter s, terminal time t, number of recursions N.

Initial condition:

(1) $\rho^{s,(0)}(x_t, x_0) = \frac{1}{\sqrt{2s}}e^{-\sqrt{2s}|x_t - x_0|}$, for movements in the entire plane.

(2) $\rho^{s,(0)}(x_t, x_0) = \frac{e^{-\sqrt{2s}|x_t - x_0|} - e^{-\sqrt{2s}|x_t + x_0|}}{\sqrt{2s}}$, for movements in positive quarter plane.

Recursion: from $i = 0$ to N:

$$\rho^{s,(m+1)}(x_t, x_0) = \rho^{s,(m)}(x_t, x_0) - \frac{\rho^{s,(m)}(x_t, a_{m+1})\rho^{s,(m)}(a_{m+1}, x_0)V(a_{m+1})(a_{m+2} - a_{m+1})}{1 + \rho^{s,(m)}(a_{m+1}, a_{m+1})V(a_{m+1})(a_{m+2} - a_{m+1})}.$$

where $\rho^s(x_t, x_0) := \int_0^{+\infty} e^{-st}\mathbb{E}^{x_0,x_t,t}\left[e^{-\int_0^t V(x)d\tau}\right] dt.$

3 Approximation of probability distribution with mixtures

The approximation of probability distribution functions is often needed in actuarial science, especially in risk theory and for calculations related to stochastic life annuities. There are mainly two approaches proposed in the literature for such kind of approximations. In Dufresne (2007a) and (2007b), the author suggests using Jacobi polynomials and logbeta distributions to generate a convergent series of exponentials for the approximation of the square root of probability density function

$$\sqrt{f_{\mathbb{T}}(t)} \approx \sum_j a_j e^{-\lambda_j t}, \quad t \geq 0, \lambda_j > 0, \quad j = 1, \cdots, n, \quad 1 \leq n < \infty.$$

The same techniques can be used for the approximation of the decumulative distribution function $\bar{F}(t)$.

Besides, an Erlang mixture is also often used in the approximation of the probability density function (pdf). It is a mixture of the form

$$f(y) = \sum_{j=1}^{\infty} q_j \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!} = \sum_{j=1}^{\infty} q_j \tau_j(y) \quad y > 0, \quad \beta > 0, \quad j = 1, 2, \cdots$$

where $\tau_j(y) = \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!}$ is the Erlang-j (E_j) random variable and $\{q_1, q_2, \cdots\}$ is a discrete probability measure. Notice that an Erlang mixture is not a combination of exponential functions. In order to simplify the calculations by implementing the recursive scheme, therefore, we concentrate on the methods of mixtures of exponentials.

To give a simple example, we consider a life following a Makeham mortality law

$$l_{x+t} = ks^{x+t} g^{c^{x+t}}.$$

Here l_x denotes the number of persons attaining age x in the chosen group under consideration, while k, s, g, c are the constants in the Makeham law. The probability that a life aged x will die within t years is

$${}_t p_x = \frac{l_{x+t}}{l_x} = \frac{ks^{x+t} g^{c^{x+t}}}{ks^x g^{c^x}} = s^t g^{c^x(c^t-1)}.$$

After some simple algebra and a Taylor expansion, we can express this probability density as

$${}_t p_x = e^{t \ln s + c^x (e^{t \ln c} - 1) \ln g} = e^{t \ln s + c^x (t \ln c + \frac{t^2 (\ln c)^2}{2}) \ln g} \approx e^{t(\ln s + c^x \ln(c+g))},$$

which is indeed a mixture of exponential functions.

3.1 Jacobi polynomials

The approximation is based on the 'shifted' Jacobi polynomials which belong to the class of orthogonal polynomials and are defined as

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1) = \sum_{j=0}^{n} \rho_{nj} x^j,$$

where

$$\rho_{nj} = \frac{(-1)^n(\beta+1)_n(-n)_j(n+\lambda)_j}{(\beta+1)_j n! j!}, \quad \lambda = \alpha+\beta+1, \quad \alpha,\beta > -1.$$

$P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomials obtained from the Gaussian hypergeometric function ${}_2F_1$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2}), \quad n = 0, 1, \cdots$$

and $(z)_n$ is Pochhammer's symbol

$$(z)_0 = 1, \quad (z)_n = z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad n \geq 1.$$

See Dufresne (2007a) for details.

3.2 Fitting probability distributions with linear combinations of exponentials

Fitting given probability distributions by combinations of exponentials often leads to simpler calculations. In the sequel, we approximate the square root of a probability density function by a mixture of exponentials

$$\sqrt{f_{\mathbb{T}}(t)} \approx \sum_j a_j e^{-\lambda_j t}, \quad t \geq 0, \quad j = 1, \cdots, n,$$

for appropriate real-valued coefficient a_j. This square root can be approximated using 'shifted' Jacobi polynomials, making use of a result in Dufresne (2007a, Theorem 3.3), stating that

$$\sqrt{f_{\mathbb{T}}(t)} = e^{-prt} \sum_{k=0}^{\infty} b_k R_k^{(\alpha,\beta)}(e^{-rt}) = \sum_{j=0}^{\infty} \underbrace{\left(\sum_{k=j}^{\infty} b_k \rho_{kj} \right)}_{a_j} e^{-\overbrace{(j+p)r}^{\lambda_j} t},$$

where

$$b_k = \frac{r}{h_k} \int_0^{\infty} e^{-(\beta-\rho+1)rt}(1-e^{-rt})^{\alpha} R_k^{(\alpha,\beta)}(e^{-rt}) \sqrt{f_{\mathbb{T}}(t)} dt,$$

and

$$\rho_{kj} = \frac{(-1)^k(\beta+1)_k(-k)_j(k+\lambda)_j}{(\beta+1)_j k! j!},$$

$$h_k = \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\lambda)k!\Gamma(k+\lambda)}.$$

Denote

$$\begin{cases} a_j = \sum_k b_k \rho_{kj} \\ \lambda_j = (j+p)r \end{cases},$$

the probability density $f_{\mathbb{T}}(t)$ could be approximated by

$$f_{\mathbb{T}}(t) \approx \left(\sum_{j=1}^{n} a_j e^{-\lambda_j t} \right)^2 = \sum_{i=1}^{N} \tilde{a}_i e^{-\tilde{\lambda}_i t}, \quad t \geq 0,$$

where $N = \frac{n(n+1)}{2}$ and it is still a combination of exponentials.

4 Mortality-linked derivative pricing

In this section, we apply the recursion algorithm to the pricing of mortality-linked derivatives, when the future remaining life time is approximated by a mixture of exponentials. Suppose the price of the financial asset X_t follows the time-homogeneous diffusion process (2), the Feynman-Kac integral of the physical probability density of X_t is given by

$$p(x_t, t|x_0, 0) = \frac{1}{\sigma(x_t)} e^{\int_{\psi(x_0)}^{\psi(x_t)} \tilde{\mu}(y)dy} k(\psi(x_t), t|\psi(x_0), 0) \tag{12}$$

where

$$Y_t := \psi(X_t) = \int^{X_t} dr/\sigma(r)$$

and

$$k(\psi(x_t), t|\psi(x_0), 0) := \mathbb{E}^{\psi(x_0),\psi(x_t),t}\left[e^{-\int_0^t V(y)d\tau}\right] = \int\limits_{(\psi(x_0),0)}^{(\psi(x_t),t)} \mathcal{D}y(\tau) e^{-\frac{1}{2}\int\limits_0^t \dot{y}^2 d\tau - \int\limits_0^t V(y)d\tau}.$$

Denote $f_{\mathbb{T}}(t)$ as the probability density function of the future life time $\mathbb{T}$, one obtains an equivalent martingale measure by omitting the exponential factor and rescaling, that is,

$$p^{\mathbb{Q}}(x_t, t|x_0, 0) = \frac{k(\psi(x_t), t|\psi(x_0), 0)}{\sigma(x_t) \cdot n(x_0)} \tag{13}$$

where $n(x_0)$ is the normalization factor which guarantees it is a martingale and is given as

$$n(x_0) = \int\limits_0^{+\infty} f_{\mathbb{T}}(t)dt \int\limits_{-\infty}^{+\infty} \frac{k(\psi(x_t), t|\psi(x_0), 0)}{\sigma(x_t)} dx_t.$$

Therefore, the risk neutral probability density function of this type of mortality-linked derivatives is given by

$$\int\limits_0^{+\infty} f_{\mathbb{T}}(t)dt \cdot p^{\mathbb{Q}}(x_t, t|x_0, 0), \tag{14}$$

while the pricing kernel under the physical probability measure reads as

$$\int\limits_0^{+\infty} f_{\mathbb{T}}(t)dt \cdot p(x_t, t|x_0, 0). \tag{15}$$

Let $\mathcal{H}(x_t)$ be the terminal payoff function. The expectation of such payoff with stochastic time under the equivalent martingale measure can be formulated as

$$\mathbb{E}_{\mathbb{T}}^{\mathbb{Q}}[\mathcal{H}(x_t)] = \int\limits_0^{+\infty} f_{\mathbb{T}}(t)dt \cdot \int\limits_{-\infty}^{+\infty} p^{\mathbb{Q}}(x_t, t|x_0, 0)\mathcal{H}(x_t)dx_t. \tag{16}$$

Similar expression also holds under the physical probability measure by replacing the corresponding pricing kernel. From the previous section, we know that the probability density function of future life time $\mathbb{T}$ could be approximated by a combination of exponential functions of the form

$$f_{\mathbb{T}}(t) \approx \sum_{i=1}^{N} \tilde{a}_i e^{-\tilde{\lambda}_i t},$$

where $N = \frac{n(n+1)}{2}$ and n is the terms of Jacobi polynomials used in the fitting. Plugging this mixture into (16), we find

$$\mathbb{E}_{\mathbb{T}}^{\mathbb{Q}}[\mathcal{H}(x_t)] \approx \int_0^{+\infty} \sum_{i=1}^{N} \tilde{a}_i e^{-\tilde{\lambda}_i t} dt \cdot \int_{-\infty}^{+\infty} p^{\mathbb{Q}}(x_t, t | x_0, 0) \mathcal{H}(x_t) dx_t \tag{17}$$

If we rearrange the terms, the probability density function (14) can be casted into the form containing the Laplace transformed transition density $\rho^{\lambda_i}(x_t, x_0)$ with parameter λ_i. The corresponding expected value of the payoff under the risk-neutral probability measure can be written as

$$\mathbb{E}_{\mathbb{T}}^{\mathbb{Q}}[\mathcal{H}(x_t)] \approx \frac{1}{n(x_0)} \int_{-\infty}^{+\infty} \frac{1}{\sigma(x_t)} \sum_{i=1}^{N} \tilde{a}_i \rho^{\tilde{\lambda}_i}(\psi(x_t), \psi(x_0)) \mathcal{H}(x_t) dx_t.$$

The recursive scheme is somehow computation expensive on the real Laplace inversion calculation of the transition densities, however, it takes only few seconds or even a fraction of a second for the calculation of the Laplace transformed transition densities $\rho^s(x_t, x_0)$. In this application, we can avoid this additional real Laplace inversion by incorporating the combinations of exponentials. To obtain the value of the mortality-linked derivative, we only have to evaluate an one-dimension integral, which significantly improves the efficiency of the algorithm.

5 Numerical examples

In this section, we consider several numerical examples. In the first example, we assume that the future remaining life time follows Makeham law. We calculate the corresponding physical and risk-neutral pricing kernels by the recursion algorithm and compare these results with the exact solutions. Next, we consider the Lee-Carter model and fit the distribution of future life time by a combination of exponentials based on the estimates from this model. Similar results as in the first example are obtained in this setting.

5.1 The Makeham law

The force of mortality or the failure rate of an individual aged x under the Makeham law is given by

$$\mu_x = A + Bc^x,$$

with constants A capturing the accident hazard and Bc^x capturing the hazard of aging. The corresponding survival function can then by expressed as

$$s(x) = e^{-Ax - \frac{B}{\log c}(c^x - 1)},$$

and the survival probabilities are given by

$${}_tp_x = \frac{s(x+t)}{s(x)} = e^{-At - \frac{B}{\log c} c^x (c^t - 1)}.$$

The probability density function for the remaining life time $\mathbb{T}$ is given by

$$f_{\mathbb{T}}(t) = \frac{d}{dt} {}_tq_x = \frac{d}{dt}\left(1 - \frac{s(x+t)}{s(x)}\right) = {}_tp_x \cdot \mu_x(t).$$

In this example, we take

$$A = 0.0007, B = 0.00005, c = 10^{0.04}.$$

The transition density function $k(x_t, t|x_0, 0)$ with quadratic potential $V(x) = x^2$ is known in closed-form

$$k(x_t, t|x_0, 0) = \frac{e^{-\frac{1}{\sqrt{2}sh(\sqrt{2}t)}\left[ch(\sqrt{2}t)(x^2+x_0^2)-2x\cdot x_0\right]}}{\sqrt{\sqrt{2}\pi sh(\sqrt{2}t)}}. \tag{18}$$

The quadratic potential corresponds to the dynamics of a special case of the Vasicek model. Note that other time-homogeneous diffusion processes can be considered in this setting as well using different forms of potential function $V(x)$. Figure 1 presents the 10-term Jacobi polynomial approximation for the probability density function of the future remaining life time $\mathbb{T}$ for an individual aged 65 which follows Makeham mortality law. The "Jacobi" refers to the shifted Jacobi polynomial approximations, and the "exact" refers to the closed-form results. The probability mass of this pdf is 0.9970 indicating that the pdf of $\mathbb{T}$ can be approximated satisfactorily by Jacobi polynomials. 7,10,20-term Jacobi polynomial approximations for the stochastic time are compared in Figure 2. Figure 3 shows the risk-neutral pricing kernel of mortality-linked derivatives with quadratic potential $V(x) = x^2$ and Makeham future remaining life time $\mathbb{T}$. The recursive approximation is given by the red dot line and the blue solid line is the exact solution for risk neutral density. The corresponding exact solution to the probability density function under the physical probability measure is shown in black dot line in Figure 3.

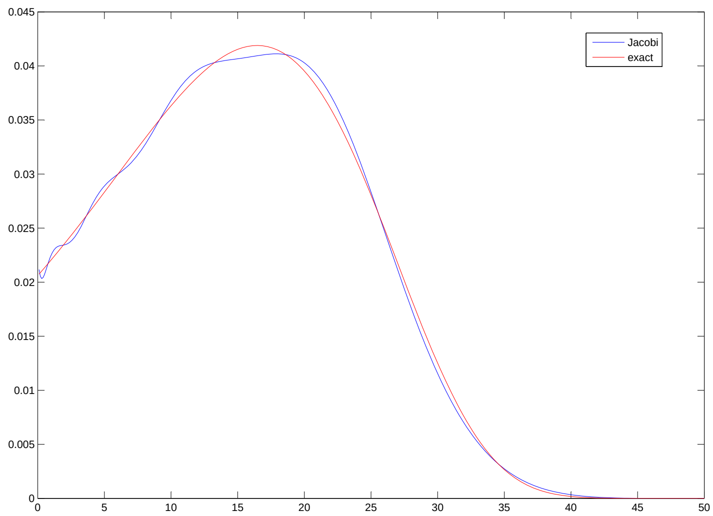

Figure 1: 10-term Jacobi polynomial approximation for the pdf of $\mathbb{T}$ for an individual aged 65 under Makeham mortality law with $\alpha = 0, \beta = 0, r = 0.08, p = 0.2$. The probability mass of this pdf is 0.9970.

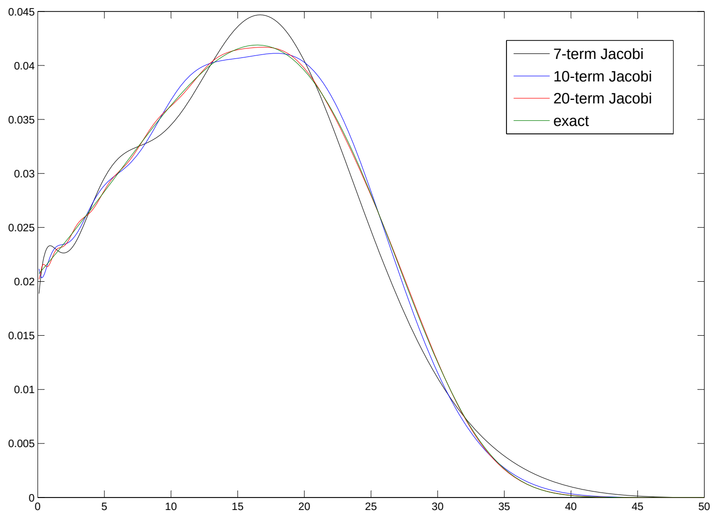

Figure 2: 7,10,20-term Jacobi polynomial approximation for the pdf of $\mathbb{T}$ for an individual aged 65 under Makeham mortality law with $\alpha = 0, \beta = 0, r = 0.08, p = 0.2$.

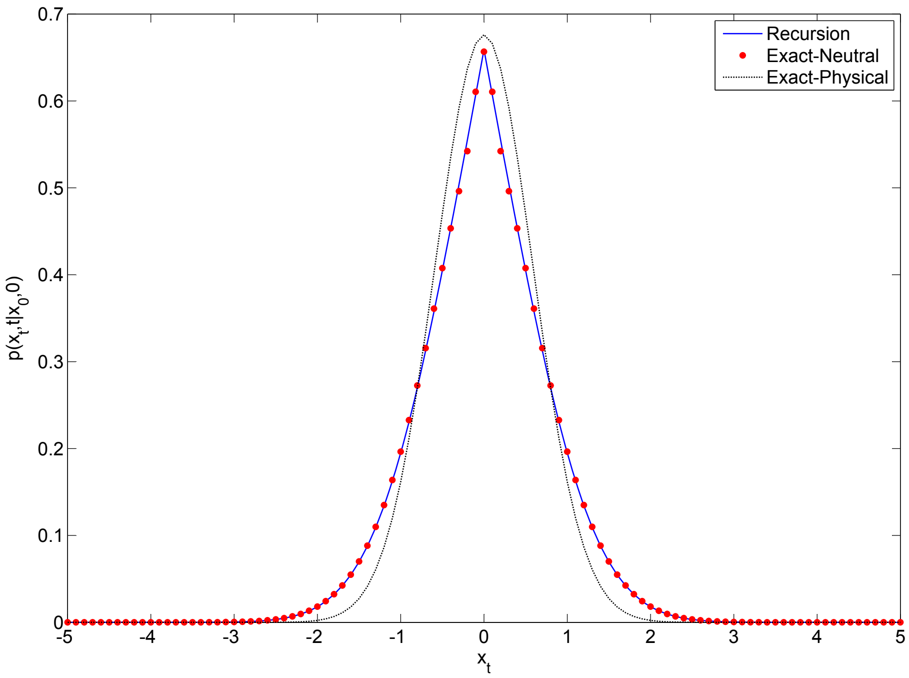

Figure 3: 10-term Jacobi polynomial approximation for the probability density function of mortality-linked derivative (15) with quadratic potential $V(x) = x^2$ and Makeham remaining life time $\mathbb{T}$ for individuals aged 65 taking $\alpha = 0, \beta = 0, r = 0.08, p = 0.2, x_0 = 0, a_0 = 5, N = 100$.

5.2 The Lee-Carter model

Lee and Carter (1992) proposed an extrapolative one-factor model for the long-term forecast of human mortality pattern based on the parameters calibrated to the historical US mortality data

$$\ln m_x(t) = \alpha_x + \beta_x \kappa(t),$$

where $m_x(t)$ is the death rate and $\kappa(t)$ is modeled as a random walk with drift c and a white noise term $u(t) \sim N(0, \sigma^2)$

$$\kappa(t) = \kappa(t-1) + c + u(t).$$

To guarantee a unique solution, the following constrains on age x and time t are usually imposed:

$$\sum_t \kappa(t) = 0 \text{ and } \sum_x \beta_x = 1.$$

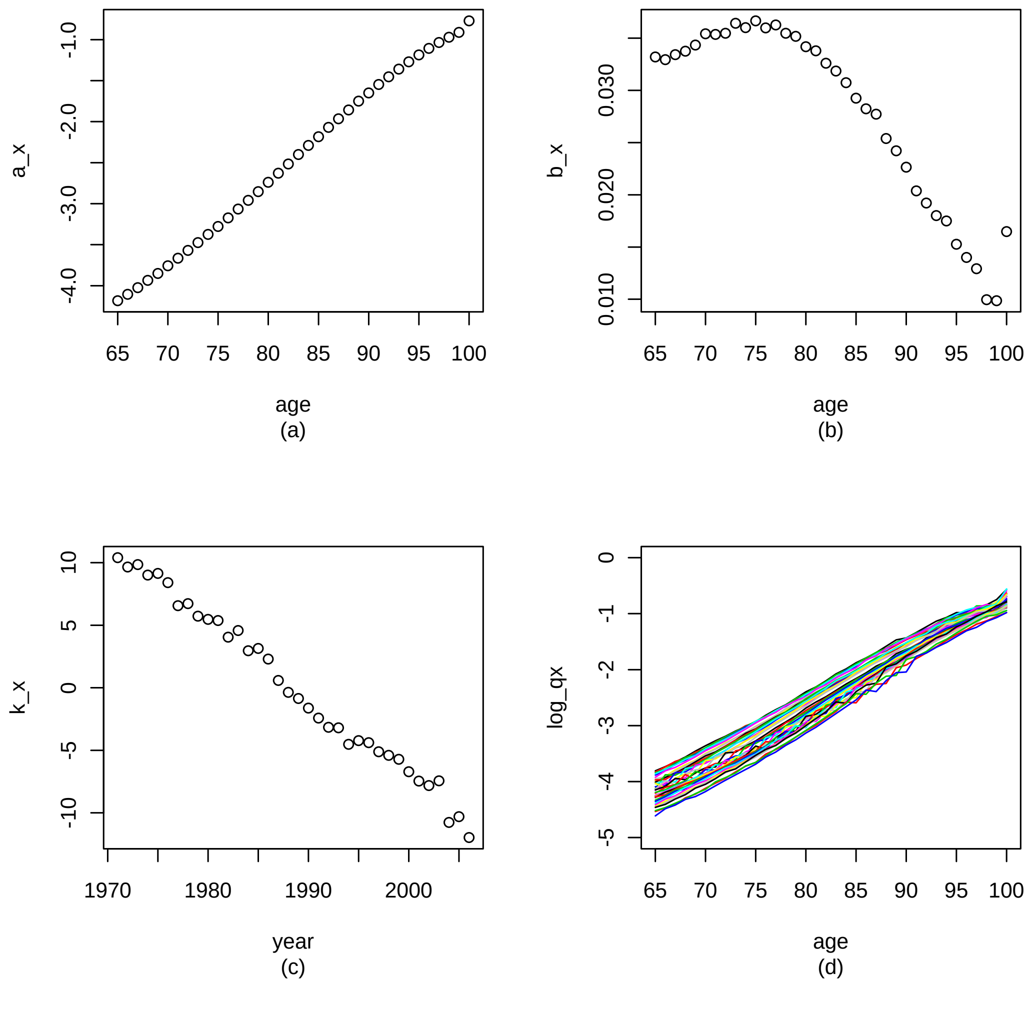

Figure 4: Parameters from Lee-Carter model for France total population data.

The data we use in this example are the France total population data from year 1971 to 2006 and ages from 65 to 100. The parameters calibrated to the historical data are given in Figure

4. Figure 6 presents the projected mortality table till year 2056 for French population aged between 65 and 100. Figure 5 gives the mortality rates from year 1971 to 2006.

The probability density function for the future remaining lifetime $\mathbb{T}$ of an individual aged x is then given by

$$f_{\mathbb{T}}(t) = \frac{d}{dt} {}_t q_x = {}_t p_x \cdot \mu_x(t).$$

The probability of dying in t years ${}_t q_x^j$ under the Lee-Carter model can be calculated by

$${}_t q_x^j = 1 - {}_1 p_x^j * {}_1 p_{x+1}^{j+i} * \cdots * {}_1 p_{x+i}^{j+i} * \cdots * {}_1 p_{x+t}^{j+i}$$

where ${}_1 p_{x+i}^{j+i}$ denotes the probability of an individual aged $x+i$ in year $j+i$ is alive in one year. The results for the probability of dying in t years ${}_t q_{65}^{2007}$ of an individual aged 65 is presented in Figure 7. The density function of the future remaining life time $f_{\mathbb{T}}(t)$ in this case can be obtained by taking the first derivative of ${}_t q_{65}^{2007}$ with respect to time t. Unlike the Makeham example with continuous survival function, ${}_t q_{65}^{2007}$ under the Lee-Carter model is only given as the yearly data. Hence, we first refine the curve for ${}_t q_{65}^{2007}$ by performing a cubic spline interpolation in order to have higher accuracy for the Jacobian polynomial approximations. Figure 8 shows the fittings of the Jacobian polynomials compared with the empirical results and Figure 9 presents the similar results as in the Makeham example for the physical and risk-neutral pricing kernels. The expected value $\mathbb{E}_{\mathbb{T}}^{\mathbb{Q}}[\mathcal{H}(x_t)]$ of the payoff function $\mathcal{H}(x_t) = (x_t - K)_+$ under the risk-neutral measure for different strike prices K is presented in Figure 10.

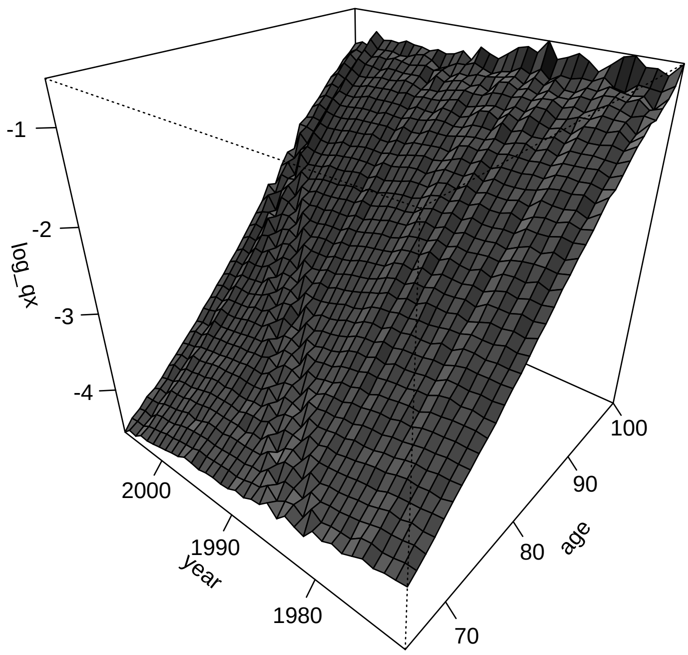

Figure 5: The surface of the logarithm of the death rate of the France total population data between year 1971 to 2006.

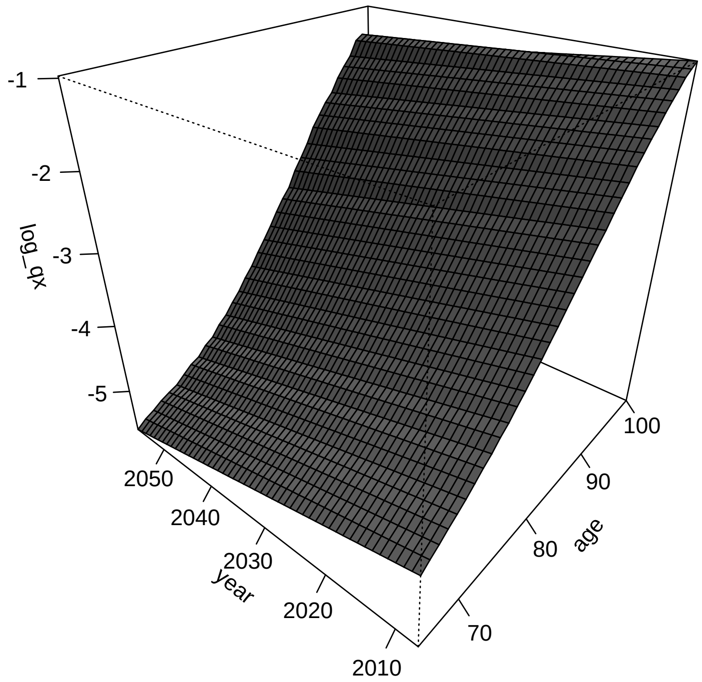

Figure 6: The projection surface of the logarithm of the death rate from Lee-Carter model for France total population data between year 2007 to 2056.

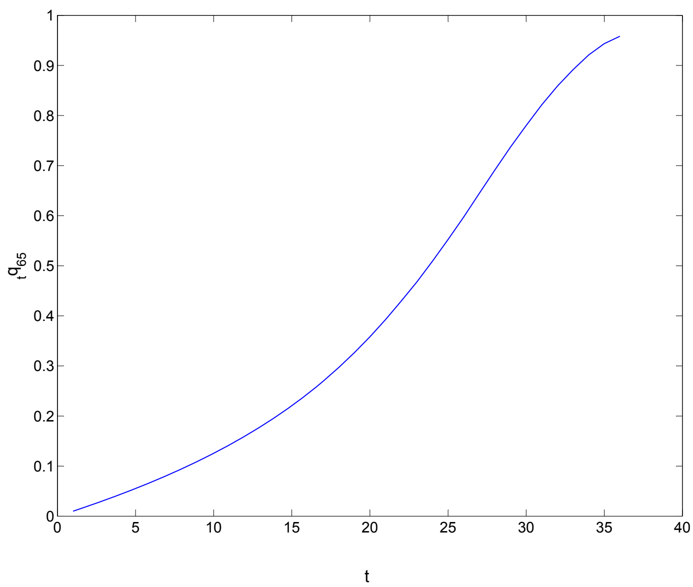

Figure 7: Probability of an individual aged 65 will die in t years $_tq_{65}^{2007}$ under the Lee-Carter model.

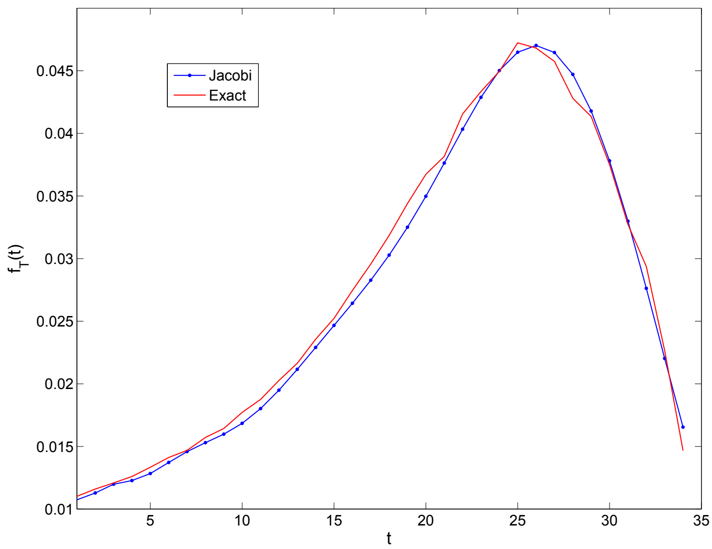

Figure 8: 10-term Jacobi polynomial approximation for the future remaining lifetime pdf $f_{\mathbb{T}}(t)$ of an individual aged 65 under the Lee-Carter model with parameters $\alpha = 0, \beta = 3, r = 0.08, p = 0.2$.

6 Conclusion

In this paper, we formulated a recursion algorithm to derive the risk-neutral probability distribution for the mortality-linked derivatives with stochastic remaining life time $\mathbb{T}$ which can be approximated by a combination of exponentials. In this application, the recursion algorithm gives very accurate results compared to the closed-form solutions and it can be easily implemented and extended to other mortality or longevity related derivatives or cash flow calculations with underlying asset governed by arbitrary time-homogeneous diffusion process combined with any type of future remaining time distributions.

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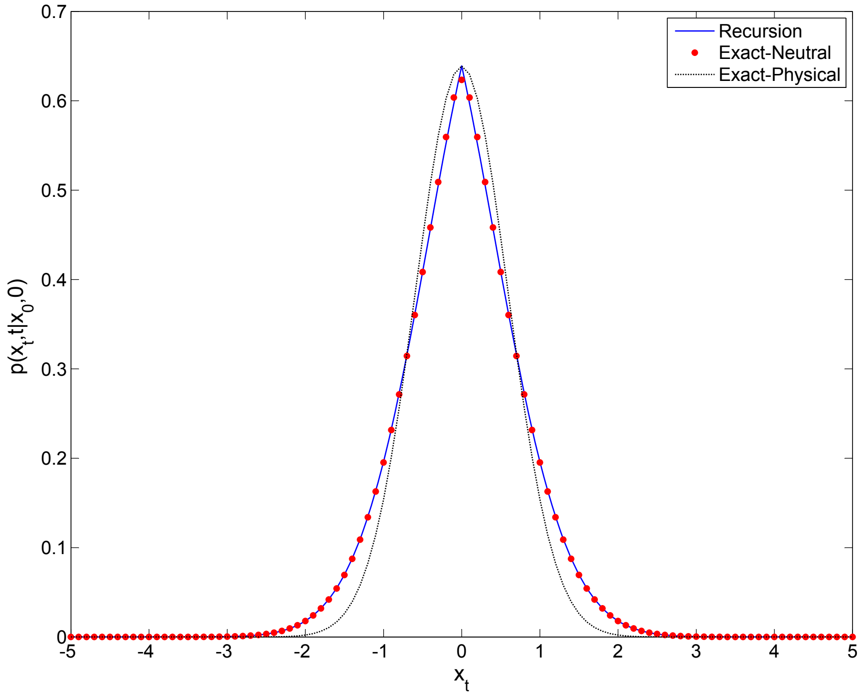

Figure 9: 10-term Jacobi polynomial approximation for the probability density function of mortality-linked derivative (15) with quadratic potential $V(x) = x^2$ for an individual aged 65 taking $\alpha = 0, \beta = 3, r = 0.08, p = 0.2, x_0 = 0, a_0 = 5, N = 100$. The distribution of the remaining life time $\mathbb{T}$ is based on the Lee-Carer model.

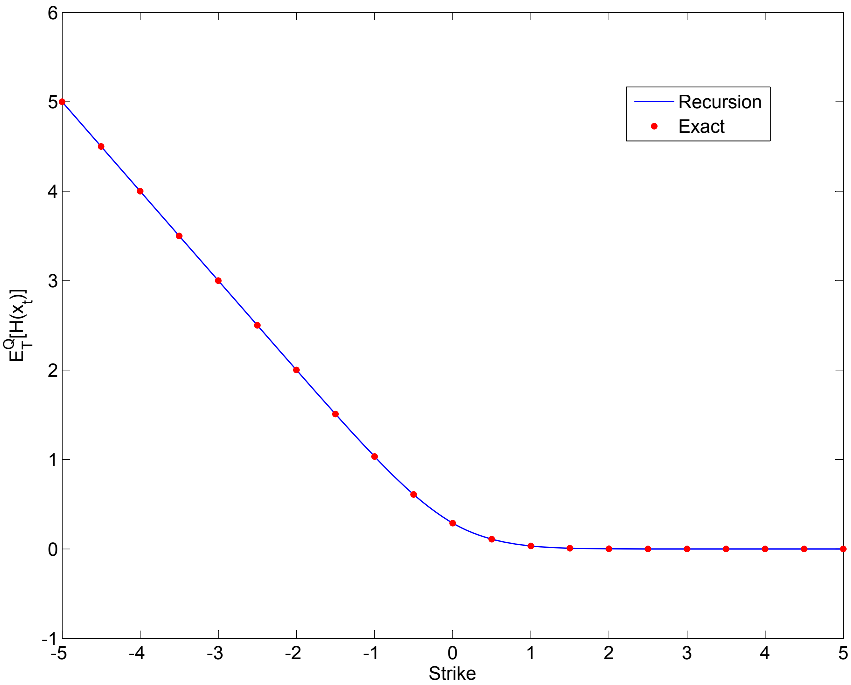

Figure 10: Expected value $\mathbb{E}_{\mathbb{T}}^{\mathbb{Q}}[\mathcal{H}(x_t)]$ of the payoff function $\mathcal{H}(x_t) = (x_t - K)_+$ under the risk-neutral measure for an individual aged 65 taking $\alpha = 0, \beta = 3, r = 0.08, p = 0.2, x_0 = 0, a_0 = 5, N = 100, K = 0$. The force of mortality is captured by the Lee-Carter model.

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A Potentials

Model	Diffusion Equation	Domain	Potential $V(x)$	$\tilde{\mu}(y)$
Wiener	$dX_t = \mu dt + \sigma dW_t$ σ positive constant	$(-\infty, \infty)$	$\frac{\mu^2}{2\sigma^2}$	$\frac{\mu}{\sigma}$
Geometric Wiener	$dX_t = \left(\mu + \frac{\sigma^2}{2}\right) X_t dt + \sigma X_t dW_t$ σ positive constant	$(0, \infty)$	$\frac{\mu^2}{2\sigma^2}$	$\frac{\mu}{\sigma}$
Vasicek	$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t$, κ, α, σ positive constants	$(-\infty, \infty)$	$\frac{\kappa^2}{2}x^2 - \frac{\kappa^2\alpha}{\sigma}x + \frac{\kappa^2\alpha^2}{2\sigma^2} - \frac{\kappa}{2}$	$\frac{\kappa\alpha}{\sigma} - \kappa y$
CIR	$dX_t = \kappa(\alpha - X_t)dt + \sigma\sqrt{X_t}dW_t$, κ, α, σ positive constants, $2\kappa\alpha \geq \sigma^2$	$(0, \infty)$	$\frac{\kappa^2}{8}x^2 + \left(\frac{2\kappa^2\alpha^2}{\sigma^4} - \frac{2\kappa\alpha}{\sigma^2} + \frac{3}{8}\right)\frac{1}{x^2} - \frac{\kappa^2\alpha}{\sigma^2}$	$\left(\frac{2\kappa\alpha}{\sigma^2} - \frac{1}{2}\right)\frac{1}{y} - \frac{\kappa}{2}y$
Adapted Geometric Wiener	$dX_t = \left(\left(\delta + \frac{\sigma^2}{2}\right) X_t - 1\right) dt + \sigma X_t dW_t$	$(0, \infty)$	$\frac{1}{2}\left(1 - \frac{2\delta}{\sigma^2}\right)e^{-\sigma x} + \frac{1}{2\sigma^2}e^{-2\sigma x} + \frac{\delta^2}{2\sigma^2}$	$\frac{\delta}{\sigma} - \frac{1}{\sigma}e^{-\sigma y}$
Bessel with Drift	$dX_t = \left(\frac{1}{X_t} - 2\right) dt + dW_t$	$(0, \infty)$	$2 - \frac{2}{x}$	$\frac{1}{y} - 2$

Table 1: Potentials in the Feynman-Kac integral representation of the transition density for popular diffusion processes. Explicit transition density available.

Model	Diffusion Equation	Domain	Potential $V(x)$	$\tilde{\mu}(x)$
CKLS	$dX_t = \kappa(\alpha - X_t)dt + \sigma X_t^{3/2} dW_t$ κ, α, σ positive constants	$(0, \infty)$	$\frac{1}{8}\left(\frac{3}{x^2} + (\kappa^2 - 3\kappa\alpha\sigma^2)x^2 - \frac{\kappa^2\alpha\sigma^2}{2}x^4 + \frac{\kappa^2\alpha^2\sigma^4}{16}x^6\right) + \kappa$	$-\frac{\kappa\alpha\sigma^2}{8}y^3 + \frac{3}{2y} + \frac{\kappa}{2}y$
Double Well Potential	$dX_t = (X_t - X_t^3)dt + dW_t$	$(-\infty, \infty)$	$\frac{1}{2}(x^6 - 2x^4 - 2x^2 + 1)$	$y - y^3$

Table 2: Potentials in the Feynman-Kac integral representation of the transition density for popular diffusion processes. Explicit transition density not available.