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Optimal allocation of policy deductibles for exchangeable risks

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ABSTRACT

Let $X_1, \ldots, X_n$ be a set of n continuous and non-negative random variables, with log-concave joint density function f, faced by a person who seeks for an optimal deductible coverage for these n risks. Let $\mathbf{d} = (d_1, \ldots d_n)$ and $\mathbf{d}^* = (d_1^*, \ldots d_n^*)$ be two vectors of deductibles such that $\mathbf{d}^*$ is majorized by $\mathbf{d}$. It is shown that $\sum_{i=1}^n (X_i \wedge d_i^*)$ is larger than $\sum_{i=1}^n (X_i \wedge d_i)$ in stochastic dominance, provided f is exchangeable. As a result, the vector $(\sum_{i=1}^n d_i, 0, \ldots, 0)$ is an optimal allocation that maximizes the expected utility of the policyholder's wealth. It is proven that the same result remains to hold in some situations if we drop the assumption that f is log-concave.

1. Introduction

Consider an insurance agreement under which a policyholder is self-insured up to a pre-specified value, known as the deductible amount. The total loss X faced by the policyholder is a non-negative random variable, hereafter called a risk. If X exceeds the deductible amount d, the remaining risk, $X - d$, will be covered by the insurer, otherwise, X is covered by the policyholder himself. This type of insurance coverage is known as policy deductible (cf. Klugman et al., 2004). Under the deductible coverage, the risk X can be expressed as $X = (X \wedge d) + (X - d)_+$, where the first part is self-insured by the policyholder and the second part is indemnified by the insurer. Now consider a situation where the policyholder faces n risks $X_1, \ldots, X_n$ which are insured under a policy deductible coverage. Suppose the amount d is the total deductible amount corresponding to all risks and the policyholder has the right to divide d into n non-negative values $d_1, \ldots, d_n$ such that $\sum_{i=1}^n d_i = d$, and for $i = 1, \ldots, n$, each d_i is the deductible corresponding to the risk X_i (cf. Cheung, 2007). The indemnified amount by the insurer is given by $\sum_{i=1}^n (X_i - d_i)_+$ and the retained risk which is not covered by the policy deductible coverage is given by $\sum_{i=1}^n (X_i \wedge d_i)$. With this set up, if w denotes the initial wealth after paying the required premium which is assumed not to depend on the choice of $(d_1, \ldots, d_n)$, the policyholder's wealth is changed into $w - \sum_{i=1}^n (X_i \wedge d_i)$. It is of importance to the policyholder to obtain the optimal vector $\mathbf{d}'$ in the set

$$s_n(d) = \left\{ \mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{R}^{+n} \mid \sum_{i=1}^n d_i = d \right\}$$

for which the amount $w - \sum_{i=1}^n (X_i \wedge d_i)$ is maximized or equivalently, $\sum_{i=1}^n (X_i \wedge d_i)$ is minimized according to a given stochastic order criterion. Several optimization criteria (such as maximizing the expected utility, minimizing the variance, minimizing the probability of ruin, etc.) have been proposed, see for example Van Heerwaarden et al. (1989) or Denuit and Vermandele (1998). In this paper, following Cheung (2007), Hua and Cheung (2008a,b), Lu and Meng (2011), Xu and Hu (2012), You and Li (2014) and Hu and Wang (2014), we use the maximization of the expected utility criterion to find an optimal deductibles allocation. That is, we are looking for an allocation that maximizes

$$E\left[u\left(w - \sum_{i=1}^n (X_i \wedge d_i) \right) \right]$$

or, equivalently, minimizes

$$E\left[\tilde{u}\left(\sum_{i=1}^n (X_i \wedge d_i) \right) \right],$$

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where $\tilde{u}(x) = -u(w - x)$ and u is a utility function which is assumed to be increasing (concave).

Let us recall the notion of majorization and various stochastic orderings which will be used to prove the main results in this paper.

Throughout this paper, we use increasing for non-decreasing and decreasing for non-increasing and assume that all the expectations of the random variables considered exist.

Let X and Y be univariate random variables with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, density functions f and g; hazard rates r_F $(= f/\bar{F})$ and r_G $(= g/\bar{G})$, respectively. Let l_X, l_Y and u_X, u_Y be the (finite or infinite) left and right endpoints of the support of X and Y, respectively. The random variable X is said to be smaller than random variable Y in the

- stochastic dominance order (denoted by $X \le_{st} Y$), if $E[\phi(X)] \le E[\phi(Y)]$ for all increasing function ϕ,
- increasing concave (convex) order (denoted by $X \le_{icv\,(icx)} Y$), if $E[\phi(X)] \le E[\phi(Y)]$ for all increasing concave (convex) function ϕ,
- hazard rate order (denoted by $X \le_{hr} Y$), if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y))$,
- likelihood ratio order (denoted by $X \le_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y))$.

It is well known that $X \le_{st} Y$ is equivalent to $\bar{F}(x) \le \bar{G}(x)$ for all x. It is easy to see that $X \le_{hr} Y$, if and only if, for every x, $r_G(x) \le r_F(x)$. Note that we have the following chain of implications among the above stochastic orderings:

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{st} Y \Rightarrow X \le_{icv,icx} Y.$$

For more details on stochastic orders see e.g. Muller and Stoyan (2002), Denuit et al. (2005) or Shaked and Shanthikumar (2007).

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $(\tau_1, \ldots, \tau_n)$ be an arbitrary permutation of $(1, \ldots, n)$ and $x_{(i)}$ and $x_{[i]}$ denote the ith smallest and the ith largest of x_i's, respectively. The notion of majorization, which is one of the basic tools in establishing various inequalities in statistics and probability, is introduced next. For more details on majorization and its properties the reader is referred to Marshall et al. (2011).

Definition 1.1. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be majorized by another vector $\mathbf{y} \in \mathbb{R}^n$, notation $\mathbf{x} \le_m \mathbf{y}$, if $\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Definition 1.2. A real valued function ϕ defined on set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathbb{A}$, if $\phi(\mathbf{x}) \le (\ge)\phi(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ such that $\mathbf{x} \le_{\mathbf{m}} \mathbf{y}$.

The following lemma which has an important role in the proof of the main result of this paper is another version of result A.2.b on page 82 of Marshall et al. (2011) and they both have the similar proof.

Lemma 1.3. *Let $\mathbb{A}$ be a set with the property*

$$\mathbf{y} \in \mathbb{A} \quad \text{and} \quad \mathbf{x} \le_m \mathbf{y} \text{ implies } \mathbf{x} \in \mathbb{A}.$$

A continuous function ϕ defined on $\mathbb{A}$ is Schur-concave on $\mathbb{A}$ if and only if ϕ is symmetric and $\phi(x_1, s - x_1, x_3, \ldots, x_n)$ is increasing in $x_1 \le \frac{s}{2}$ for each fixed $s, x_3, \ldots, x_n$.

Next, we define log-concave functions.

Definition 1.4. A real valued function ϕ defined on set $\mathbb{A} = \{\mathbf{x} \in \mathbb{R}^n : \phi(\mathbf{x}) \ge 0\}$ is said to be log-concave, if for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ and $\alpha \in [0, 1]$,

$$\phi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \ge [\phi(\mathbf{x})]^{\alpha}[\phi(\mathbf{y})]^{1-\alpha}.$$

The class of log-concave probability density functions includes many common parametric families. For univariate, examples include normal densities, gamma densities with shape parameter greater than or equal to one, Weibull densities with exponents greater than or equal to one, beta densities with both parameters greater than or equal to one and logistic densities. For a more comprehensive list of univariate examples, see e.g. Bagnoli and Bergstrom (2005). Multivariate examples include the multivariate normal densities, Wishart densities and Dirichlet densities. Finally, we define exchangeable random variables.

Definition 1.5. The random variables $X_1, \ldots, X_n$ are said to be exchangeable, if the joint distribution function of $(X_{\tau_1}, \ldots, X_{\tau_n})$ is not dependent on $(\tau_1, \ldots, \tau_n)$.

For example, in the background risk models (BRM) it is common for random risks $X_1, \ldots, X_n$ to assume $X_i = h(Y_i, Z)$ for $i = 1, \ldots, n$ where $Y_1, \ldots, Y_n$ are stand-alone risks and Z is a background risk. In special cases, X_i can be defined as $X_i = ZY_i$ or $X_i = Y_i + Z$ for $i = 1, \ldots, n$. In BRM, if for every z, $Y_1|Z = z, \ldots, Y_n|Z = z$ are independent and identically distributed, then $X_1, \ldots, X_n$ will be exchangeable random risks. For more information about BRM see Pratt (1988), Finkelshtain et al. (1999), Tsanakas (2008), Franke et al. (2011) and Asimit et al. (2013). Furthermore, when analyzing real data it is of interest to have an arbitrary but given risks distribution, therefore the issue is to construct an exchangeable risks with given marginal distribution. Clearly the easiest but not the best case is that of independent and identically distributed random risks. However, the point here is, to allow a model having a possible dependence structure among the risks.

In the following, we assume that all considered random variables are absolutely continuous and non-negative.

Let $X_1, \ldots, X_n$ be a set of n risks faced by a policyholder and $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathbf{d}^* = (d_1^*, \ldots, d_n^*) \in S_n(d)$. Cheung (2007) proved that if either all X_i's are independent and $X_1 \le_{hr} \ldots \le_{hr} X_n$, or all X_i's are comonotonic and $X_1 \le_{st} \ldots \le_{st} X_n$, then

$$\sum_{i=1}^{n} (X_i \wedge d_{[i]}) \le_{icx} \sum_{i=1}^{n} (X_i \wedge d_{\tau_i}).$$

For more details on comonotonic random vectors, see e.g. Dhaene et al. (2002). Hua and Cheung (2008b) proved that for any random vector $(X_1, \ldots, X_n)$, we have that

$$\sum_{i=1}^{n} (X_i \wedge d_i) \le_{st} \left(\sum_{i=1}^{n} X_i\right) \wedge d,$$

which means that for any policyholder who has a deductible coverage for each risk with a fixed total deductible, the global insurance is the worst case. If the X_i's are independent with X_i, $i = 1, \ldots, n$, having log-concave density function, then Lu and Meng (2011) proved that if $X_1 \le_{lr} \ldots \le_{lr} X_n$, then

$$\mathbf{d} \ge_m \mathbf{d}^* \Longrightarrow \sum_{i=1}^{n} (X_i \wedge d_{[i]}) \le_{st} \sum_{i=1}^{n} (X_i \wedge d_{[i]}^*). \tag{1.1}$$

Hu and Wang (2014) proved that (1.1) holds under the weaker condition that $X_1 \le_{hr} \ldots \le_{hr} X_n$.

In this paper, we drop the independence assumption of $X_1, \ldots, X_n$. In Theorem 2.4, we prove that under appropriate conditions, we have that

$$\mathbf{d} \ge_m \mathbf{d}^* \Longrightarrow \sum_{i=1}^{n} (X_i \wedge d_i) \le_{st} \sum_{i=1}^{n} (X_i \wedge d_i^*).$$

We show that the result holds in particular if the X_i's are exchangeable and the joint density function f is log-concave; see Theorem 2.6. The main consequence of this result (Corollary 2.7) is that the vector $(d, 0, \ldots, 0)$ is the best allocation of the total deductible d in the sense that it maximizes the expected utility of the policyholder's wealth.

2. Optimal allocation of deductibles

For two random variables X_1 and X_2, let $X_{a,c} = (X_1 \wedge a) + (X_2 \wedge (c-a))$, where $c > 0$ and $0 \le a \le \frac{c}{2}$. In order to be able to prove the main results of this paper, we first need to prove the following three lemmas. The first two lemmas might be of independent interest.

Lemma 2.1. *Let X_1 and X_2 be two continuous and non-negative risks with joint distribution function F_{X_1,X_2} and density function f_{X_1,X_2}. Then, for $0 \le a \le \frac{c}{2}$, we have that*

$$\bar{F}_{X_{a,c}}(t) = \begin{cases} 1 - P(X_1 + X_2 \le t), \quad t < a; \\ 1 - F_{X_2}(t-a) - P(X_1 \le a, t-a \le X_2 \le t - X_1), \\ \quad a \le t < c-a; \\ 1 - F_{X_1}(t-c+a) - F_{X_2}(t-a) \\ \quad + F_{X_1,X_2}(t-c+a, t-a) \\ -P(t-c+a \le X_1 \le a, t-a \le X_2 \le t-X_1), \\ \quad c-a \le t < c; \\ 0, \quad t \ge c. \end{cases}$$

Proof. It is easy to see that for $x_1, x_2 \in \mathbb{R}^+$,

$$x_{a,c} = \begin{cases} x_1 + x_2, & x_1 \le a, x_2 \le c-a; \\ x_1 + (c-a), & x_1 \le a, x_2 > c-a; \\ x_2 + a, & x_1 > a, x_2 \le c-a; \\ c, & x_1 > a, x_2 > c-a. \end{cases}$$

Therefore, the survival function of $X_{a,c}$ at $t \in \mathbb{R}$ can be expressed as

$$\begin{aligned} \bar{F}_{X_{a,c}}(t) &= \int_0^\infty \int_0^\infty 1_{((x_1 \wedge a)+(x_2 \wedge (c-a))>t)} f(x_1,x_2) dx_2 dx_1 \\ &= \int_0^a \int_0^{c-a} 1_{(x_1+x_2>t)} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_0^a \int_{c-a}^\infty 1_{(x_1+c-a>t)} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_a^\infty \int_0^{c-a} 1_{(x_2+a>t)} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_a^\infty \int_{c-a}^\infty 1_{(c>t)} f(x_1,x_2) dx_2 dx_1. \end{aligned}$$

Now, if $0 \le t < a$, then

$$\begin{aligned} \bar{F}_{X_{a,c}}(t) &= \int_0^a \int_0^{c-a} 1_{(x_1+x_2>t)} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_0^a \int_{c-a}^\infty f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_a^\infty \int_0^{c-a} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_a^\infty \int_{c-a}^\infty f(x_1,x_2) dx_2 dx_1. \\ &= 1 - P(X_1 + X_2 \le t). \end{aligned}$$

Next, if $a \le t < c-a$, then

$$\begin{aligned} \bar{F}_{X_{a,c}}(t) &= \int_0^a \int_0^{c-a} 1_{(x_1+x_2>t)} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_0^a \int_{c-a}^\infty f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_a^\infty \int_{t-a}^{c-a} f(x_1,x_2) dx_2 dx_1 \\ &\quad + \int_a^\infty \int_{c-a}^\infty f(x_1,x_2) dx_2 dx_1 \\ &= 1 - P(X_1 \le a, t-a \le X_2 \le t - X_1) \\ &\quad - F_{X_2}(t-a). \end{aligned}$$

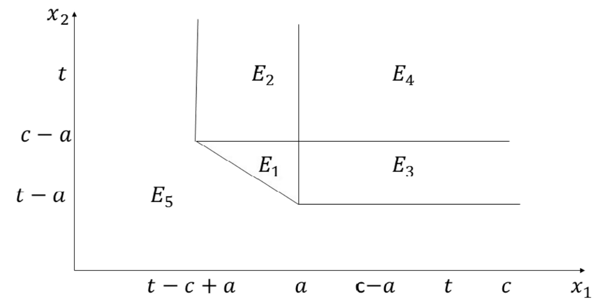

Fig. 2.1. Subsets of $E_1, \ldots, E_5$, corresponding to Lemma 2.1.

Finally, if $c-a \le t < c$, then

$$\begin{aligned} \bar{F}_{X_{a,c}}(t) &= P(E_1) + P(E_2) + P(E_3) + P(E_4) \\ &= 1 - P(E_5) \\ &= 1 - F_{X_1}(t-c+a) - F_{X_2}(t-a) \\ &\quad + F_{X_1,X_2}(t-c+a, t-a) \\ &\quad - P(t-c+a \le X_1 \le a, t-a \le X_2 \le t - X_1) \end{aligned}$$

where the subsets of $E_1, \ldots, E_5$ are shown in Fig. 2.1. This completes the proof of the stated result. □

Lemma 2.2. *Let X_1 and X_2 be two continuous and non-negative risks with joint distribution function F_{X_1,X_2} and density function f_{X_1,X_2}. Then, for $0 \le a \le a^* \le \frac{c}{2}$ and $t < c$, we have that*

$$\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t) = \begin{cases} P\left(t - a^* \le X_2 \le t - a, X_2 + X_1 > t\right) \\ -P\left(t-c+a \le X_1 \le t-c+a^*, X_2 + X_1 > t\right), & t < c, \\ 0, & t \ge c. \end{cases}$$

Proof. Let $k(t, a, a^*, c)$ be defined by

$$\begin{aligned} k(t, a, a^*, c) &= P\left(t - a^* \le X_2 \le t - a, X_2 + X_1 > t\right) \\ &\quad - P\left(t-c+a \le X_1 \le t-c+a^*, X_2 + X_1 > t\right). \end{aligned}$$

Hereafter, we will prove the stated result by considering the cases $t < a$, $a \le t < a^*$, $a^* \le t < c-a^*$, $c-a^* \le t < c-a$, $c-a \le t < c$ and $t \ge c$, sequentially.

(a) For $t < a$, it is easy to see that $\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t) = k(t, a, a^*, c) = 0$.
(b) For $a \le t < a^*$, we have that

$$\begin{aligned} &\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t) \\ &= F_{X_2}(t-a) + P(X_1 \le a, t-a \le X_2 \le t - X_1) \\ &\quad - P(X_2 + X_1 \le t) \\ &= P(A_2 \cup A_3) + P(A_1) - P(A_1 \cup A_2) \\ &= P(A_3) \\ &= P(X_2 \le t-a, X_1 + X_2 > t) \\ &= k(t, a, a^*, c), \qquad (2.1) \end{aligned}$$

where the subsets A_1, A_2 and A_3 are shown in Fig. 2.2. The equality (2.1) follows since for $t < a^*$, $P(t-c+a \le X_1$

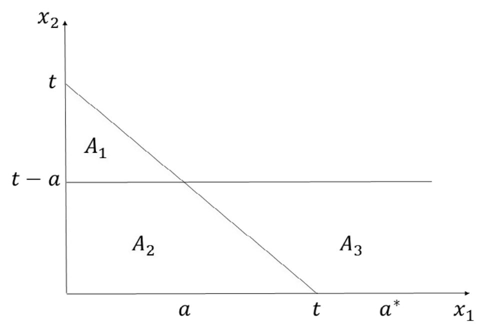

Fig. 2.2. Subsets of A_1, A_2 and A_3, corresponding to case (b) in Lemma 2.2.

$$\leq t-c+a^*) = 0 \text{ and}$$

$$P\left(X_2 \leq t-a, X_2+X_1 > t\right)$$
$$= P\left(t-a^* \leq X_2 \leq t-a, X_2+X_1 > t\right).$$

(c) For $a^* \leq t < c-a^*$,

$$\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t)$$
$$= F_{X_2}(t-a) + P\left(X_1 \leq a, t-a \leq X_2 \leq t-X_1\right)$$
$$- F_{X_2}(t-a^*) - P\left(X_1 \leq a^*, t-a^* \leq X_2 \leq t-X_1\right)$$
$$= P\left(t-a^* \leq X_2 \leq t-a\right)$$
$$+ P\left(X_1 \leq a, t-a \leq X_2 \leq t-X_1\right)$$
$$- P\left(X_1 \leq a^*, t-a^* \leq X_2 \leq t-X_1\right)$$
$$= P(B_1 \cup B_3) + P(B_2) - P(B_1 \cup B_2)$$
$$= P(B_3)$$
$$= P\left(t-a^* \leq X_2 \leq t-a, X_1+X_2 > t\right)$$
$$= k(t, a, a^*, c), \tag{2.2}$$

where the subsets B_1, B_2 and B_3 are shown in Fig. 2.3.

The equality (2.2) follows since for $t \leq c-a^*$, $P\left(t-c+a \leq X_1 \leq t-c+a^*\right) = 0$.

(d) For $c-a^* \leq t < c-a$, we have that

$$\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t)$$
$$= F_{X_2}(t-a) + P\left(X_1 \leq a, t-a \leq X_2 \leq t-X_1\right)$$
$$- F_{X_1}(t-c+a^*) - F_{X_2}(t-a^*)$$
$$+ F_{X_1,X_2}(t-c+a^*, t-a^*)$$
$$- P\left(t-c+a^* \leq X_1 \leq a^*, t-a^* \leq X_2 \leq t-X_1\right)$$
$$= P(\cup_{i=1}^{6} C_i) + P(C_7 \cup C_8)$$
$$- P(\cup_{\{i=1,4,7,9\}} C_i) - P(\cup_{i=1}^{3} C_i)$$
$$+ P(C_1) - P(C_5 \cup C_8)$$
$$= P(C_6) - P(C_9)$$
$$= P\left(t-a^* \leq X_2 \leq t-a, X_2+X_1 > t\right)$$
$$- P\left(X_1 \leq t-c+a^*, X_2+X_1 > t\right),$$
$$= k(t, a, a^*, c), \tag{2.3}$$

where the subsets $C_1, \ldots, C_9$ are shown in Fig. 2.4. Equality (2.3) follows since for $t < c-a$, it follows that

$$P\left(X_1 \leq t-c+a^*, X_2+X_1 > t\right)$$
$$= P\left(t-c+a \leq X_1 \leq t-c+a^*, X_2+X_1 > t\right).$$

(e) Similarly, for $c-a \leq t < c$, we can show that $\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t) = k(t, a, a^*, c)$.

(f) Finally, for $t > c$, $\bar{F}_{X_{a^*,c}}(t) - \bar{F}_{X_{a,c}}(t) = 0$.

This completes the proof of the stated results. □

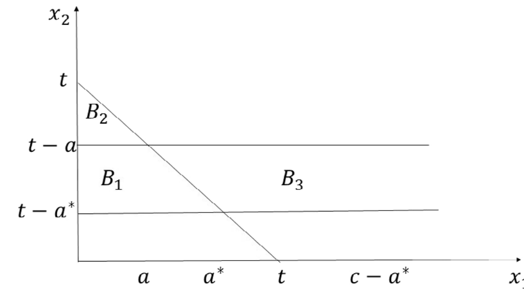

Fig. 2.3. Subsets of B_1, B_2 and B_3, corresponding to case (c) in Lemma 2.2.

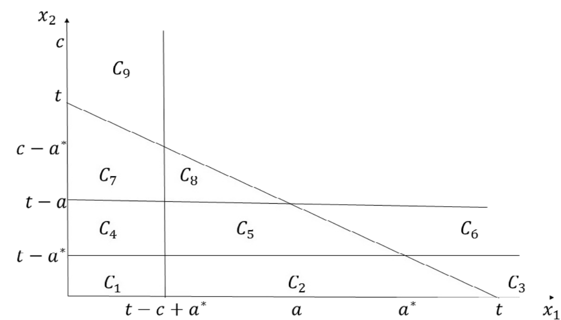

Fig. 2.4. Subsets of $C_1, \ldots, C_9$, corresponding to case (d) in Lemma 2.2.

Lemma 2.3. *Let X_1 and X_2 be two continuous and non-negative risks with joint distribution function F_{X_1,X_2} and density function f_{X_1,X_2}. Then, for $0 \leq a \leq a^* \leq \frac{c}{2}$ and $t < c$, we have that*

$$X_{a,c} \leq_{st} X_{a^*,c} \tag{2.4}$$

if and only if

$$\int_0^\infty f_{X_1,X_2}(t-c+a, y+c-a)dy$$
$$\leq \int_0^\infty f_{X_1,X_2}(y+a, t-a)dy. \tag{2.5}$$

Proof. From Lemma 2.2, it follows that (2.4) is equivalent to

$$k(t, a, a^*, c) \geq 0,$$

which in turn is equivalent to stating that

$$P\left(X_1 \leq t-c+a, X_2+X_1 > t\right) + P\left(X_2 \leq t-a, X_2+X_1 > t\right)$$

is decreasing in a. Let us denote the sum of two probabilities above by $h(t, a, c)$. Hence, the condition (2.4) is equivalent to

$$\frac{\partial\left(h(t,a,c)\right)}{\partial a} = \int_{c-a}^\infty f_{X_1,X_2}(t-c+a, x_2)dx_2$$
$$- \int_a^\infty f_{X_1,X_2}(x_1, t-a)dx_1 \leq 0.$$

This inequality can be transferred into (2.5), which completes the proof of the stated result. □

Having proved the three previous lemmas, we are now ready to prove our main result.

Theorem 2.4. *Let $X_1, \ldots, X_n$ be a set of continuous, non-negative and exchangeable risks with joint density function $f_{X_1,\ldots,X_n}$. Then, for $t < c$ and $0 \leq a \leq \frac{c}{2}$, under the condition*

$$\int_0^\infty f_{X_1,X_2|X_3,\ldots,X_n}(t-c+a, y+c-a \mid x_3, \ldots, x_n)dy$$
$$\leq \int_0^\infty f_{X_1,X_2|X_3,\ldots,X_n}(y+a, t-a \mid x_3, \ldots, x_n)dy, \tag{2.6}$$

we have that the following implication holds:

$$\mathbf{d} \geq_m \mathbf{d}^* \Longrightarrow \sum_{i=1}^n (X_i \wedge d_i) \leq_{st} \sum_{i=1}^n (X_i \wedge d_i^*).$$

Proof. We should show that $P\left(\sum_{i=1}^n (X_i \wedge d_i) > t\right)$ is a Schur-concave function in $(d_1, \ldots, d_n)$. Due to continuity and exchangeability of $X_1, \ldots, X_n$, $P\left(\sum_{i=1}^n (X_i \wedge d_i) > t\right)$ is a continuous and symmetric function in $(d_1, \ldots, d_n)$, so using Lemma 1.3, it suffices to show that the implication holds for the case when $\mathbf{d} = (d_1, c - d_1, d_3, \ldots, d_n)$, $\mathbf{d}^* = (d_1^*, c - d_1^*, d_3, \ldots, d_n)$ where $0 \leq d_1 \leq d_1^* \leq \frac{c}{2}$. Using assumption (2.6), it follows from Lemma 2.3 that

$$\left[X_{d_1,c} \mid X_3 = x_3, \ldots, X_n = x_n\right] \leq_{st} \left[X_{d_1^*,c} \mid X_3 = x_3, \ldots, X_n = x_n\right],$$

which in turn implies that

$$\begin{aligned} &P\left(\sum_{i=1}^n (X_i \wedge d_i) > t\right) \\ &= \int_0^\infty \cdots \int_0^\infty P\Big((X_1 \wedge d_1) + (X_2 \wedge (c - d_1)) \\ &\quad + \sum_{i=3}^n (x_i \wedge d_i) > t \mid X_3 = x_3, \ldots, X_n = x_n\Big) \\ &\quad \times f(x_3, \ldots, x_n) dx_n \ldots dx_3 \\ &\leq \int_0^\infty \cdots \int_0^\infty P\Big((X_1 \wedge d_1^*) + (X_2 \wedge (c - d_1^*)) \\ &\quad + \sum_{i=3}^n (x_i \wedge d_i) > t \mid X_3 = x_3, \ldots, X_n = x_n\Big) \\ &\quad \times f(x_3, \ldots, x_n) dx_n \ldots dx_3 \\ &= P\left((X_1 \wedge d_1^*) + (X_2 \wedge (c - d_1^*)) + \sum_{i=3}^n (X_i \wedge d_i) > t\right). \end{aligned}$$

This completes the proof of the stated result. □

The following lemma will be used to prove the next theorem.

Lemma 2.5. *(Marshall et al., 2011, p. 98). If ϕ is symmetric and log concave, then ϕ is Schur-concave.*

Theorem 2.6. *Let $X_1, \ldots, X_n$ be a set of exchangeable, continuous and non-negative risks with log-concave density function $f_{X_1,\ldots,X_n}$. Then, for $\mathbf{d}, \mathbf{d}^* \in S_n(d)$,*

$$\mathbf{d} \geq_m \mathbf{d}^* \Longrightarrow \sum_{i=1}^n (X_i \wedge d_i) \leq_{st} \sum_{i=1}^n (X_i \wedge d_i^*). \tag{2.7}$$

Proof. Since $f_{\mathbf{X}}(x_1, \ldots, x_n)$ is log-concave and exchangeable in $(x_1, \ldots, x_n)$, it follows from Lemma 2.5 that $f_{\mathbf{X}}(x_1, \ldots, x_n)$ is Schur-concave, from which it follows that for each $(x_3, \ldots, x_n)$, $f_{X_1,X_2|X_3,\ldots,X_n}(x_1, x_2 \mid x_3, \ldots, x_n)$ is also Schur-concave. That is, for $t < c$ and $a \leq \frac{c}{2}$, $f_{X_1,X_2|X_3,\ldots,X_n}(t - c + a, y + c - a \mid x_3, \ldots, x_n) \leq f_{X_1,X_2|X_3,\ldots,X_n}(y + a, t - a \mid x_3, \ldots, x_n)$ which implies that (2.6) holds. Now, the required result follows from Theorem 2.6. □

As discussed in the introduction, if a policyholder, after having paid the premium, has initial wealth w, then his resulting wealth, considering the retained risk, is $w - \sum_{i=1}^n (X_i \wedge d_i)$. Thus, the policyholder might be interested in finding a vector $(d_1', \ldots, d_n')$ which satisfies

$$P\left(w - \sum_{i=1}^n (X_i \wedge d_i') > t\right) = \max_{\mathbf{d} \in S_n(\mathbf{d})} P\left(w - \sum_{i=1}^n (X_i \wedge d_i) > t\right) \qquad t \in \mathbb{R}, \tag{2.8}$$

or, equivalently,

$$P\left(\sum_{i=1}^n (X_i \wedge d_i') > t\right) = \min_{\mathbf{d} \in S_n(\mathbf{d})} P\left(\sum_{i=1}^n (X_i \wedge d_i) > t\right) \quad t \in \mathbb{R},$$

where $X_1, \ldots, X_n$ are exchangeable risks. This optimization problem is discussed in the next corollary.

Corollary 2.7. *Consider the exchangeable, continuous and non-negative risks $X_1, \ldots, X_n$.*

If $f(x_1, \ldots, x_n)$ is log-concave, then

$$(X_i \wedge d) \leq_{st} \sum_{i=1}^n (X_i \wedge d_i) \leq_{st} \sum_{i=1}^n (X_i \wedge \bar{d}) \quad \textit{for } i = 1, \ldots, n$$

where $\bar{d} = \frac{1}{n}\sum_{i=1}^n d_i$.

Proof. Using Theorem 2.6, the survival function of $P\left(\sum_{i=1}^n (X_i \wedge d_i) > t\right)$ is Schur concave. The stated result follows from this observation and from the fact that

$$(\bar{d}, \ldots, \bar{d}) \leq_m (d_1, \ldots, d_n) \leq_m (d, 0, \ldots, 0). \quad \square$$

From Corollary 2.7, we conclude that in a policy deductible agreement with the total deductible amount d, the retained risks (part of the risks that is self insured by the policyholder) for different allocation of deductibles can be ordered by the sense of stochastic dominance.

Since $X \leq_{st} Y$, implies that $w - X \geq_{st} w - Y$, from Corollary 2.7 we have that

$$w - \sum_{i=1}^n (X_i \wedge \bar{d}) \leq_{st} w - \sum_{i=1}^n (X_i \wedge d_i) \leq_{st} w - (X_i \wedge d) \quad \text{for } i = 1, \ldots, n \tag{2.9}$$

which means that for different allocations of deductibles, the final wealth also can be ordered by the sense of stochastic dominance. Considering the concept of stochastic dominance, the ordering relation (2.9) implies

$$E\left[u\left(w - \sum_{i=1}^n (X_i \wedge \bar{d})\right)\right] \leq E\left[u\left(w - \sum_{i=1}^n (X_i \wedge d_i)\right)\right] \leq E\left[u\left(w - (X_i \wedge d)\right)\right] \tag{2.10}$$

for $i = 1, \ldots, n$. It follows from (2.10) that for every policyholder (whether he is risk neutral, risk averse or risk preferent) the best allocation which maximizes the expectation of the utility function is $\mathbf{d}' = (d, 0, \ldots, 0)$ and the worst allocation which minimizes the expectation of the utility function is $\mathbf{d} = (\bar{d}, \ldots, \bar{d})$. Therefore, a sample solution of the above mentioned optimization problem in (2.8) is $\mathbf{d}' = (d, 0, \ldots, 0)$. Indeed, due to the exchangeability of the risks, $(d, 0, \ldots, 0)$, $(0, d, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, d)$ are equivalent.

Remark 2.8. The class of density functions that satisfy (2.7) is not limited to the class of log-concave density functions. For example, let (X_1, X_2) be a random vector with joint density function given by

$$f_{X_1,X_2(x_1,x_2)} = a(a+1)(\theta_1\theta_2)^{(a+1)}(\theta_1 x_1 + \theta_2 x_2 - \theta_1\theta_2)^{-(a+2)}$$
$$x_i \geq \theta_i > 0, \; i = 1, 2; \; a > 0,$$

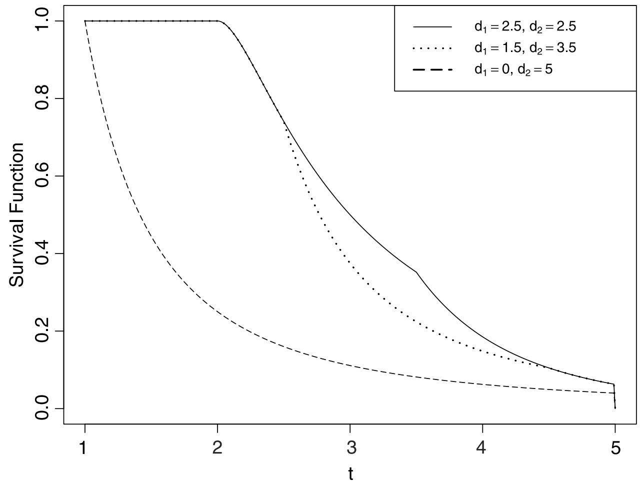

Fig. 2.5. Survival functions of retained risk in several deductible agreement, in case of bivariate Pareto distribution.

which is known as the bivariate Pareto distribution (cf. Johnson and Kotz, 1972, p.285). It is easy to check that when $\theta_1 = \theta_2$, although f is not log-concave, it satisfies the relation (2.5). Therefore, if $(d_1, d_2) \geq_m (d_1^*, d_2^*)$, then $(X_1 \wedge d_1) + (X_2 \wedge d_2) \leq_{st} (X_1 \wedge d_1^*) + (X_2 \wedge d_2^*)$. Now, let $\theta_1 = \theta_2 = 1$ and $a = 2$. In order to justify the results of Lemma 2.3, in Fig. 2.5 we plot $p((X_1 \wedge d_1) + (X_2 \wedge d_2) > t)$, for three cases $(d_1 = 2.5, d_2 = 2.5)$, $(d_1 = 1.5, d_2 = 3.5)$ and $(d_1 = 0, d_2 = 5)$. As expected, in the allocation $(d_1 = 0, d_2 = 5)$, the survival function of the retained risk is smaller than the other allocations.

Remark 2.9. It is well-known that the product of log-concave functions is also log-concave. This implies that Theorem 2.6 holds in particular for the case when $X_1, \ldots, X_n$ are independent and identically distributed and have log-concave density functions.

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