Fair dynamic valuation of insurance liabilities: Merging actuarial judgement with market- and time-consistency

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Abstract

In this paper, we investigate the fair valuation of insurance liabilities in a dynamic multi-period setting. We define a fair dynamic valuation as a valuation which is actuarial (mark-to-model for claims independent of financial market evolutions), market-consistent (mark-to-market for any hedgeable part of a claim) and time-consistent, extending the work of Dhaene et al. (2017) and Barigou and Dhaene (2019). We provide a complete hedging characterization for fair dynamic valuations. Moreover, we show how to implement fair dynamic valuations through a backward iterations scheme combining risk minimization methods from mathematical finance with standard actuarial techniques based on risk measures.

Keywords: Fair dynamic valuation, time-consistency, Solvency II, market-consistent valuation, actuarial valuation.

1 Introduction

Fair valuation of insurance liabilities has become a fundamental feature of modern solvency regulations in the insurance industry, such as the Swiss Solvency Test, Solvency II and C-ROSS (Chinese solvency regulation). A fair valuation method combines techniques from financial mathematics and actuarial science, in order to take into account and be

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consistent with information provided by the financial markets on the one hand and actuarial judgement based on generally available data about non-financial risks on the other hand. Moreover, for the determination of the solvency capital requirement (SCR), each insurance company is required to determine the fair value of its liabilities, not only today but also in future points in time.

An actuarial valuation is typically based on a diversification argument which justifies the mitigation of the risk borne by an individual by averaging out its consequences over a large pool of individuals exposed to the same risk. This valuation is performed under the real-world measure $\mathbb{P}$ and is defined as the expectation plus an additional risk margin to cover any adverse economic-demographic development that is not diversified. Based on historical data, the actuarial valuation involves a subjective actuarial judgement on the choice of the model to be chosen, see e.g. Kaas et al. (2008) for non-life and Laurent et al. (2016) for life insurance.

A market-consistent valuation is based on the minimal requirement that the value of a purely hedgeable financial payoff should be equal to the amount necessary to hedge it. A large branch of literature investigated valuations in this so-called market-consistent setting, trying to extend the arbitrage-free pricing operators (initially defined in a complete market) to the general set of unhedgeable claims. Several approaches were considered such as utility indifference pricing (Hodges and Neuberger (1989)) or risk-minimization techniques (Föllmer and Schweizer (1988) and Černỳ and Kallsen (2009)). The notion of market-consistency has been recently formalized by diverse authors as an extension of the notion of cash-invariance to all hedgeable claims, see e.g. Malamud et al. (2008), Pelsser and Stadje (2014) and Dhaene et al. (2017).

An important question in a dynamic setting is how risk valuations at different times are interrelated. In this context, time-consistency is a natural approach to glue together static valuations. It means that the same value is assigned to a position regardless of whether it is calculated over two time periods at once or in two-steps backwards in time. Time-consistent valuations have been largely studied and we refer to Acciaio and Penner (2011) for an overview.

In this paper, we investigate the fair valuation of insurance liabilities in a dynamic multi-period setting. We define a fair dynamic valuation as a valuation which is actuarial (mark-to-model for claims independent of financial market evolutions), market-consistent (mark-to-market for hedgeable parts of claims) and time-consistent, and study their properties. In particular, we provide a complete hedging characterization for fair dynamic valuations, extending the work of Dhaene et al. (2017) and Barigou and Dhaene (2019) in a dynamic setting. Moreover, we show how we can implement fair dynamic valuations through a backward iterations scheme combining risk minimization methods from mathematical finance with standard actuarial techniques based on risk measures. We remark that Pelsser and Stadje (2014) proposed time-consistent and market-consistent valuations via a so-called 'two-step market evaluation'. Compared to their paper which characterizes time-consistent and market-consistent valuations in a complete financial market by operator splitting, our valuation framework is hedge-based and allows for financial market incompleteness.

The paper is organized as follows. In Section 2, we describe the combined financial-actuarial world and the notions of orthogonal and hedgeable claims. In Section 3, fair $t-$valuations and the related notion of fair $t-$hedgers are introduced. In particular, we show that any fair $t-$valuation can be characterized in terms of a fair $t-$hedger. In Section 4, we extend the results in a time-consistent setup and provide a time-consistent hedging characterization for time-consistent and market-consistent valuations. Section 5 presents a practical approach to apply our framework and some numerical illustrations. Section 6 concludes the paper.

2 The combined financial-actuarial world

Consider a combined financial-actuarial world which is home to tradable as well as non-tradable claims. Now we are at time 0, while the time horizon is $T \in \{1, 2, \ldots\}$. The set of trading dates is denoted by $\tau = \{0, 1, ..., T\}$. The financial-actuarial world is modeled by the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, equipped with the finite and discrete time filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t \in \tau}$, such that $\mathcal{G}_0$ is equal to $\{\emptyset, \Omega\}$ and $\mathcal{G}_T = \mathcal{G}$. The $\sigma-$algebra $\mathcal{G}_t$, $t \in \tau$, represents the general information available up to and including time t in the combined world. Further, $\mathbb{P}$ is the measure attaching a physical probability to any event in that world. All random variables (r.v.'s) and stochastic processes in this paper are defined on this filtered probability space and all equalities between r.v.'s are understood in the $\mathbb{P}-$almost sure sense. Throughout the paper, we assume that the second moments of all r.v.'s that we consider exist under $\mathbb{P}$. Furthermore, we will denote the set of all $t-$claims defined on $(\Omega, \mathbb{G}, \mathcal{G})$, that is the set of all $\mathcal{G}_t-$measurable r.v.'s, by $\mathcal{C}_t$. Hereafter, when considering a $t-$claim, we will always silently assume that it is payable at time t, except if stated otherwise.

The combined financial-actuarial world hosts a number of insurance liabilities. An insurance liability due at time T will be generally denoted by $S(T)$ or simply S if no confusion is possible. A simple example of an insurance liability related to the remaining lifetime of an insured (x) observed at time 0 is the r.v. $1_{(x)} \in \mathcal{C}_T$ defined by

$$1_{(x)} = \begin{cases} 0 & : (x) \text{ dies before or at time } T \\ 1 & : (x) \text{ dies after time } T \end{cases} \tag{1}$$

The combined financial-actuarial world $(\Omega, \mathbb{G}, \mathbb{P})$ is also home to a financial market of $n+1$ tradable (non-dividend paying[1]) assets. The tradable assets can be stocks, bonds, mutual funds, options, etc. We introduce the notation $Y^{(i)}(t)$ for the $t-$claim denoting the market price of risky asset i at time $t \in \tau$. Moreover, we assume that any tradable asset can be bought and/or sold in any quantities in a deep, liquid and transparent market with negligible transactions costs and other market frictions.

The price processes of the traded assets are described by the $(n+1)-$dimensional stochastic process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\}_{t \in \tau}$. Here, $\boldsymbol{Y}(t)$, $t \in \tau$, is the vector of time$-t$ prices of

[1]Without loss of generality, we assume that there are no dividends. Otherwise, one can replace the traded asset by the gain process of the traded asset, which is the sum of its price process and the process describing its accumulated dividends.

all tradable assets, i.e. $\boldsymbol{Y}(t) = \left(Y^{(0)}(t), Y^{(1)}(t), \ldots, Y^{(n)}(t)\right)$. We assume that the price process $\boldsymbol{Y}$ is adapted to the filtration $\mathbb{G}$, which means that

$$\boldsymbol{Y}(t) \text{ is } \mathcal{G}_t - \text{measurable}, \qquad \text{for any } t = 0, 1, , \ldots, T.$$

The filtration $\mathbb{G}$ may simply coincide with the filtration generated by the price process $\boldsymbol{Y}$. In this paper however, we will consider a more general setting, where $\mathbb{G}$ is not only related to the price history of traded assets, but may also contain additional information, such as information related to non-tradable claims or a survival index of a particular population.

A *time−t trading strategy* (also called a *time−t dynamic portfolio*), $t \in \{0, \ldots, T-1\}$, is an $(n+1)$ −dimensional *predictable* process $\boldsymbol{\theta}_t = \{\boldsymbol{\theta}_t(u)\}_{u \in \{t+1, \ldots, T\}}$ with respect to the filtration $\mathbb{G}$. The predictability requirement means that

$$\boldsymbol{\theta}_t(u) \text{ is } \mathcal{G}_{u-1} - \text{measurable}, \qquad \text{for any } u = t+1, \ldots, T.$$

Notice that a time−t trading strategy is only set up at time t by acquiring a portfolio $\boldsymbol{\theta}_t(t+1)$ at that time. Introducing the notations $\boldsymbol{\theta}_t(u) = \left(\theta_t^{(0)}(u), \theta_t^{(1)}(u) \ldots, \theta_t^{(n)}(u)\right)$ for the components of $\boldsymbol{\theta}_t(u)$, we interpret the quantity $\theta_t^{(i)}(u)$ as the number of units invested in asset i in time period u, that is in the time interval $(u-1, u]$. The $\mathcal{G}_{u-1}$−measurability requirement means that the portfolio composition $\boldsymbol{\theta}_t(u)$ for the time period u follows from the general information available up to and including time $u-1$. This information includes, but is broader than the price history of traded assets in that time interval.

The *initial investment* or the *endowment* at time t of the trading strategy $\boldsymbol{\theta}_t$ can be expressed as

$$\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = \sum_{i=0}^{n} \theta_t^{(i)}(t+1) \times Y^{(i)}(t).$$

The value of the trading strategy $\boldsymbol{\theta}_t$ at time u, just before rebalancing, is given by

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \sum_{i=0}^{n} \theta_t^{(i)}(u) \times Y^{(i)}(u), \qquad \text{for any } u = t+1, \ldots, T,$$

whereas its value at time u, just after rebalancing, is given by

$$\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u) = \sum_{i=0}^{n} \theta_t^{(i)}(u+1) \times Y^{(i)}(u), \qquad \text{for any } u = t+1, \ldots, T-1.$$

Obviously, $\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u)$ and $\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u)$ are $\mathcal{G}_u$−measurable.

A time−t trading strategy $\boldsymbol{\theta}_t$ is said to be *self-financing* if

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u), \qquad \text{for any } u = t+1, ..., T-1. \tag{2}$$

This means that no capital is injected or withdrawn at any rebalancing moment $u = t+1, ..., T-1$. We denote the set of self-financing time−t trading strategies by Θ_t.

Taking into account (2), the time$-T$ value of any self-financing time$-t$ strategy $\boldsymbol{\theta}_t \in \Theta_t$ can be expressed as

$$\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) = \boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) + \sum_{u=t+1}^{T} \boldsymbol{\theta}_t(u) \cdot \Delta \boldsymbol{Y}(u), \tag{3}$$

with $\Delta \boldsymbol{Y}(u) = \boldsymbol{Y}(u) - \boldsymbol{Y}(u-1)$. In this formula, $\boldsymbol{\theta}_t(u) \cdot \Delta \boldsymbol{Y}(u)$ is the change of the market value of the investment portfolio in the time period u, i.e. between time $u-1$ (just after rebalancing) and time u (just before rebalancing).

We assume that the market of traded assets is *arbitrage-free* in the sense that there is no self-financing strategy $\boldsymbol{\theta}_0 \in \Theta_0$ with the following properties:

$$\boldsymbol{\theta}_0(1) \cdot \boldsymbol{Y}(0) = 0,\ \mathbb{P}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) \geq 0\right] = 1 \text{ and } \mathbb{P}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) > 0\right] > 0. \tag{4}$$

In our discrete-time setting, the absence of arbitrage is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$ (further abbreviated as EMM), under which the discounted price process $\boldsymbol{Y}$ is a $\mathcal{G}-$martingale:

$$\boldsymbol{Y}(t-1) = \mathbb{E}_{t-1}^{\mathbb{Q}}\left[e^{-\int_{t-1}^{t} r_s ds} \boldsymbol{Y}(t)\right], \qquad \text{for any } t = 1, ..., T, \tag{5}$$

for some (possibly stochastic) interest rate r_s. For the rest of the paper, we will use the notation $\mathbb{E}_t^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot | \mathcal{G}_t]$. For a proof of this equivalence, we refer to Delbaen and Schachermayer (2006).
Consider a time$-t$ self-financing strategy $\boldsymbol{\theta}_t \in \Theta_t$. From (5) it follows that its time$-u$ price is given by

$$\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u) = \mathbb{E}_u^{\mathbb{Q}}\left[e^{-\int_u^T r_s ds} \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T)\right], \qquad \text{for any } u = t, ..., T-1. \tag{6}$$

In the remainder of the paper, we assume that the asset 0 is the zero-coupon bond paying an amount of 1 at maturity T. Its price at time t, denoted by $B(t,T)$, is given by

$$Y^{(0)}(t) = B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right], \qquad \text{for any } t = 0, 1, ..., T-1.$$

A simple example of a self-financing time$-t$ trading strategy is the static trading strategy $\boldsymbol{\beta}_t$ consisting of buying one unit of the zero-coupon bond $B(t,T)$ at time t and holding it until maturity T. The value of this strategy at time u is given by

$$\boldsymbol{\beta}_t(u) \cdot \boldsymbol{Y}(u) = \mathbb{E}_u^{\mathbb{Q}}\left[e^{-\int_u^T r_s ds}\right], \qquad \text{for any } u = t+1, ..., T.$$

Definition 1 ($t-$hedgeable $T-$claim) *A $t-$hedgeable $T-$claim S^h is an element of $\mathcal{C}_T$ which can be replicated by a time$-t$ self-financing strategy $\boldsymbol{\theta}_t \in \Theta_t$:*

$$S^h = \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T),$$

where $\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T)$ is the time$-T$ value of the hedging portfolio $\boldsymbol{\theta}_t$.

We introduce the notation $\mathcal{H}_T^t$ for the set of all time$-t$ hedgeable $T-$claims. For any time$-t$ hedgeable T-claim S^h, a time$-t$ trading strategy which replicates S^h is called a *replicating $t-$hedge* of S^h.

The time$-t$ price of S^h is given by

$$\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T)],$$

where $\mathbb{Q}$ is a generic member of the class of EMM's and $\boldsymbol{\theta}_t$ is a replicating $t-$hedge of S^h.

Notice that $\mathcal{H}_T^t$ is increasing in t. The $T-$claim

$$S = Y^{(1)}(t)\, Y^{(2)}(T),$$

is an example of a $T-$claim which will in general not be an element of $\mathcal{H}_T^s$ for any $s = 0, 1, \ldots, t-1$, while $S \in \mathcal{H}_T^s$ for any $s = t, t+1, \ldots, T-1$.

Next, we introduce the notion of $t-$orthogonal $T-$claims.

Definition 2 ($t-$orthogonal $T-$claim) *A $t-$orthogonal $T-$claim $S^{\perp}$ is an element of $\mathcal{C}_T$ which is $\mathbb{P}$-independent of the stochastic process $\mathbf{Y}_{t+1} = \{\boldsymbol{Y}(u)\}_{u \in \{t+1,\ldots,T\}}$ describing the evolution of the traded assets from $t+1$ onwards:*

$$S^{\perp} \perp \mathbf{Y}_{t+1}.$$

Hereafter, we will denote the set of all $t-$orthogonal T-claims by $\mathcal{O}_T^t$. We remark that the set $\mathcal{O}_T^t$ is also increasing in t. An example of a $T-$claim which does not belong to the initial set of orthogonal claims $\mathcal{O}_T^0$, but which is an element of $\mathcal{O}_T^t$ is given by

$$S = \frac{1}{t} \sum_{i=1}^{t} Y^{(1)}(i)\ 1_{(x)}$$

where $1_{(x)}$ is the indicator variable which equals 1 if (x) survives until time T and 0 otherwise. Hence, in case of survival, the claim guarantees the average price of asset 1 between time 1 and time t. Under independence between mortality and the traded assets, we have that $S \notin \mathcal{O}_T^u$, for $u = 0, 1, \ldots, t-1$, while $S \in \mathcal{O}_T^u$ for $u = t, t+1, \ldots, T$.

3 $t-$valuations

In this section, we define different classes of $t-$valuations. In a dynamic multiperiod setting, a $t-$valuation ρ_t assigns to each $T-$claim a $\mathcal{G}_t-$measurable random variable $\rho_t[S]$ that represents the value of the $T-$claim given the available information at time t. In Dhaene et al. (2017) fair valuations of insurance claims in a static one-period setting are considered. The authors showed that any fair valuation can be characterized in terms of a fair hedger. In this section, we generalize this result in a dynamic setting by showing that any fair $t-$valuation can be characterized in terms of a fair $t-$hedger.

3.1 Fair $t-$valuations

In this subsection, we define the notion of $t-$valuation. Furthermore, we introduce the notions of actuarial, market-consistent and fair $t-$valuations, respectively.

Definition 3 ($t-$valuation) *A $t-$valuation, $t = 0, 1, \ldots, T-1$, is a mapping $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$, attaching a $t-$claim to any T-claim $S \in \mathcal{C}_T$:*

$$S \to \rho_t [S],$$

such that

- *ρ_t is normalized:*
$$\rho_t [0] = 0.$$
- *ρ_t is translation invariant:*
$$\rho_t [S + a] = \rho_t [S] + B(t,T)a, \qquad \text{for any } S \in \mathcal{C}_T \text{ and } a \in \mathcal{C}_t \text{ payable at } T.$$

For any $T-$claim, the value $\rho_t [S]$ is a $t-$claim and hence, seen from the perspective of time 0, it is a random variable. On the other hand, having arrived at time t, $\rho_t [S]$ is clearly deterministic. In Pelsser and Stadje (2014), $t-$valuations are called $\mathcal{G}_t-$conditional evaluations.

Important subclasses of $t-$valuations include the class of actuarial and market-consistent $t-$valuations, which are defined hereafter.

Definition 4 (Actuarial and market-consistent $t-$valuations) *Consider a $t-$valuation $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$.*

- *ρ_t is actuarial if any $t-$orthogonal T-claim is marked-to-model:*
$$\rho_t \left[S^{\perp}\right] = B(t,T)\pi_t \left[S^{\perp}\right], \qquad \text{for any } S^{\perp} \in \mathcal{O}_T^t, \tag{7}$$
where the $t-$valuation $\pi_t : \mathcal{O}_T^t \to \mathcal{C}_t$ is $\mathbb{P}-$law invariant and $\mathbb{P}$-independent of time$-t$ and future asset prices $\mathbf{Y}_t = \{\boldsymbol{Y}(u)\}_{u\in\{t,\ldots,T\}}$.
- *ρ_t is market-consistent (MC) if any $t-$hedgeable part of any $T-$claim is marked-to-market:*
$$\rho_t \left[S + S^h\right] = \rho_t[S] + \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} S^h\right], \tag{8}$$
for any $S \in \mathcal{C}_T$ and $S^h \in \mathcal{H}_T^t$.

The mark-to-model condition (7) corresponds to the traditional valuation of orthogonal (i.e. non-equity-linked) claims in an insurance context. It postulates that any $t-$orthogonal claim is valuated by a $\mathbb{P}-$law invariant $t-$valuation π_t (e.g. standard deviation principle, mean-variance principle,...) multiplied by the time-t zero-coupon bond price $B(t,T)$. For instance, in case π_t is the standard deviation principle, we find that

$$\rho_t\left[S^{\perp}\right] = \left(\mathbb{E}_t^{\mathbb{P}}\left[S^{\perp}\right] + \alpha\sigma_t^{\mathbb{P}}\left[S^{\perp}\right]\right)B(t,T),$$

with $\sigma_t^{\mathbb{P}}\left[S^{\perp}\right] := \sqrt{Var^{\mathbb{P}}\left[S^{\perp} \mid \mathcal{G}_t\right]}$ and $\alpha > 0$. As another example, one may consider a distorted expectation as an actuarial valuation:

$$\rho_t\left[S^{\perp}\right] = \mathbb{E}_t^{\mathbb{P}^*}\left[S^{\perp}\right]B(t,T).$$

In this case, the distorted probability measure $\mathbb{P}^* \sim \mathbb{P}$ is there to take into account the uncertainty in the orthogonal claims, see e.g. Chapter 2.6 in Wüthrich (2016).

Moreover, we make the technical requirement that $\pi_t\left[S^{\perp}\right]$ is $\mathbb{P}$-independent of asset prices at time t and beyond: $\mathbf{Y}_t = \{\boldsymbol{Y}(u)\}_{u\in\{t,\ldots,T\}}$ for any $S^{\perp} \in \mathcal{O}_T^t$. Otherwise stated, the actuarial value of a claim independent of future asset prices is independent of time$-t$ *and* future asset prices. This intuitive requirement will be used in the proof of Theorem 2.

In the literature, market-consistency is usually defined via a condition identical or similar to the condition (8), see e.g. Kupper et al. (2008), Malamud et al. (2008), Artzner and Eisele (2010) and Pelsser and Stadje (2014). This mark-to-market condition extends the notion of cash-invariance to all $t-$hedgeable claims by postulating that any $t-$hedgeable claim should be valuated at the price of its replicating $t-$hedge. We remark that the mark-to-market condition can also be expressed as follows:

$$\rho_t\left[S + S^h\right] = \rho_t[S] + \boldsymbol{\theta}_t(t+1)\cdot\boldsymbol{Y}(t), \tag{9}$$

for any $S \in \mathcal{C}_T$ and $S^h \in \mathcal{H}_T^t$, with $\boldsymbol{\theta}_t$ a replicating $t-$hedge of S^h.

Combining these notions leads to the definition of a fair $t-$valuation.

Definition 5 (Fair $t-$valuation) *A fair $t-$valuation is a $t-$valuation which is both actuarial and market-consistent.*

Hereafter, we provide a simple example of a fair $t-$valuation for equity-linked life-insurance contracts.

Example 1 [**Fair $t-$valuation of product claims**]
Consider a $T-$claim S for which we want to determine the fair valuation at time t. We assume that we can decompose the claim as follows

$$S = S^{\perp} \times S^h,$$

where $S^{\perp}$ is a $t-$orthogonal $T-$claim and S^h is a $t-$hedgeable $T-$claim.
Such *product* claims often arise in insurance as payoffs of equity-linked life-insurance

contracts. In such payoffs, S^h is typically a hedgeable claim contingent on the price history of traded assets such as stock, mutual funds, options or bonds while $S^\perp$ is contingent on the survival or death of a policyholder. For any product $T-$claim S, we define the $t-$valuation

$$\rho_t\left[S\right] = \mathbb{E}_t^{\mathbb{P}}\left[S^\perp\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}S^h\right], \qquad \text{for any } S^\perp \in \mathcal{O}_T^t \text{ and } S^h \in \mathcal{H}_T^t.$$

Hence, the $t-$valuation ρ_t appears as a product of two expectations. The non-equity linked part $S^\perp$ is valuated under the physical measure $\mathbb{P}$ modeling the non-hedgeable risks and the hedgeable part S^h is valuated under a risk-neutral measure $\mathbb{Q}$ modeling hedgeable risks.
One can easily verify that the $t-$valuation ρ_t is actuarial:

$$\rho_t\left[S^\perp\right] = \mathbb{E}_t^{\mathbb{P}}\left[S^\perp\right]B(t,T),$$

and market-consistent:

$$\rho_t\left[S + S^h\right] = \rho_t[S] + \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}S^h\right].$$

3.2 Fair $t-$hedgers

In this section, we introduce the class of $t-$hedgers, as well as the subclasses of actuarial, market-consistent and fair $t-$hedgers. These notions are generalizations of the time-0 hedgers which were defined in Dhaene et al. (2017). In the forthcoming sections of this paper, we will use these notions to express our main results.

Definition 6 ($t-$hedger) *A $t-$hedger is a function $\boldsymbol{\theta}_t : \mathcal{C}_T \to \Theta_t$ which maps any $T-$claim S into a self-financing time$-t$ trading strategy $\boldsymbol{\theta}_{t,S} \in \Theta_t$ such that*

- *$\boldsymbol{\theta}_t$ is normalized:*

$$\boldsymbol{\theta}_{t,0} = \mathbf{0}_t,$$

  where $\mathbf{0}_t$ is the self-financing time$-t$ trading strategy corresponding to the null investment at time t, i.e. $\mathbf{0}_t(u) = (0, 0, \ldots, 0)$ for all $u = t+1, \ldots, T$.

- *$\boldsymbol{\theta}_t$ is translation invariant:*

$$\boldsymbol{\theta}_{t,S+a} = \boldsymbol{\theta}_{t,S} + a\boldsymbol{\beta}_t, \qquad \text{for any } S \in \mathcal{C}_T \text{ and } a \in \mathcal{C}_t \text{ payable at } T,$$

  where $\boldsymbol{\beta}_t$ is the static trading strategy which consists in buying one unit of the zero-coupon bond $B(t,T)$ and holding it until maturity T.

The mapping $\boldsymbol{\theta}_t : \mathcal{C}_T \to \Theta_t$ is called a $t-$hedger, whereas for any T-claim S, the self-financing trading strategy $\boldsymbol{\theta}_{t,S}$ is called a $t-$hedge for S. The value of the hedge $\boldsymbol{\theta}_{t,S}$ of S at time $u = t+1, \ldots, T$, before rebalancing, is given by $\boldsymbol{\theta}_{t,S}(u) \cdot \boldsymbol{Y}(u)$, while after rebalancing, it is $\boldsymbol{\theta}_{t,S}(u+1) \cdot \boldsymbol{Y}(u)$.

Hereafter, we introduce the subclasses of actuarial, market-consistent and fair $t-$hedgers.

Definition 7 (Actuarial and market-consistent $t-$hedgers) *Consider a $t-$hedger $\boldsymbol{\theta}_t$.*

- $\boldsymbol{\theta}_t$ *is actuarial in case any $t-$orthogonal $T-$claim $S^\perp$ is hedged via an actuarial $t-$valuation ρ_t in zero-coupon bonds:*

$$\boldsymbol{\theta}_{t,S^\perp} = \frac{\rho_t\left[S^\perp\right]}{B(t,T)}\boldsymbol{\beta}_t \qquad \text{for any } S^\perp \in \mathcal{O}_T^t. \tag{10}$$

- $\boldsymbol{\theta}_t$ *is market-consistent (MC) in case any $t-$hedgeable part S^h of any T-claim S is hedged by a replicating hedge:*

$$\boldsymbol{\theta}_{t,S+S^h} = \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h}, \qquad \text{for any } S \in \mathcal{C}_T \text{ and any } S^h \in \mathcal{H}_T^t, \tag{11}$$

  where $\boldsymbol{\theta}_{t,S^h}$ is a replicating $t-$hedge of S^h.

We remark that an actuarial $t-$hedger $\boldsymbol{\theta}_t$ is defined in terms of an actuarial $t-$valuation ρ_t. Hereafter, we will call ρ_t the underlying actuarial $t-$valuation of the actuarial $t-$hedger $\boldsymbol{\theta}_t$.

Combining the definitions of actuarial and market-consistent $t-$hedgers leads to the definition of fair $t-$hedgers.

Definition 8 (Fair $t-$hedger) *A $t-$hedger is fair in case it is actuarial and market-consistent.*

In the remainder of the paper, we often consider the trading strategy which consists in investing (at time t) $\rho_t[S]$ in the zero-coupon bond $B(t,T)$, for $t = 0, 1, ..., T-1$. It is clear that the initial investment at time t of this trading strategy is $\rho_t[S]$ and its time$-T$ value, denoted by $\widetilde{\rho}_t$, is given by

$$\widetilde{\rho}_t[S] = \frac{\rho_t[S]}{B(t,T)}. \tag{12}$$

Hereafter, we provide an example of a fair $t-$hedger. This will be used later in the proof of Theorem 1.

Example 2 Fix $t \in \{0, \dots, T-1\}$ and define the $t-$hedger $\boldsymbol{\theta}_t$ as follows:

1. For any $t-$orthogonal $T-$claim $S^\perp \in \mathcal{O}_T^t$, we define the $t-$hedger $\boldsymbol{\theta}_t$ by
$$\boldsymbol{\theta}_{t,S^\perp} = \mathbb{E}_t^{\mathbb{P}}\left[S^\perp\right]\boldsymbol{\beta}_t.$$
2. For all other $T-$claims $S \notin \mathcal{O}_T^t$, the $t-$hedger $\boldsymbol{\theta}_t$ is defined as the mean-variance hedger:
$$\boldsymbol{\theta}_{t,S} = \arg\min_{\boldsymbol{\theta}\in\Theta_t} \mathbb{E}_t^{\mathbb{P}}\left[\left(S - \boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right)^2\right]. \tag{13}$$
   As we assume that the time$-T$ value of any time$-t$ trading strategy is square-integrable, a solution to the optimization problem (13) exists (see for instance Černỳ and Kallsen (2009)). It is then easy to verify that $\boldsymbol{\theta}_t$ is well defined and a fair $t-$hedger.

3.3 Characterization of $t-$valuations

In the following lemma, we consider properties of a $t-$hedger $\boldsymbol{\mu}_{t,S}$ which is defined as the sum of another $t-$hedger $\boldsymbol{\theta}_{t,S}$ and an investment in zero-coupon bonds of the remaining risk $S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)$. The proof of a forthcoming theorem is based on the construction of such hedgers.

Lemma 1 *Consider a $t-$hedger $\boldsymbol{\theta}_t$ and a $t-$valuation ρ_t. Define the $t-$hedger $\boldsymbol{\mu}_t$ by*

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,S} + \widetilde{\rho}_t \left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right] \ \boldsymbol{\beta}_t, \qquad \textit{for any } S \in \mathcal{C}_T, \tag{14}$$

(a) If $\boldsymbol{\theta}_t$ is an actuarial $t-$hedger and ρ_t is an actuarial $t-$valuation, then $\boldsymbol{\mu}_t$ is an actuarial $t-$hedger with underlying actuarial $t-$valuation ρ_t.

(b) If $\boldsymbol{\theta}_t$ is a MC $t-$hedger, then $\boldsymbol{\mu}_t$ is a MC $t-$hedger and $\boldsymbol{\mu}_{t,S^h} = \boldsymbol{\theta}_{t,S^h}$ for any $t-$hedgeable $T-$claim S^h.

(c) If $\boldsymbol{\theta}_t$ is a fair $t-$hedger and ρ_t is an actuarial $t-$valuation, then $\boldsymbol{\mu}_t$ is a fair $t-$hedger with underlying actuarial $t-$valuation ρ_t.

Proof: It is a straightforward exercise to verify that $\boldsymbol{\mu}_t$ is a $t-$hedger.
(a) Suppose that $\boldsymbol{\theta}_t$ is an actuarial $t-$hedger with underlying actuarial $t-$valuation ψ_t. Further, suppose that ρ_t is an actuarial $t-$valuation. For any $t-$orthogonal $T-$claim $S^\perp$, we have

$$\begin{aligned}
\boldsymbol{\mu}_{t,S^\perp} &= \boldsymbol{\theta}_{t,S^\perp} + \widetilde{\rho}_t \left[S^\perp - \boldsymbol{\theta}_{t,S^\perp}(T) \cdot \boldsymbol{Y}(T)\right] \ \boldsymbol{\beta}_t \\
&= \widetilde{\psi}_t \left[S^\perp\right] \ \boldsymbol{\beta}_t + \widetilde{\rho}_t \left[S^\perp - \widetilde{\psi}_t \left[S^\perp\right]\right] \ \boldsymbol{\beta}_t \\
&= \widetilde{\rho}_t \left[S^\perp\right] \ \boldsymbol{\beta}_t,
\end{aligned}$$

where in the last step, we used the translation invariance of ρ_t. We can conclude that $\boldsymbol{\mu}_t$ is an actuarial $t-$hedger with underlying actuarial $t-$valuation ρ_t.
(b) Suppose that $\boldsymbol{\theta}_t$ is a MC $t-$hedger. By definition of $\boldsymbol{\mu}_t$, we have that

$$\boldsymbol{\mu}_{t,S+S^h} = \boldsymbol{\theta}_{t,S+S^h} + \widetilde{\rho}_t \left[S + S^h - \boldsymbol{\theta}_{t,S+S^h}(T) \cdot \boldsymbol{Y}(T)\right] \ \boldsymbol{\beta}_t, \qquad \text{for any } S^h \in \mathcal{H}_T^t.$$

Given that $\boldsymbol{\theta}_t$ is a MC $t-$hedger, we find

$$\begin{aligned}
\boldsymbol{\mu}_{t,S+S^h} &= \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h} + \widetilde{\rho}_t \left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right] \ \boldsymbol{\beta}_t \\
&= \boldsymbol{\mu}_{t,S} + \boldsymbol{\theta}_{t,S^h}.
\end{aligned}$$

We can conclude that $\boldsymbol{\mu}_t$ is a MC $t-$hedger.
(c) Finally, suppose that $\boldsymbol{\theta}_t$ is a fair $t-$hedger with underlying actuarial $t-$valuation ψ_t,

while ρ_t is an actuarial $t-$valuation. From (a) and (b) it follows immediately that $\boldsymbol{\mu}_t$ is a fair $t-$hedger with underlying actuarial $t-$valuation ρ_t. ■

In the following theorem it is shown that any actuarial $t-$valuation ρ_t can be represented as the time$-t$ price of an actuarial $t-$hedger. Similar properties hold for market-consistent and fair $t-$valuations.

Theorem 1 *Consider a $t-$valuation $\rho_t : \mathcal{C}_T \longrightarrow \mathcal{C}_t$.*
(a) ρ_t is an actuarial $t-$valuation if and only if there exists an actuarial $t-$hedger $\boldsymbol{\theta}_t^a$ such that

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}^a(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{15}$$

(b) ρ_t is a MC $t-$valuation if and only if there exists a MC $t-$hedger $\boldsymbol{\theta}_t^m$ such that

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}^m(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{16}$$

(c) ρ_t is a fair $t-$valuation if and only if there exists a fair $t-$hedger $\boldsymbol{\theta}_t^f$ such that

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}^f(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{17}$$

Proof: (a) Let ρ_t be an actuarial $t-$valuation. For any $S \in \mathcal{C}_T$, we can write $\rho_t[S]$ as

$$\begin{aligned} \rho_t[S] &= \widetilde{\rho}_t[S]\, B(t,T) \\ &= \boldsymbol{\theta}_{t,S}^a(t+1) \cdot \boldsymbol{Y}(t), \end{aligned}$$

with $\boldsymbol{\theta}_{t,S}^a$ defined by

$$\boldsymbol{\theta}_{t,S}^a = \widetilde{\rho}_t[S]\ \boldsymbol{\beta}_t.$$

Obviously, $\boldsymbol{\theta}_t^a$ is an actuarial $t-$hedger.
(a') Suppose that the $t-$valuation ρ_t is defined by (15) for some actuarial $t-$hedger $\boldsymbol{\theta}_t^a$ with underlying actuarial $t-$valuation π_t. For any $t-$orthogonal $T-$claim $S^\perp$, we have

$$\rho_t\left[S^\perp\right] = \boldsymbol{\theta}_{t,S^\perp}^a(t+1) \cdot \boldsymbol{Y}(t) = \pi_t\left[S^\perp\right].$$

We can conclude that the valuation ρ_t is an actuarial $t-$valuation.
(b) Let ρ_t be a MC $t-$valuation. Consider a MC $t-$hedger $\boldsymbol{\theta}_t$, e.g. the $t-$hedger defined in Example 2. For any $T-$claim S, we find from (8) that

$$\begin{aligned} \rho_t[S] &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right] + \rho_t\left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right] \\ &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t\left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right] \\ &= \boldsymbol{\theta}_{t,S}^m(t+1) \cdot \boldsymbol{Y}(t), \end{aligned}$$

with

$$\boldsymbol{\theta}_{t,S}^m = \boldsymbol{\theta}_{t,S} + \widetilde{\rho}_t\left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t. \tag{18}$$

From Lemma 1 we know that $\boldsymbol{\theta}^m$ is a MC $t-$hedger.
(b') Consider the $t-$valuation ρ_t defined by (16) for some MC $t-$hedger $\boldsymbol{\theta}_t^m$. For any $T-$claim S and any $t-$hedgeable $T-$claim S^h, we find that

$$\begin{aligned}\rho_t\left[S+S^h\right] &= \boldsymbol{\theta}^m_{t,S+S^h}(t+1)\cdot \boldsymbol{Y}(t)\\ &= \boldsymbol{\theta}^m_{t,S}(t+1)\cdot \boldsymbol{Y}(t) + \boldsymbol{\theta}^m_{t,S^h}(t+1)\cdot \boldsymbol{Y}(t)\\ &= \rho_t\left[S\right] + \rho_t\left[S^h\right].\end{aligned}$$

We can conclude that ρ_t is a MC $t-$valuation.
(c) Let ρ_t be a fair $t-$valuation. Consider a fair $t-$hedger $\boldsymbol{\theta}_t$, e.g. the $t-$hedger defined in Example 2, with underlying actuarial $t-$valuation ψ_t. From (a) we know that for any $T-$claim S, $\rho_t[S]$ can be expressed as

$$\rho_t\left[S\right] = \boldsymbol{\theta}^m_{t,S}(t+1)\cdot \boldsymbol{Y}(t),$$

with the MC $t-$hedger $\boldsymbol{\theta}_t^m$ given by (18). For any $t-$orthogonal $T-$claim $S^\perp$, we find that

$$\begin{aligned}\boldsymbol{\theta}^m_{t,S^\perp} &= \boldsymbol{\theta}_{t,S^\perp} + \widetilde{\rho}_t\left[S^\perp - \boldsymbol{\theta}^m_{t,S^\perp}(T)\cdot \boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t\\ &= \widetilde{\psi}_t\left[S^\perp\right]\boldsymbol{\beta}_t + \widetilde{\rho}_t\left[S^\perp - \widetilde{\psi}_t\left[S^\perp\right]\right]\ \boldsymbol{\beta}_t\\ &= \widetilde{\rho}_t\left[S^\perp\right]\ \boldsymbol{\beta}_t.\end{aligned}$$

As ρ_t is an actuarial valuation, we can conclude that the $t-$hedger $\boldsymbol{\theta}_t^m$ is not only market-consistent but also actuarial and hence, a fair $t-$hedger.
(c') Suppose that the $t-$valuation ρ_t is defined by (17) for some fair $t-$hedger $\boldsymbol{\theta}_t^f$. From (a) and (b) we can conclude that the $t-$valuation ρ_t is actuarial and market-consistent, which means that it is fair. ■

Taking into account (6), we have that the relation (17) for a fair $t-$valuation can be rewritten as follows:

$$\rho_t\left[S\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\boldsymbol{\theta}^f_{t,S}(T)\cdot \boldsymbol{Y}(T)\right], \qquad \text{for any } S\in\mathcal{C}_T. \tag{19}$$

The fair valuation at time t of any $T-$claim can then be expressed as a conditional expectation of the time$-T$ value of a fair hedge for S, $\boldsymbol{\theta}^f_{t,S}$, under an equivalent martingale measure $\mathbb{Q}$. Actuarial considerations are implicitly involved since any fair valuation is an actuarial valuation, implying actuarial judgement on the valuation of orthogonal claims.

4 Dynamic valuations

In the previous section, we introduced the concept of $t-$valuations which assess a time$-t$ value for any $T-$claim, taking into account the available information at time t, for any time $t = 0, 1, \ldots, T-1$. This approach was static in the sense that we considered the value of a $T-$claim at different times $t < T$, without specifying the interconnection between the $t-$valuations. Bringing the $t-$valuations together leads to the concepts of *time-consistent* and *dynamic* valuations, which are defined hereafter.

4.1 Fair dynamic valuations

In the following definition, we introduce the notion of dynamic valuation. See for instance Acciaio and Penner (2011), Artzner et al. (2007) or Riedel (2004) for similar notions.

Definition 9 (Dynamic valuation) *A dynamic valuation is a sequence* $(\rho_t)_{t=0}^{T-1}$ *where for each* $t = 0, 1, ..., T-1$, ρ_t *is a* $t-$*valuation.*

After having introduced the concept of dynamic valuation, we now define actuarial, market-consistent and time-consistent dynamic valuations. Notice that a $t-$valuation ρ_t is defined for $T-$claims S payable at time T. In order to compare $t-$valuations at different times, we consider the $t-$valuation $\widetilde{\rho}_t\left[S\right]$ introduced in (12) which corresponds to the value at time T of the investment of the $t-$valuation $\rho_t\left[S\right]$ in the zero-coupon bond $B(t,T)$.

Definition 10 (Actuarial, MC and TC dynamic valuations) *Consider the dynamic valuation* $(\rho_t)_{t=0}^{T-1}$.

- $(\rho_t)_{t=0}^{T-1}$ *is* <u>*actuarial*</u> *in case any* $t-$*valuation* ρ_t *is actuarial.*
- $(\rho_t)_{t=0}^{T-1}$ *is* <u>*market-consistent*</u> *(MC) in case any* $t-$*valuation* ρ_t *is market-consistent.*
- $(\rho_t)_{t=0}^{T-1}$ *is* <u>*time-consistent*</u> *(TC) in case all* $t-$*valuations involved are connected in the following way:*

$$\rho_t\left[S\right] = \rho_t\left[\widetilde{\rho}_{t+1}\left[S\right]\right], \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, ..., T-2. \tag{20}$$

Actuarial and market-consistent dynamic valuations are natural generalizations of actuarial and market-consistent $t-$valuations. Time-consistency is a concept that couples the different static $t-$valuations. It means that the same time$-t$ value is assigned to a $T-$claim regardless of whether it is calculated in one step or in two steps backwards in time. Some weaker notions of time-consistency have been proposed in the literature, see e.g. Roorda et al. (2005) and Kriele and Wolf (2014). The definition (20) is often named the "recursiveness" or "tower property" definition. In the literature, an alternative definition of time-consistency is often used: if a claim is preferred to another claim at time $t+1$ in almost all states of nature, then the same conclusions should be drawn at time t:

$$\rho_{t+1}\left[S_1\right] \leq \rho_{t+1}\left[S_2\right] \Longrightarrow \rho_t\left[S_1\right] \leq \rho_t\left[S_2\right] \quad \text{for all } S_1, S_2 \in \mathcal{C}_T \text{ and } t < T. \tag{21}$$

Under monotonicity of the dynamic valuation $(\rho_t)_{t=0}^{T-1}$, it is well-known that both notions of time-consistency are equivalent (see for instance Acciaio and Penner (2011)). Since (21) implies monotonicity, the advantage of using the definition (20) is that we can also apply time-consistency to non-monotone dynamic valuations.

Time-consistent valuations have been discussed extensively in recent years. For the discrete time case, we refer to Cheridito and Kupper (2011), Acciaio and Penner (2011) and Föllmer and Schied (2011). For the continuous case, we refer to Frittelli and Gianin (2004), Delbaen et al. (2010), Pelsser and Stadje (2014) and Feinstein and Rudloff (2015).

Merging the notions of actuarial, market-consistent and time-consistent valuations leads to the concept of fair dynamic valuations.

Definition 11 (Fair dynamic valuations) *A fair dynamic valuation is a dynamic valuation which is actuarial, market-consistent and time-consistent.*

4.2 Fair dynamic hedgers

After having defined the class of $t-$hedgers in the previous section, we introduce the notion of a dynamic hedger.

Definition 12 (Dynamic hedger) *A dynamic hedger is a sequence* $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ *where for each* $t = 0, 1, \ldots, T-1$, $\boldsymbol{\theta}_t$ *is a* $t-$*hedger.*

Hereafter, we introduce natural definitions of actuarial, market-consistent and time-consistent dynamic hedgers in accordance with Definition 10.

Definition 13 (Actuarial, MC and TC dynamic hedgers) *Consider the dynamic hedger* $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$.

- $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ *is actuarial in case any* $t-$*hedger* $\boldsymbol{\theta}_t$ *is actuarial.*
- $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ *is market-consistent (MC) in case any* $t-$*hedger* $\boldsymbol{\theta}_t$ *is market-consistent.*
- $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ *is time-consistent (TC) in case all* $t-$*hedgers involved are connected in the following way:*

$$\boldsymbol{\theta}_{t,S} = \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}, \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, ..., T-2, \tag{22}$$

where $\rho_{t+1}[S]$ *is the initial investment of* $\boldsymbol{\theta}_{t+1}$*:*

$$\rho_{t+1}[S] = \boldsymbol{\theta}_{t+1,S}(t+2) \cdot \boldsymbol{Y}(t+1).$$

The definition of a time-consistent dynamic hedger should be compared with the definition of a time-consistent dynamic valuation. It means that the same hedger is assigned to a $T-$claim regardless of whether it is hedged in one step (i.e. directly over $T-t$ periods) or in two steps backwards in time.

Similarly to the concept of fair dynamic valuations, we introduce the concept of fair dynamic hedgers.

Definition 14 (Fair dynamic hedgers) *A fair dynamic hedger is a dynamic hedger which is actuarial, market-consistent and time-consistent.*

4.3 Characterization of fair dynamic valuations

In the following theorem we show that a fair dynamic valuation can be characterized in terms of a fair dynamic hedger.

Theorem 2 *A dynamic valuation* $(\rho_t)_{t=0}^{T-1}$ *is fair if and only if there exists a fair dynamic hedger* $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ *such that*

$$\rho_t[S] = \boldsymbol{\mu}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \qquad \textit{for any } S \in \mathcal{C}_T. \tag{23}$$

Proof: (a) Suppose that $(\rho_t)_{t=0}^{T-1}$ is a fair dynamic valuation. From Theorem 1, we have that for any $t = 0, 1, \ldots, T-1$, there exists a fair $t-$hedger $\boldsymbol{\theta}_t$ such that

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{24}$$

The dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is actuarial and market-consistent but is a priori not time-consistent. Based on the dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$, we construct a dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ which is actuarial, market-consistent *and* time-consistent. First, we set $\boldsymbol{\mu}_{T-1} = \boldsymbol{\theta}_{T-1}$. Obviously, $\boldsymbol{\mu}_{T-1}$ is a fair $(T-1)-$hedger and

$$\rho_{T-1}[S] = \boldsymbol{\mu}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1), \qquad \text{for any } S \in \mathcal{C}_T.$$

Second, we define the $(T-2)-$hedger $\boldsymbol{\mu}_{T-2}$ via

$$\boldsymbol{\mu}_{T-2,S} = \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]}, \qquad \text{for any } S \in \mathcal{C}_T.$$

Let us prove that $\boldsymbol{\mu}_{T-2}$ is a fair $(T-2)-$hedger.

- Actuarial hedger: for any $(T-2)-$orthogonal $T-$claim $S^{\perp}$, we have

  $$\boldsymbol{\mu}_{T-2,S^{\perp}} = \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}\left[S^{\perp}\right]}.$$

  Given that ρ_{T-1} is an actuarial $(T-1)-$valuation, $\widetilde{\rho}_{T-1}\left[S^{\perp}\right]$ equals $\pi_{T-1}\left[S^{\perp}\right]$, which is by definition a $(T-2)-$orthogonal $T-$claim. Given that $\boldsymbol{\theta}_{T-2}$ is actuarial, we have

  $$\begin{aligned}\boldsymbol{\mu}_{T-2,S^{\perp}} &= \boldsymbol{\theta}_{T-2,\pi_{T-1}\left[S^{\perp}\right]} \\ &= \pi_{T-2}\left[\pi_{T-1}\left[S^{\perp}\right]\right] \boldsymbol{\beta}_{T-2} \\ &= \pi_{T-2}\left[S^{\perp}\right] \boldsymbol{\beta}_{T-2},\end{aligned}$$

  where we used the time-consistency of $(\rho_t)_{t=0}^{T-1}$.
  Hence, $\boldsymbol{\mu}_{T-2}$ is an actuarial $(T-2)-$hedger.

- Market-consistent hedger: for any $(T-2)-$hedgeable $T-$claim S^h, we have

$$\begin{aligned}\boldsymbol{\mu}_{T-2,S+S^h} &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}\left[S+S^h\right]} \\ &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]+\widetilde{\rho}_{T-1}\left[S^h\right]} \\ &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]} + \boldsymbol{\theta}_{T-2,S^h} \\ &= \boldsymbol{\mu}_{T-2,S} + \boldsymbol{\theta}_{T-2,S^h},\end{aligned}$$

where we used the fact that any $t-$hedgeable claim is $(t+1)-$hedgeable as well (remark that the inverse is not true) and the market-consistency of $\boldsymbol{\theta}_{T-2}$.
Hence, $\boldsymbol{\mu}_{T-2}$ is a market-consistent $(T-2)-$hedger.

Moreover, by (24), we have

$$\begin{aligned}\rho_{T-2}\left[S\right] &= \boldsymbol{\theta}_{T-2,S}(T-1)\cdot\boldsymbol{Y}(T-2) \\ &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]}(T-1)\cdot\boldsymbol{Y}(T-2) \text{ by time-consistency of } (\rho_t)_{t=0}^{T-1} \\ &= \boldsymbol{\mu}_{T-2,S}(T-1)\cdot\boldsymbol{Y}(T-2) \text{ by definition of } \boldsymbol{\mu}_{T-2}.\end{aligned}$$

Iteratively, starting from a fair $t-$hedger $\boldsymbol{\theta}_t$, we construct the time-consistent adaptation

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}, \qquad \text{for any } S\in\mathcal{C}_T.$$

Similarly to $\boldsymbol{\mu}_{T-2}$, one can verify that $\boldsymbol{\mu}_t$ is a fair $t-$hedger. Moreover, $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ is time-consistent by construction and we have

$$\begin{aligned}\rho_t\left[S\right] &= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \\ &= \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t) \text{ by time-consistency of } (\rho_t)_{t=0}^{T-1} \\ &= \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \text{ by definition of } \boldsymbol{\mu}_t,\end{aligned}$$

which ends the proof.
(b) Suppose that there exists a fair dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that

$$\rho_t\left[S\right] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t), \qquad \text{for any } S\in\mathcal{C}_T. \tag{25}$$

From Theorem 1, we know that any $t-$valuation ρ_t is fair. Moreover, we have

$$\begin{aligned}\rho_t\left[S\right] &= \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \\ &= \boldsymbol{\mu}_{t,\widetilde{\rho}_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t) \text{ given } (\boldsymbol{\mu}_t)_{t=0}^{T-1} \text{ is time-consistent} \\ &= \rho_t\left[\widetilde{\rho}_{t+1}\left[S\right]\right] \text{ by definition of } \rho_t,\end{aligned}$$

which ends the proof. ■

5 Fair dynamic valuations: A practical approach

This section is dedicated to the practical application of the concepts introduced above. In Section 5.1, we present a general procedure to determine the fair dynamic valuation of insurance liabilities. The procedure is based on a backward iterations scheme combining risk minimization methods from mathematical finance and standard actuarial techniques. In Section 5.2, we apply the procedure to a portfolio of equity-linked life insurance contracts via a Least Square Monte Carlo (LSMC) implementation. We provide numerical results illustrating the impact of time-consistency on the fair valuation in Section 5.3.

5.1 Fair dynamic valuation problem

We study the problem of an insurer who needs to determine a *fair* (actuarial, market-consistent and time-consistent) dynamic valuation for an insurance liability S which matures at time T. We assume that the financial market consists of a risk-free asset $Y^{(0)}(t) = e^{rt}$ and a risky asset $Y^{(1)}(t)$, $t = 0, 1, ..., T$. This objective is achieved by a backward procedure in which the constructed hedger $\boldsymbol{\theta}_t$ is optimal (in the quadratic hedging sense) for the fair value $\rho_{t+1}[S]$ for any time $t = 0, ..., T-1$. Moreover, for each time step, the residual non-hedged risk is valuated via an actuarial $t-$valuation π_t, implying that the dynamic valuation is actuarial as well.

Consider a $T-$claim S. The optimal hedger at time $T-1$ is defined by

$$\boldsymbol{\theta}_{T-1,S}(T) = \arg\min_{\boldsymbol{\theta}\in\Theta_{T-1}} \mathbb{E}^{\mathbb{P}}_{T-1}\left[\left(S - \theta^{(0)}_{T-1,S}(T)\cdot e^{rT} - \theta^{(1)}_{T-1,S}(T)\cdot Y^{(1)}(T)\right)^2\right]$$

Hence, the hedging strategy is determined at time $T-1$ such that the value of the hedger at time T is as close as possible to S in the quadratic hedging sense. Once the hedging strategy is set up, we value the non-hedged risk via an actuarial $(T-1)-$valuation π_{T-1}. The fair value of S at time $T-1$ is then defined as the sum of the financial value of the optimal hedge and the actuarial value of the remaining risk:

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T)\cdot \boldsymbol{Y}(T-1) + \pi_{T-1}\left[S - \boldsymbol{\theta}_{T-1,S}(T)\cdot \boldsymbol{Y}(T)\right].$$

Iteratively, the optimal hedge at time t for $\rho_{t+1}[S]$ is determined by

$$\boldsymbol{\theta}_{t,S}(t+1) = \arg\min_{\boldsymbol{\theta}\in\Theta_t} \mathbb{E}^{\mathbb{P}}_t\left[\left(\rho_{t+1}[S] - \theta^{(0)}_{t,S}(t+1)\cdot e^{r(t+1)} - \theta^{(1)}_{t,S}(t+1)\cdot Y^{(1)}(t+1)\right)^2\right].$$

After some direct derivations (see also Föllmer and Schweizer (1988) and Černý and Kallsen (2009)), we find that

$$\theta^{(1)}_{t,S}(t+1) = \frac{Cov^{\mathbb{P}}_t\left[\rho_{t+1}[S], Y^{(1)}(t+1)\right]}{Var^{\mathbb{P}}_t\left[Y^{(1)}(t+1)\right]}, \tag{26}$$

$$\theta^{(0)}_{t,S}(t+1) = \left(\mathbb{E}^{\mathbb{P}}_t\left[\rho_{t+1}[S]\right] - \theta^{(1)}_{t,S}(t+1)\cdot \mathbb{E}^{\mathbb{P}}_t\left[Y^{(1)}(t+1)\right]\right)\cdot e^{-r(t+1)}. \tag{27}$$

Then, the fair value at time t is obtained via

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \pi_t\left[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)\right],$$

with π_t an actuarial $t-$valuation.

The procedure is quite intuitive: for each time period, an optimal hedge is set up by quadratic hedging and the remaining risk is valuated via an actuarial valuation, combining actuarial judgement and market-consistency. Moreover, the scheme is iterated backward in time to make it time-consistent. Since the hedger $\boldsymbol{\theta}_t$ is fair, by Theorem 2, ρ_t is a fair dynamic valuation.

5.2 Application to a portfolio of equity-linked life-insurance contracts

The backward recursive scheme presented above is similar to the one solving the local quadratic hedging problem and can be implemented by dynamic programming. Since the optimal hedger is a function of conditional expectations, a popular technique consists of constructing a Markov grid with the use of a multinomial tree model (see e.g. Černỳ (2004), Coleman et al. (2006)). However, in order to decrease the calculation volume, we follow a LSMC approach. This regression-based method was proposed by Carriere (1996) and Longstaff and Schwartz (2001) for the valuation of American-type options. The key idea is to regress the conditional expectations on the cross-sectional information of the underlying risk drivers (in our case, mortality and equity risks). The LSMC technique will be used in order to determine the dynamic hedger in the expressions (26)-(27).

For the remainder of this section, we assume that the insurance liability which matures at time T has the following form

$$S = N(T) \times \max\left(Y^{(1)}(T), K\right), \tag{28}$$

with $N(t)$ a mortality process, $Y^{(1)}(t)$ a risky asset process and K is a fixed guarantee level.

For simplicity of illustration[2], we assume that the stock follows a geometric Brownian motion:

$$dY^{(1)}(t) = Y^{(1)}(t)\left(\mu dt + \sigma dW_1(t)\right)$$

with parameters $\mu, \sigma > 0$. The conditional expectation and variance are then given by

$$\mathbb{E}_t^{\mathbb{P}}\left[Y^{(1)}(t+1)\right] = Y^{(1)}(t)e^{\mu+\frac{\sigma^2}{2}}, \tag{29}$$

$$Var_t^{\mathbb{P}}\left[Y^{(1)}(t+1)\right] = \left(Y^{(1)}(t)\right)^2 e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right). \tag{30}$$

[2]The presented approach can be easily adapted to other stock dynamics, e.g. stochastic volatility or Lévy models.

We assume that the mortality process $N(t)$ counts the number of survivals among an initial population of l_x policyholders of age x. The mortality intensity is assumed to be stochastic and follows the dynamics under $\mathbb{P}$ given by

$$d\lambda_x(t) = c\lambda_x(t)dt + \xi dW_2(t),$$

with $c, \xi > 0$ and $W_2(t)$ a standard Brownian motion, independent of $W_1(t)$. The survival function is then defined by

$$S_x(t) := \mathbb{P}\left(T_x > t\right) = \exp\left(-\int_x^{x+t} \lambda_x(s)ds\right),$$

where T_x is the remaining lifetime of an individual who is aged x at time 0.

Moreover, deaths of individuals are assumed to be independent events conditional on knowing population mortality (see Milevsky et al. (2006) for similar assumptions). Further, if we denote $D(t+1)$ the number of deaths during year $t+1$, the dynamics of the number of active contracts can be described as a nested binomial process as follows: $N(t+1) = N(t) - D(t+1)$ with $D(t+1)|N(t), q_{x+t} \sim Bin(N(t), q_{x+t})$. Here, q_{x+t} represents the one-year death probability

$$q_{x+t} := \mathbb{P}\left(T_x \leq t+1 | T_x > t\right) = 1 - \frac{S_x(t+1)}{S_x(t)}, \text{ for } t = 0, \ldots, T-1.$$

Knowing the dynamics of $N(t)$ and $Y^{(1)}(t)$, one can simulate n scenarios for the mortality and the equity risk factors for $t = 1, \ldots, T$. Finally, the conditional expectations at time t are regressed over the risk drivers at time t via a second-order[3] least-squares regression:

$$\mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}\left[S\right]\right] \approx \alpha_0 + \alpha_1 N(t) Y^{(1)}(t) + \alpha_2 \left(N(t) Y^{(1)}(t)\right)^2, \tag{31}$$

$$\mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}\left[S\right] Y^{(1)}(t+1)\right] \approx \beta_0 + \beta_1 N(t)\left(Y^{(1)}(t)\right)^2 + \beta_2 \left(N(t)\left(Y^{(1)}(t)\right)^2\right)^2. \tag{32}$$

By inserting (29), (30), (31) and (32) into (26) and (27), we obtain a LSMC approximation for $\boldsymbol{\theta}_{t,S}(t+1)$ based on simulations of $N(t)$ and $Y^{(1)}(t)$. For the one-year actuarial $t-$valuation, we consider a standard deviation principle:

$$\begin{aligned}\pi_t\left[\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right] = {} & e^{-r}\ \mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right] \\ & + e^{-r}\ \alpha\sigma_t^{\mathbb{P}}\left[\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right],\end{aligned}$$

with $\alpha > 0$. From (26) and (27), one can find that $\mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right] = 0$. Therefore, the standard deviation of the residual risk is given by

$$\sigma_t^{\mathbb{P}}\left[\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right] = \sqrt{\mathbb{E}_t^{\mathbb{P}}\left[\left(\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right)^2\right]}$$

and we use the following LSMC approximation:

$$\mathbb{E}_t^{\mathbb{P}}\left[\left(\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right)^2\right] \approx \gamma_0 + \gamma_1 N(t) Y^{(1)}(t) + \gamma_2\left(N(t) Y^{(1)}(t)\right)^2.$$

[3] The choice of type and number of basis functions was based on an equilibrium between bias and complexity and the payoff structure in (28). For a discussion of the basis functions and its implications on robustness and convergence, we refer to Areal et al. (2008), Moreno and Navas (2003) and Stentoft (2012).

5.3 Numerical analysis

In this section, we provide a numerical analysis for the fair dynamic valuation of the insurance liability S introduced above. Our numerical results are obtained by generating 50000 sample paths for $N(t)$ and $Y^{(1)}(t)$, for $t = 1, \dots, T$. The benchmark parameters for the financial market are $r = 0.01$, $\mu = 0.02$, $\sigma = 0.1$, $K = 1$ and $Y^{(1)}(0) = 1$. The mortality parameters $(\lambda_x(0) = 0.0087, c = 0.0750, \xi = 0.000597)$ follow from Luciano et al. (2017) and correspond to UK male individuals who are aged 55 at time 0. We assume that there are $l_x = 1000$ initial contracts at time 0 with a maturity of $T = 10$ years.

5.3.1 The effect of a time-consistent and actuarial dynamic valuation

First, we assess the effect of valuating the non-hedgeable risk in each step of our dynamic valuation. To do so, we compare two situations:

- Situation 1: We determine the optimal hedger in each step by quadratic hedging without adding an actuarial valuation for the remaining risk. In this case, the dynamic valuation is market-consistent and time-consistent but not actuarial in the sense that there is no risk margin for the mortality risk. Indeed, under this approach, one can prove that
$$\rho_t\left[N(T)\right] = \mathbb{E}_t^{\mathbb{P}}\left[N(T)\right] \cdot e^{-r(T-t)}.$$
- Situation 2: We determine the optimal hedger in each step as explained above by valuating the remaining risk through a dynamic standard deviation principle
$$\pi_t\left[S\right] = e^{-r}\left[\mathbb{E}_t^{\mathbb{P}}\left[S\right] + \alpha\sigma_t^{\mathbb{P}}\left[S\right]\right],$$
with $\alpha = 0.15$. In that case, the dynamic valuation is market-consistent, time-consistent and actuarial as well.

Figure 1 compares the dynamic valuations in situations 1 and 2 through time. Since $\rho_t\left[S\right]$ is random from the view point of time 0, we consider the evolution of the *expected* dynamic valuation $\mathbb{E}^{\mathbb{P}}\left[\rho_t\left[S\right]\right]$.

In situation 1, we observe that the dynamic valuation is steadily increasing over time to reach the expected payoff at maturity. This was expected since it is market-consistent, the dynamic valuation follows the trend of the risky asset. We remark that given there is no risk margin for the non-hedgeable risk (in particular the mortality risk), the insurer will suffer losses in case policyholders live longer than expected.

On the other hand, in situation 2, we observe a slightly decreasing trend of the dynamic valuation. This can be explained by two adverse effects: while the upward trend of the stock increases the dynamic valuation through time, the value of the non-hedgeable risk decreases over time (a shorter time horizon reduces the uncertainty). From Figure 1, we observe that this latter effect decreases at a higher rate than the increase of the former effect.

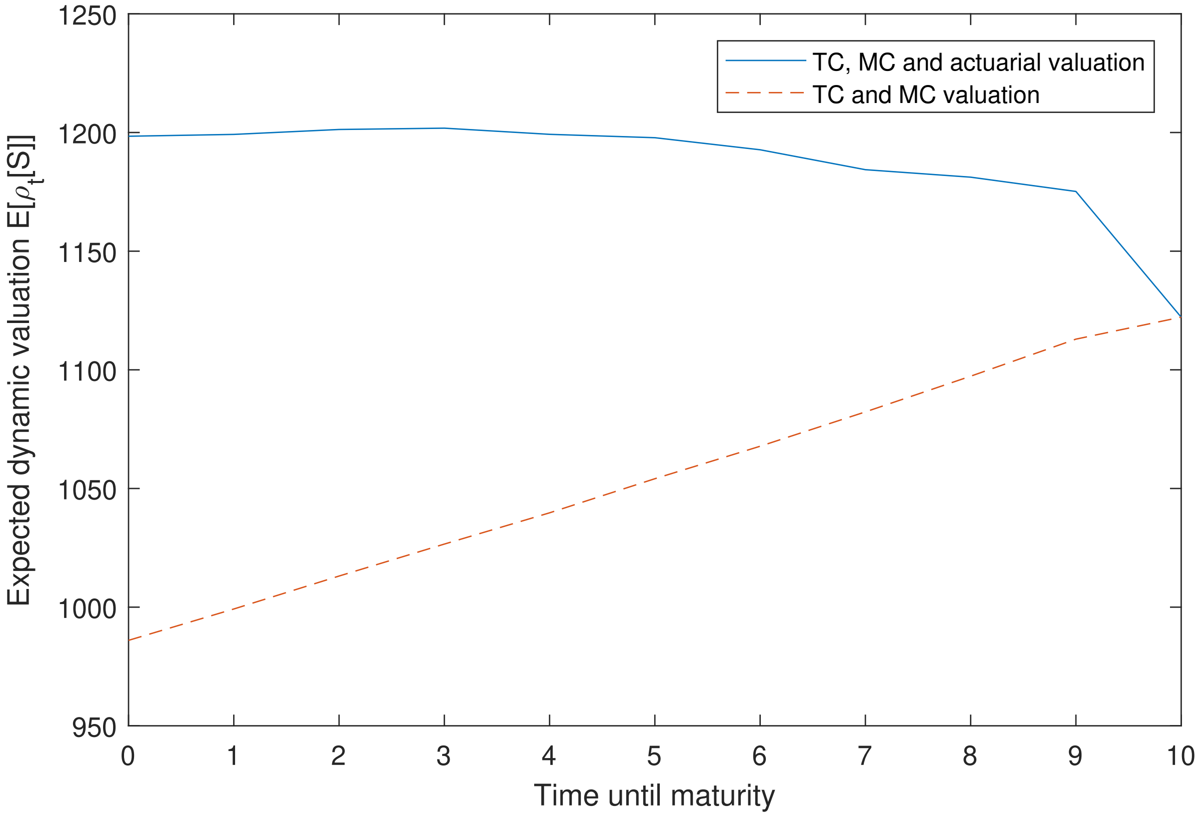

Figure 1: Expected dynamic valuation for the life-insurance portfolio with and without actuarial valuation for the non-hedgeable risk.

5.3.2 The effect of a static versus dynamic actuarial valuation for different maturities

Now, we take another perspective: instead of considering the evolution of the fair valuation until a fixed maturity, we consider the fair valuation at time 0 for different maturities. Moreover, compared to the previous case, we add an intermediate situation in which the non-hedgeable risk until maturity is valuated via a static actuarial valuation. The three situations can be summarized as follows:

- Situation 1: We follow the situation 1 above, i.e. the optimal hedger in each step is determined by quadratic hedging without adding an actuarial valuation for the remaining risk. Hence, there is no risk margin for the non-hedgeable risk.
- Situation 2: We introduce an intermediate situation in which we follow the situation 1 but add a *static* risk margin at time 0 for the non-hedgeable risk

$$RM\left[S\right]=\sum_{t=0}^{T-1}\pi\left[\rho_{t+1}\left[S\right]-\ \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}\left(t+1\right)\right] \tag{33}$$

with π is a static standard deviation principle.

- <u>Situation 3</u>: We consider the fair (actuarial, MC and TC) valuation in which the non-hedgeable risk is valuated via a *dynamic* standard deviation principle. This corresponds to the situation 2 in Section 5.3.1.

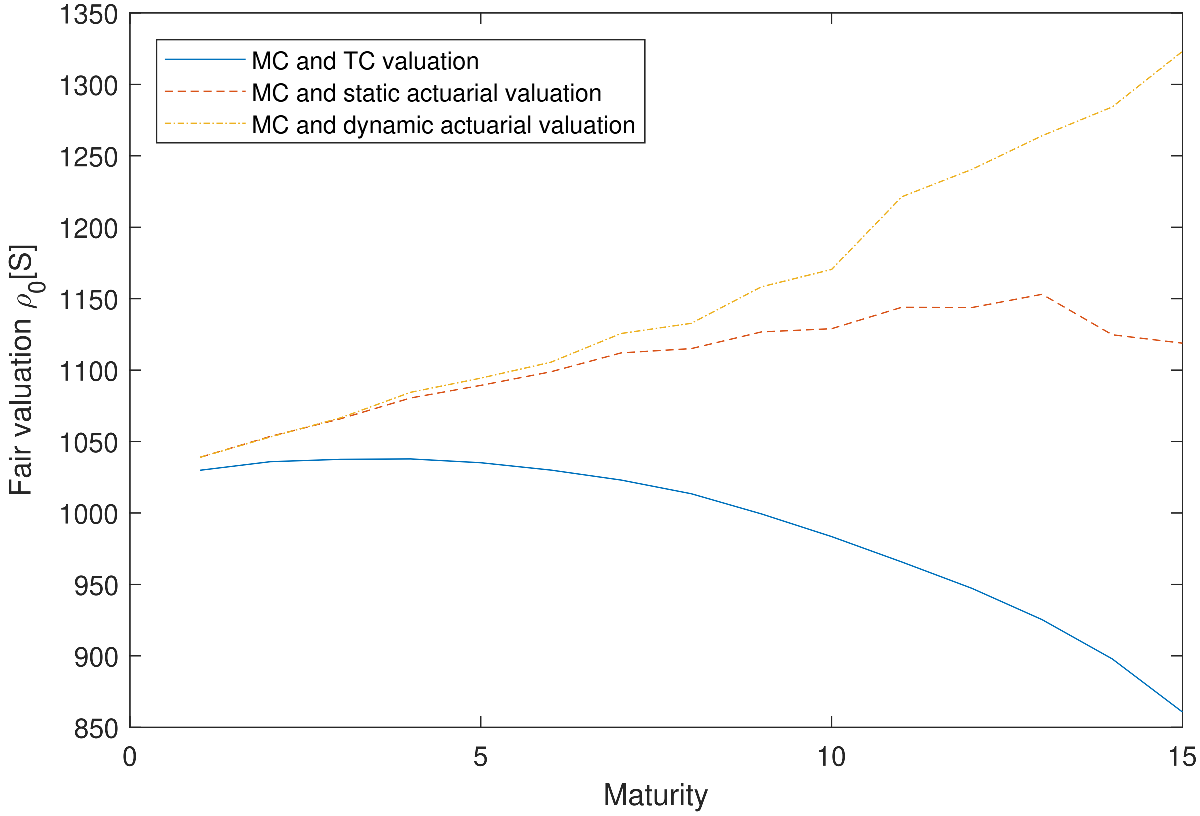

Figure 2: Fair valuation at time 0 with a static versus a dynamic actuarial valuation for the non-hedgeable risk.

Figure 2 compares the three situations for different maturities $T = 1, ..., 15$ years. In situation 1, we observe that the valuation decreases with the maturity increase. This follows from two adverse effects: the longer the maturity, the fewer the number of survivals $N(T)$. But at the same time, the longer the maturity, the higher the financial guarantee $\max\left(Y^{(1)}(T), K\right)$. Figure 2 shows that the mortality effect is stronger that the effect of the financial guarantee.

Not surprisingly, the fair valuation in situations 2 and 3 are higher than the pure market-consistent valuation because of the inclusion of a risk margin for the non-hedgeable risk. Moreover, the fair valuation with dynamic actuarial valuation dominates the one with static actuarial valuation. This difference is due to the iterating effect of the time-consistent valuation. While in situation 2, the one-year remaining risks are added up (see the relation (33)), the time-consistent valuation has a multiplicative effect since $\rho_t [S]$ contains all non-hedgeable risks from time t until maturity T.

5.3.3 The effect of dependence between financial and actuarial risks

Finally, we study the impact of a dependence structure between mortality and equity risks on the fair dynamic valuation of the insurance liability S. We assume that under $\mathbb{P}$ the dynamics of the stock process and the population force of mortality are given by

$$dY^{(1)}(t) = Y^{(1)}(t)\left(\mu dt + \sigma dW_1(t)\right) \tag{34}$$

$$d\lambda_x(t) = c\lambda_x(t)dt + \xi dW_2(t), \tag{35}$$

with c, ξ, μ and σ are positive constants, and $W_1(t) = \rho W_2(t) + \sqrt{1-\rho^2}Z(t)$. Here, $W_2(t)$ and $Z(t)$ are independent standard Brownian motions.

We consider three levels of correlation, namely $\rho = \{-1, 0, 1\}$. The case $\rho = 0$ corresponds to the independence case of our previous analysis while the extreme cases $\rho = 1$ and $\rho = -1$ represents the comonotonic (respectively countermonotonic) situation in which stock and force of mortality are driven by the same random source in the same direction (respectively in the opposite direction). Intuitively, given the payoff

$$S = N(T) \times \max\left(Y^{(1)}(T), K\right),$$

we could expect that if $N(T)$ and $Y^{(1)}(T)$ move in the same direction, this is synonymous with a better hedging and hence a reduction of the non-hedgeable risk.

Figure 3 represents the expected value for the non-hedgeable risk until maturity, computed as the difference between the time-consistent valuation with and without inclusion of an actuarial valuation for the non-hedgeable risk. The figure confirms our intuition: if the number of survivals and the stock are moving in the same direction (i.e. force of mortality and stock are moving in the *opposite* direction), the non-hedgeable risk is reduced. Moreover, as expected, the non-hedgeable risk decreases when we come closer to maturity. We remark that even in extreme cases, the non-hedgeable risk is not null given that the financial guarantee $\max\left(Y^{(1)}(T), K\right)$ and the number of survivals $N(T)$ are not completely hedgeable.

6 Concluding remarks

The determination of the fair valuation for insurance liabilities, which are often a combination of hedgeable and unhedgeable risks, has become a challenging task. Information about prices of traded assets provided by the financial market should be combined with information about mortality experience to provide a reliable market-consistent and actuarial valuation. Moreover, for the determination of future solvency capitals, the fair valuations have to be determined at future points in time in a consistent way, leading to time-consistent valuations.

In this paper, we have studied the fair valuation of insurance liabilities in a dynamic discrete-time setting. We have proposed a new framework to merge actuarial, market-consistent and time-consistent considerations in a set of so-called *fair dynamic valuations*,

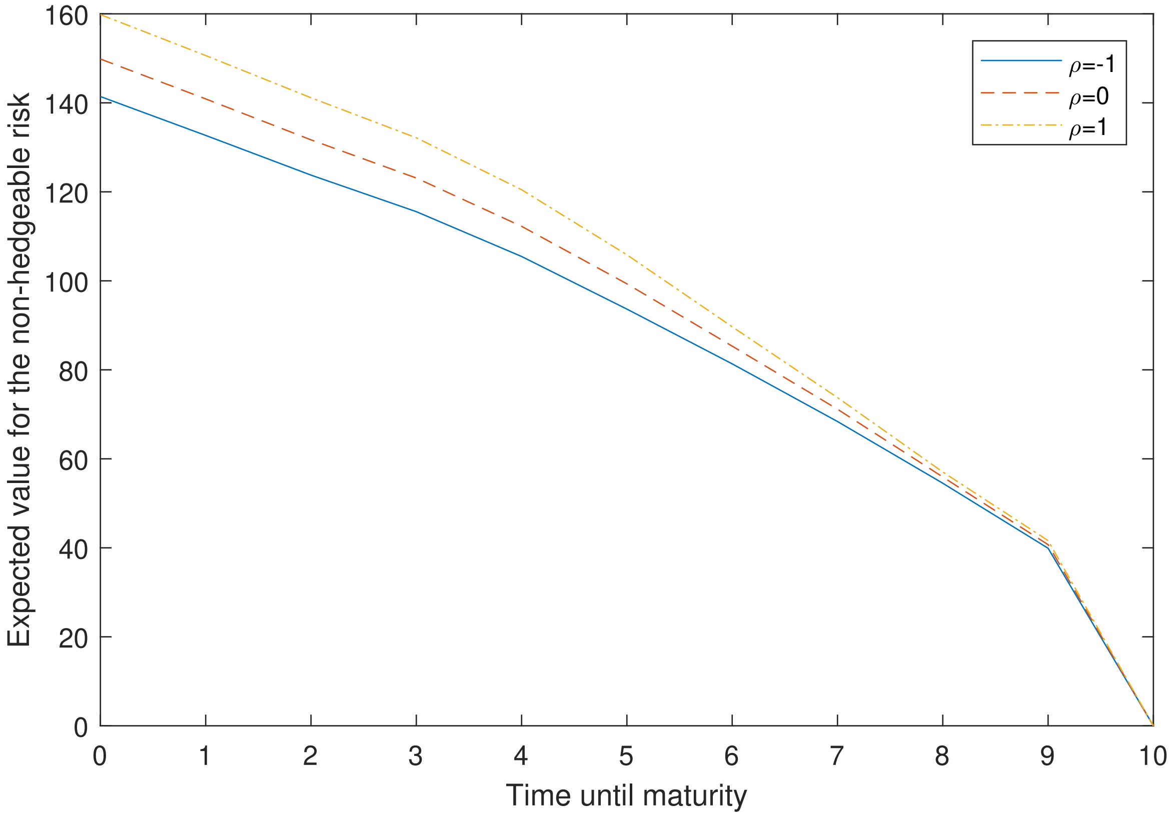

Figure 3: Expected value for the non-hedgeable risk under different dependence levels $\rho = \{-1, 0, 1\}$.

extending the framework of Dhaene et al. (2017) and Barigou and Dhaene (2019). We have provided a complete hedging characterization in Theorem 2 and illustrated how these fair dynamic valuations can be implemented through a backward iterations scheme combining risk minimization techniques with standard actuarial principles.

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