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Fair valuation of insurance liabilities via mean-variance hedging in a multi-period setting

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ABSTRACT

A general class of fair valuations which are both market-consistent (mark-to-market for any hedgeable part of a claim) and actuarial (mark-to-model for any claim that is independent of financial market evolutions) was introduced in Dhaene et al. [*Insurance: Mathematics & Economics*, **76**, 14–27 (2017)] in a single period framework. In particular, the authors considered *mean-variance hedge-based (MVHB)* valuations where fair valuations of insurance liabilities are expressed in terms of mean-variance hedges and actuarial valuations. In this paper, we generalize this MVHB approach to a multi-period dynamic investment setting. We show that the classes of fair valuations and MVHB valuations are equivalent in this generalized setting. We derive tractable formulas for the fair valuation of equity-linked contracts and show how the actuarial part of their MVHB valuation decomposes into a diversifiable and a non-diversifiable component.

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1. Introduction

Insurance liabilities may be partially replicable by traded assets. This may be due to the fact that the payoffs of the underlying insurance contracts are defined in terms of a combination of hedgeable and unhedgeable claims (e.g. unit-linked insurance) or due to the existence of traded insurance-linked securities of which the payoff is correlated with the payoff of the insurance liability (e.g. CAT bonds). Recent insurance regulations (Swiss Solvency Test, Solvency II) require insurers to take into account the market prices of the hedgeable parts of insurance liabilities when determining technical provisions for these liabilities. Loosely speaking, any hedgeable (part of a) claim has to be valuated at the price of its hedge. Otherwise, the value of the claim is determined by its expected present value (called the best estimate), augmented by an appropriate risk loading (called the risk margin, e.g. based on cost-of-capital arguments). For insurance claims which are not completely hedgeable, the hedgeable part of the claim is usually not uniquely determined, implying that different feasible valuation approaches are possible.

In this paper, we extend the approach proposed by Dhaene et al. (2017). These authors introduced a so-called *fair* valuation in a single period setting. A fair valuation is a valuation which is market-consistent and actuarial. It is market-consistent in the sense that any hedgeable part of a claim is valuated at the price of the underlying hedge. Moreover, it is actuarial in the sense that a claim which is independent of the financial market evolutions is valuated using a mark-to-model approach based on actuarial judgement. As an example of a fair valuation, the authors introduced the so-called *mean-variance hedge-based* valuation (hereafter abbreviated as MVHB valuation) which is a two-stage valuation procedure. In a first step, a mean-variance hedge is set up for the claim, based on the

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available traded assets. In a second step, an actuarial valuation is applied to the remaining non-hedged part of the claim. The fair value is then defined as the sum of the price of the mean-variance hedge and the actuarial value of the residual claim. In this paper, we will generalize the MVHB valuation approach in a dynamic investment setting and investigate properties of this valuation framework.

Several authors have investigated the market-consistent valuation of insurance liabilities, see e.g. Salzmann and Wüthrich (2010), Moehr (2011), Wüthrich and Merz (2013) and Pelsser and Stadje (2014). Assuming that the financial market is complete, Pelsser and Stadje (2014) proposed a 'two-step' valuation which extends standard actuarial valuations into time-consistent and market-consistent valuations. Mean-variance hedging in relation to the market-consistent valuation of partially hedgeable insurance claims is considered e.g. in Thomson (2005), Dahl and Møller (2006) and Tsanakas et al. (2013).

In this paper, we will show that the classes of fair valuations and MVHB valuations are identical. We will illustrate how in the MVHB valuation framework applied to the valuation of equity-linked insurance claims in a stochastic mortality setting, the actuarial part of the valuation decomposes into a diversifiable and a non-diversifiable component. As another illustration, we will consider the fair valuation of a portfolio of equity-linked contracts where the self-financing trading strategy depends on the number of survivors in the insured population, a case which is rarely considered in the literature.

Throughout the paper, we will give particular attention to time-T claims of the form

$$S = S^{\perp} \times S^{f},$$

where $S^{\perp}$ is a T-claim which is independent of the financial market evolutions, while S^{f} is a financial T-claim. Such *product claims* often arise in insurance as payoffs of equity-linked life-insurance contracts. For local risk minimization of such payoffs, see e.g. Pansera (2012) and Gaillardetz and Moghtadai (2017).

The paper is structured as follows. In Section 2 we define the combined financial-actuarial world. In this world, we introduce the concepts of orthogonal claims, financial trading strategies and financially hedgeable claims. In Section 3 we consider mean-variance hedging in discrete time. We investigate the mean-variance hedge for product claims, as well as the mean-variance hedge for general claims in a linear subset of self-financing trading strategies available to the valuator. In Section 4, fair valuations and MVHB valuations are introduced. In particular, we show that these two classes of valuations are equivalent and provide some detailed illustrative examples. Section 5 concludes the paper.

2. The combined financial-actuarial world

Consider a combined financial-actuarial world which is home to tradable as well as non-tradable claims. The time horizon is given by T, which is an element of the set $\{1, 2, \ldots\}$. The financial-actuarial world is modeled by the probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, equipped with the finite and discrete-time filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in\tau}$ with $\tau = \{0, 1, \ldots, T\}$. The initial σ-algebra $\mathcal{G}_0$ is set equal to $\{\emptyset, \Omega\}$ while the σ-algebra $\mathcal{G}_T$ is identical to $\mathcal{G}$. The σ-algebra $\mathcal{G}_t$, $t \in \tau$, represents the general information available up to and including time t in the combined world. Further, $\mathbb{P}$ is the measure attaching physical probabilities to all events in that world. All random variables (r.v.'s) and stochastic processes in this paper are defined on this filtered probability space. Throughout the paper, we assume that all r.v.'s that we consider have finite second order moments under $\mathbb{P}$. Equalities and inequalities between r.v.'s have to be understood in the $\mathbb{P}$—almost sure sense. Furthermore, we will denote the set of all *t-claims* defined on $(\Omega, \mathbb{G}, \mathcal{G})$, that is the set of all $\mathcal{G}_t$-measurable r.v.'s, by $\mathcal{C}_t$.

The combined financial-actuarial world hosts a number of insurance liabilities, which are due at time T. Any insurance liability is represented by a T-claim, which will be generally denoted by $S(T)$ or simply by S if no confusion is possible. A simple example of an insurance liability related to the remaining lifetime T_x of an insured (x) observed at time 0 is the indicator variable S defined by

$$S = \begin{cases} 0 & : T_x \leq T \\ 1 & : T_x > T \end{cases} \tag{1}$$

The combined financial-actuarial world $(\Omega, \mathcal{G}, \mathbb{P})$ is also home to a financial market of $n \in \{1, 2, \ldots\}$ tradable (non-dividend paying) risky assets and a risk-free bank account. For any $i = 1, 2, \ldots, n$, we introduce the notation $Y^{(i)}(t)$ for the market price of 1 unit of risky asset i at time $t \in \tau$. The risky assets can be stocks, bonds, mutual funds, etc. The time-t value of an investment of amount 1 at time 0 in the risk-free bank account is given by $Y^{(0)}(t) = \mathrm{e}^{rt}$, where $r \geq 0$ is the deterministic and constant risk-free interest rate. We assume that any tradable asset can be bought and/or sold in any quantities in a deep, liquid and transparent market with negligible transactions costs and other market frictions.

The price processes of the traded assets are described by the $(n+1)$-dimensional stochastic process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\}_{t \in \tau}$. Here, $\boldsymbol{Y}(t)$, $t \in \tau$, is the vector of time$-t$ prices of all tradable assets, i.e. $\boldsymbol{Y}(t) = (Y^{(0)}(t), Y^{(1)}(t), \ldots, Y^{(n)}(t))$. We assume that the price process $\boldsymbol{Y}$ is adapted to the filtration $\mathbb{G}$:

$$\boldsymbol{Y}(t) \text{ is } \mathcal{G}_t - \text{measurable}, \quad \text{for any } t \in \tau.$$

The filtration $\mathbb{G}$ may simply coincide with the filtration generated by the price process $\boldsymbol{Y}$. In this paper however, we will consider a more general setting, where $\mathbb{G}$ is not only related to the price history of traded assets, but may also contain information related to non-tradable claims such as a survival index of a particular population.

A *trading strategy* (also called a *dynamic portfolio*) $\boldsymbol{\theta} = \{\boldsymbol{\theta}(t)\}_{t \in \{1,2,\ldots,T\}}$ is a *predictable* $(n+1)$-dimensional process with respect to the filtration $\mathbb{G}$:

$$\boldsymbol{\theta}(t) \text{ is } \mathcal{G}_{t-1} - \text{measurable}, \quad \text{for any } t \in \{1, 2, \ldots, T\}.$$

The vector $\boldsymbol{\theta}(t) = (\theta^{(0)}(t), \theta^{(1)}(t) \ldots, \theta^{(n)}(t))$ represents the number of units $\theta^{(i)}(t)$ invested in each asset i in time period t, that is in the time interval $(t-1, t]$. The $\mathcal{G}_{t-1}$-measurability requirement means that the portfolio composition $\boldsymbol{\theta}(t)$ for the period $(t-1, t]$ follows from the general information available up to and including time $t-1$, i.e. the information collected in time interval $[0, t-1]$. This information includes in particular the price history of traded assets in that time interval.

The value at time t of the trading strategy $\boldsymbol{\theta}$ is denoted by $V^{\boldsymbol{\theta}}(t)$:

$$V^{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}(t) \cdot \boldsymbol{Y}(t) = \sum_{i=0}^{n} \theta^{(i)}(t) Y^{(i)}(t), \quad \text{for any } t = 1, 2, \ldots, T,$$

while

$$V^{\boldsymbol{\theta}}(0) = \boldsymbol{\theta}(1) \cdot \boldsymbol{Y}(0) = \sum_{i=0}^{n} \theta^{(i)}(1) Y^{(i)}(0).$$

Obviously, $V^{\boldsymbol{\theta}}(t)$ is $\mathcal{G}_t$-measurable. For any $t > 0$, we have that $V^{\boldsymbol{\theta}}(t)$ is the value of the trading strategy at time t, just before eventual rebalancing, whereas $V^{\boldsymbol{\theta}}(0)$ is the *initial investment* or the *endowment* of the trading strategy $\boldsymbol{\theta}$.

Fair valuation in the single period case $T = 1$ is investigated in detail in Dhaene et al. (2017). Hereafter, we will always assume that $T \geq 2$, implying that there is at least one rebalancing moment. A trading strategy $\boldsymbol{\theta}$ is said to be *self-financing* if

$$\boldsymbol{\theta}(t) \cdot \boldsymbol{Y}(t) = \boldsymbol{\theta}(t+1) \cdot \boldsymbol{Y}(t), \quad \text{for any } t = 1, \ldots, T-1. \tag{2}$$

This means that no capital is injected or withdrawn at any rebalancing moment $t = 1, \ldots, T-1$. We denote the set of self-financing trading strategies by Θ. Taking into account (2), the time$-T$ value of any self-financing strategy $\boldsymbol{\theta} \in \Theta$ with initial investment $V^{\boldsymbol{\theta}}(0)$ can be expressed as

$$V^{\boldsymbol{\theta}}(T) = \boldsymbol{\theta}(T) \cdot \boldsymbol{Y}(T) = V^{\boldsymbol{\theta}}(0) + \sum_{t=1}^{\mathrm{T}} \boldsymbol{\theta}(t) \cdot \Delta \boldsymbol{Y}(t), \tag{3}$$

with $\Delta \boldsymbol{Y}(t) = \boldsymbol{Y}(t) - \boldsymbol{Y}(t-1)$. In this formula, $\boldsymbol{\theta}(t) \cdot \Delta \boldsymbol{Y}(t)$ is the change of the market value of the investment portfolio in the time period $(t-1, t]$, i.e. between time $t-1$ (just after rebalancing) and time t (just before rebalancing).

We will always assume that the market of traded assets is *arbitrage-free* in the sense that there is no self-financing strategy $\boldsymbol{\theta} \in \Theta$ with the following properties:

$$\mathbb{P}\left[V^{\boldsymbol{\theta}}(0) = 0\right] = 1, \quad \mathbb{P}\left[V^{\boldsymbol{\theta}}(T) \geq 0\right] = 1 \quad \text{and} \quad \mathbb{P}\left[V^{\boldsymbol{\theta}}(T) > 0\right] > 0. \tag{4}$$

In our discrete-time setting, the absence of arbitrage is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$, such that the price $Y^{(i)}(t)$ of any traded asset i at any trading date t can be expressed as

$$Y^{(i)}(t) = \mathrm{e}^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[Y^{(i)}(T) \mid \mathcal{G}_t\right]. \tag{5}$$

For a proof of this equivalence, we refer to Chapter 6 in Delbaen and Schachermayer (2006).

Definition 2.1 (Hedgeable *T*-claim): A hedgeable T-claim S is an element of $\mathcal{C}_T$ that can be replicated by a self-financing strategy $\boldsymbol{\theta} \in \Theta$:

$$S = V^{\boldsymbol{\theta}}(T).$$

We will denote the set of all hedgeable T-claims by $\mathcal{H}_T$. The no-arbitrage assumption guarantees that the time-t price $S(t)$ of a hedgeable T-claim S is equal to the time$-t$ price of the underlying self-financing strategy $\boldsymbol{\theta}$:

$$S(t) = V^{\boldsymbol{\theta}}(t) = \mathrm{e}^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[S \mid \mathcal{G}_t\right]. \tag{6}$$

In this paper, we consider an incomplete market setting. This means that apart from the hedgeable T-claims, of which the valuation is straightforward, there are also T-claims that cannot be perfectly replicated. A possible example of an unhedgeable T-claim is the r.v. S defined in (1).

A self-financing strategy is by definition $\mathbb{G}-$predictable. Hence, the rebalancing of the portfolio at any time t may depend on all information available up to time t, not only including observed asset prices, but also actuarial information such as survival indices, earthquake indices, etc. Hereafter, we will often consider the smaller set of self-financing strategies which are predictable with respect to the financial information. For this purpose, we introduce the financial filtration $\mathbb{F}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \tau}$ contains all information about financial events. This filtration may coincide with the filtration $\mathbb{F}^{\boldsymbol{Y}}$ generated by the price process $\mathbf{Y}$ but may include additional financial information, such as economic barometers and/or information about non-traded securities. Hence, in general we have that

$$\mathbb{F}^{\boldsymbol{Y}} \subseteq \mathbb{F} \subseteq \mathbb{G}.$$

We will denote the set of all *financial t-claims*, that is the set of all $\mathcal{F}_t$-measurable r.v.'s, by $\mathcal{C}_t^{\mathcal{F}}$. It is obvious that

$$\mathcal{C}_t^{\mathcal{F}} \subseteq \mathcal{C}_t.$$

Furthermore, we introduce the notation $\Theta^{\mathcal{F}}$ for the set of self-financing strategies which are predictable with respect to $\mathbb{F}$ and call its elements *financial* self-financing trading strategies, as they are based on the financial filtration. We have that

$$\Theta^{\mathcal{F}} \subseteq \Theta.$$

For any financial self-financing strategy $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$, the investor selects his period $(t-1, t]$ portfolio, based on the financial information observed in the time period $[0, t-1]$, including asset prices and other additional financial information.

Next, we define the set of financially hedgeable T-claims.

Definition 2.2 (Financially hedgeable T-claim): A financially hedgeable T-claim S is an element of $\mathcal{C}_T^{\mathcal{F}}$ which can be replicated by a financial trading strategy $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$:

$$S = V^{\boldsymbol{\theta}}(T).$$

We introduce the notation $\mathcal{H}_T^{\mathcal{F}}$ for the set of all *financially* hedgeable T-claims. One has that

$$\mathcal{H}_T^{\mathcal{F}} \subseteq \mathcal{H}_T.$$

Finally, we introduce orthogonal T-claims. We will use the term $\mathbb{P}$-independence for independence between r.v.'s under the measure $\mathbb{P}$.

Definition 2.3 (Orthogonal claim): An orthogonal T-claim S is an element of $\mathcal{C}_T$ which is $\mathbb{P}$-independent of the financial filtration $\mathbb{F}$.

Hereafter, we will denote the set of all orthogonal T-claims by $\mathcal{O}_T$. Hence, $S \in \mathcal{O}_T$ means that S is $\mathbb{P}$-independent of any $\mathcal{F}_T$-measurable random variable. An example of an orthogonal T-claim is the indicator variable S defined in (1), provided T_x is independent of the financial market evolution. In case $\mathbb{F} \equiv \mathbb{F}^{\boldsymbol{Y}}$, one has that $S \in \mathcal{O}_T$ if and only if S is $\mathbb{P}$-independent of any r.v. which can be expressed as $f(\boldsymbol{Y})$ for some measurable function f.

We remark that

$$\mathbb{E}^{\mathbb{P}}\left[S^{\perp} \times S^{f}\right] = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \mathbb{E}^{\mathbb{P}}\left[S^{f}\right], \quad \text{for any } S^{\perp} \in \mathcal{O}_T \quad \text{and} \quad S^{f} \in \mathcal{C}_T^{\mathcal{F}}.$$

In particular, we find that

$$\mathbb{E}^{\mathbb{P}}\left[S^{\perp} \times V^{\boldsymbol{\theta}}(T)\right] = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}}(T)\right], \quad \text{for any } S^{\perp} \in \mathcal{O}_T \quad \text{and} \quad \boldsymbol{\theta} \in \Theta^{\mathcal{F}}.$$

This follows immediately from the fact that $V^{\boldsymbol{\theta}}(T) \in \mathcal{C}_T^{\mathcal{F}}$.

3. Mean-variance hedging of insurance liabilities

3.1. Some general results on mean-variance hedging

Mean-variance hedging (further abbreviated as MV hedging) is the technique of approximating, with minimal mean squared error, a given T-claim by the time-T value of a self-financing trading strategy. The literature on MV hedging is extensive. We refer to Schweizer (2010) for a survey. Two main approaches are considered in the literature: the first one uses martingale measures and projection arguments, see e.g. Černý and Kallsen (2007), while the second one describes the problem in terms of a linear backward stochastic differential equation, see e.g. Delong (2013).

In this section, we introduce MV hedging to determine the 'closest' hedge of a combined financial-actuarial claim. This hedge will constitute the first step of the mean-variance hedge-based valuation which will be considered in Section 4. Hereafter, whenever we consider a subset Θ' of the set of all self-financing trading strategies Θ, we assume that Θ' is a linear subspace (closed under addition and scalar multiplication) of Θ. This assumption implies, in particular, that the set $\{V^{\boldsymbol{\theta}}(T) \mid \boldsymbol{\theta} \in \Theta'\}$ is a linear subspace of $\mathcal{C}_T$.

Definition 3.1 (Mean-variance hedging): Consider a T-claim S. The MV hedge of S in $\Theta' \subseteq \Theta$ is the self-financing strategy $\boldsymbol{\theta}_S^{MV} \in \Theta'$ for which the expected quadratic hedging error at time T is minimized:

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\theta}\in\Theta'} \mathbb{E}^{\mathbb{P}}\left[\left(S - V^{\boldsymbol{\theta}}(T)\right)^2\right]. \tag{7}$$

The existence of a solution to the minimization problem (7) is tantamount to the condition that $\{V^{\boldsymbol{\theta}}(T) \mid \boldsymbol{\theta} \in \Theta'\}$ is a closed set. In this paper, we will always assume that this condition is satisfied.[1] Uniqueness of this solution holds under the additional condition of non-redundancy of Θ'. Non-redundancy of Θ' means that for any $\boldsymbol{\theta} \in \Theta'$, one has that $V^{\boldsymbol{\theta}}(T) = 0$ implies that $\boldsymbol{\theta} = \mathbf{0}$, where $\mathbf{0}$ is the trivial zero investment strategy with all components equal to 0. Hereafter, we will not require non-redundancy of Θ'. This means that a T-claim S can have several mean-variance hedges. Notice however that the time$-T$ values of all these self-financing strategies are identical. In the remainder of the paper, we will denote the unique time$-T$ value of all the mean-variance hedges of S by $V_S^{MV}(T)$.

The determination of the solution of the discrete time minimization problem (7) for the set Θ of $\mathbb{G}$-predictable self-financing strategies is considered in Černý and Kallsen (2009), see also Schweizer (2010) and the references therein.

It is well-known that MV hedging in the linear subspace Θ' has the following properties:

$$V_{\alpha\times S}^{MV}(T) = \alpha \times V_S^{MV}(T), \quad \text{for any scalar } \alpha \geq 0, \tag{8}$$

and

$$V_{S_1+S_2}^{MV}(T) = V_{S_1}^{MV}(T) + V_{S_2}^{MV}(T), \quad \text{for any } S_1 \text{ and } S_2 \in \mathcal{C}_T. \tag{9}$$

A no-arbitrage argument leads to

$$V_{S_1+S_2}^{MV}(0) = V_{S_1}^{MV}(0) + V_{S_2}^{MV}(0), \quad \text{for any } S_1 \text{ and } S_2 \in \mathcal{C}_T. \tag{10}$$

As a special case of the additivity property (9), we have that

$$V_{S+S^h}^{MV}(T) = V_S^{MV}(T) + S^h, \quad \text{for any } S \in \mathcal{C}_T \quad \text{and} \quad S^h = V^{\boldsymbol{\theta}}(T), \quad \text{with } \boldsymbol{\theta} \in \Theta'. \tag{11}$$

In the following subsection, we will consider mean-variance hedging of claims which can be expressed as the product of an orthogonal claim and the time-T value of a financial self-financing strategy.

3.2. MV hedging of product claims

The benefit payment of an insurance contract at contract termination date T can often be expressed as

$$S = S^{\perp} \times S^f, \quad \text{with } S^{\perp} \in \mathcal{O}_T \quad \text{and} \quad S^f \in \mathcal{C}_T^{\mathcal{F}}. \tag{12}$$

This situation occurs for unit-linked contracts in case the corresponding claim S is the product of an actuarial and a financial component, where the actuarial component is independent of the financial information flow over time. In the following theorem, we determine the MV hedge of T-claims of the form (12) in a subset of financial trading strategies: $\Theta' \subseteq \Theta^{\mathcal{F}}$.

[1] The closedness assumption is satisfied for $\Theta' = \Theta$ and for $\Theta' = \Theta^{\mathcal{F}}$, see Černý and Kallsen (2009) for technical details. It is also satisfied for the set $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)} = \{\sum_{j=1}^m \alpha_j \boldsymbol{\theta}_j \mid (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m\}$, with $\boldsymbol{\theta}_j, j = 1, \ldots, m, \in \Theta$, which is considered in Section 3.3.

Theorem 3.1: *Consider the T-claim S defined in* (12). *The MV hedge $\boldsymbol{\theta}_S^{MV}$ of S in the subset Θ' of the set $\Theta^{\mathcal{F}}$ of financial self-financing strategies is given by*

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \boldsymbol{\theta}_{S^f}^{MV}, \tag{13}$$

where $\boldsymbol{\theta}_{S^f}^{MV}$ is the MV hedge of S^f in Θ'. Moreover, the time-T value of the MV hedge of S equals

$$V_S^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times V_{S^f}^{MV}(T). \tag{14}$$

Proof: For any financial self-financing strategy $\boldsymbol{\mu} \in \Theta'$, we find that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left(S - V^{\boldsymbol{\mu}}(T)\right)^2\right] &= \mathbb{E}^{\mathbb{P}}\left[\left(\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]S^f - V^{\boldsymbol{\mu}}(T)\right) + (S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right])S^f\right)^2\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]S^f - V^{\boldsymbol{\mu}}(T)\right)^2\right] + \mathbb{E}^{\mathbb{P}}\left[\left((S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right])S^f\right)^2\right],\end{aligned}$$

where the last step follows from taking into account that $S^{\perp} \in \mathcal{O}_T$, which is independent of S^f and $V^{\boldsymbol{\mu}}(T)$.

As $\mathbb{E}^{\mathbb{P}}[((S^{\perp} - \mathbb{E}^{\mathbb{P}}[S^{\perp}])S^f)^2]$ does not depend on $\boldsymbol{\mu}$, we find that the MV hedge $\boldsymbol{\theta}_S^{MV}$ in the set Θ' follows from

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta'} \mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]S^f - V^{\boldsymbol{\mu}}(T))\right)^2\right].$$

Taking into account that Θ' is a linear space, we can conclude that $V_S^{MV}(T)$ is given by (14) and the self-financing strategy $\boldsymbol{\theta}_S^{MV}$ defined in (13) is a solution of the minimization problem (7). ■

A special case of Theorem 3.1 arises when the financial part of the payoff at time T is equal to the time$-T$ value of a financial self-financing trading strategy in $\Theta^{\mathcal{F}}$. This case is considered in the following corollary, with $\Theta' = \Theta^{\mathcal{F}}$.

Corollary 3.1: *Let $S^{\perp} \in \mathcal{O}_T$ and consider the following T-claim:*

$$S = S^{\perp} \times V^{\boldsymbol{\theta}}(T), \quad \text{with } S^{\perp} \in \mathcal{O}_T \text{ and } \boldsymbol{\theta} \in \Theta^{\mathcal{F}}. \tag{15}$$

The MV hedge of S in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \boldsymbol{\theta},$$

while the time-T value of this MV hedge equals

$$V_S^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times V^{\boldsymbol{\theta}}(T). \tag{16}$$

In the following corollary, we consider the special case of Corollary 3.1, where the financial part of the payoff at time T is equal to the time$-T$ price of a traded asset.

Corollary 3.2: *Let $S^{\perp} \in \mathcal{O}_T$ and consider the following T-claim:*

$$S = S^{\perp} \times Y^{(i)}(T), \quad i = 0, 1, \ldots, n. \tag{17}$$

The MV hedge of S in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \boldsymbol{\theta}^{(i)},$$

where $\boldsymbol{\theta}^{(i)} \in \Theta^{\mathcal{F}}$ is the static financial investment strategy consisting of buying 1 unit of asset i at time 0 and holding it until time T. The time-T value of $\boldsymbol{\theta}_S^{MV}$ is given by

$$V_S^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times Y^{(i)}(T). \tag{18}$$

The MV hedge of an orthogonal claim is considered in the following corollary.

Corollary 3.3: *The MV hedge of $S^{\perp} \in \mathcal{O}_T$ in the set $\Theta^{\mathcal{F}}$ is given by*

$$\boldsymbol{\theta}_{S^{\perp}}^{MV} = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\boldsymbol{\theta}^{(0)},$$

where $\boldsymbol{\theta}^{(0)} \in \Theta^{\mathcal{F}}$ is the static investment strategy consisting of a risk-free investment of 1 at time 0, which is maintained until time T. The time-T value of $\boldsymbol{\theta}_{S^{\perp}}^{MV}$ is given by

$$V_{S^{\perp}}^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]. \tag{19}$$

The proof of this corollary follows immediately from Corollary 3.2.

3.3. MV hedging of general claims

3.3.1. MV hedging in the set of linear combinations of self-financing strategies

In this subsection, we consider a general T-claim S and search for the self-financing strategy with the minimal expected quadratic hedging error at time T, where we restrict our search to the set of all strategies which can be expressed as linear combinations of a number of given self-financing trading strategies which are available to the decision maker. More specifically, we consider a vector of m self-financing trading strategies $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m)$, with any $\boldsymbol{\theta}_j \in \Theta$, and the following set of self-financing investment strategies:

$$\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)} = \left\{ \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j \mid (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m \right\}. \tag{20}$$

Notice that $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)} \subseteq \Theta$, but it is not necessary a subset of $\Theta^{\mathcal{F}}$. In the following theorem, we determine the MV hedge of a general T-claim S in the set of trading strategies defined in (20). The MV hedge $\boldsymbol{\theta}_S^{MV}$ of S in $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)}$ is determined from

$$\min_{\alpha \in \mathbb{R}^m} \mathbb{E}^{\mathbb{P}}\left[\left(S - \sum_{j=1}^{m} \alpha_j V^{\boldsymbol{\theta}_j}(T)\right)^2\right]. \tag{21}$$

Hereafter, we use the notation $\mathbf{A}^{\mathsf{T}}$ for the transpose of a matrix $\mathbf{A}$ and the notation $\times$ for the product of 2 matrices.

Theorem 3.2: *Consider the vector* $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m)$ *of self-financing investment strategies* $\boldsymbol{\theta}_i \in \Theta$ *and assume that their time-T values* $V^{\boldsymbol{\theta}_i}(T)$, $i = 1, 2, \ldots, m$, *are linearly independent. The MV hedge* $\boldsymbol{\theta}_S^{MV}$ *of the T-claim S in the set* $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)}$ *is given by*

$$\boldsymbol{\theta}_S^{MV} = \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j,$$

where the m-vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ *is given by*

$$\boldsymbol{\alpha}^{\intercal} = \mathbf{W}^{-1} \times \mathbf{V}^{\intercal}. \tag{22}$$

In this expression, $\mathbf{W}$ *is the* $(m \times m)$*-matrix with elements* (i, j) *defined by*

$$(\mathbf{W})_{ij} = \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_i}(T) V^{\boldsymbol{\theta}_j}(T)\right], \tag{23}$$

while $\mathbf{V}$ *is the* $(1 \times m)$*-matrix with jth element given by*

$$(\mathbf{V})_j = \mathbb{E}^{\mathbb{P}}\left[S V^{\boldsymbol{\theta}_j}(T)\right]. \tag{24}$$

Proof: Taking the derivatives of the objective function in (21) with respect to the α_i and setting them equal to 0 leads to the following set of equations:

$$\sum_{j=1}^{m} (\mathbf{W})_{ij}\, \alpha_j = \mathbb{E}^{\mathbb{P}}[S V^{\boldsymbol{\theta}_i}(T)], \quad i = 1, \ldots, m,$$

where the elements $(\mathbf{W})_{ij}$ are defined in (23). This set of equations can be rewritten as follows:

$$\mathbf{W} \times \boldsymbol{\alpha}^{\intercal} = \mathbf{V}^{\intercal}.$$

The assumption of linear independence of the r.v.'s $V^{\boldsymbol{\theta}_1}(T)$, $V^{\boldsymbol{\theta}_2}(T), \ldots, V^{\boldsymbol{\theta}_m}(T)$ is equivalent to the non-singularity of the matrix $\mathbf{W}$. This proves (22). ∎

The MV hedge of the T-claim S in the set $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)}$ takes into account the mutual dependency structure between the time T-values of the m self-financing strategies via the components $\mathbb{E}^{\mathbb{P}}[V^{\boldsymbol{\theta}_i}(T)V^{\boldsymbol{\theta}_j}(T)]$ of the matrix $\mathbf{W}$, while the dependency between the time T-values of these m strategies and the claim S is captured by the components $\mathbb{E}^{\mathbb{P}}[SV^{\boldsymbol{\theta}_j}(T)]$ of the vector $\mathbf{V}$.

Remark that the optimization problem solved in Theorem 3.2 is very similar to the general MV hedging problem in the single period setting. This problem is strongly related to portfolio replication where one searches for a linear combination of traded assets which generates cash-flows which approximate the cash-flows of a given T-claim. For further details, we refer to Pelsser and Schweizer (2016) and Natolski and Werner (2017).

3.3.2. MV hedging in the set of linear combinations of a risk-free and risky self-financing strategies

In this subsection, we consider the special case of Theorem 3.2, where apart from a risk-free investment, there is a number of risky self-financing strategies. This case is considered in the following corollary.

Corollary 3.4: *Consider the vector* $(\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m)$, *where* $\boldsymbol{\theta}^{(0)}$ *is the static strategy consisting of a risk-free investment of* 1 *at time* 0, *while* $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m$ *are self-financing investment strategies in* Θ.

Assume that the m+1 time−T values of the self-financing strategies are linearly independent. The MV hedge $\boldsymbol{\theta}_S^{MV}$ of any T-claim S in the set $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$ is then given by

$$\boldsymbol{\theta}_S^{MV} = \alpha_0 \boldsymbol{\theta}^{(0)} + \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j,$$

with the α_j determined by the set of equations

$$\sum_{j=1}^{m} cov^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_i}(T), V^{\boldsymbol{\theta}_j}(T)\right]\alpha_j = \mathrm{cov}^{\mathbb{P}}\left[S, V^{\boldsymbol{\theta}_i}(T)\right], \quad i = 1, 2, \ldots, m, \tag{25}$$

while α_0 follows from

$$\alpha_0 = \mathrm{e}^{-rT}\left(\mathbb{E}^{\mathbb{P}}[S] - \sum_{j=1}^{m} \alpha_j \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_j}(T)\right]\right) \tag{26}$$

Moreover, we have that

$$\mathbb{E}^{\mathbb{P}}\left[V_S^{MV}(T)\right] = \mathbb{E}^{\mathbb{P}}[S]. \tag{27}$$

Proof: The proof follows from (22). ■

As a special case of the previous corollary, consider the case where $m = 1$. Then we find that the MV hedge of S is given by

$$\boldsymbol{\theta}_S^{MV} = \alpha_0 \boldsymbol{\theta}^{(0)} + \alpha_1 \boldsymbol{\theta}_1 \tag{28}$$

with

$$\alpha_1 = \frac{\mathrm{cov}^{\mathbb{P}}\left[S, V^{\boldsymbol{\theta}_1}(T)\right]}{\mathrm{Var}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_1}(T)\right]} \tag{29}$$

and

$$\alpha_0 = \mathrm{e}^{-rT}\left(\mathbb{E}^{\mathbb{P}}[S] - \alpha_1 \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_1}(T)\right]\right). \tag{30}$$

For the particular case of a single-period setting, i.e. $T = 1$, these equations can be found e.g. in Tsanakas et al. (2013) and Černý and Kallsen (2009). In the particular case that S is $\mathbb{P}$-independent of the time−T value $V^{\boldsymbol{\theta}_1}(T)$ of the risky self-financing strategy $\boldsymbol{\theta}_1$, we find that

$$\alpha_1 = 0$$

and

$$\alpha_0 = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{P}}[S].$$

Hence, in this particular case, the MV hedge is given by $\mathrm{e}^{-rT}\,\mathbb{E}^{\mathbb{P}}[S]\,\boldsymbol{\theta}^{(0)}$, which is a static investment strategy of amount $\mathrm{e}^{-rT}\,\mathbb{E}^{\mathbb{P}}[S]$ in the risk-free asset at time 0.

3.3.3. MV hedging with a single self-financing strategy

In this subsection, we consider a self-financing strategy $\boldsymbol{\theta}_1 \in \Theta$ and determine the MV hedge of the T-claim S in the set

$$\Theta^{(\boldsymbol{\theta}_1)} = \{\alpha\boldsymbol{\theta}_1 \mid \alpha \in \mathbb{R}\}.$$

From Theorem 3.2, we immediately find the following corollary.

Corollary 3.5: *Consider the self-financing investment strategy $\boldsymbol{\theta}_1 \in \Theta$. The MV hedge of the T-claim S in the set $\Theta^{(\boldsymbol{\theta}_1)}$ is given by*

$$\boldsymbol{\theta}_S^{MV} = \alpha\boldsymbol{\theta}_1,$$

with α determined by

$$\alpha = \frac{\mathbb{E}^{\mathbb{P}}\left[S V^{\boldsymbol{\theta}_1}(T)\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(V^{\boldsymbol{\theta}_1}(T)\right)^2\right]}. \tag{31}$$

Hereafter, we consider two special cases of this corollary.

First, suppose the strategy $\boldsymbol{\theta}_1$ coincides with the static risk-free investment strategy $\boldsymbol{\theta}^{(0)}$. In this case, we have that $V^{\boldsymbol{\theta}^{(0)}}(T) = \mathrm{e}^{rT}$, which leads to

$$\alpha = \mathbb{E}^{\mathbb{P}}[S]\,\mathrm{e}^{-rT}.$$

Hence, the MV hedge of S in $\Theta^{(\boldsymbol{\theta}^{(0)})}$ consists of buying $\mathbb{E}^{\mathbb{P}}[S]\,\mathrm{e}^{-rT}$ zero-coupon bonds at time 0 and holding this portfolio until time T. Obviously, the time T-value of this hedge is given by

$$V_S^{\boldsymbol{\theta}^{(0)}}(T) = \mathbb{E}^{\mathbb{P}}[S].$$

Next, suppose that $\boldsymbol{\theta}_1 \in \Theta^{\mathcal{F}}$ and consider the T-claim $S = S^{\perp} \times V^{\boldsymbol{\theta}_1}(T)$, where $S^{\perp} \in \mathcal{O}_T$. In this case, we find that α is given by

$$\alpha = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right].$$

This means that the MV hedge of S in $\Theta^{(\boldsymbol{\theta}_1)}$ equals

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\boldsymbol{\theta}_1.$$

This result was to be expected, taking into account Theorem 3.1.

3.4. Examples

In this subsection, we consider two examples illustrating the calculation of MV hedges of insurance liabilities. In a first example, we consider the MV hedge of an equity-linked life insurance contract with payment guarantee.

Example 3.1 (MV hedging of equity-linked liabilities): Consider a portfolio of equity-linked life insurance contracts underwritten at time 0 on l_x persons of age x. Each contract specifies that at time T the financial T-claim $S^f \in \mathcal{C}_T^{\mathcal{F}}$ is paid out, provided the underlying insured is still alive at that time. Let T_i be the remaining lifetime of insured i, $i = 1, 2, \ldots, l_x$, at contract initiation. The time$-T$ payoff for policy i is given by

$$S_i = 1_{\{T_i > T\}} \times S^f, \quad i = 1, 2, \ldots, l_x, \tag{32}$$

where $1_{\{T_i>T\}}$ is the indicator variable which equals 1 in case $T_i > T$ and 0 otherwise. We assume that the remaining lifetimes of all insureds follow the same distribution and introduce the notation ${}_Tp_x$

for the survival probability $\mathbb{P}[T_i > T]$. The average claim per policy at time T is given by the time$-T$ claim

$$S = \frac{L_{x+T}}{l_x} \times S^f, \tag{33}$$

with

$$L_{x+T} = \sum_{i=1}^{l_x} 1_{\{T_i > T\}}. \tag{34}$$

Furthermore, we assume that the policyholders' remaining lifetimes $T_1, \ldots, T_{l_x}$ are independent of the financial market evolution in the sense that any $T_i \in \mathcal{O}_T$. This implies that the indicator variables $1_{\{T_i>T\}} \in \mathcal{O}_T$ and also that $L_{x+T} \in \mathcal{O}_T$.

In case mortality is fully diversifiable and the portfolio is sufficiently large, we can substitute L_{x+T}/l_x by ${}_Tp_x$ in (33) and we have that the claim S is a financial T-claim: $S = {}_Tp_x \times S^f \in \mathcal{C}_T^{\mathcal{F}}$, see Brennan and Schwartz (1976) and Boyle and Schwartz (1977).

An example of a payoff S^f is given by

$$S^f = \max\left(f\left(Y^{(1)}(T)\right), K\right). \tag{35}$$

Here, $Y^{(1)}(T)$ is the market price of 1 unit of risky asset 1 at time T, while f is a real-valued non-negative non-decreasing function, e.g. $f(x) = (1-\varepsilon)^T x$, where ε is an annual fee rate. Furthermore, $K \geq 0$ is the guaranteed minimal survival benefit. It is well-known that the payoff (35) can be split into a deterministic payment and a call option payoff:

$$\max\left(f\left(Y^{(1)}(T)\right), K\right) = K + \max(0, f\left(Y^{(1)}(T)\right) - K). \tag{36}$$

Hereafter, we investigate the valuation of the claim S defined in (33) in case the actuarial risk L_{x+T}/l_x is not necessarily fully diversified.

(a) Let us first consider the case where the payoff upon survival is a financially hedgeable T-claim, i.e. $S^f \in \mathcal{H}_T^{\mathcal{F}}$. This means that

$$S^f = V^{\boldsymbol{\theta}}(T), \quad \text{for some } \boldsymbol{\theta} \in \Theta^{\mathcal{F}}. \tag{37}$$

From Corollary 3.1 it follows that the MV hedge of the equity-linked payoff S in $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = {}_Tp_x \times \boldsymbol{\theta}, \tag{38}$$

while the time-0 value of the MV hedge of S equals

$$V_S^{MV}(0) = {}_Tp_x \times V^{\boldsymbol{\theta}}(0). \tag{39}$$

(b) Usually, the time horizon for equity-linked life insurance policies (typically 5 to 10 years) is different from the time horizon of standard call options (typically less than a few years). This makes that the claim S^f is often unhedgeable. Therefore, let us now assume that $S^f \notin \mathcal{H}_T^{\mathcal{F}}$. In this case, one could determine the MV hedge of the claim S in the set $\Theta^{(\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m)}$, where $\boldsymbol{\theta}^{(0)}$ is the static zero-coupon bond investment and each $\boldsymbol{\theta}_i \in \Theta^{\mathcal{F}}$ is a financial self-financing strategy. We assume that the time$-T$ values of the m+1 self-financing strategies are linearly independent. Taking into account

Theorem 3.1 and Corollary 3.4, we find that

$$\boldsymbol{\theta}_S^{MV} = {}_Tp_x \times \boldsymbol{\theta}_{S^f}^{MV} = {}_Tp_x \times \left(\alpha_0 \boldsymbol{\theta}^{(0)} + \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j \right) \tag{40}$$

and

$$V_S^{MV}(0) = {}_Tp_x \times \left(\alpha_0 V^{\boldsymbol{\theta}^{(0)}}(0) + \sum_{j=1}^{m} \alpha_j V^{\boldsymbol{\theta}_j}(0) \right), \tag{41}$$

with the α_j determined by the set of Equations (25) and (26). From (27), if follows that

$$\mathbb{E}^{\mathbb{P}}\left[V_S^{MV}(T) \right] = \mathbb{E}^{\mathbb{P}}[S]. \tag{42}$$

Notice that in case S^f can be replicated by a hedge in the set $\Theta^{(\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m)}$, the two values (39) and (41) are equal. For instance, in the one-period binomial model (Cox et al. 1979), one can verify that (α_0, α_1) is a perfect hedge for $\max(f(Y^{(1)}(T)), K)$.

In the following example, we consider a two-period binomial setting and apply Theorem 3.2 to derive the MV hedge of a financial-actuarial liability with a survival benefit at time 2 equal to the maximum of two dynamic investment strategies.

Example 3.2 (Equity-linked contract with the maximum of two dynamic strategies): Suppose that the combined financial-actuarial world is home to a financial market where a risk-free and two risky assets are traded in a two-period setting. The current value of risky asset $i = 1,2$, is given by $Y^{(i)}(0) = 1$, while its price dynamics follows a binomial tree over 2 periods. At each time $t = 1,2$, the value of asset i can go up to $Y^{(i)}(t-1)\ u^{(i)}$ or down to $Y^{(i)}(t-1)\ 1/u^{(i)}$, with $u^{(i)} > 1$. We assume that $u^{(1)} \neq u^{(2)}$.

First, consider a *constant-mix strategy*, which is defined as the self-financing strategy $\boldsymbol{\theta}_1 = \{\boldsymbol{\theta}_1(t)\}_{t=1,2}$ with

$$\begin{aligned} \boldsymbol{\theta}_1(1) &= (1, 1) \\ \boldsymbol{\theta}_1(2) &= \left(\frac{Y^{(1)}(1) + Y^{(2)}(1)}{2Y^{(1)}(1)}, \frac{Y^{(1)}(1) + Y^{(2)}(1)}{2Y^{(2)}(1)} \right). \end{aligned}$$

At time 0, 1 unit of each risky asset is bought. Hence, the time-0 market price of the strategy is 2. At time 1, the portfolio is rebalanced such that the initial proportions of 50% investment of the available capital in each risky asset, are restored. The time-2 value of this 50% / 50% mix portfolio is given by

$$\begin{aligned} V^{\boldsymbol{\theta}_1}(2) &= \boldsymbol{\theta}_1(2) \cdot \boldsymbol{Y}(2) \\ &= \sum_{i=1}^{2} \frac{Y^{(1)}(1) + Y^{(2)}(1)}{2Y^{(i)}(1)} Y^{(i)}(2). \end{aligned}$$

Next, consider the *buy-and-hold strategy* which keeps the number of units constant over time:

$$\boldsymbol{\theta}_2(t) = (1, 1), \quad \text{for } t = 1, 2.$$

The time-0 price of this strategy is 2, while its time-2 value is given by

$$V^{\boldsymbol{\theta}_2}(2) = \sum_{i=1}^{2} Y^{(i)}(2).$$

Suppose that the combined financial-actuarial world is also home to the indicator variable $S^{\perp} \in \mathcal{O}_2$ defined by

$$S^{\perp} = \begin{cases} 0 & : T_x \leq 2, \\ 1 & : T_x > 2 \end{cases} \tag{43}$$

where T_x is the remaining lifetime of (x).

Consider an insurance liability $S \in \mathcal{C}_2$ which guarantees the maximum payoff of the two self-financing strategies defined above, provided the insured (x) is still alive at time 2:

$$S = S^{\perp} \times \max\left(V^{\boldsymbol{\theta}_1}(2), V^{\boldsymbol{\theta}_2}(2)\right). \tag{44}$$

By Theorem 3.2, we know that the MV hedge of S in the set $\Theta^{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \alpha_1 \boldsymbol{\theta}_1 + \alpha_2 \boldsymbol{\theta}_2,$$

with the α_i determined by

$$\alpha_i = \sum_{j=1}^{2} \left(\mathbf{W}^{-1}\right)_{ij} \mathbb{E}^{\mathbb{P}}\left[S V^{\boldsymbol{\theta}_j}(2)\right], \quad i = 1, 2, \tag{45}$$

where $\mathbf{W}$ is the (2×2)-matrix with elements (i, j) defined by

$$(\mathbf{W})_{ij} = \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_i}(2) V^{\boldsymbol{\theta}_j}(2)\right]. \tag{46}$$

Apart from the claim S defined in (44), we also consider claims $\tilde{S} \in \mathcal{C}_2$ of the form

$$\tilde{S} = S^{\perp} \times \left(\beta_1 V^{\boldsymbol{\theta}_1}(2) + \beta_2 V^{\boldsymbol{\theta}_2}(2)\right),$$

for given real numbers β_1 and β_2.

From Theorem 3.1, we find that the MV hedge of $\tilde{S}$ in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_{\tilde{S}}^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] (\beta_1 \boldsymbol{\theta}_1 + \beta_2 \boldsymbol{\theta}_2).$$

As $\boldsymbol{\theta}_{\tilde{S}}^{MV} \in \Theta^{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)} \subseteq \Theta^{\mathcal{F}}$, it is obvious that the MV hedge of $\tilde{S}$ in $\Theta^{\mathcal{F}}$ is equal to the MV hedge of $\tilde{S}$ in $\Theta^{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)}$.

Let us now suppose that $u^{(1)} = \frac{4}{3}$ and $u^{(2)} = \frac{8}{3}$, indicating that the second asset is more volatile. Furthermore, we suppose that the $\mathbb{P}-$probability of an up-movement equals 1/2 for each risky asset and each time period. Finally, suppose that $\mathbb{P}[T_x > 2] = 0.9$, implying that $\mathbb{E}^{\mathbb{P}}[S^{\perp}] = 0.9$. The numerical values of the MV hedges of different time-2 claims are summarized in the following table.

Time-2 claim	MV hedge
$S^{\perp} \times V^{\boldsymbol{\theta}_1}(2)$	$0.9\,\boldsymbol{\theta}_1$
$S^{\perp} \times V^{\boldsymbol{\theta}_2}(2)$	$0.9\,\boldsymbol{\theta}_2$
$S^{\perp} \times (0.5V^{\boldsymbol{\theta}_1}(2) + 0.5V^{\boldsymbol{\theta}_2}(2))$	$0.45\,\boldsymbol{\theta}_1 + 0.45\boldsymbol{\theta}_2$
$S^{\perp} \times \max(V^{\boldsymbol{\theta}_1}(2), V^{\boldsymbol{\theta}_2}(2))$	$0.52\,\boldsymbol{\theta}_1 + 0.46\boldsymbol{\theta}_2$

The claims considered in the previous examples were in general not perfectly hedgeable. In the next section, we introduce an approach which values unhedgeable claims as the sum of the time-0 price of their MV hedge and an actuarial value for the remaining (unhedged) part of the claim.

4. Fair valuation of insurance liabilities

In this section, we define the class of fair valuations as well as the class of mean-variance hedge-based (MVHB) valuations in a multi-period setting. These concepts were introduced and investigated in Dhaene et al. (2017) in a single period framework. In Section 4.2, we show that the classes of fair valuations and MVHB valuations are equal. In Section 4.3, we provide some detailed examples illustrating the MVHB valuation.

4.1. Fair valuations

Solvency II, the European regulatory framework for insurance and reinsurance companies, focuses on the fair valuation of insurance liabilities. A distinction is made between hedgeable and non-hedgeable claims. For a hedgeable claim, the fair value equals the market value of the underlying hedging portfolio. The fair value of a non-hedgeable claim is defined as the sum of the expected present value (called best estimate) and a risk margin, see CEIOPS (2010). The application of this regulatory principle is not always straightforward as insurance liabilities are often partially replicable and it is usually not clear how the regulatory valuation principle should be applied in such a case.

Dhaene et al. (2017) define a general class of fair valuations which meet the fundamental regulatory requirements by merging actuarial judgement and market-consistency. Hereafter, we first define the class of valuations and then introduce the classes of market-consistent, actuarial and fair valuations.

Definition 4.1 (Valuation): A valuation is a mapping $\rho : \mathcal{C}_T \to \mathbb{R}$, attaching a real number to any claim $S \in \mathcal{C}_T$:

$$S \to \rho\left[S\right],$$

such that ρ is normalized:

$$\rho\left[0\right] = 0,$$

and ρ is translation invariant:

$$\rho\left[S + a\right] = \rho\left[S\right] + \mathrm{e}^{-rT} a, \quad \text{for any } S \in \mathcal{C}_T \text{ and } a \in \mathbb{R}.$$

Our convention of identifying r.v.'s which are equal in the $\mathbb{P}$-almost sure sense implies that $\rho[S_1] = \rho[S_2]$ in case S_1 and S_2 are equal in that sense.

Definition 4.2 (Market-consistent valuation): A valuation $\rho : \mathcal{C}_T \to \mathbb{R}$ is market-consistent (MC) if any financially hedgeable part of any claim is marked-to-market:

$$\rho\left[S + S^h\right] = \rho[S] + V^{\boldsymbol{\theta}}(0), \quad \text{for any } S \in \mathcal{C}_T \text{ and any } S^h = V^{\boldsymbol{\theta}}(T) \text{ with } \boldsymbol{\theta} \in \Theta^{\mathcal{F}}. \tag{47}$$

In the literature, market-consistency is usually defined via a condition equal or similar to condition (47), see e.g. Artzner and Eisele (2010), Malamud et al. (2008) and Pelsser and Stadje (2014). The mark-to-market condition (47) postulates that any financially replicable part of a claim is valuated by the price of its hedge. The MC condition (47) can be seen as an extension of translation invariance from scalars to financially hedgeable claims. We remark that the condition (47) is closely related to the market-consistent property defined in Pelsser and Stadje (2014).

In order to define actuarial valuations, we first have to introduce the notions of $\mathbb{P}$-law invariant and market-invariant mappings on the set of orthogonal claims $\mathcal{O}_T$.

Definition 4.3 ($\mathbb{P}$-law invariant mapping): A mapping $\rho : \mathcal{O}_T \to \mathbb{R}$ is $\mathbb{P}$-law invariant if for any $S_1^{\perp}$ and $S_2^{\perp} \in \mathcal{O}_T$ with the same $\mathbb{P}$-distribution, one has that $\rho[S_1^{\perp}] = \rho[S_2^{\perp}]$.

The $\mathbb{P}$-law invariance property stems from the fact that changing the r.v. $S_1^{\perp}$ into $S_2^{\perp}$ does not change the value of the mapping, provided both have the same $\mathbb{P}$-distribution. In other words, a $\mathbb{P}$-law invariant mapping $\rho : \mathcal{O}_T \to \mathbb{R}$ is in fact a mapping from the set of all $\mathbb{P}$-distributions of orthogonal claims to the real line. In this sense, one can say that $\rho[S^{\perp}]$ only depends on the $\mathbb{P}$-distribution of the orthogonal claim $S^{\perp}$.

Definition 4.4 (Market-invariant mapping): A mapping $\rho : \mathcal{O}_T \to \mathbb{R}$ is market-invariant if for any $S^{\perp} \in \mathcal{O}_T$, the value $\rho[S^{\perp}]$ is independent of the current risky asset prices $Y^{(1)}(0), \ldots, Y^{(n)}(0)$.

In this case, the market-invariance property results from the observation that $\rho[S^{\perp}]$ is constant with respect to any change in the current risky asset prices.

Definition 4.5 (Actuarial valuation): A valuation $\rho : \mathcal{C}_T \to \mathbb{R}$ is actuarial if any orthogonal claim is marked-to-model:

$$\rho\left[S^{\perp}\right] = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] + \mathrm{RM}\left[S^{\perp}\right], \quad \text{for any } S^{\perp} \in \mathcal{O}_T, \tag{48}$$

where $\mathrm{RM} : \mathcal{O}_T \to \mathbb{R}$ is a $\mathbb{P}$—law invariant and market-invariant mapping.

The mark-to-model (or actuarial) condition (48) introduces actuarial aspects in the valuation of claims by stating that for claims that are independent of the financial market information that will become available over time, the valuation does not depend on the current prices of traded risky assets and hence, also does not depend on $\mathbb{Q}$.

Notice that all results that we will derive hereafter in this paper remain valid if we define an actuarial valuation as a member of a given subset of the broad class of valuations considered in the definition above. For instance, we could define an actuarial valuation as a valuation of the form (48) where RM $[S^{\perp}] = \mathrm{e}^{-rT}\beta \mathrm{var}^{\mathbb{P}}[S^{\perp}]$, for some deterministic $\beta \geq 0$.

Definition 4.6 (Fair valuation): A fair valuation is a valuation which is both market-consistent and actuarial.

Our definition of a fair valuation in a multi-period setting is in line with current insurance solvency regulations which impose mark-to-market as well as mark-to-model requirements for the fair valuation of assets and liabilities.[2] Definition 4.6 combines market-consistency considerations concerning financially hedgeable parts of claims with the traditional actuarial view involving actuarial judgement of insurance claims. We remark that our definition of a fair valuation is generic and does not necessarily fully correspond to any particular definition of fair value in a particular regulation.

4.2. Mean-variance hedge-based valuations

Valuating a T-claim S via MV hedging starts with finding the optimal self-financing trading strategy $\boldsymbol{\theta}_S^{MV}$ which hedges the claim S with minimal expected quadratic hedging error in a linear subspace Θ' of Θ. Defining the value of the claim S as the initial cost $V_S^{MV}(0)$ of the MV hedging strategy $\boldsymbol{\theta}_S^{MV}$ leads to the same value for the T-claim S and for $V_S^{MV}(T)$, neglecting the part of S which is not hedged, i.e. $S - V_S^{MV}(T)$. In order to solve this issue, Dhaene et al. (2017) considered a class of fair valuations, the members of which are called mean-variance hedge-based (MVHB) valuations. Determining the

[2] In the 'Solvency II Glossary' of the 'Comité Européen des Assurances' and the 'Groupe Consultatif Actuariel Européen' of 2007, Fair Value is defined as 'the amount for which ⋯ a liability could be settled between knowledgeable, willing parties in an arm's length transaction. This is similar to the concept of Market Value, but the Fair Value may be a mark-to-model price if no actual market price for the ⋯ liability exists.'

MVHB value of a T-claim S departs from splitting this claim into the time-T value of its MV hedge and the remaining claim:

$$S = V_S^{MV}(T) + \left(S - V_S^{MV}(T)\right).$$

The MVHB value of S is then defined as the sum of the financial market price of the MV hedge and an actuarial value of the remaining claim.

Definition 4.7 (MVHB valuation): A mapping $\rho : \mathcal{C}_T \to \mathbb{R}$ is a mean-variance hedge-based (MVHB) valuation in case there exists a linear subspace Θ' of Θ and an actuarial valuation π such that

$$\rho\,[S] = V_S^{MV}(0) + \pi\,[S - V_S^{MV}(T)], \quad \text{for any } S \in \mathcal{C}_T, \tag{49}$$

where $V_S^{MV}(0)$ and $V_S^{MV}(T)$ are the time-0 and time-T values of the MV hedge $\boldsymbol{\theta}_S^{MV}$ of S in Θ', respectively.

It is straightforward to prove that a MVHB valuation is normalized and translation invariant, and hence, a valuation in the sense of Definition 4.1. Moreover, a MVHB valuation is positive homogeneous, provided the underlying actuarial valuation is positive homogeneous.

In the following lemma, a MVHB valuation formula is derived for product claims, taking into account Theorem 3.1.

Lemma 4.1: *Consider the MVHB valuation with underlying MV hedging in the linear space of self-financing trading strategies $\Theta' \subseteq \Theta^{\mathcal{F}}$ and actuarial valuation π. For any $S^{\perp} \in \mathcal{O}_T$ and any $S^f \in \mathcal{C}_T^{\mathcal{F}}$, the MVHB value of $S = S^{\perp} \times S^f$ is given by*

$$\rho\,[S] = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] V_{S^f}^{MV}(0) + \pi\left[S^{\perp} \times S^f - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] V_{S^f}^{MV}(T)\right]. \tag{50}$$

In the following theorem, we prove that the class of MVHB valuations is identical to the class of fair valuations.[3]

Theorem 4.1: *A mapping $\rho : \mathcal{C}_T \to \mathbb{R}$ is a MVHB valuation with underlying MV hedging in the set $\Theta^{\mathcal{F}}$ if and only if it is a fair valuation.*

Proof: (a) Consider the MVHB valuation ρ defined in (49). In order to show that ρ is a fair valuation, we have to verify whether ρ is a market-consistent and actuarial valuation.

(i) Let $S \in \mathcal{C}_T$ and $S^h = V^{\boldsymbol{\theta}}(T)$ with $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$. We have that

$$V_{S+S^h}^{MV}(T) = V_S^{MV}(T) + S^h,$$

and hence, also

$$V_{S+S^h}^{MV}(0) = V_S^{MV}(0) + V^{\boldsymbol{\theta}}(0),$$

see (11). Taking into account these additivity relations, we find that

$$\begin{aligned} \rho\left[S + S^h\right] &= V_{S+S^h}^{MV}(0) + \pi\left[S + S^h - V_{S+S^h}^{MV}(T)\right] \\ &= V_S^{MV}(0) + V^{\boldsymbol{\theta}}(0) + \pi\,[S - V_S^{MV}(T)] \\ &= \rho[S] + V^{\boldsymbol{\theta}}(0). \end{aligned}$$

Hence, ρ is market-consistent.

[3] This paper focuses on valuations based on the MV hedge but we remark that Definition 4.7 can be restated in terms of more general hedges. Notice that Theorem 4.1 remains to hold in this generalized setting. Dhaene et al. (2017) consider this more general setting in a one-period model.

(ii) Let $S^{\perp} \in \mathcal{O}_T$. From Corollary 3.3, we know that $\boldsymbol{\theta}_{S^{\perp}}^{MV} = \mathrm{e}^{-rT}\,\mathbb{E}^{\mathbb{P}}[S^{\perp}]\,\boldsymbol{\theta}^{(0)}$. Taking into account the translation-invariance of π leads to

$$\begin{aligned}\rho\left[S^{\perp}\right] &= V_{S^{\perp}}^{MV}(0) + \pi[S^{\perp} - V_{S^{\perp}}^{MV}(T)] \\ &= \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] + \pi[S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]] \\ &= \pi[S^{\perp}].\end{aligned}$$

Given that π is an actuarial valuation, we find that ρ is also an actuarial valuation.

(b) Consider the fair valuation ρ. Let $V_S^{MV}(T)$ be the time-T value of the MV hedge of the T-claim S in $\Theta^{\mathcal{F}}$. By the market-consistency property, we immediately find that

$$\begin{aligned}\rho[S] &= \rho\left[V_S^{MV}(T) + \left(S - V_S^{MV}(T)\right)\right] \\ &= V_S^{MV}(0) + \rho\left[S - V_S^{MV}(T)\right].\end{aligned}$$

Given that ρ is fair, it is also actuarial. Hence, we can conclude that the fair valuation ρ is a MVHB valuation. ■

Theorem 4.1 holds for MVHB valuations with MV hedge determined in the set of financial self-financing strategies $\Theta^{\mathcal{F}}$ whereas the MC condition (47) in the definition of a fair valuation has to hold for all $S^h = V^{\boldsymbol{\theta}}(T)$ with $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$. Important to notice is that Theorem 3 remains to hold if we replace $\Theta^{\mathcal{F}}$ by a linear subspace Θ' of $\Theta^{\mathcal{F}}$ which includes $\boldsymbol{\theta}^{(0)}$ as one of its elements, provided we redefine a MC valuation as a valuation which satisfies the MC property only for claims which are hedgeable with a self-financing strategy in Θ', while we redefine a MVHB valuation as a valuation of the form (49), where the MV hedge is determined in the set Θ'. We remark that the self-financing strategy $\boldsymbol{\theta}^{(0)}$ is required to be an element of Θ' in order to guarantee that the MVHB valuation is actuarial.

Moreover, Theorem 4.1 is a generalization of Theorem 3 in Dhaene et al. (2017) in the MV hedging case as it allows periodic rebalancing (for instance yearly) for long term T-claims. Obviously, this cannot be achieved within a single period model.

4.3. Examples

We end this section with two examples illustrating the fair valuation of insurance liabilities.

In Example 4.1, we consider the fair value of the liabilities related to a portfolio of equity-linked life insurance contracts by applying the MVHB valuation with a standard deviation actuarial valuation principle for the non-hedged part of the claims. Under the assumption of diversifiability of mortality, the actuarial value of the non-hedged part per policy converges to zero due to the law of large numbers (LLN). In case of conditional independence, instead of independence of the remaining lifetimes of the insureds, the LLN breaks down and the actuarial value in the MVHB valuation converges to a non-zero constant, giving rise to a risk margin for non-diversifiable mortality risk, see also Milevsky et al. (2006).

Example 4.1 (Valuation of equity-linked liabilities): Consider the portfolio of l_x contracts underwritten at time 0 as described in Example 3.1. Each contract guarantees to its beneficiary the payment $S^f \in \mathcal{C}_T^{\mathcal{F}}$ at time T, provided the insured is still alive at that time. All insureds are assumed to be x years old at policy issue. As in Example 3.1, we assume that the policyholders' remaining lifetimes $T_1, \ldots, T_{l_x}$ are identically distributed and independent of the financial market evolution in the sense that any $T_i \in \mathcal{O}_T$. As before, we use the notation ${}_Tp_x$ for $\mathbb{E}^{\mathbb{P}}[L_{x+T}/l_x]$.

The average claim per policy at time T is given by (33):

$$S = \frac{L_{x+T}}{l_x} \times S^f,$$

with

$$L_{x+T} = \sum_{i=1}^{l_x} 1_{\{T_i > T\}}.$$

Suppose that we apply the MVHB valuation (49) with underlying MV hedging in the space of of self-financing trading strategies $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$ defined in Example 3.1, and as actuarial valuation π the standard deviation principle, i.e.

$$\pi\,[S] = \mathrm{e}^{-rT}\left(\mathbb{E}^{\mathbb{P}}\,[S] + \beta\sigma^{\mathbb{P}}[S]\right), \quad \text{for any } S \in \mathcal{C}_T,$$

with β a given non-negative real number.

From (42), we know that

$$\boldsymbol{\theta}_S^{MV} = {}_T p_x \times \boldsymbol{\theta}_{S^f}^{MV},$$

with

$$\boldsymbol{\theta}_{S^f}^{MV} = \left(\alpha_0\boldsymbol{\theta}^{(0)} + \sum_{j=1}^{m}\alpha_j\boldsymbol{\theta}_j\right),$$

where the coefficients α_j follow from (25) and (26).

Taking into account Lemma 4.1 and (42), we find that the MVHB value of S is given by

$$\rho\,[S] = {}_T p_x \times V_{S^f}^{MV}(0) + \mathrm{e}^{-rT}\beta\sigma^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x} \times S^f - {}_T p_x \times V_{S^f}^{MV}(T)\right]. \tag{51}$$

After some straightforward calculations, this value can be rewritten as follows:

$$\rho\,[S] = {}_T p_x \times V_{S^f}^{MV}(0) + \mathrm{e}^{-rT}\beta\sigma \tag{52}$$

with

$$\sigma^2 = \left({}_T p_x\right)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right]. \tag{53}$$

The actuarial premium for the non-hedged part of the claim, i.e. $\mathrm{e}^{-rT}\ \beta\ \sigma$, can be interpreted as a 'risk loading' composed of two components. The first component is related to the fact that S^f is not perfectly hedgeable, whereas the second component is due to the fact that the survival risk is not fully diversified. In case S^f is perfectly hedgeable in $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$, the first term of σ^2 vanishes, whereas in case of full diversification of the survival risk, its second term disappears. Due to (27), we remark that

$$\mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] = \mathbb{E}^{\mathbb{P}}\left[\left(S^f - V_{S^f}^{MV}(T)\right)^2\right]$$
$$= \min_{\boldsymbol{\mu}\in\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}} \mathbb{E}^{\mathbb{P}}\left[\left(S^f - V^{\boldsymbol{\mu}}(T))\right)^2\right].$$

(a) Let us additionally assume that $T_1, \ldots, T_{l_x}$ are i.i.d under $\mathbb{P}$. In this case, we find that

$$\mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] = \frac{{}_Tp_x(1 - {}_Tp_x)}{l_x}.$$

The MVHB value $\rho[S]$ of the average claim per policy is then given by (52) with

$$\sigma^2 = \left({}_Tp_x\right)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \frac{{}_Tp_x(1 - {}_Tp_x)}{l_x}.$$

Increasing the number of policies leads to a decrease of the value of the average claim per policy for the non-hedged part of the claim. Moreover, we have that

$$\lim_{l_x\to\infty} \rho\left[S\right] = {}_Tp_x\left(V_{S^f}^{MV}(0) + \mathrm{e}^{-rT}\beta\sigma^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right]\right).$$

Therefore, when l_x goes to infinity, the actuarial value per policy for the non-hedged part of the claim is only due to the hedging error.

(b) Instead of assuming that $T_1, \ldots, T_{l_x}$ are $\mathbb{P}$-i.i.d, let us now assume that there exists a r.v. P with $\mathbb{P}$-cdf given by $F_P^{\mathbb{P}}(p)$, $p \in [0, 1]$, such that given $P = p$, the remaining lifetimes $T_1, \ldots, T_{l_x}$ are $\mathbb{P}$-i.i.d., with

$$\mathbb{P}\left[T_i > T \mid P = p\right] = p, \quad p \in [0, 1].$$

P can be interpreted as the 'stochastic survival probability' and we find that

$$\mathbb{P}\left[T_i > T\right] = \mathbb{E}^{\mathbb{P}}\left[P\right].$$

Due to the random nature of P, the remaining lifetimes T_i are not mutually independent anymore. Instead they have become conditionally independent.

The expectation and the variance of L_{x+T}/l_x are now given by

$$\mathbb{E}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] = \mathbb{E}^{\mathbb{P}}\left[P\right]$$

and

$$\mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[P(1 - P)\right]}{l_x} + \mathrm{Var}^{\mathbb{P}}\left[P\right].$$

Inserting these expressions in (53), we find that the MVHB value $\rho[S]$ of the average claim per policy is given by (52) with

$$\sigma^2 = \left({}_Tp_x\right)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \left(\frac{\mathbb{E}^{\mathbb{P}}\left[P(1 - P)\right]}{l_x} + \mathrm{Var}^{\mathbb{P}}\left[P\right]\right).$$

Again, we can conclude that increasing the number of policies leads to a decrease of the actuarial value per policy for the non-hedged part of the claim. Moreover, we have that

$$\lim_{l_x\to\infty}\sigma^2=\left({}_Tp_x\right)^2\times\mathrm{Var}^{\mathbb{P}}\left[S^f-V_{S^f}^{MV}(T)\right]+\mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right]\times\mathrm{Var}^{\mathbb{P}}\left[P\right].$$

Hence, in case $\mathrm{Var}^{\mathbb{P}}[P]\neq 0$, the survival risk is not fully diversifiable: even if the number of insureds becomes infinitely large, the actuarial premium of the unhedged risk contains a term related to the undiversifiable survival risk.

In the following example, we investigate the fair value of a product claim of the form $S=S^{\perp}\times V^{\boldsymbol{\theta}}(T)$ where the trading strategy $\boldsymbol{\theta}$ depends on $S^{\perp}$ (hence, $\boldsymbol{\theta}\notin\Theta^{\mathcal{F}}$). In particular, we consider the fair value of a pool of equity-linked contracts in which the investment portfolio depends on the number of survivors. We further quantify the impact on the fair value when the aggregate longevity risk is transferred from the pool to the insurer.

Example 4.2 (Transfer of the longevity risk in equity-linked liabilities): (a) Consider a portfolio of T-year equity-linked policies underwritten at time 0 on a cohort of l_x insureds aged x at policy initiation. The random number of survivors a time t is denoted by L_{x+t}, $t=0,1,\ldots,T$. At time T, the value $V^{\boldsymbol{\theta}}(T)$ of a self-financing investment strategy $\boldsymbol{\theta}\in\Theta$, set up at time 0, is equally distributed among the survivors in the portfolio. Hence, the payoff per policy in force at time T is given by

$$S_i=\frac{V^{\boldsymbol{\theta}}(T)}{L_{x+T}}1_{\{T_i>T\}},\quad i=1,2,\ldots,l_x,\tag{54}$$

where T_i is the remaining lifetime of insured i. Any policyholder i faces three sources of risk: investment risk (caused by the random nature of the final value $V^{\boldsymbol{\theta}}(T)$ of the investment strategy), individual longevity risk (due to the randomness of the remaining lifetime T_i of the insured) and aggregate longevity risk (because of the random nature of the number of survivors L_{x+T}). The aggregate portfolio liability at time T equals

$$S=\sum_{i=1}^{l_x}S_i=V^{\boldsymbol{\theta}}(T).$$

As S is a hedgeable claim, its fair value at time 0 is given by the cost of the initial investment of the trading strategy $\boldsymbol{\theta}$:

$$\rho[S]=V^{\boldsymbol{\theta}}(0).\tag{55}$$

The insurer who charges a single premium of $V^{\boldsymbol{\theta}}(0)/l_x$ per underwritten contract and sets up the investment strategy $\boldsymbol{\theta}$ at time 0 does not take any risk: all sources of risk are born by the pool of policyholders. The portfolio can be considered as a pool of tontine-like policies. For the reader interested in pooled funds and tontines, we refer to Milevsky and Salisbury (2015), Bräutigam et al. (2017) and the interesting book on the 1693 tontine by Milevsky (2015).

(b) Let us now in addition assume that $\boldsymbol{\theta}\in\Theta^{\mathcal{F}}$ and that all T_i, and hence also L_{x+t}, are orthogonal claims. Furthermore, we adapt the contract payoff (54) in the sense that the random number of survivors L_{x+T} is replaced by its deterministic estimate $l_{x+t} = \mathbb{E}^{\mathbb{P}}[L_{x+T}]$ in the payoff per policy:

$$S_i = \frac{V^{\boldsymbol{\theta}}(T)}{l_{x+T}} 1_{\{T_i > T\}}. \tag{56}$$

In this adapted contract, the aggregate longevity risk is transferred to the insurer, i.e. he bears the uncertainty on the number of survivors at maturity. The aggregate portfolio liability is now given by

$$S = V^{\boldsymbol{\theta}}(T) \frac{L_{x+T}}{l_{x+T}}. \tag{57}$$

As the aggregate liability S is no longer hedgeable, the insurer determines the fair value of S via a MVHB valuation.

From Corollary 3.1, it follows that the MV hedge of S in $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \boldsymbol{\theta}.$$

From Lemma 4.1, we find that the MVHB value of S equals

$$\rho[S] = V^{\boldsymbol{\theta}}(0) + \pi\left[\left(\frac{L_{x+T} - l_{x+T}}{l_{x+T}}\right) V^{\boldsymbol{\theta}}(T)\right].$$

Let us now choose, as actuarial valuation, the standard deviation principle, i.e.

$$\pi[S] = \mathrm{e}^{-rT}\left(\mathbb{E}^{\mathbb{P}}[S] + \beta\sigma^{\mathbb{P}}[S]\right),$$

for some $\beta > 0$. Taking into account that

$$\mathrm{Var}^{\mathbb{P}}\left[\left(\frac{L_{x+T} - l_{x+T}}{l_{x+T}}\right) V^{\boldsymbol{\theta}}(T)\right] = \mathbb{E}^{\mathbb{P}}\left[\left(V^{\boldsymbol{\theta}}(T)\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_{x+T}}\right],$$

we find that

$$\rho[S] = V^{\boldsymbol{\theta}}(0) + \beta\,\mathrm{e}^{-rT}\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(V^{\boldsymbol{\theta}}(T)\right)^2\right]} \times \sigma^{\mathbb{P}}\left[\frac{L_{x+T}}{l_{x+T}}\right].$$

In case the insurer charges a premium of $\rho[S]/l_x$ per policy, we observe that the first part of the premium corresponds with the premium charged in (a), whereas the second part is the extra loading per policy for the transfer of the aggregate longevity risk to the insurer. This extra loading is caused by the volatility of both $V^{\boldsymbol{\theta}}(T)$ and L_{x+T}.

(c) As a special case of (b), suppose that $\boldsymbol{\theta} = l_{x+T}\,\mathrm{e}^{-rT}\,\boldsymbol{\theta}^{(0)}$. The time$-T$ payoff per policy is then given by

$$S_i = 1_{\{T_i > T\}}. \tag{58}$$

In this case, the policyholder only bears the individual longevity risk. The aggregate portfolio liability is now given by

$$S = \sum_{i=1}^{l_x} S_i = L_{x+T}, \tag{59}$$

while the fair value of the portfolio liability is given by

$$\rho[S] = \mathrm{e}^{-rT}\left(l_{x+T} + \beta\sigma^{\mathbb{P}}[L_{x+T}]\right).$$

Notice that the insurance contract considered in (c) corresponds to a classical pure endowment.

(d) Let us go back to (a) and consider the portfolio of l_x contracts with time$-T$ benefits given by (54), with trading strategy $\boldsymbol{\theta} \in \Theta$ defined in the following way:

At time 0, for any underwritten policy, an amount A/l_x is fully invested in the risk-free bank account. Furthermore, any time an insured dies in any year $(j-1, j)$, the amount $(A/l_x)\, \mathrm{e}^{rj}$ is withdrawn from the bank account at time j and is fully invested in asset 1, from time j until time T. The aggregate portfolio liability is then equal to the time$-T$ value of the investment strategy $\boldsymbol{\theta}$:

$$S = V^{\boldsymbol{\theta}}(T) = \frac{A}{l_x}\left(L_{x+T}\, \mathrm{e}^{rT} + \sum_{j=1}^{\mathrm{T}} D_{x+j-1}\, \mathrm{e}^{rj} \frac{Y^{(1)}(T)}{Y^{(1)}(j)}\right),$$

$$= \frac{A}{l_x} L_{x+T}\, \mathrm{e}^{rT} + S',$$

where D_{x+j-1} is the number of people who died during the year $(j-1, j)$ and S' denotes the part of the aggregate survival benefits which was invested in the risky asset 1 (after the death of the respective insureds). Obviously, $\boldsymbol{\theta}$ is a self-financing trading strategy with $V^{\boldsymbol{\theta}}(0) = A$. Moreover, $\boldsymbol{\theta} \notin \Theta^{\mathcal{F}}$ as the investment strategy depends on the number of survivors at each time j. As in (a), we have that S is a hedgeable claim and the fair value of the portfolio is given by

$$\rho\,[S] = A.$$

From (56), it follows that the time$-T$ payoff per policy S_i is given by

$$S_i = \left(\frac{A}{l_x}\mathrm{e}^{rT} + \frac{S'}{L_{x+T}}\right) 1_{\{T_i > T\}},$$

which clearly shows that the policyholder bears the risky investment risk, as well as the individual and the aggregate longevity risk.

(e) Let us consider the self-financing strategy $\boldsymbol{\theta}$ introduced in (d). Suppose now that the time$-T$ payoff per policy is defined by

$$S_i = \left(\frac{A}{l_x}\mathrm{e}^{rT} + \frac{\bar{S}'}{l_{x+T}}\right) 1_{\{T_i > T\}}, \tag{60}$$

where

$$\bar{S}' = \frac{A}{l_x} \sum_{j=1}^{\mathrm{T}} d_{x+j-1}\, \mathrm{e}^{rj} \frac{Y^{(1)}(T)}{Y^{(1)}(j)},$$

with $d_{x+j-1} = \mathbb{E}^{\mathbb{P}}[L_{x+j-1} - L_{x+j}]$, the *expected* number of people who will die during the year $(j-1, j)$. The aggregate portfolio liability is now given by

$$S = \left(\frac{A}{l_x}\mathrm{e}^{rT} + \frac{\bar{S}'}{l_{x+T}}\right) L_{x+T}. \tag{61}$$

From the expression (61), we observe that the aggregate longevity risk is transferred to the insurer and also that the aggregate portfolio liability is no longer hedgeable. Notice that S can be written as

$$S = \frac{A}{l_x}\left(l_{x+T}\, \mathrm{e}^{rT} + \sum_{j=1}^{\mathrm{T}} d_{x+j-1}\, \mathrm{e}^{rj} \frac{Y^{(1)}(T)}{Y^{(1)}(j)}\right) \frac{L_{x+T}}{l_{x+T}}$$

$$= V^{\boldsymbol{\mu}}(T)\, \frac{L_{x+T}}{l_{x+T}},$$

where $\boldsymbol{\mu}$ is similar to the strategy $\boldsymbol{\theta}$ introduced in (d), but with the real numbers of deaths and survivors (D_{x+t} and L_{x+t}) replaced by their respective expectations (d_{x+t} and l_{x+t}). As $\boldsymbol{\mu} \in \Theta^{\mathcal{F}}$, we can follow the same approach as in (b) to determine the MVHB value of S.

From Corollary 3.1, the MV hedge of S in $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \boldsymbol{\mu},$$

and from Lemma 4.1, we find that the MVHB value of S equals

$$\rho\,[S] = V^{\boldsymbol{\mu}}(0) + \pi\left[\left(\frac{L_{x+T} - l_{x+T}}{l_{x+T}}\right) V^{\boldsymbol{\mu}}(T)\right].$$

Since $V^{\boldsymbol{\mu}}(0) = V^{\boldsymbol{\theta}}(0) = A$, the second term in this expression for $\rho[S]$ can be interpreted as the fair value for the transfer of the aggregate longevity risk to the insurer.

5. Summary and concluding remarks

The fair value of an insurance liability, which is often neither completely hedgeable nor orthogonal, is in general not uniquely determined. This is not only due to the involvement of actuarial judgement, but also at an earlier stage in the valuation process due to the ambiguity in the current regulatory directives on how to determine the hedgeable part of such a liability.

In this paper, we investigated the fair valuation of insurance liabilities based on mean-variance hedging and extended the results of Dhaene et al. (2017) to a multi-period dynamic investment setting. We focused on product claims, i.e. claims which can be expressed as the product of an actuarial and a financial claim. Under independence between the actuarial claim (typically a mortality-related claim) and the financial market, we derived the MV hedge in Theorem 3.1 and obtained tractable formulas for the fair valuation of such product claims. For general claims, we derived the MV hedge in the set of all strategies which can be expressed as linear combinations of a number of given self-financing trading strategies. The obtained results have been illustrated with numerous examples.

In Section 4, we showed that the class of fair valuations is identical to the class of mean-variance hedge-based valuations. Under the MVHB approach, we showed that the risk margin in equity-linked contracts can be decomposed into a risk loading for non-diversifiable mortality risk and a risk loading for non-hedgeable financial risk. Moreover, we determined the extra loading in the fair value when the longevity risk in pooled equity-linked contracts is transferred to the insurer.

As considered in Dhaene et al. (2017), one can also define a two-step valuation based on a conditional actuarial valuation, which extends the two-step valuation of Pelsser and Stadje (2014) by introducing actuarial considerations. In the same manner as in Dhaene et al. (2017), one can show that the set of two-step valuations coincides with the set of fair valuations and hence, also with the set of MVHB valuations. These results can be seen as generalizations of the equivalences which hold in a one-period static setting in Dhaene et al. (2017).

Under the MVHB approach, the valuation gives an explicit hedge and an additive decomposition of the claim into a financial hedgeable part and an actuarial non-hedgeable part. Therefore, we believe that the MVHB valuation provides a relevant framework to determine the hedgeable part and the fair valuation of insurance liabilities which involve both actuarial and financial components.

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