EXERCISE 1

Q1 $\boxed{\exists \text{ an EMM } \mathbb{Q} \implies \text{no-arbitrage}}$

PROOF: • Suppose there exist an EMM $\mathbb{Q}$ such that

$$y^{(m)} = e^{-2} E^{\mathbb{Q}}[Y^{(m)}] \qquad m = 0, 1, \dots, n \qquad (1)$$

• Suppose there exists an arbitrage opportunity denoted by $\underline{\theta}$:

$$\begin{cases} \underline{\theta} \cdot \underline{y} = 0 & (2) \\ \mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1 & (3) \\ \mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] > 0 & (4) \end{cases}$$

• As $\mathbb{Q} \sim \mathbb{P}$, we find from (3) and (4):

$$\begin{cases} \mathbb{Q}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1 & (5) \\ \mathbb{Q}[\underline{\theta} \cdot \underline{Y} > 0] > 0 & (6) \end{cases}$$

• Hence, from (5) and (6), we find that

$$E^{\mathbb{Q}}[\underline{\theta} \cdot \underline{Y}] > 0$$

• On the other hand,

$$e^{-2} E^{\mathbb{Q}}[\underline{\theta} \cdot \underline{Y}] = \sum_{m=0}^{n} \theta^{(m)} \cdot \left\{ e^{-2} E^{\mathbb{Q}}[Y^{(m)}] \right\}$$

$$\overset{(1)}{=} \sum_{m=0}^{n} \theta^{(m)} \cdot y^{(m)} = \underline{\theta} \cdot \underline{y} \overset{(2)}{=} 0$$

• We found a contradiction. Hence, there can be no arbitrage opportunity ☒

EXERCISE 1

Q2

$\boxed{\text{Arbitrage-free market complete} \implies \text{There is a unique } \mathbb{Q}}$

PROOF: • Market is assumed to be complete. Hence, all S_{ij} can be replicated:

$$S_{ij} = \underline{\theta_{ij}} \cdot \underline{I} \quad \text{for any } i \text{ and } j$$

↳ investment strategy replicating S_{ij}

• Suppose that there exist 2 EMM's $\mathbb{Q}^*$ and $\mathbb{Q}^{**}$

• Then:

$$\text{Price of } S_{ij} \text{ at time } 0 = e^{-2} E^{\mathbb{Q}^*}[S^{ij}] = e^{-2} E^{\mathbb{Q}^{**}}[S^{ij}$$

↳ see slide 18

• Hence, $e^{-2} q^*_{ij} = e^{-2} q^{**}_{ij}$, $\forall i, j$

$\implies \mathbb{Q}^* = \mathbb{Q}^{**}$

$\implies$ The EMM is unique ∎

$\boxed{\text{Example 1}}$

There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\begin{cases} E^{\mathbb{Q}}\left[Y^{(0)}\right] = 1 \\ E^{\mathbb{Q}}\left[Y^{(1)}\right] = 100 \end{cases} \qquad (1)$$

$\iff$

There exist $q_{50,0} > 0$; $q_{50,1} > 0$; $q_{150,0} > 0$, $q_{150,1} > 0$ with

$$\begin{cases} q_{50,0} + q_{50,1} + q_{150,0} + q_{150,1} = 1 \\ (q_{50,0} + q_{50,1}) \times 50 + (q_{150,0} + q_{150,1}) \times 150 = 100 \end{cases}$$

$\iff$

There exist $q_{50,0} > 0$; $q_{50,1} > 0$; $q_{150,0} > 0$, $q_{150,1} > 0$

with $$\begin{cases} q_{50\bullet} + q_{150\bullet} = 1 \\ q_{50\bullet} \times 50 + q_{150\bullet} \times 150 = 100 \end{cases}$$

$\iff$

There exist $q_{50,0} > 0$; $q_{150,0} > 0$; $q_{50,1} > 0$; $q_{150,1} > 0$

with
$R_2 - 50 R_1$
$$\begin{cases} q_{50\bullet} + q_{150\bullet} = 1 \\ q_{150\bullet} \times 100 = 50 \end{cases}$$

$\Longleftrightarrow$

There exist $q_{50,0} > 0$; $q_{50,1} > 0$; $q_{150,0} > 0$, $q_{150,1} > 0$

such that
$$\begin{cases} q_{50\bullet} = 0.5 \\ q_{150\bullet} = 0.5 \end{cases}$$

Exercise 2

Q1

$$\mathbb{Q} \text{ is EMM} \iff \begin{cases} q_{50,0} > 0,\ q_{50,1} > 0,\ q_{150,0} > 0,\ q_{150,1} > 0 \\ q_{50\cdot} = 1/2 \\ q_{150\cdot} = \frac{1}{2} \end{cases}$$

$$\iff \begin{cases} q_{50,1} = q \\ q_{50,0} = 1/2 - q \\ q_{150,1} = q' \\ q_{150,0} = 1 - q' \end{cases} \quad \text{for some} \quad \begin{matrix} 0 < q < \frac{1}{2} \\ 0 < q' < 1/2 \end{matrix}$$

Q2 (*) $\underline{\theta} = (\theta^{(0)}, \theta^{(1)})$ is a hedge for $(100 - Y^{(1)})_+$

$$\iff \underline{\theta} \cdot \underline{Y} = (100 - Y^{(1)})_+$$

$$\iff \theta^{(0)} \cdot Y^{(0)} + \theta^{(1)} \cdot Y^{(1)} = (100 - Y^{(1)})_+$$

$$\iff \begin{cases} \theta^{(0)} + \theta^{(1)} \times 50 = 50 & : Y^{(1)} = 50 \\ \theta^{(0)} + \theta^{(1)} \times 150 = 0 & : Y^{(1)} = 150 \end{cases}$$

$$\begin{matrix} -R_1 + R_2 \\ \iff \\ R_2 - 3R_1 \end{matrix} \begin{cases} 100\,\theta^{(1)} = -50 \\ -2\theta^{(0)} = -150 \end{cases} \iff \begin{cases} \theta^{(0)} = 75 \\ \theta^{(1)} = -0.5 \end{cases}$$

(*) Hence, $\underline{\theta} = (75, -0.5)$ is a hedge for $(100 - Y^{(1)})_+$

(*) Price of $(100 - Y^{(1)})_+$ $= \underline{\theta} \cdot \underline{y} = 75 \times y^{(0)} - 0.5 \times y^{(1)}$

$$= 75 - 0.5 \times 100$$

$$= 25$$

Exercise 2

Q3

(*). The set of admissable prices of I is given by $\{ \varrho^{\mathbb{Q}}[I] \mid \mathbb{Q} \text{ is an EMM} \}$

- $\varrho^{\mathbb{Q}}[I] = e^{-2} E^{\mathbb{Q}}[I] = q_{\cdot 1} = q_{50,1} + q_{150,1}$

$= q + q'$ for some $0 < q < \frac{1}{2}$ and $0 < q' < \frac{1}{2}$

- Hence, $\boxed{\varrho^{\mathbb{Q}}[I] \in (0,1)}$

(*). The set of admissable prices of $(100 - Y^{(1)})_+ \times I$ is given by $\{ \varrho^{\mathbb{Q}}[(100 - Y^{(1)})_+ \times I] \mid \mathbb{Q} \text{ is an EMM} \}$

- $\varrho^{\mathbb{Q}}[(100 - Y^{(1)})_+ \times I]$

$= e^{-2} E^{\mathbb{Q}}[(100 - Y^{(1)})_+ \times I]$

$= q_{50,1} \times 50$

$= 50\, q$ for some $0 < q < \frac{1}{2}$.

- Hence, $\boxed{\varrho^{\mathbb{Q}}[(100 - Y^{(1)})_+ \times I] \in (0, 25)}$

Example 1'

There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\begin{cases} E^{\mathbb{Q}}[Y^{(0)}] = 1 \\ E^{\mathbb{Q}}[Y^{(1)}] = 100 \\ E^{\mathbb{Q}}[Y^{(2)}] = 70 \end{cases} \qquad (2)$$

$\iff$

There exist $q_{50,0} > 0;\ q_{50,1} > 0;\ q_{150,0} > 0;\ q_{150,1} > 0$ such that

$$\begin{cases} q_{50\cdot} = 0.5 \\ q_{150\cdot} = 0.5 \\ E^{\mathbb{Q}}[Y^{(2)}] = 70 \end{cases}$$

$\iff$ There exist $q_{50,0} > 0;\ \ldots$ such that

$$\begin{cases} q_{50\cdot} = 0.5 \\ q_{150\cdot} = 0.5 \\ q_{\cdot 0} \times 100 = 70 \end{cases}$$

$\iff$ There exist $q_{50,0} > 0;\ q_{50,1} > 0;\ q_{150,0} > 0;\ q_{150,1} > 0$

such that $$\begin{cases} q_{50\cdot} = 0.5 \\ q_{150\cdot} = 0.5 \\ q_{\cdot 0} = 0.7 \\ q_{\cdot 1} = 0.3 \end{cases}$$

Exercise 3

Q1

- Suppose that $(100-Y^{(1)})_+ \times I$ can be replicated by $\underline{\theta} = (\theta^{(0)}, \theta^{(1)}, \theta^{(2)})$.

- Then: $\theta^{(0)} \times Y^{(0)} + \theta^{(1)} \times Y^{(1)} + \theta^{(2)} \times Y^{(2)} = (100-Y^{(1)})_+ \times I$

- Or, equivalently,

$$\begin{cases} \theta^{(0)} \times 1 + \theta^{(1)} \times 50 + \theta^{(2)} \times 0 = 0 & : (50,0) \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 150 + \theta^{(2)} \times 0 = 0 & : (150,0) \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 50 + \theta^{(2)} \times 100 = 50 & : (50,1) \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 150 + \theta^{(2)} \times 100 = 0 & : (150,1) \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta^{(0)} + \theta^{(1)} \times 50 = 0 \\ \theta^{(0)} + \theta^{(1)} \times 150 = 0 \\ \theta^{(0)} + \theta^{(1)} \times 50 + \theta^{(2)} \times 100 = 50 \\ \theta^{(0)} + \theta^{(1)} \times 150 + \theta^{(2)} \times 100 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dots \\ \dots \\ \theta^{(2)} \times 100 = 50 & R_3 - R_1 \\ \theta^{(2)} \times 100 = 0 & R_4 - R_2 \end{cases} \Leftrightarrow \begin{cases} \dots \\ \dots \\ \theta^{(2)} = 1/2 \\ \theta^{(2)} = 0 \end{cases} \lightning$$

- CONCLUSION: $(100 - Y^{(1)})_+ \times I$ CANNOT be hedged.

↳ we cannot find a $\underline{\theta}$.

Exercise 3

Q2

- Admissable arbitrage-free prices for $(100-Y^{(1)})_+ \times I$:

$$\rho^{\mathbb{Q}}\left[(100-Y^{(1)})_+ \times I\right]$$

$$= e^{-2} E^{\mathbb{Q}}\left[(100-Y^{(1)})_+ \times I\right]$$

$$= q_{50,1} \times 50$$

$$= q \quad \text{with} \quad q \in (0;0.3)$$

- Hence, $\rho^{\mathbb{Q}}\left[(100-Y^{(1)})_+ \times I\right]$ ~~(...)~~

$\in (0;15)$

Example 1''

There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\left\{\begin{array}{l} E^{\mathbb{Q}}\left[Y^{(0)}\right] = 1 \\ E^{\mathbb{Q}}\left[Y^{(1)}\right] = 100 \\ E^{\mathbb{Q}}\left[Y^{(2)}\right] = 70 \\ E^{\mathbb{Q}}\left[Y^{(3)}\right] = y^{(3)} \end{array}\right. \qquad (3)$$

$\Longleftrightarrow$

There exist $q_{50,0} > 0$; $q_{50,1} > 0$; $q_{150,0} > 0$; $q_{150,1} > 0$

such that
$$\left\{\begin{array}{l} q_{50\cdot} = 0.5 \\ q_{150\cdot} = 0.5 \\ q_{\cdot 0} = 0.7 \\ q_{\cdot 1} = 0.3 \\ q_{50,1} \times 50 = y^{(3)} \end{array}\right.$$

$\Longleftrightarrow$

There exist $q_{50,0} > 0, \ldots$ such that

$$\begin{array}{l} R_1 - R_5/50 \\ \\ \\ R_4 - R_5/50 \\ \\ \end{array} \left\{\begin{array}{l} q_{50,0} = \dfrac{25 - y^{(3)}}{50} \\ q_{150\cdot} = 0.5 \\ q_{\cdot 0} = 0.7 \\ q_{150,1} = \dfrac{15 - y^{(3)}}{50} \\ q_{50,1} = \dfrac{y^{(3)}}{50} \end{array}\right.$$

$\Longleftrightarrow$

There exist $q_{50,0} > 0, \ldots$ such that

$$R_2 - R_4 \quad \begin{cases} q_{50,0} = \dfrac{25 - y^{(3)}}{50} \\ q_{150,0} = \dfrac{10 + y^{(3)}}{50} \\ \cancel{q_{\cdot 0} = 0.1} \\ q_{50,1} = \dfrac{y^{(3)}}{50} \\ q_{150,1} = \dfrac{15 - y^{(3)}}{50} \end{cases}$$

$\Longleftrightarrow$

There exist $q_{50,0} > 0$; $q_{150,0} > 0$; $q_{50,1} > 0$; $q_{150,1} > 0$

such that
$$\begin{cases} q_{50,0} = \dfrac{25 - y^{(3)}}{50} \\ q_{150,0} = \dfrac{10 + y^{(3)}}{50} \\ q_{50,1} = \dfrac{y^{(3)}}{50} \\ q_{150,1} = \dfrac{15 - y^{(3)}}{50} \end{cases}$$

EXERCISE 4

- We have to prove that buying

$\underline{\theta} = (100, 0, -1, -2)$

is an arbitrage opportunity.

- <u>Time-0 price of $\underline{\theta}$:</u>

$$\underline{\theta} \cdot \underline{y} = 100 \times y^{(0)} - 1 \times y^{(2)} - 2 \times y^{(3)}$$

$$= 100 \times 1 - 1 \times 70 - 2 \times 15 = 0$$

- <u>Time-1 value of $\underline{\theta}$:</u>

$$\underline{\theta} \cdot \underline{Y} = 100 \times Y^{(0)} - 1 \times Y^{(2)} - 2 \times Y^{(3)}$$

$$= 100 \times 1 - 1 \times 100(1 - I) - 2 \times (100 - Y^{(1)})_+ \times I$$

$$= 100 \times I - 2(100 - Y^{(1)})_+ \times I$$

$$= \begin{cases} 100 I - 100 I = 0 & ; Y^{(1)} = 50 \\ 100 I \geq 0 & ; Y^{(1)} = 150 \end{cases}$$

$\hookrightarrow \mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1$

$\hookrightarrow \mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] = \mathbb{P}[Y^{(1)} = 150, I = 1] = \pi_{150,1} > 0$

- <u>Conclusion:</u> $\underline{\theta} \cdot \underline{y} = 0$; $\mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1$, $\mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] > 0$

$\Rightarrow \underline{\theta}$ is an arbitrage opportunity.

EXERCISE 5

Q. • The market is arbitrage-free and complete

$\Rightarrow 100 \times [Y^{(2)}]^{I}$ is hedgeable.

$\Rightarrow \exists\, \underline{\theta} = (\theta^{(0)}, \theta^{(1)}, \theta^{(2)}, \theta^{(3)})$ such that

$$\boxed{\underline{\theta} \cdot \underline{Y} = 100 \times \left[Y^{(2)}\right]^{I}} \qquad (*)$$

• <u>Determining $\underline{\theta}$</u>:

ω

$$(*) \Leftrightarrow \begin{cases} \theta^{(0)} \times 1 + \theta^{(1)} \times 50 + \theta^{(2)} \times 100 + \theta^{(3)} \times 0 = 100 & : (50, 0) \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 150 + \theta^{(2)} \times 100 + \theta^{(3)} \times 0 = 100 & : (150, 0) \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 50 + \theta^{(2)} \times 0 + \theta^{(3)} \times 50 = 5000 & : (50, 1) \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 150 + \theta^{(2)} \times 0 + \theta^{(3)} \times 0 = 15000 & : (150, 2) \end{cases}$$

$-R_1 + R_2$

$$\Leftrightarrow \begin{cases} 100\, \theta^{(1)} = 0 \\ \theta^{(0)} + \cancel{\theta^{(1)} \times 150} + \theta^{(2)} \times 100 = 100 \\ \theta^{(0)} + \cancel{\theta^{(1)} \times 50} + \theta^{(3)} \times 50 = 5000 \\ \theta^{(0)} + \cancel{\theta^{(1)} \times 150} = 15000 \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta^{(1)} = 0 \\ \theta^{(0)} = 15000 \\ \theta^{(2)} = -149 \\ \theta^{(3)} = -200 \end{cases} \Leftrightarrow \cancel{\underline{\theta} = (15000, 0, -148, -200)}$$

$$\boxed{\underline{\theta} = (15000, 0, -148, -200)}$$

- Price replicating portfolio:

$\underline{\theta} = (15000, 0, -149, -200)$

↳ - buy 15000 zero-coupon bonds
- sell 149 pandemic bonds
- sell 200 combined securities.

↳ $\underline{\theta} \cdot y = 15.000 \times 1 - 149 \times 40 - 200 \times y^{(3)}$

$\Rightarrow \boxed{\underline{\theta} \cdot y = 4.570 - 200\, y^{(3)}}$

- REMARK: other derivation of price of $100 \times [Y^{(1)}]^I$:

- $100 \times [Y^{(1)}]^I = \begin{cases} 100 & : (50,0) \\ 100 & : (150,0) \\ 5000 & : (50,1) \\ 15000 & : (150,1) \end{cases}$

- Price $[100 \times [Y^{(1)}]^I] = \boxed{E^{Q}[100\, [Y^{(1)}]^I]}$

$= 100 \times q_{50,0} + 100 \times q_{150,0} + 5000 \times q_{50,1} + 15000 \times q_{150,1}$

$= 100 \times \frac{25 - y^{(3)}}{50} + 100 \times \frac{10 + y^{(3)}}{50} + 5000 \times \frac{y^{(3)}}{50} + 15000 \times \frac{15 - y^{(3)}}{50}$

$= (50 - 2y^{(3)}) + (20 + 2y^{(3)}) + 100y^{(3)} + 4500 - 300y^{(3)}$

$\boxed{= 4570 - 200\, y^{(3)}}$

Escample 2

(4) $\Longleftrightarrow$

There exist $q_{B,0} > 0 ; \dots ; q_{R,1} > 0$ such that

$$\cancel{q} + \begin{cases} q_{B,0} + q_{B,1} + q_{M,0} + q_{M,1} + q_{R,0} + q_{R,1} = 1 \\ 100 \times q_{B.} = 50 \\ 100 \times q_{.1} = 70 \\ 100 \times q_{B,0} = y^{(3)} \end{cases}$$

$\Longleftrightarrow$ There exist $q_{B,0} > 0, \dots, q_{R,1} > 0$ such that

$$\begin{array}{l} 100R_1 - R_3 - R_4 \\ R_2 - R_4 \\ \\ \\ \end{array} \begin{cases} 100 \times q_{M,0} + 100 \times q_{R,0} = 30 - y^{(3)} \\ 100 \times q_{B,1} = 50 - y^{(3)} \\ 100 \times q_{.1} = 70 \\ 100 \times q_{B,0} = y^{(3)} \end{cases}$$

$\Longleftrightarrow$ There exist $q_{B,0} > 0, \dots, q_{R,1} > 0$ such that

$$\begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array} \begin{cases} 100 \times q_{M,0} + 100 \times q_{R,0} = 30 - y^{(3)} \\ 100 \times q_{B,1} = 50 - y^{(3)} \\ 100 \times q_{M,1} + 100 \times q_{R,1} = 20 + y^{(3)} \\ 100 \times q_{B,0} = y^{(3)} \end{cases}$$

$\Leftrightarrow$ There exist $q_{B,0} > 0, \dots, q_{R,1} > 0$ such that

$$\begin{cases} q_{B,0} = \dfrac{y^{(3)}}{100} \\ q_{B,1} = \dfrac{50 - y^{(3)}}{100} \\ q_{M,0} + q_{R,0} = \dfrac{30 - y^{(3)}}{100} \\ q_{M,1} + q_{R,1} = \dfrac{20 + y^{(3)}}{100} \end{cases}$$

EXERCISE 6

Q1 • <u>No-arbitrage condition:</u>

• There exist a $\mathbb{Q} \sim \mathbb{P}$ with
$$\begin{cases} E^{\mathbb{Q}}[Y^{(0)}] = 1 \\ E^{\mathbb{Q}}[Y^{(1)}] = \\ E^{\mathbb{Q}}[Y^{(2)}] = \\ E^{\mathbb{Q}}[Y^{(3)}] = \\ E^{\mathbb{Q}}[Y^{(4)}] = 14 \end{cases}$$

$\Longleftrightarrow$ There exist $q_{B,0} > 0, \dots, q_{R,1} > 0$ such that

$$\begin{cases} q_{B,0} = y^{(3)}/100 \\ q_{B,1} = (50 - y^{(3)})/100 \\ q_{M,0} + q_{R,0} = (30 - y^{(3)})/100 \\ q_{M,1} + q_{R,1} = (20 + y^{(3)})/100 \\ 100 \times q_{R,1} \stackrel{!}{=} 14 \end{cases}$$

$\Longleftrightarrow$ There exist $q_{B,0} > 0, \dots, q_{R,1} > 0$ such that

$$\begin{cases} q_{B,0} = \dfrac{y^{(3)}}{100} \\ q_{B,1} = \dfrac{50 - y^{(3)}}{100} \\ q_{M,0} + q_{R,0} = \dfrac{30 - y^{(3)}}{100} \\ q_{M,1} = \dfrac{6 + y^{(3)}}{100} \\ q_{R,1} = \dfrac{14}{100} \end{cases}$$

$100 R_4 - R_5$

$\hookrightarrow$ For any $0 < y^{(3)} < 30$ there are ∞ solutions

$\Longrightarrow$ The market is arbitrage-free and incomplete

EXERCISE 6

Q2 (a) If there exists an EMM $\mathbb{Q}$ under which "Barometer" and "Index" are $\mathbb{Q}$ independent $\Rightarrow y^{(3)} = 15$

PROOF: We must have $q_{B,0} = q_{B\cdot} \times q_{\cdot 0}$

$$\Rightarrow \frac{y^{(3)}}{100} = (q_{B0} + q_{B1}) \times (q_{B,0} + q_{M,0} + q_{R,0})$$

$$\Rightarrow \frac{y^{(3)}}{100} = \left(\frac{y^{(3)}}{100} + \frac{50 - y^{(3)}}{100}\right) \times \left(\frac{y^{(3)}}{100} + \frac{30 - y^{(3)}}{100}\right)$$

$$\Rightarrow \frac{y^{(3)}}{100} = \frac{50}{100} \times \frac{30}{100}.$$

$$\Rightarrow y^{(3)} = 15 \quad \blacksquare$$

(b) If $y^{(3)} = 15 \Rightarrow$ There exists an EMM $\mathbb{Q}$ under which "Barometer" and "index" are $\mathbb{Q}$-indep.

PROOF

- $y^{(3)} = 15 \Longrightarrow \begin{cases} q_{B,0} = 0.15 \\ q_{B,1} = 0.35 \\ q_{M,0} + q_{R,0} = 0.15 \\ q_{M,1} = 0.21 \\ q_{R,1} = 0.14 \end{cases}$ (*)

- Define $\hat{\mathbb{Q}}$ with $\hat{q}_{B,0} = 0.15$, $\hat{q}_{M,0} = 0.09$, $\hat{q}_{R,0} = 0.06$, $\hat{q}_{B,1} = 0.35$, $\hat{q}_{M,1} = 0.21$, $\hat{q}_{R,1} = 0.14$

- $\hat{\mathbb{Q}}$ is a solution of (*)

- Moreover, 'Barometer' and 'Index' are $\hat{\mathbb{Q}}$-independ. Indeed,

$$\hat{q}_{B\cdot} \times \hat{q}_{\cdot 0} = 0.5 \times 0.3 = 0.15 = \hat{q}_{B,0}$$

$$\hat{q}_{B\cdot} \times \hat{q}_{\cdot 1} = 0.5 \times 0.7 = 0.35 = \hat{q}_{B,1}$$

...

... $\blacksquare$

EXERCISE 7

Q

- We will show that buying

$\underline{\theta} = (200, 0, -2, -1, 0)$

is an arbitrage opportunity

- Price at time 0:

$$\underline{\theta} \cdot \underline{y} = \theta^{(0)} y^{(0)} + \theta^{(1)} y^{(1)} + \theta^{(2)} y^{(2)} + \theta^{(3)} y^{3} + \theta^{(4)} y^{(4)}$$

$$= 200 \times 1 - 2 \times 70 - 1 \times 60 = 0$$

- Payoff at time 1:

$$\underline{\theta} \cdot \underline{Y} = 200 \times Y^{(0)} - 2 \times Y^{(2)} - 1 \times Y^{(3)}$$

$$= 200 - 2 \times 100 \times I - Y^{(1)}(1 - I)$$

$$= 200 - 200\, I - \underbrace{Y^{(1)}}_{= 100 \times 1_B}(1 - I)$$

$$= \begin{cases} 200 - 0 - 100 = 100 & : (B, 0) \\ 200 - 200 - 0 = 0 & : (B, 1) \\ 200 - 0 - 0 = 200 & : (M, 0) \\ 200 - 200 - 0 = 0 & : (M, 1) \\ 200 - 0 - 0 = 200 & : (R, 0) \\ 200 - 200 - 0 = 0 & : (R, 1) \end{cases}$$

↳ $\mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1$

↳ $\mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] = 1_{B,0} + 1_{M,0} + 1_{R,0} > 0$

↳ because all states-of-world have pos. prob.

- Conclusion: $\underline{\theta}$ is an arbitrage opportunity

EXERCISE 8

Q1 $\mathbb{Q}$ is EMM $\iff$ $\mathbb{Q} \sim \mathbb{P}$ such that $E^{\mathbb{Q}}[Y^{(m)}] = y^{(m)}$ $(m=0,1)$

$\iff q_0 > 0,\ q_{50} > 0,\ q_{100} > 0$

such that $\begin{cases} q_0 + q_{50} + q_{100} = 1 \\ (q_0 + q_{50}) \times 100 = 90 \end{cases}$

$\iff q_0 > 0,\ q_{50} > 0,\ q_{100} > 0$ such that

$100 R_1 - R_2$ $\begin{cases} 100 \times q_{100} = 10 \\ (q_0 + q_{50}) \times 100 = 90 \end{cases}$

$\iff q_0 > 0,\ q_{50} > 0,\ q_{100} > 0$ such that

$\begin{cases} q_{100} = 0.1 \\ q_0 + q_{50} = 0.9 \end{cases}$

$$\iff \boxed{\begin{cases} q_0 = q \\ q_{50} = 0.9 - q \quad \text{for some } q \in (0;0.9) \\ q_{100} = 0.1 \end{cases}}$$

Q2 • $y^{(2)} = E^{\mathbb{Q}}[Y^{(2)}] = 100 \times q_0 + 60 \times q_{50} + 20 \times q_{100}$

$= 100 \times q + 60 \times (0.9 - q) + 20 \times 0.1$

$= 40q + 56$ for some $q \in (0;0.9)$

• Hence, $\boxed{y^{(2)} \in (56;92)}$

EXERCISE 8

Q3 . $\underline{\theta}$ is arbitrage if
$$\begin{cases} \underline{\theta} \cdot y = 0 & (1) \\ \mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1 & (2) \\ \mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] > 0 & (3) \end{cases}$$

- $\underline{\theta} \cdot y = 0$

$\iff \theta^{(0)} \times y^{(0)} + \theta^{(1)} \times y^{(1)} + \theta^{(2)} \times y^{(2)} = 0$

$\iff \theta^{(0)} \times 1 - 2 \times 90 + 1 \times 40 = 0$

$\iff \boxed{\theta^{(0)} = 140}$

- We have to prove that $\underline{\theta} = (140, -2, 1)$ is an arbitrage.

(1) $\underline{\theta} \cdot y = 0$: see above

(2). $\underline{\theta} \cdot \underline{Y} = 140 \times \cancel{Y}^{(0)} - 2 \times Y^{(1)} + 1 \times Y^{(2)}$

$= 140 - 2 \times Y^{(1)} + Y^{(2)}$

$$= \begin{cases} 140 - 2 \times 100 + 100 & : I = 0 \\ 140 - 2 \times 100 + 60 & : I = 50 \\ 140 - 0 + 20 & : I = 100 \end{cases}$$

$$= \begin{cases} 40 & : I = 0 \\ 0 & : I = 50 \\ 160 & : I = 100 \end{cases}$$

↳ Hence, $\mathbb{P}[\underline{\theta} \cdot \underline{Y}] \geq 0 = 1$

(3) $\mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] = \mathbb{P}[I = 0 \text{ or } I = 100] = 1_0 + 1_{100} > 0$

. CONCLUSION: $\underline{\theta} = (140, -2, 1)$ is an arbitrage.

$\mathbb{Q}$ is EMM

Q4

$\iff q_0 > 0 \ , \ q_{50} > 0 \ , \ q_{100} > 0$ such that

$$E^{\mathbb{Q}}[Y^{(m)}] = y^{(m)} \quad \text{for } m = 0,1,2$$

$\iff q_0 > 0 \ , \ q_{50} > 0, \ q_{100} > 0$ such that

$$\begin{cases} q_0 = q \\ q_{50} = 0.9 - q \\ q_{100} = 0.1 \\ 100 \times q_0 + 60 \times q_{50} + 20 \times q_{100} = y^{(2)} \end{cases} \quad \text{for some } 0 < q < 0.9$$

$\iff q_0 > 0 \ , \ q_{50} > 0 \ , \ q_{100} > 0$ such that

$$\begin{cases} q_0 = q \\ q_{50} = 0.9 - q \\ q_{100} = 0.1 \\ 100 \times q + 60 \times (0.9 - q) + 20 \times 0.1 = y^{(2)} \end{cases} \quad \text{for some } q \in (0;0.9)$$

$\iff q_0 > 0 \ , \ q_{50} > 0, \ q_{100} > 0$ such that

$$\begin{cases} q_0 = q \\ q_{50} = 0.9 - q \\ q_{100} = 0.1 \\ 40q = y^{(2)} - 56 \end{cases} \quad \text{for some } q \in (0;0.9)$$

$$\iff \boxed{\begin{cases} q_0 = \dfrac{y^{(2)} - 56}{40} \\ q_{50} = \dfrac{92 - y^{(2)}}{40} \\ q_{100} = 0.1 \end{cases}} \quad \text{with } y^{(2)} \in (56;92)$$

EXERCISE 8

Q5 • Let $\underline{\theta}$ be the hedge for I. Then:

$$\underline{\theta} \cdot \underline{Y} = I$$

$$\Leftrightarrow \begin{cases} \theta^{(0)} \times 1 + \theta^{(1)} \times 100 + \theta^{(2)} \times 100 = 0 & : I = 0 \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 100 + \theta^{(2)} \times 60 = 50 & : I = 50 \\ \theta^{(0)} \times 1 + \theta^{(1)} \times 0 + \theta^{(2)} \times 20 = 100 & : I = 100 \end{cases}$$

$$\begin{matrix} R_1 - R_2 \\ R_2 - R_3 \\ \Leftrightarrow \end{matrix} \begin{cases} \cdot\ 40 \times \theta^{(2)} = -50 \\ \cdot\ 100 \times \theta^{(1)} + 40 \times \theta^{(2)} = -50 \\ \cdot\ \theta^{(0)} + 20 \times \theta^{(2)} = 100 \end{cases}$$

$$\begin{matrix} R_2 - R_1 \\ \Leftrightarrow \\ 2R_3 - R_1 \end{matrix} \begin{cases} 40\theta^{(2)} = -50 \\ 100\theta^{(1)} = 0 \\ 2\theta^{(0)} = 250 \end{cases} \Leftrightarrow \begin{cases} \theta^{(0)} = 125 \\ \theta^{(1)} = 0 \\ \theta^{(2)} = -5/4 \end{cases}$$

• $\underline{\text{Conclusion}}$: $\underline{\theta} = (125; 0, -\frac{5}{4})$ is the hedge of I.

• $\text{Price}[I] = \underline{\theta} \cdot \underline{y} = 125 \times 1 - \frac{5}{4} \times y^{(2)}$

$$= 125 - \frac{5}{4} y^{(2)}$$

• $\underline{\text{CONTROL}}$: $\text{Price}[I] = E^{Q}[I]$

$$= q_0 \times 0 + q_{50} \times 50 + q_{100} \times 100$$

$$= \frac{92 - y^{(2)}}{40} \times 50 + 0.1 \times 100$$

$$= 125 - \frac{5}{4} y^{(2)}.$$

EXERCISE 9

Q1 The market is arbitrage-free

$\iff$

There exist $q_0 > 0, q_1 > 0, q_2 > 0$ such that
$E^{\mathbb{Q}}[Y^{(m)}] = y^{(m)}$, for $m = 0, 1, 2$

$\iff$ There exist $q_0 > 0, q_1 > 0, q_2 > 0$ such that

$$\begin{cases} q_0 + q_1 + q_2 = 1 \\ (q_0 + q_1) \cdot 100 + q_2 \cdot 150 = y^{(1)} \\ q_0 \cdot 150 + (q_1 + q_2) \cdot 100 = y^{(2)} \end{cases}$$

$\iff$ There exist $q_0 > 0, q_1 > 0, q_2 > 0$ such that

$$\begin{cases} q_0 + q_1 + q_2 = 1 \\ 50\, q_2 = y^{(1)} - 100 \\ 50\, q_0 = y^{(2)} - 100 \end{cases}$$

$R_2 - 100 R_1$
$R_3 - 100 R_1$

$\iff$ There exist $q_0 > 0, q_1 > 0, q_2 > 0$ such that

$$\begin{cases} q_0 = \dfrac{y^{(2)} - 100}{50} \\ q_1 = \dfrac{250 - y^{(1)} - y^{(2)}}{50} \\ q_2 = \dfrac{y^{(1)} - 100}{50} \end{cases}$$

$$\iff \begin{cases} y^{(2)} - 100 > 0 \\ 250 - y^{(1)} - y^{(2)} > 0 \\ y^{(1)} - 100 > 0 \end{cases} \iff \begin{cases} y^{(2)} > 100 \\ y^{(1)} + y^{(2)} < 250 \\ y^{(1)} > 100 \end{cases}$$

$$\iff \boxed{100 < y^{(1)} < 250 - y^{(2)} < 150}$$

EXERCISE 5

Q2 If market is arbitrage - free, then $100 < y^{(1)} < 250 - y^{(2)} < 150$ and the ~~unique~~ EMM $\mathbb{Q} = (q_0, q_1, q_2)$ is given by

$$\boxed{\begin{cases} q_0 = \dfrac{y^{(2)} - 100}{50} \\ q_1 = \dfrac{250 - y^{(1)} - y^{(2)}}{50} \\ q_2 = \dfrac{y^{(1)} - 100}{50} \end{cases}}$$

Hence, $\mathbb{Q}$ is unique and the market is complete

Q3 • Hedge of I is denoted by $\underline{\theta} = (\theta^{(0)}, \theta^{(1)}, \theta^{(2)})$

$$\underline{\theta} \cdot \underline{Y} = I \iff \theta^{(0)} \cdot Y^{(0)} + \theta^{(1)} \cdot Y^{(1)} + \theta^{(2)} \cdot Y^{(2)} = I$$

$$\iff \begin{cases} \theta^{(0)} + \theta^{(1)} \cdot 100 + \theta^{(2)} \cdot 150 = 0 & : I = 0 \\ \theta^{(0)} + \theta^{(1)} \cdot 100 + \theta^{(2)} \cdot 100 = 1 & : I = 1 \\ \theta^{(0)} + \theta^{(1)} \cdot 150 + \theta^{(2)} \cdot 100 = 2 & : I = 2 \end{cases}$$

$$\begin{matrix} \\ R_2 - R_1 \\ R_3 - R_2 \end{matrix} \begin{cases} \theta^{(0)} + 100\,\theta^{(1)} + 150\,\theta^{(2)} = 0 \\ -50\,\theta^{(2)} = 1 \\ 50\,\theta^{(1)} = 1 \end{cases}$$

$$\iff \begin{cases} \theta^{(0)} = 1 \\ \theta^{(1)} = 1/50 \\ \theta^{(2)} = -1/50 \end{cases}$$

Hence, $\boxed{\underline{\theta} = (1, 1/50, -1/50)}$ is the hedge of I

EXERCISE 8

Q3 (cont'd)

- $\boxed{\text{Price of } I} = E^{Q}[\underline{\theta} \cdot \underline{Y}]$

$= E^{Q}[\theta^{(0)} \cdot Y^{(0)} + \theta^{(1)} \cdot Y^{(1)} + \theta^{(2)} \cdot Y^{(2)}]$

$= E^{Q}[1 \times Y^{(0)} + \frac{1}{50} Y^{(1)} - \frac{1}{50} Y^{(2)}]$

$= E^{Q}[Y^{(0)}] + \frac{1}{50} E^{Q}[Y^{(1)}] - \frac{1}{50} E^{Q}[Y^{(2)}]$

$$\boxed{= 1 + \frac{1}{50}(y^{(1)} - y^{(2)})}$$

- Other way to calculate price of I:

Price of I $= E^{Q}[I]$

$= q_0 \times 0 + q_1 \times 1 + q_2 \times 2$

$= \underbrace{\frac{250 - y^{(1)} - y^{(2)}}{50}}_{>0} + 2 \times \underbrace{\frac{y^{(1)} - 100}{50}}_{>0}$

$= 1 + \frac{1}{50}(y^{(1)} - y^{(2)}) > 0$

EXERCISE 5

Q4 • $(\theta^{(0)}, -1, -1)$ is arbitrage if

(1) $\underline{\theta} \cdot \underline{y} = 0$
(2) $\mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1$
(3) $\mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] > 0$

• Condition (1):

$$\underline{\theta} \cdot \underline{y} = 0 \iff \theta^{(0)} \times 1 - 1 \times y^{(1)} - 1 \times y^{(2)} = 0$$
$$\iff \theta^{(0)} = y^{(1)} + y^{(2)}$$
$$\iff \boxed{\theta^{(0)} = 250}$$

Hence, condition (1) is satisfied if we chose $\theta^{(0)} = 250$.
Hereafter, we chose $\theta^{(0)} = 250$.

• $\underline{\theta} \cdot \underline{Y} = 250 \times Y^{(0)} - 1 \times Y^{(1)} - 1 \times Y^{(2)}$
$= 250 - Y^{(1)} - Y^{(2)}$

$$= \begin{cases} 250 - 100 - 150 = 0 & : I = 0 \\ 250 - 100 - 100 = 50 & : I = 1 \\ 250 - 150 - 100 = 0 & : I = 2 \end{cases}$$

• Condition (2):

$\mathbb{P}[\underline{\theta} \cdot \underline{Y} \geq 0] = 1$, see

• Condition (3): $\mathbb{P}[\underline{\theta} \cdot \underline{Y} > 0] = \mathbb{P}[I = 1] > 0$

• Conclusion: $\boxed{\underline{\theta} = (250, -1, -1)}$ is an arbitrage opportunity