

EXERCISE 1

Q: PROOF:

• Suppose $S^A = \underline{y}_1 \cdot \underline{y} = \underline{y}_2 \cdot \underline{y}$

• Then $(\underline{y}_1 - \underline{y}_2) \cdot \underline{y} = 0$

$\Rightarrow \underline{y}_1 - \underline{y}_2 = 0$: due to non-redundancy assumption

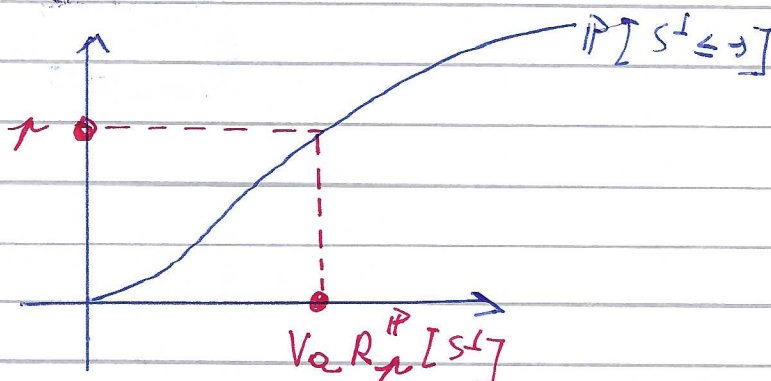
$\Rightarrow \underline{y}_1 = \underline{y}_2$ □

EXERCISE 2

Q: • $S[S^\perp] = e^{-2} \left\{ E^P[S^\perp] + i \left(\text{Var}_P^P[S^\perp] - E^P[S^\perp] \right) \right\}$
 • $E^P[S^\perp] = N \cdot \mu$
 • $\text{Var}_P^P[S^\perp] = N \cdot \sigma^2$
 • $\boxed{\text{Var}_P^P[S^\perp]} \stackrel{\text{def.}}{=} \inf \left\{ s \mid P[S^\perp \leq s] \geq \alpha \right\}$

by assumption:
 $Z = \frac{S^\perp - N\mu}{\sqrt{N} \cdot \sigma} \sim N(0,1)$

$$\begin{aligned}
 &= \inf \left\{ s \mid P \left[\frac{S^\perp - N\mu}{\sqrt{N} \cdot \sigma} \leq \frac{s - N\mu}{\sqrt{N} \cdot \sigma} \geq \alpha \right] \right\} \\
 &= \inf \left\{ s \mid P \left[Z \leq \frac{s - N\mu}{\sqrt{N} \cdot \sigma} \right] \geq \alpha \right\} \\
 &= \inf \left\{ s \mid \Phi \left[\frac{s - N\mu}{\sqrt{N} \cdot \sigma} \right] \geq \alpha \right\} \\
 &= \inf \left\{ s \mid \frac{s - N\mu}{\sqrt{N} \cdot \sigma} \geq \Phi^{-1}(\alpha) \right\} \\
 &= \inf \left\{ s \mid s \geq N\mu + \sqrt{N} \cdot \sigma \Phi^{-1}(\alpha) \right\} \\
 &= \boxed{N\mu + \sqrt{N} \cdot \sigma \Phi^{-1}(\alpha)}
 \end{aligned}$$



• $S[S^\perp] = e^{-2} \left\{ N\mu + i \left(\cancel{N\mu} + \sqrt{N} \sigma \Phi^{-1}(\alpha) - \cancel{N\mu} \right) \right\}$
 $= e^{-2} \left\{ N\mu + i \cdot \sqrt{N} \sigma \Phi^{-1}(\alpha) \right\}$

$\Rightarrow \boxed{\frac{S[S^\perp]}{N} = e^{-2} \left\{ \mu + \frac{i}{\sqrt{N}} \sigma \Phi^{-1}(\alpha) \right\}}$

• Other way to prove that $\text{Var}_n^P[S^1]$
 $= N \cdot \mu + \sqrt{N} \cdot \sigma \Phi^{-1}(\alpha)$:

PROOF:

$$S^1 = N \cdot \mu + \sqrt{N} \cdot \sigma \cdot Z \quad (\text{by assumption})$$

$$\begin{aligned} \Rightarrow \text{Var}_n^P[S^1] &= \text{Var}_n^P[N \cdot \mu + \sqrt{N} \cdot \sigma \cdot Z] \\ &= N \cdot \mu + \sqrt{N} \cdot \sigma \cdot \text{Var}_n^P[Z] \\ &= N \cdot \mu + \sqrt{N} \cdot \sigma \cdot \Phi^{-1}(\alpha) \end{aligned}$$

EXERCISE 3

Q1: $\text{Var}^P[Y_N^d] = \text{Var}^P\left[S^d \times \left(\frac{S_N^d}{N} - E^P[X_1]\right)\right]$

$$= \text{Var}^P\left[E^P\left\{S^d \times \left(\frac{S_N^d}{N} - E^P[X_1]\right) \mid Y\right\}\right]$$

$$+ E^P\left[\text{Var}^P\left\{S^d \times \left(\frac{S_N^d}{N} - E^P[X_1]\right) \mid Y\right\}\right]$$

$$= \text{Var}^P\left[S^d \times E^P\left\{\frac{S_N^d}{N} - E^P[X_1]\right\}\right]$$

$$+ E^P\left[(S^d)^2 \times \text{Var}^P\left\{\frac{S_N^d}{N}\right\}\right]$$

$$= \cancel{\text{Var}^P[S^d \times 0]} + E^P[(S^d)^2] \times \frac{\text{Var}^P[X_1]}{N}$$

$$= \frac{1}{N} \times E^P[(S^d)^2] \times \text{Var}^P[X_1]$$

$$= \frac{A}{N} \text{ with } A = E^P[(S^d)^2] \times \text{Var}^P[X_1]$$

Q2: CHEBYSHEV:

$$P\left[\left|Y_N^d - E^P[Y_N^d]\right| \geq k \cdot \sigma^P[Y_N^d]\right] \leq \frac{1}{k^2}$$

for all $k \in \mathbb{R}^+$

$$\bullet \quad P\left[\left|Y_N^d\right| \geq k \cdot \sqrt{\frac{A}{N}}\right] \leq \frac{1}{k^2}$$

• For $k = \varepsilon \cdot \sqrt{\frac{N}{A}}$ with $\varepsilon > 0$, we find:

$$P\left[\left|Y_N^d\right| \geq \varepsilon\right] \leq \frac{A}{N\varepsilon^2}$$

• Hence, $\lim_{N \rightarrow \infty} P\left[\left|Y_N^d\right| \geq \varepsilon\right] = 0$ for all $\varepsilon > 0$

$$\Leftrightarrow \left|Y_N^d\right| \xrightarrow{P} 0$$

Exercise 3

Q3: $\text{Var}^P \left[\bar{y}_N^d \right] = \frac{1}{N} E^P \left[(\xi^d)^2 \right] \times \text{Var}^P \left[X_1 \right]$

$$\cdot \text{Var}^P \left[X_1 \right] = E^P \left[(X_1)^2 \right] - \left(E^P \left[X_1 \right] \right)^2$$

$$= \mu - \mu^2$$

$$= \mu (1 - \mu)$$

$$\cdot \boxed{\text{Var}^P \left[\bar{y}_N^d \right] = \frac{1}{N} E^P \left[(\xi^d)^2 \right] \times \mu \times (1 - \mu)}$$

EXERCISE 3'

Q1: $\mathbb{P}[X_i \leq x, Y \leq y]$

$$= \int_A \mathbb{P}[X_i \leq x, Y \leq y | Z=z] dF_Z(z)$$

$$= \int_A \mathbb{P}[X_i \leq x | Z=z] \times \mathbb{P}[Y \leq y] dF_Z(z)$$

$$= \left(\int_A \mathbb{P}[X_i \leq x | Z=z] dF_Z(z) \right) \times \mathbb{P}[Y \leq y]$$

$$= \mathbb{P}[X_i \leq x] \times \mathbb{P}[Y \leq y], \forall (x, y)$$

$$\Rightarrow X_i \in C^\perp$$

Q2:

$$\text{Var}^P[Y^d] = \text{Var}^P \left[S^d \times \left(\frac{S_N^L}{N} - E^P[X_1|Z] \right) \right]$$

$$\begin{aligned} \text{Eve's Law} &= \text{Var}^P \left[E^P \left\{ S^d \times \left(\frac{S_N^L}{N} - E^P[X_1|Z] \right) \mid Y \right\} \right] \\ &+ E^P \left[\text{Var}^P \left\{ S^d \times \left(\frac{S_N^L}{N} - E^P[X_1|Z] \right) \mid Y \right\} \right] \\ &= \text{Var}^P \left[S^d \times E^P \left\{ \frac{S_N^L}{N} - E^P[X_1|Z] \mid Y \right\} \right] \\ &+ E^P \left[(S^d)^2 \times \text{Var}^P \left\{ \frac{S_N^L}{N} - E^P[X_1|Z] \mid Y \right\} \right] \\ &= 0 + E^P[(S^d)^2] \times \text{Var}^P \left[\frac{S_N^L}{N} - E^P[X_1|Z] \right] \end{aligned}$$

$$\begin{aligned} \text{Eve's Law} &= E^P[(S^d)^2] \times \text{Var}^P \left[E^P \left\{ \frac{S_N^L}{N} - E^P[X_1|Z] \mid Z \right\} \right] \\ &+ \cancel{E^P[(S^d)^2] \times E^P \left[\frac{S_N^L}{N} - E^P[X_1|Z] \right]} \\ &+ E^P[(S^d)^2] \times E^P \left[\text{Var}^P \left\{ \frac{S_N^L}{N} - E^P[X_1|Z] \mid Z \right\} \right] \\ &= E^P[(S^d)^2] \times \text{Var}^P \left[E^P \{ X_1 | Z \} - E^P[X_1|Z] \right] \\ &+ E^P[(S^d)^2] \times E^P \left[\text{Var}^P \left\{ \frac{S_N^L}{N} \mid Z \right\} \right] \\ &= 0 + E^P[(S^d)^2] \times E^P \left[\frac{1}{N} \text{Var}^P \{ X_1 | Z \} \right] \\ &= E^P[(S^d)^2] \times \frac{1}{N} \times E^P[\text{Var}^P[X_1|Z]] \end{aligned}$$

Exercise 3'

Q3: $\text{Var}^P[Y^2] = \text{Var}^P[S^Q \times (E^P[X_1|Z] - E^P[X_2])]$

Eve's Law = $\text{Var}^P[E^P\{S^Q \times (E^P[X_1|Z] - E^P[X_2]) | Y\}]$
 $+ E^P[\text{Var}^P\{S^Q \times (E^P[X_1|Z] - E^P[X_2]) | Y\}]$
 $= \text{Var}^P[S^Q \times E^P\{E^P[X_1|Z] - E^P[X_2]\}]$
 $+ E^P[(S^Q)^2 \times \text{Var}^P\{E^P[X_1|Z] - E^P[X_2]\}]$
 $= 0 + E^P[(S^Q)^2] \times \text{Var}^P[E^P[X_1|Z]]$

Q4: $\text{Var}^P[Y^0 + Y^2] = \text{Var}^P[S^Q \times (\frac{S_N^L}{N} - E^P[X_1])]$

Eve's Law = $\text{Var}^P[E^P\{S^Q \times (\frac{S_N^L}{N} - E^P[X_1]) | Y\}]$
 $+ E^P[\text{Var}^P\{S^Q \times (\frac{S_N^L}{N} - E^P[X_1]) | Y\}]$
 $= \text{Var}^P[S^Q \times E^P\{\frac{S_N^L}{N} - E^P[X_1]\}]$
 $+ E^P[(S^Q)^2 \times \text{Var}^P\{\frac{S_N^L}{N} - E^P[X_1]\}]$
 $= 0 + E^P[(S^Q)^2] \times \text{Var}^P\{\frac{S_N^L}{N}\}$

Eve's Law = $E^P[(S^Q)^2] \times \text{Var}^P[E^P\{\frac{S_N^L}{N} | Z\}]$
 $+ E^P[(S^Q)^2] \times E^P[\text{Var}^P\{\frac{S_N^L}{N} | Z\}]$
 $= E^P[(S^Q)^2] \times \text{Var}^P[E^P[X_1|Z]]$
 $+ E^P[(S^Q)^2] \times \frac{1}{N} E^P[\text{Var}^P[X_1|Z]]$

see Q2 and Q3 = $\text{Var}^P[Y^0] + \text{Var}^P[Y^2]$

$\Rightarrow \text{Cov}^P[Y^0, Y^2] = 0$

Exercise 3'

Q5:

• CHEBYSHEV:
$$P[|Y_N^d - E^P[Y_N^d]| \geq k \cdot \sigma^P[Y_N^d]] \leq \frac{1}{k^2}, \forall k \in \mathbb{R}^+$$

• $E^P[Y_N^d] = 0$

•
$$\text{Var}^P[Y_N^d] = \frac{1}{N} \times E^P[\beta^q]^2 \times E^P[\text{Var}^P[X_i|Z]] = \frac{A}{N}$$

• Hence,

$$P[|Y_N^d| \geq k \cdot \sqrt{\frac{A}{N}}] \leq \frac{1}{k^2}$$

• For $k = \varepsilon \sqrt{\frac{N}{A}}$, we find

$$P[|Y_N^d| \geq \varepsilon] \leq \frac{A}{N\varepsilon^2}$$

• Hence, $\lim_{N \rightarrow \infty} P[|Y_N^d| \geq \varepsilon] = 0, \forall \varepsilon > 0$

• This means that $Y_N^d \xrightarrow{P} 0$

Q6:

Exercise 3'

$$\cdot E^P[X_1 | Z = \omega] = \mu(\omega) \Rightarrow E^P[X_1 | Z] = \mu(Z)$$

$$\cdot E^P[X_1 \cdot X_2 | Z = \omega] = E^P[X_1 | Z = \omega] \times E^P[X_2 | Z = \omega] \\ = (\mu(\omega))^2$$

$$\Rightarrow E^P[X_1 \cdot X_2 | Z] = (\mu(Z))^2 \quad (*)$$

$$\cdot \text{Cov}^P[X_1, X_2] = E^P[X_1 \cdot X_2] - (E^P[X_1])^2 \\ = E^P[E^P\{X_1 \cdot X_2 | Z\}] \\ = (E^P[E^P\{X_1 | Z\}])^2 \\ \stackrel{(*)}{=} E^P[(\mu(Z))^2] - (E^P[\mu(Z)])^2 \\ = \text{Var}^P[\mu(Z)]$$

Q7:

$$P[X_2 = 1 | X_1 = 1] = \frac{P[X_1 = 1, X_2 = 1]}{P[X_1 = 1]}$$

$$= \frac{E^P[X_1 \cdot X_2]}{\mu}$$

$$= \frac{\text{Cov}^P[X_1, X_2] + \mu^2}{\mu}$$

$$= \mu + \frac{\text{Var}^P[\mu(Z)]}{\mu}$$

Q8

Exercice 3'

$$\bullet \text{Var}^P [X_1 | Z = z] = p(z) \times (1 - p(z))$$

$$\Rightarrow \boxed{\text{Var}^P [X_1 | Z] = p(Z) \times (1 - p(Z))} \quad (**)$$

$$\bullet \boxed{\text{Var}^P [Y_N^o]} \stackrel{Q2}{=} \frac{1}{N} E^P [(S^Q)^2] \times E^P [\text{Var}^P [X_1 | Z]]$$

$$\stackrel{(**)}{=} \frac{1}{N} E^P [(S^Q)^2] \times E^P [p(Z) \times (1 - p(Z))]$$

$$\bullet \boxed{\text{Var}^P [Y^2]} \stackrel{Q3}{=} E^P [(S^Q)^2] \times \text{Var}^P [E^P [X_1 | Z]]$$

$$\stackrel{(*)}{=} E^P [(S^Q)^2] \times \text{Var}^P [p(Z)]$$

EXERCISE 4

MAC = market-consistent
MOC = model-consistent

(1) • S_1 is MAC because it is a financial valuation

- S_1 is not MOC: S_1 restricted to C^+ cannot coincide with an actuarial valuation (except if P is an EMM)
- S_1 is not fair.

(2) • S_2 is not MAC:

$$\begin{aligned} S_2 [S + S^h] &= e^{-2} E^P[S] + e^{-2} E^P[S^h] \\ &= S_2 [S] + e^{-2} E^P[S^h] \\ &\quad \xrightarrow{L} = \underline{r} \cdot \underline{Y} \\ &\neq S_2 [S] + \underline{r} \cdot Y \end{aligned}$$

(except if P is an EMM)

- S_2 is MOC: because S_2 is an actuarial valuation
- S_2 is not fair.

(3) • $\mathcal{P}_3$ is MAC:

$$\begin{aligned}
 \mathcal{P}_3 [S + S^h] &= e^{-2} E^{\mathbb{Q}} \left[E^{\mathbb{P}} [S + S^h | \underline{Y}] \right] \\
 &= e^{-2} E^{\mathbb{Q}} \left[E^{\mathbb{P}} [S | \underline{Y}] \right] \\
 &\quad + e^{-2} E^{\mathbb{Q}} \left[E^{\mathbb{P}} [S^h | \underline{Y}] \right] \\
 &= \mathcal{P}_3 [S] + e^{-2} E^{\mathbb{Q}} [S^h] \\
 &= \mathcal{P}_3 [S] + \underline{y} \cdot \underline{y} \quad \text{L} \rightarrow = \underline{y} \cdot \underline{y}
 \end{aligned}$$

• $\mathcal{P}_3$ is MOC:

$$\begin{aligned}
 - \mathcal{P}_3 [S^\perp] &= e^{-2} E^{\mathbb{Q}} \left[E^{\mathbb{P}} [S^\perp | \underline{Y}] \right] \\
 &= e^{-2} E^{\mathbb{P}} [S^\perp]
 \end{aligned}$$

$$- \pi [S] = e^{-2} E^{\mathbb{P}} [S], \quad \forall S \in \mathcal{C}$$

$\hookrightarrow \pi$ is an actuarial valuation

• $\mathcal{P}_3$ is fair.

EXERCISE 5

$$\begin{aligned}
 \text{Q1: } \underline{\theta}_a &= \underline{\theta}_{0+a} \stackrel{\text{tr. inv.}}{=} \underline{\theta}_0 + (e^{-2a}, 0, \dots, 0) \\
 &\stackrel{\text{norm.}}{=} (0, 0, 0, \dots, 0) + (e^{-2a}, 0, \dots, 0) \\
 &= (e^{-2a}, 0, \dots, 0)
 \end{aligned}$$

$$\begin{aligned}
 \text{Q2: } \underline{\theta}_{S^\perp} &= \underline{\theta}_{0+S^\perp} \stackrel{M_{\sigma}C}{=} \underline{\theta}_0 + \underline{\gamma} \\
 &\stackrel{\text{norm.}}{=} (0, 0, \dots, 0) + \underline{\gamma} \\
 &= \underline{\gamma}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q3: } \underline{\theta}_{S^\perp+S^\perp} &\stackrel{M_{\sigma}C}{=} \underline{\theta}_{S^\perp} + \underline{\gamma} \\
 &\stackrel{M_{\sigma}C}{=} (\pi[S^\perp], 0, \dots, 0) + \underline{\gamma}
 \end{aligned}$$

EXERCISE 6

Q1: $\underline{\Theta}_S^{MV}$ follows from:

$$\frac{\partial}{\partial \Theta^{(\lambda)}} E^P \left[(S - \underline{\Theta} \cdot \underline{Y})^2 \right] = 0$$

$\lambda = 0, 1, \dots, m$

$$\Leftrightarrow E^P \left[(S - \underline{\Theta} \cdot \underline{Y}) \times (-Y^{(\lambda)}) \right] = 0$$

$$\Leftrightarrow E^P \left[Y^{(\lambda)} \times \underline{\Theta} \cdot \underline{Y} \right] = E^P \left[S \times Y^{(\lambda)} \right]$$

$$\Leftrightarrow E^P \left[Y^{(\lambda)} \times \sum_{m=0}^n \Theta^{(m)} \times Y^{(m)} \right] = E^P \left[S \times Y^{(\lambda)} \right]$$

$$\Leftrightarrow \sum_{m=0}^n E^P \left[Y^{(\lambda)} \times Y^{(m)} \right] = E^P \left[S \times Y^{(\lambda)} \right]$$

$(\lambda = 0, 1, \dots, m)$

NOTE: This set of $(m+1)$ equations to determine $\underline{\Theta}_S^{MV}$ can be written in matrix notation:

$$E^P \left[\underline{Y}^T \cdot \underline{Y} \right] \times \left(\underline{\Theta}_S^{MV} \right)^T = E^P \left[S \times \underline{Y}^T \right]$$

$$\text{with } E^P \left[\underline{Y}^T \cdot \underline{Y} \right] = E^P \left[\begin{pmatrix} Y^{(0)} \\ Y^{(1)} \\ \vdots \\ Y^{(n)} \end{pmatrix} \cdot (Y^{(0)} \ Y^{(1)} \dots Y^{(n)}) \right]$$

$$= \begin{pmatrix} E^P [Y^{(0)} Y^{(0)}] & \dots & E^P [Y^{(0)} Y^{(n)}] \\ \vdots & & \vdots \\ E^P [Y^{(n)} Y^{(0)}] & \dots & E^P [Y^{(n)} Y^{(n)}] \end{pmatrix}$$

Exercise 6

Q2:

$$\sum_{m=0}^n E^P[Y^{(k)} \times Y^{(m)}] \times \theta_S^{(m)} = E^P[S \times Y^{(k)}] \quad k=0,1,\dots,n$$

$$\Leftrightarrow \begin{cases} \sum_{m=0}^n E^P[\cancel{e^{\lambda}} \times Y^{(m)}] \times \theta_S^{(m)} = E^P[S \times \cancel{e^{\lambda}}] : \lambda=0 \\ \sum_{m=0}^n E^P[Y^{(k)} \times Y^{(m)}] \times \theta_S^{(m)} = E^P[S \times Y^{(k)}] : \lambda=1,\dots,n \end{cases}$$

$$\Rightarrow \begin{cases} e^{\lambda} \theta_S^{(0)} + \sum_{m=1}^n E^P[Y^{(m)}] \times \theta_S^{(m)} = E^P[S] : \lambda=0 \\ e^{\lambda} E^P[Y^{(k)}] \theta_S^{(0)} + \sum_{m=1}^n E^P[Y^{(k)} \times Y^{(m)}] \times \theta_S^{(m)} = E^P[S \times Y^{(k)}] : \lambda=1,\dots,n \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta_S^{(0)} = e^{-\lambda} \left\{ E^P[S] - \sum_{m=1}^n E^P[Y^{(m)}] \times \theta_S^{(m)} \right\} \\ e^{\lambda} E^P[Y^{(k)}] \times e^{-\lambda} \left\{ E^P[S] - \sum_{m=1}^n E^P[Y^{(m)}] \times \theta_S^{(m)} \right\} \\ + \sum_{m=1}^n E^P[Y^{(k)} \times Y^{(m)}] \times \theta_S^{(m)} = E^P[S \times Y^{(k)}] : \lambda=1,\dots,n \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta_S^{(0)} = e^{-\lambda} \left\{ E^P[S] - \sum_{m=1}^n E^P[Y^{(m)}] \times \theta_S^{(m)} \right\} \\ \sum_{m=1}^n \text{Cov}^P[Y^{(k)}, Y^{(m)}] = \text{Cov}^P[Y^{(k)}, S] \end{cases}$$

EXERCISE 7

(a) From the expression for $\theta_S^{(0)}$, we find

$$\theta_S^{(0)} \times e^r + \sum_{m=1}^n E^P[Y^{(m)}] \times \theta_S^{(m)} = E^P[S]$$

$$\Leftrightarrow \sum_{m=0}^n E^P[Y^{(m)}] \times \theta_S^{(m)} = E^P[S]$$

$$\Leftrightarrow E^P \left[\sum_{m=0}^n Y^{(m)} \times \theta_S^{(m)} \right] = E^P[S]$$

$$\Leftrightarrow \cancel{E^P \left[\sum_{m=0}^n Y^{(m)} \times \theta_S^{(m)} \right]}$$

$$\Leftrightarrow \boxed{E^P \left[\underline{\theta}_S^{MV} \cdot \underline{Y} \right] = E^P[S]}$$

(b) We have: $E^P \left[\underline{Y}^T \cdot \underline{Y} \right] \times \left(\underline{\theta}_S^{MV} \right)^T = E^P \left[S \times \underline{Y}^T \right]$

$$\text{Hence, } \left(\underline{\theta}_S^{MV} \right)^T = \left(E^P \left[\underline{Y}^T \cdot \underline{Y} \right] \right)^{-1} \cdot E^P \left[S \times \underline{Y}^T \right]$$

$\forall S \in C$

$$\begin{aligned} \left(\underline{\theta}_{S_1+S_2}^{MV} \right)^T &= \left(E^P \left[\underline{Y}^T \cdot \underline{Y} \right] \right)^{-1} \cdot E^P \left[(S_1+S_2) \times \underline{Y}^T \right] \\ &= (\underline{A})^{-1} \left\{ E^P \left[S_1 \times \underline{Y}^T \right] + E^P \left[S_2 \times \underline{Y}^T \right] \right\} \\ &= \left(\underline{\theta}_{S_1}^{MV} \right)^T + \left(\underline{\theta}_{S_2}^{MV} \right)^T \end{aligned}$$

Let us denote $E^P \left[\underline{Y}^T \cdot \underline{Y} \right]$ by A

$$\begin{aligned} (c) \quad \left(\underline{\theta}_{aS}^{MV} \right)^T &= (\underline{A})^{-1} \cdot E^P \left[aS \cdot \underline{Y}^T \right] \\ &= a (\underline{A})^{-1} \cdot E^P \left[S \cdot \underline{Y}^T \right] \\ &= a \left(\underline{\theta}_S^{MV} \right)^T \end{aligned}$$

Exercise 7

$$\begin{aligned}
 (d) \quad \begin{pmatrix} \Theta^{MV} \\ -S^Q \times S^L \end{pmatrix}^T &= (A)^{-1} \cdot E^P [S^Q \times S^L \cdot \underline{y}^T] \\
 &= (A)^{-1} \cdot E^P [S^L] \times E^P [S^Q \cdot \underline{y}^T] \\
 &= E^P [S^L] \times \begin{pmatrix} \Theta^{MV} \\ -S^Q \end{pmatrix}^T \\
 &\quad \downarrow \\
 &= \begin{pmatrix} E^P [S^L] \times \underline{y}^T \end{pmatrix}
 \end{aligned}$$

$S^Q = \underline{y} \cdot \underline{y}$

$$\begin{aligned}
 (e) \quad \begin{pmatrix} \Theta^{MV} \\ -f(\underline{y}) \times S^L \end{pmatrix}^T &= (A)^{-1} \cdot E^P [f(\underline{y}) \times S^L \cdot \underline{y}^T] \\
 &= (A)^{-1} \cdot E^P [S^L] \cdot E^P [f(\underline{y}) \cdot \underline{y}^T] \\
 &= E^P [S^L] \cdot \begin{pmatrix} \Theta^{MV} \\ -f(\underline{y}) \end{pmatrix}^T
 \end{aligned}$$

$$\begin{aligned}
 (f) \quad \begin{pmatrix} \Theta^{MV} \\ -S \end{pmatrix}^T &= (A)^{-1} \cdot E^P [S \times \underline{y}] \\
 &= (A)^{-1} \cdot E^P [E^P [S \times \underline{y} | \underline{y}]]
 \end{aligned}$$

~~wrong~~

$$\begin{aligned}
 &= (A)^{-1} \cdot E^P [E^P [S | \underline{y}] \cdot \underline{y}] \\
 &= \begin{pmatrix} \Theta^{MV} \\ E^P [S | \underline{y}] \end{pmatrix}^T
 \end{aligned}$$

EXERCISE 3"

Q1: We have that $\frac{S_N}{N} \xrightarrow{P} E^P[X_1|Z]$

and $S \xrightarrow{P} S$

$$\Rightarrow S \times \frac{S_N}{N} \xrightarrow{P} S \times E^P[X_1|Z]$$

$$\Leftrightarrow S \times \left(\frac{S_N}{N} - E^P[X_1|Z] \right) \xrightarrow{P} 0$$

$$\Leftrightarrow \gamma_N^d \xrightarrow{P} 0$$

Q2: γ^h : hedgeable claim
 γ^d : diversifiable claim
 γ^2 : residual claim

($\hookrightarrow$ not hedgeable, not orthogonal)

Q3: $\gamma^h = \frac{\Theta^{MV}}{f(Y) \times \frac{S_N}{N}} \cdot \underline{Y} = \frac{\Theta^{MV}}{f(Y)} \cdot \underline{Y} \times E^P[X_1]$ ↗ see Exercise 7

$$\gamma_N^d = f(Y) \times \left(\frac{S_N}{N} - E^P[X_1|Z] \right)$$

$$\begin{aligned} \gamma^2 &= f(Y) \times E^P[X_1|Z] - \frac{\Theta^{MV}}{f(Y)} \cdot \underline{Y} \times E^P[X_1] \\ &= f(Y) \times \left[E^P[X_1|Z] - E^P[X_1] \right] \\ &\quad + \left[f(Y) - \frac{\Theta^{MV}}{f(Y)} \cdot \underline{Y} \right] \times E^P[X_1] \\ &= \gamma_{act}^2 + \gamma_{fin}^2 \end{aligned}$$

Q4: γ^h and γ^d : see Q3

γ_{act}^2 = residual risk due to actuarial systemic risk

γ_{fin}^2 = residual risk due to imperfect hedge

5. Linking valuations and hedgers

Lemma:

- ▶ Consider a hedger θ and a valuation ρ . Define the hedger μ by

$$\mu_S = \theta_S + (\rho[S - \theta_S \cdot \mathbf{Y}], 0, \dots, 0) \quad \text{for any } S \in \mathcal{C}$$

- ① ▶ If θ is a market-consistent hedger, then μ is a market-consistent hedger.
- ② ▶ If θ is a model-consistent hedger¹⁰ and ρ is a model-consistent valuation with underlying actuarial valuation π , then μ is a model-consistent hedger with underlying actuarial valuation π .

¹⁰This condition can be weakened to ' θ hedges any orthogonal claim by a zero coupon bond trading strategy'.

Linking valuations and hedgers

Lemma:

- ▶ Let θ be a fair hedger¹¹.
- ▶ Let ρ be a model-consistent valuation with underlying actuarial valuation π .
- ③ ▶ Then the hedger μ defined by

$$\mu_S = \theta_S + (\rho[S - \theta_S \cdot \mathbf{Y}], 0, \dots, 0) \quad \text{for any } S \in \mathcal{C}$$

is a fair hedger with underlying actuarial valuation π .

Proofs of ①, ②, ③ : see next page

¹¹This condition can be weakened to ' θ is a market-consistent hedger which hedges any orthogonal claim by a zero coupon bond trading strategy'.

PROOF OF LEMMA:

①

$$\begin{aligned}
 \text{Proof: } \mu_{s+s^h} &= \frac{\theta}{s+s^h} + \left(\beta [s+s^h - \frac{\theta}{s+s^h} \cdot \underline{y}] ; 0, \dots, 0 \right) \\
 &\stackrel{\text{Qu. the C}}{=} \frac{\theta}{s} + \underline{y} + \left(\beta [s+s^h - (\frac{\theta}{s} + \underline{y}) \cdot \underline{y}] ; 0, \dots, 0 \right) \\
 &= \frac{\theta}{s} + \underline{y} + \left(\beta [s - \frac{\theta}{s} \cdot \underline{y}] ; 0, \dots, 0 \right) \\
 &= \mu_s + \underline{y}
 \end{aligned}$$

②

$$\begin{aligned}
 \text{Proof: } &\text{Suppose } \frac{\theta}{s^+} = (f[s^+] ; 0, \dots, 0) \text{ for any } s^+ \in C \\
 \mu_{s^+} &\stackrel{\text{def.}}{=} \frac{\theta}{s^+} + \left(\beta [s^+ - \frac{\theta}{s^+} \cdot \underline{y}] ; 0, \dots, 0 \right) \\
 &\stackrel{\text{Oss}}{=} (f[s^+] ; 0, \dots, 0) + \left(\beta [s^+ - (f[s^+] \cdot \underline{y})] ; 0, \dots, 0 \right) \\
 &= (f[s^+] ; 0, \dots, 0) + \left(\beta [s^+ - f[s^+] \cdot \underline{y}] ; 0, \dots, 0 \right) \\
 &\stackrel{\text{Satz. nr.}}{=} (\beta [s^+] ; 0, \dots, 0) \\
 &= (\pi[s^+] ; 0, \dots, 0) \\
 &\quad \hookrightarrow \text{act. valuation}
 \end{aligned}$$

③

From previous lemma, we find that μ is MoC and MoC, hence fair

Linking valuations and hedgers

Theorem:

► Consider the valuation $\rho : \mathcal{C} \rightarrow \mathbb{R}$.

- ① ► ρ is a market-consistent valuation if and only if there exists a market-consistent hedger μ^{mac} such that

$$\boxed{\rho[S] = \mu_S^{mac} \cdot y} \quad \text{for any } S \in \mathcal{C}$$

- ② ► ρ is a model-consistent valuation if and only if there exists a model-consistent hedger μ^{moc} such that

$$\boxed{\rho[S] = \mu_S^{moc} \cdot y} \quad \text{for any } S \in \mathcal{C}$$

- ③ ► ρ is a fair valuation if and only if there exists a fair hedger μ^f such that

$$\boxed{\rho[S] = \mu_S^f \cdot y} \quad \text{for any } S \in \mathcal{C}$$

Proofs of ①, ②, ③ : see next pages.

PROOF OF THEOREM:

①

→ (a) $\boxed{S \text{ is MAC valuation} \Rightarrow S[S] = \mu_S^{\text{MAC}} \cdot \gamma}$
↳ MAC hedger

PROOF: Define $S'[S]$ by:

$$S'[S] = \left\{ \underline{\Theta}_S^{\text{MV}} + \left(S[S - \underline{\Theta}_S^{\text{MV}} \cdot \underline{\gamma}]; 0, \dots, 0 \right) \right\} \cdot \gamma$$
$$= \mu_S^{\text{MAC}} \cdot \gamma$$

↳ is a MAC hedger (see lemma)

$$\begin{aligned} S'[S] &= \underline{\Theta}_S^{\text{MV}} \cdot \gamma + S[S - \underline{\Theta}_S^{\text{MV}} \cdot \underline{\gamma}] \\ &\stackrel{S \text{ is MAC}}{=} S[S - \underline{\Theta}_S^{\text{MV}} \cdot \underline{\gamma} + \underline{\Theta}_S^{\text{MV}} \cdot \underline{\gamma}] \\ &= S[S] \end{aligned}$$

• Conclusion: $S[S] = \mu_S^{\text{MAC}} \cdot \gamma$ ■

(b) $\boxed{S[S] = \mu_S^{\text{MAC}} \cdot \gamma \Rightarrow S \text{ is a MAC valuation}}$
↳ MAC hedger

PROOF:

$$\begin{aligned} S[S + \underbrace{S^{\perp}}_{\underline{\gamma} \cdot \underline{\gamma}}] &= \mu_{S+S^{\perp}}^{\text{MAC}} \cdot \gamma \\ &\stackrel{\mu_S^{\text{MAC}} + \mu_{S^{\perp}}^{\text{MAC}}}{=} \left(\mu_S^{\text{MAC}} + \underline{\gamma} \right) \cdot \gamma \\ &= S[S] + \underline{\gamma} \cdot \gamma \end{aligned}$$

$\Rightarrow S$ is MAC valuation ■

(2) $\rightarrow$ (a) S is MOC valuation $\Rightarrow S[S] = \underline{\mu}_S^{MOC} \cdot \mathbb{1}$

$\hookrightarrow$ MOC hedger

PROOF: Define $S'[S]$ by

$$\begin{aligned} S'[S] &= (\underbrace{S[S]}_{\mu_S^{MOC}}, 0, \dots, 0) \cdot \mathbb{1} \\ &= \mu_S^{MOC} \cdot \mathbb{1} \end{aligned}$$

μ_S^{MOC} is a MOC hedger.

Inoked: $\mu_{S^\perp}^{MOC} = (S[S^\perp], 0, \dots, 0)$

$$= (\pi[S^\perp], 0, \dots, 0)$$

$\hookrightarrow$ underlying actuarial valuation of S

$S'[S] = S[S]$

Conclusion: $S[S] = \mu_S^{MOC} \cdot \mathbb{1}$

(b) $S[S] = \mu_S^{MOC} \cdot \mathbb{1} \Rightarrow S$ is MOC valuation
 $\hookrightarrow$ MOC hedger.

PROOF: $S[S^\perp] = \mu_{S^\perp}^{MOC} \cdot \mathbb{1}$

$$= \mu_{S^\perp}^{MOC} (\pi[S^\perp], 0, \dots, 0) \cdot \mathbb{1}$$

$$= \pi[S^\perp]$$

π is underlying actuarial valuation of $\mu_{S^\perp}^{MOC}$

③ $\rightarrow$ (a) S is fair valuation $\Rightarrow S[S] = \mu_S^f \cdot f$

$\hookrightarrow$ fair hedger

PROOF: • Consider the valuation S' :

$$S'[S] = \left\{ \frac{Q^{MV}}{S} + (S[S - \frac{Q^{MV}}{S} \cdot f]; 0; \dots; 0) \right\} \cdot f$$

$$= \mu_S^f \cdot f$$

• μ^f is a fair hedger (see Lemma)

• $S'[S] = S[S]$ (see proof ① a)

• Conclusion: $S[S] = \mu_S^f \cdot f$

□

(b) $S[S] = \mu_S^f \cdot f \Rightarrow S$ is fair valuation

$\hookrightarrow$ fair hedger

PROOF: follows immediately from ① b and ① c

□

EXERCISE 8

Q1: • $\mathcal{S}$ is normalised:

$$\boxed{\mathcal{S}[0]} = \underline{\underline{Q}}_0 \cdot \underline{y} + \pi [0 - \underline{\underline{Q}}_0 \cdot \underline{y}]$$

$\pi, \underline{\underline{Q}}$ is normalised $\rightarrow (0, 0, \dots, 0) \cdot \underline{y} + \pi [0 - (0, 0, \dots, 0) \cdot \underline{y}] = \underline{\underline{0}}$

• $\mathcal{S}$ is translation invariant:

$$\begin{aligned} \boxed{\mathcal{S}[S+a]} &= \underline{\underline{Q}}_{S+a} \cdot \underline{y} + \pi [S+a - \underline{\underline{Q}}_{S+a} \cdot \underline{y}] \\ \text{Q is tr. inv.} \quad &\downarrow \\ &= (\underline{\underline{Q}}_S + (e^{-2}a, 0, \dots, 0)) \cdot \underline{y} \\ &\quad + \pi [S+a - (\underline{\underline{Q}}_S + (e^{-2}a, 0, \dots, 0)) \cdot \underline{y}] \\ &= \underline{\underline{Q}}_S \cdot \underline{y} + e^{-2}a + \pi [S + \cancel{a} - \underline{\underline{Q}}_S \cdot \underline{y} - \cancel{a}] \\ &= \boxed{\mathcal{S}[S] + e^{-2}a} \end{aligned}$$

$$\begin{aligned} \text{Q2: } \boxed{\mathcal{S}[S^q]} &= \underline{\underline{Q}}_{S^q} \cdot \underline{y} + \pi [S^q - \underline{\underline{Q}}_{S^q} \cdot \underline{y}] \\ S^q = \underline{x} \cdot \underline{y} \quad &\downarrow \\ &= \underline{x} \cdot \underline{y} + \pi [S^q - \underline{x} \cdot \underline{y}] \end{aligned}$$

$$\begin{aligned} &= e^{-2} E^Q [\underline{x} \cdot \underline{y}] + \pi [0] \\ \pi \text{ is normalised} \quad &\downarrow \\ &= e^{-2} E^Q [S^q] + 0 \end{aligned}$$

$$= \boxed{e^{-2} E^Q [S^q]}$$

Exercise 8

$$\bullet \quad \boxed{\mathcal{L}[S^\perp]} = \underline{\Theta}_{S^\perp} \cdot \underline{y} + \pi[S^\perp - \underline{\Theta}_{S^\perp} \cdot \underline{y}]$$

π' is underlying
act. val. of $\underline{\Theta}$

$$= (\pi'[S^\perp], 0, \dots, 0) \cdot \underline{y} + \pi[S^\perp - (\pi'[S^\perp], 0, \dots, 0) \cdot \underline{y}]$$

$$\begin{aligned} &= \pi'[S^\perp] + \pi[S^\perp - \pi'[S^\perp] \cdot e^2] \\ &= \cancel{\pi'[S]} + \pi[S^\perp] - e^{-2} \cancel{\pi'[S^\perp]} \cdot e^2 \\ &= \boxed{\pi[S^\perp]} \end{aligned}$$

π' is translation
invariant

Hedge-based valuations

► **Theorem:**

- ① ► A **HB valuation** ρ is **positive homogeneous**:

$$\rho[a S] = a \rho[S] \quad \text{for any } a > 0 \text{ and } S \in \mathcal{C}$$

if its underlying θ and π are positive homogeneous.

- A **HB valuation** ρ is **subadditive**:

$$\rho[S_1 + S_2] \leq \rho[S_1] + \rho[S_2] \quad \text{for any } S_1, S_2 \in \mathcal{C}$$

- ② if its underlying θ is additive and π is subadditive.

► **Theorem:**

③ ρ is a **HB valuation** $\Leftrightarrow \rho$ is a **fair valuation**

Proofs: See next pages

PROOF OF FIRST THEOREM (① + ②)

- Consider the HB valuation ρ :

$$\rho[S] = \underbrace{\frac{\theta}{S} \cdot y}_{\text{fair ledger}} + \underbrace{\pi[S - \frac{\theta}{S} \cdot y]}_{\text{MOC valuation}}$$

- PROOF OF ①: suppose θ and π are pos. hom.

$$\begin{aligned} \rho[a \cdot S] &\stackrel{\text{def. } \rho}{=} \frac{\theta}{a \cdot S} \cdot y + \pi[a \cdot S - \frac{\theta}{a \cdot S} \cdot y] \\ &\stackrel{\theta \text{ pos. hom.}}{=} a \cdot \frac{\theta}{S} \cdot y + \pi[a \cdot S - a \cdot \frac{\theta}{S} \cdot y] \\ &\stackrel{\pi \text{ pos. hom.}}{=} a \cdot \left\{ \frac{\theta}{S} \cdot y + \pi[S - \frac{\theta}{S} \cdot y] \right\} \\ &\stackrel{\text{def. } \rho}{=} a \cdot \rho[S] \quad \square \end{aligned}$$

- PROOF OF ②: suppose θ is additive, π is subadditive

$$\begin{aligned} \rho[S_1 + S_2] &\stackrel{\text{def. } \rho}{=} \frac{\theta}{S_1 + S_2} \cdot y + \pi[S_1 + S_2 - \frac{\theta}{S_1 + S_2} \cdot y] \\ &\stackrel{\theta \text{ add.}}{=} \left(\frac{\theta}{S_1} + \frac{\theta}{S_2} \right) \cdot y + \pi[S_1 + S_2 - \frac{\theta}{S_1} \cdot y - \frac{\theta}{S_2} \cdot y] \\ &= \left(\frac{\theta}{S_1} + \frac{\theta}{S_2} \right) \cdot y + \pi \left[\left(S_1 - \frac{\theta}{S_1} \cdot y \right) + \left(S_2 - \frac{\theta}{S_2} \cdot y \right) \right] \\ &\stackrel{\pi \text{ subadd.}}{\leq} \frac{\theta}{S_1} \cdot y + \frac{\theta}{S_2} \cdot y + \pi \left[S_1 - \frac{\theta}{S_1} \cdot y \right] + \pi \left[S_2 - \frac{\theta}{S_2} \cdot y \right] \\ &\stackrel{\text{def. } \rho}{=} \rho[S_1] + \rho[S_2] \quad \square \end{aligned}$$

PROOF OF SECOND THEOREM: (3)

(a) Suppose g is HB valuation

$$\begin{aligned}\Rightarrow g[S] &= \underbrace{\underline{Q}_S \cdot \underline{I}}_{\text{fair}} + \underbrace{\pi[S - \underline{Q}_S \cdot \underline{I}]}_{\text{MOC}} \\ &= \left\{ \underline{Q}_S + (\pi[S - \underline{Q}_S \cdot \underline{I}], 0, \dots, 0) \right\} \cdot \underline{I} \\ &= \underline{\mu}_S^{\#} \cdot \underline{I} \quad (\text{see Lemma}) \\ &\quad \downarrow \\ &\quad \text{fair ledger}\end{aligned}$$

$\Rightarrow g$ is a fair valuation (see Theorem)

(b) Suppose g is a fair valuation

- Define $g'[S] = \underbrace{\underline{Q}_S' \cdot \underline{I}}_{\text{fair ledger}} + \underbrace{g[S - \underline{Q}_S' \cdot \underline{I}]}_{\text{MOC valuation}}$

- g' is a HB valuation

- Moreover, $g'[S] = \underbrace{g[S - \underline{Q}_S' \cdot \underline{I}]}_{\text{MOC}} + \underline{Q}_S' \cdot \underline{I}$
 $= g[S]$

- Conclusion: g is a HB valuation

~~Exercise 8~~

EXERCISE 8

- Consider the valuation s' defined by

$$s'[S] = \frac{\Theta^{MV}}{S} \cdot Y + s[S - \frac{\Theta^{MV}}{S} \cdot Y], \quad \forall S \in C$$

$\hookrightarrow \frac{\Theta^{MV}}{S}$ is fair
hedger

$\hookrightarrow s$ is fair valuation
 $\Rightarrow s$ is MOC valuation

- s' is a MVHB valuation

~~with the independence~~

$$\begin{aligned} s'[S] &= e^{-rT} E^Q \left[\frac{\Theta^{MV}}{S} \cdot Y + E^P \left[S - \frac{\Theta^{MV}}{S} \cdot Y \mid Y \right] \right] \\ &= e^{-rT} E^Q \left[E^P \left[\cancel{\frac{\Theta^{MV}}{S} \cdot Y} + S - \cancel{\frac{\Theta^{MV}}{S} \cdot Y} \mid Y \right] \right] \\ &= e^{-rT} E^Q \left[E^P [S \mid Y] \right] \\ &= s[S], \quad \forall S \in C \end{aligned}$$

- Conclusion:

s is a MVHB valuation

Hedge-based valuations

Brennan-Schwartz formula for MVHB valuations

- Consider the **MVHB valuation** ρ defined by

$$\rho[X] = \theta_X^{MV} \cdot y + \pi[X - \theta_X^{MV} \cdot Y] \quad \text{for any } X \in \mathcal{C}$$

- Consider the **product liability** S :

$$S = S^h \times S^\perp \quad \text{with } S^h \in \mathcal{C}^h \text{ and } S^\perp \in \mathcal{C}^\perp$$

- Brennan-Schwartz formula for MVHB valuations:**

$$\rho[S] = e^{-r} \mathbb{E}^Q[S^h] \times \mathbb{E}^P[S^\perp] + \pi[S^h \times (S^\perp - \mathbb{E}^P[S^\perp])]$$

PROOF:

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$$\begin{aligned} \rho[S] &= \rho[S^h \times S^\perp] \\ &= \theta_{S^h \times S^\perp}^{MV} \cdot y + \pi[S^h \times S^\perp - \theta_{S^h \times S^\perp}^{MV} \cdot y] \\ &= \mathbb{E}^P[S^\perp] \times y \cdot \cancel{y} + \pi[S^h \times S^\perp - \mathbb{E}^P[S^\perp] \times y \cdot \cancel{y}] \\ &= \mathbb{E}^P[S^\perp] \times e^{-2} \mathbb{E}^Q[y \cdot y] + \pi[S^h \times S^\perp - \mathbb{E}^P[S^\perp] \times S^h] \\ &= \mathbb{E}^P[S^\perp] \times e^{-2} \mathbb{E}^Q[S^h] + \pi[S^h \times (S^\perp - \mathbb{E}^P[S^\perp])] \end{aligned}$$

□

EXERCISE 10

→ B-S formula for MVHB valuations

Q1: • $\rho[S] = e^{-2} E^Q[S^Q] \times E^P[S^L] + \pi [S^Q \times (S^L - E^P[S^L])]$

$$= e^{-2} E^Q[S^Q] \times E^P[S^L]$$

$$+ e^{-2} E^P[S^Q \times (S^L - E^P[S^L])]$$

$$+ e^{-2} \cdot d \cdot \sigma^P [S^Q \times (S^L - E^P[S^L])]$$

• $\text{Var}^P [S^Q \times (S^L - E^P[S^L])]$

$$= \text{Var}^P \left\{ E^P [S^Q \times (S^L - E^P[S^L]) | Y] \right\}$$

$$+ E^P \left\{ \text{Var}^P [S^Q \times (S^L - E^P[S^L]) | Y] \right\}$$

$$= \text{Var}^P \left\{ S^Q \times E^P [S^L - E^P[S^L]] \right\}$$

$$+ E^P \left\{ (S^Q)^2 \text{Var}^P [S^L] \right\}$$

$$= E^P [(S^Q)^2] \times \text{Var}^P [S^L]$$

• $\rho[S] = e^{-2} E^Q [S^Q] \times E^P [S^L]$

$$+ d \cdot e^{-2} \sigma^P [S^L] \times \sqrt{E^P [(S^Q)^2]}$$

Q2: • $\text{Var}^P [S^L] = N \cdot \text{Var}^P [X_i^L]$

• $\frac{\rho[S^Q \times S^L]}{N} = e^{-2} E^Q [S^Q] \times E^P [X_i^L]$

$$+ \frac{d \cdot e^{-2}}{\sqrt{N}} \times \sigma^P [X_i^L] \times \sqrt{E^P [(S^Q)^2]}$$

Q3:

Exercise 10

$$\bullet \quad \mathbb{P}[S^L \times S^L] = e^{-2} E^Q[S^L] \times E^P[S^L] + \alpha e^{-2} \sigma^P[S^L] \times \sqrt{E^P[(S^L)^2]}$$

L = Loss at time 1

$$= S^L \times S^L - \frac{\mathbb{P}[S^L \times S^L]}{e^{-2} E^Q[S^L]} \times S^L$$

↓
aggregate claim at time 1

units of S^L bought at 0.

time-1 value of invested premiums

insert expression for $\mathbb{P}[S^L \times S^L]$ in expression for L

~~$\mathbb{P}[S^L \times S^L]$~~

$$L = S^L \times \left\{ S^L - E^P[S^L] - \alpha \frac{\sigma^P[S^L]}{E^Q[S^L]} \sqrt{E^P[(S^L)^2]} \right\}$$

$$\bullet \quad \mathbb{P}[L \leq 0] = \mathbb{P}\left[\frac{S^L - E^P[S^L]}{\sigma^P[S^L]} \leq \alpha \frac{\sqrt{E^P[(S^L)^2]}}{E^Q[S^L]} \right]$$

EXERCISE 11

Q1: $S^q = \max(Y^{(1)}, K) = Y^{(2)} + (K - Y^{(1)})_+ = Y^{(1)} + Y^{(2)}$

$$= (0, 1, 1) \cdot \underline{Y} \Rightarrow \underline{\Theta}_{S^q}^{MV} = (0, 1, 1)$$

$$\begin{aligned} \underline{\Theta}_{S^q \times S_N^L}^{MV} &= \underline{\Theta}_{S^q}^{MV} \times E^P[S_N^L] = (0, 1, 1) \times N \times E^P[X_1^L] \\ &= (0, 1, 1) \times N \times \mu \end{aligned}$$

Q2: From Exercise 10-Q2, we have:

$$\frac{g[S^q \times S_N^L]}{N} = e^{-2} E^Q[S^q] \times E^P[X_1^L] + \frac{\sigma e^{-2} E^P[X_1^L]}{\sqrt{N}} \sqrt{E^P[(S^q)^2]}$$

$$e^{-2} E^Q[S^q] = y^{(1)} + y^{(2)}$$

$$E^P[X_1^L] = \mu \text{ and } \text{Var}^P[X_1^L] = \mu \times (1 - \mu)$$

Hence,

$$\frac{g[S^q \times S_N^L]}{N} = (y^{(1)} + y^{(2)}) \times \mu + \frac{\sigma e^{-2} \sqrt{\mu(1-\mu)}}{\sqrt{N}} \sqrt{E^P[(S^q)^2]}$$

Q3: $\overline{P[L \leq 0]} \stackrel{\text{Ex. 10-Q3}}{=} \overline{P\left[Z \leq d \frac{\sqrt{E^P[(S^q)^2]}}{E^Q[S^q]}\right]}$

$$= \Phi\left[d \frac{e^{-2} \sqrt{E^P[(S^q)^2]}}{e^{-2} E^Q[S^q]}\right]$$

$$= \Phi\left[d \frac{e^{-2} \sqrt{E^P[(S^q)^2]}}{y^{(1)} + y^{(2)}}\right]$$

EXERCISE 12

Q1:

Exercise 10 - Q1

$$\begin{aligned}
 \bullet \quad \mathbb{E}[S^Q \times S_N^L] &= e^{-2} \mathbb{E}^Q[S^Q] \times \mathbb{E}^P[S_N^L] + d e^{-2} \sqrt{\mathbb{E}^P[(S^Q)^2]} \times \sigma^P[S_N^L] \\
 \bullet \quad \text{Var}^P[S_N^L] &= \mathbb{E}^P[\text{Var}^P[S_N^L | Z]] + \text{Var}^P[\mathbb{E}^P[S_N^L | Z]] \\
 &= \mathbb{E}^P[N \times \text{Var}^P[X_1 | Z]] + \text{Var}^P[N \times \mathbb{E}^P[X_1 | Z]] \\
 &= N \times \mathbb{E}^P[\text{Var}^P[X_1 | Z]] + N^2 \times \text{Var}^P[\mathbb{E}^P[X_1 | Z]] \\
 &= N \times A + N^2 \times B
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad \mathbb{E}[S^Q \times S_N^L] &= e^{-2} \mathbb{E}^Q[S^Q] \times \mathbb{E}^P[S_N^L] \\
 &\quad + d e^{-2} \sqrt{\mathbb{E}^P[(S^Q)^2]} \times \sqrt{N \times A + N^2 \times B}
 \end{aligned}$$

$\Rightarrow$

$$\frac{\mathbb{E}[S^Q \times S_N^L]}{N} = e^{-2} \mathbb{E}^Q[S^Q] \times \mathbb{E}^P[X_1] + d e^{-2} \sqrt{\mathbb{E}^P[(S^Q)^2]} \times \sqrt{\frac{A}{N} + B}$$

Q2:

- From Exercise 3', we find

$$\frac{S^Q \times S_N^L}{N} = Y^Q + Y_N^d + Y^2$$

with : $E^Q[Y^Q] = E^Q[S^Q] \times E^P[X_1]$

$\text{Var}^P[Y_N^d] = \frac{1}{N} E^P[(S^Q)^2] \times A$

$\text{Var}^P[Y^2] = E^P[(S^Q)^2] \times B$

$$\frac{S[S^Q \times S_N^L]}{N} = e^{-2} E^Q[Y^Q] + d e^{-2} \sqrt{\text{Var}^P[Y_N^d] + \text{Var}^P[Y^2]}$$

Q3:

$$\bullet X_1 | Z = z = \begin{cases} 0 & : 1 - p(z) \\ 1 & : p(z) \end{cases}$$

$$\bullet \text{Var}^P [X_1 | Z = z] = p(z) \times (1 - p(z))$$

$$\Rightarrow \text{Var}^P [X_1 | Z] = p(Z) \times (1 - p(Z))$$

$$\Rightarrow \boxed{A} = E^P [\text{Var}^P [X_1 | Z]]$$

$$= E^P [p(Z) \times (1 - p(Z))]$$

$$\bullet E^P [X_1 | Z = z] = p(z)$$

$$\Rightarrow E^P [X_1 | Z] = p(Z)$$

$$\Rightarrow \boxed{B} = \text{Var}^P [E^P [X_1 | Z]] = \text{Var}^P [p(Z)]$$

EXERCISE 13-1

Q1 • We have to prove that I and $Y^{(n)}$ are P -independent. Or, equivalently,

$$P[Y^{(n)}=y, I=i] = P[Y^{(n)}=y] \times P[I=i]$$

$$\forall y \in \{0,1\} \text{ and } i \in \{0,1\}$$

$$• P_{0\cdot} = P_{00} + P_{01} = 1/6 + 1/6 = 1/3$$

$$P_{1\cdot} = P_{10} + P_{11} = 2/6 + 2/6 = 2/3$$

$$P_{\cdot 0} = P_{00} + P_{10} = 1/6 + 2/6 = 1/2$$

$$P_{\cdot 1} = P_{01} + P_{11} = 1/6 + 2/6 = 1/2$$

$$• P_{0\cdot} \times P_{\cdot 0} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = P_{00}$$

$$P_{0\cdot} \times P_{\cdot 1} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = P_{01}$$

$$P_{1\cdot} \times P_{\cdot 0} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} = P_{10}$$

$$P_{1\cdot} \times P_{\cdot 1} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} = P_{11}$$

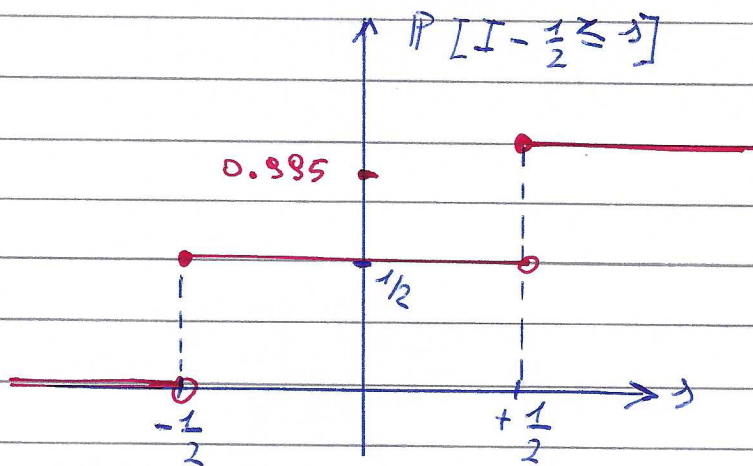
• Conclusion: I and $Y^{(n)}$ are P -independent

$$\Rightarrow I \in C^+$$

Q2: $I \in C^1 \Rightarrow \underline{\Theta}_I^{MV} = \left(e^{-\lambda} E^P[I], 0 \right)$
 $= (1.1, 0) = \left(\frac{1}{2}, 0 \right)$

Q3: $S[I] = \underline{\Theta}_I^{MV} \cdot \underline{y} + \pi \left[I - \underline{\Theta}_I^{MV} \cdot \underline{y} \right]$
 $= \left(\frac{1}{2}, 0 \right) \cdot \left(1, \frac{1}{2} \right) + \pi \left[I - \left(\frac{1}{2}, 0 \right) \cdot \left(1, \frac{1}{2} \right) \right]$
 $= \frac{1}{2} + \pi \left[I - \frac{1}{2} \right]$

Q4: $I - \frac{1}{2} = \begin{cases} -\frac{1}{2} & : I=0 \\ +\frac{1}{2} & : I=1 \end{cases} = \begin{cases} -\frac{1}{2} & : \frac{1}{2} \\ +\frac{1}{2} & : \frac{1}{2} \end{cases}$



$Var^P \left[I - \frac{1}{2} \right] = \frac{1}{2}$
 0.995

$S[I] = \frac{1}{2} + \pi \left[I - \frac{1}{2} \right]$
 $= \frac{1}{2} + 0.06 Var^P_{0.995} \left[I - \frac{1}{2} \right]$
 $= \frac{1}{2} + 0.06 \times \frac{1}{2}$
 $= 0.53$

Q5:

First way

$$* S = (1 - Y^{(2)}) (1 - I) = S^h \times S^L$$

$$\hookrightarrow \text{with } S^h = (1 - Y^{(2)}) = (Y^{(0)} - Y^{(2)})$$

$$= (1, -1) \cdot (Y^{(0)}, Y^{(2)})$$

$$= (1, -1) \cdot \underline{Y} \Rightarrow \underline{Q}_{S^h}^{MV} = (1, -1)$$

$$\bullet \hookrightarrow \text{with } S^L = 1 - I$$

$$E^P[S^L] = 1 - E^P[I] = \frac{1}{2}$$

$$(*) \underline{Q}_S^{MV} = \underline{Q}_{S^h \times S^L}^{MV}$$

$$= \underline{Q}_{S^h}^{MV} \times E^P[S^L]$$

$$= (1, -1) \times \frac{1}{2}$$

$$= \left(\frac{1}{2}, -\frac{1}{2} \right)$$

Q5: Second way

- $\underline{\Theta}_{S}^{MV} = (\Theta_{S}^{(0)}, \Theta_{S}^{(1)})$ follows from:

$$\begin{cases} \sum_{m=1}^m \text{Cov}^P[Y^{(2)}, Y^{(m)}] \times \Theta_{S}^{(m)} = \text{Cov}^P[Y^{(2)}, S] & \text{, } 2=1,2,\dots,m \\ \Theta_{S}^{(0)} = e^{-1} \left\{ E^P[S] - \sum_{m=1}^m E^P[Y^{(m)}] \times \Theta_{S}^{(m)} \right\} \end{cases}$$

$\begin{matrix} \text{, } 2=0 \\ m=1 \end{matrix}$

$$\Leftrightarrow \begin{cases} \text{Cov}^P[Y^{(1)}, Y^{(1)}] \times \Theta_{S}^{(1)} = \text{Cov}^P[Y^{(1)}, S] \\ \Theta_{S}^{(0)} = E^P[S] - E^P[Y^{(1)}] \times \Theta_{S}^{(1)} \end{cases}$$

- $E^P[S] = E^P[(1 - Y^{(1)}) (1 - I)] = 1_{00} = 1/6$

- $E^P[Y^{(1)}] = \mu_{10} = 2/3$

- $\text{Cov}^P[Y^{(1)}, Y^{(1)}] = \text{Var}^P[Y^{(1)}]$

$$= E^P[(Y^{(1)})^2] - (E^P[Y^{(1)}])^2$$

$$= 1_{10} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = 2/9$$

- $\text{Cov}^P[Y^{(1)}, S] = E^P[Y^{(1)} \times S] - E^P[Y^{(1)}] \times E^P[S]$

$$= E^P[Y^{(1)} \times (1 - Y^{(1)}) \times (1 - I)] - \frac{2}{3} \times \frac{1}{6}$$

$$= 0 - \frac{1}{9}$$

$$= -\frac{1}{9}$$

• $\underline{\theta}^{MV}_S$ follows from

$$\begin{cases} \frac{2}{3} \theta^{(1)}_S = -\frac{1}{3} \\ \theta^{(0)}_S = \frac{1}{6} - \frac{2}{3} \times \theta^{(1)}_S \end{cases}$$

$$\begin{cases} \theta^{(1)}_S = -\frac{1}{2} \\ \theta^{(0)}_S = \frac{1}{6} - \frac{2}{3} \times \left(-\frac{1}{2}\right) = \frac{1}{2} \end{cases}$$

$$\underline{\theta}^{MV}_S = (\theta^{(0)}_S, \theta^{(1)}_S) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

↳ $\begin{cases} \text{buy } 1/2 \text{ zero-coupon bonds} \\ \text{sell } 1/2 \text{ stock} \end{cases}$

Q6: $\underline{g}[S] = \underline{\theta}^{MV}_S \cdot \underline{y} + \pi \left[S - \underline{\theta}^{MV}_S \cdot \underline{y} \right]$

$$= \left(\frac{1}{2}, -\frac{1}{2}\right) \cdot \left(1, \frac{1}{2}\right) + \pi \left[S - \left(\frac{1}{2}, -\frac{1}{2}\right) \cdot \left(1, \frac{1}{2}\right) \right]$$

$$= \frac{1}{4} + \pi \left[S - \frac{1}{2} + \frac{1}{2} y^{(1)} \right]$$

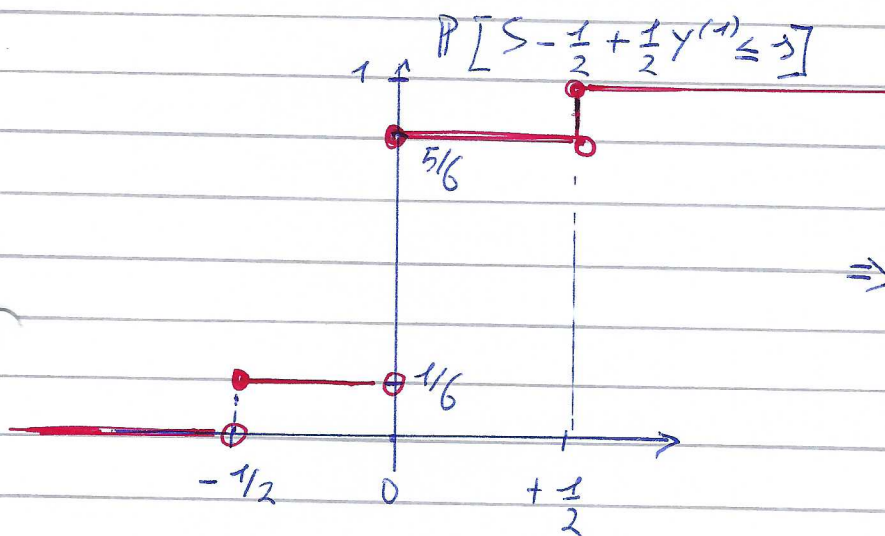
Q7:

$$P[S] = \underbrace{\frac{1}{4}}_{\frac{QMV}{S} \cdot f} + \pi \left[\underbrace{S - \frac{1}{2} + \frac{1}{2} Y^{(1)}}_{\frac{QMV}{S} \cdot Y} \right]$$

$$\pi \left[S - \frac{1}{2} + \frac{1}{2} Y^{(1)} \right] = 0.06 \times \text{VaR}_{0.995}^{\text{IP}} \left[S - \frac{1}{2} + \frac{1}{2} Y^{(1)} \right]$$

$$S - \frac{1}{2} + \frac{1}{2} Y^{(1)} = (1 - Y^{(1)}) \times (1 - I) - \frac{1}{2} + \frac{1}{2} Y^{(1)}$$

$$= \begin{cases} 1/2 & : (0,0) \text{ with IP} = 1/6 \\ -1/2 & : (0,1) & 1/6 \\ 0 & : (1,0) & 1/3 \\ 0 & : (1,1) & 1/3 \end{cases}$$



$$\Rightarrow \text{VaR}_{0.995}^{\text{IP}} [\dots] = 1/2$$

$$\begin{aligned} P[S] &= \frac{1}{4} + 0.06 \text{VaR}_{0.995}^{\text{IP}} [\dots] \\ &= \frac{1}{4} + 0.06 \times \frac{1}{2} \\ &= 0.28 \end{aligned}$$

EXERCISE 13-2

Q1: $\underline{\theta}_{MV} = (\theta^{(0)}, \theta^{(1)}, \theta^{(2)})$ follows from

$$\begin{cases} \text{cov}^P[Y^{(1)}, Y^{(1)}] \theta_S^{(1)} + \text{cov}^P[Y^{(1)}, Y^{(2)}] \theta_S^{(2)} = \text{cov}^P[Y^{(1)}, S] \\ \text{cov}^P[Y^{(2)}, Y^{(1)}] \theta_S^{(1)} + \text{cov}^P[Y^{(2)}, Y^{(2)}] \theta_S^{(2)} = \text{cov}^P[Y^{(2)}, S] \\ \theta_S^{(0)} = E^P[S] - E^P[Y^{(1)}] \theta_S^{(1)} - E^P[Y^{(2)}] \theta_S^{(2)} \end{cases}$$

- $E^P[S] = 1/6$ (see Exc. 13-1)

- $E^P[Y^{(1)}] = 2/3$ (see Exc. 13-1)

- $E^P[Y^{(2)}] = E^P[I] = 1/2 = 1/2$

- $\text{cov}^P[Y^{(1)}, Y^{(1)}] = 2/8$ (see Ex. 13-1)

- $\text{cov}^P[Y^{(1)}, Y^{(2)}] = E^P[Y^{(1)} \times I] - E^P[Y^{(1)}] \times E^P[I]$

$$= 1/11 - \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} - \frac{2}{6} = 0$$

$Y^{(1)}$ and I are
IP-independent!

- $\text{cov}^P[Y^{(1)}, S] = -1/8$ (see Exc. 13-1)

- $\text{cov}^P[Y^{(2)}, S] = E^P[I \times S] - E^P[I] \times E^P[S]$

$$= E^P[I \times (1 - Y^{(1)}) \times (1 - I)] - \frac{1}{2} \times \frac{1}{6}$$

$$= 0 - \frac{1}{12}$$

$$= -\frac{1}{12}$$

• $\underline{\Theta}^{MV}_S$ follows from

Exercise 13-2

$$\begin{cases} \frac{2}{3} \Theta_S^{(1)} = -1/8 \\ \frac{1}{4} \Theta_S^{(2)} = -1/12 \\ \Theta_S^{(0)} = \frac{1}{6} - \frac{2}{3} \Theta_S^{(1)} - \frac{1}{2} \Theta_S^{(2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} \Theta_S^{(1)} = -1/2 \\ \Theta_S^{(2)} = -1/3 \\ \Theta_S^{(0)} = \frac{1}{6} - \frac{2}{3} \times \left(-\frac{1}{2}\right) - \frac{1}{2} \times \left(-\frac{1}{3}\right) = \frac{2}{3} \end{cases}$$

$$\Leftrightarrow \underline{\Theta}^{MV}_S = \left(\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}\right)$$

↳ Buy $2/3$ zero-coupon bonds
 Sell $1/2$ stock
 Sell $1/3$ survival indices

Q2:

$$\underline{P}[S] = \underline{\Theta}^{MV}_S \cdot \underline{y} + \pi \left[S - \underline{\Theta}^{MV}_S \cdot \underline{I} \right]$$

$$= \left(\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}\right) \cdot \left(1, \frac{1}{2}, \frac{2}{3}\right)$$

$$+ \pi \left[S - \left(\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}\right) \cdot (1, Y^{(1)}, I) \right]$$

$$= \frac{2}{3} \times 1 - \frac{1}{2} \times \frac{1}{2} - \frac{1}{3} \times \frac{2}{3} + \pi \left[S - \frac{2}{3} + \frac{1}{2} Y^{(1)} + \frac{1}{3} I \right]$$

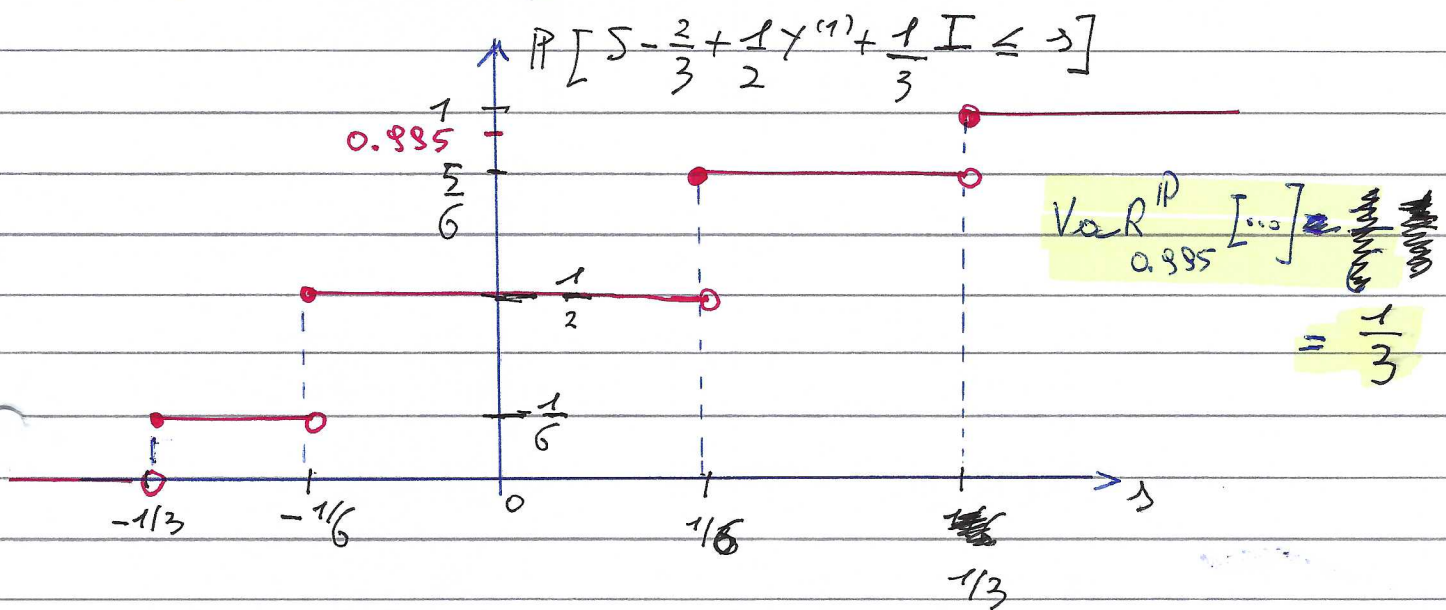
$$= \frac{2}{3} + \pi \left[S - \frac{2}{3} + \frac{1}{2} Y^{(1)} + \frac{1}{3} I \right]$$

Q3:

$$P[S] = \frac{7}{36} + 0.06 \text{Var}^P \left[S - \frac{2}{3} + \frac{1}{2} Y^{(1)} + \frac{1}{3} I \right]$$

$$S = \frac{2}{3} + \frac{1}{2} Y^{(1)} + \frac{1}{3} I = (1 - Y^{(1)})(1 - I) - \frac{2}{3} + \frac{1}{2} Y^{(1)} + \frac{1}{3} I$$

$$\begin{cases} 1 - \frac{2}{3} = \frac{1}{3} & : (0,0) \text{ with } P = 1/6 \\ -\frac{2}{3} + \frac{1}{3} = -\frac{1}{3} & : (0,1) = 1/3 \\ -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6} & : (1,0) = 1/6 \\ -\frac{2}{3} + \frac{1}{2} + \frac{1}{3} = \frac{1}{6} & : (1,1) = 1/3 \end{cases}$$



$$P[S] = \frac{7}{36} + 0.06 \times \frac{1}{3} = \frac{175}{900} + \frac{18}{900}$$

$$= \frac{193}{900}$$

EXERCISE 13-2'

do NOT depend on y !

Q: $\boxed{g[S]} = \frac{\theta^{MV}}{S} y + \pi [S - \frac{\theta^{MV}}{S} \cdot y]$

see Ex. 13-2 = $\left(\frac{2}{3}, -\frac{1}{2}, -\frac{1}{2}\right) \cdot \left(1, \frac{1}{2}, y^{(2)}\right) + 0.061\frac{1}{3}$

$$= \frac{2}{3} - \frac{1}{4} - \frac{1}{3} y^{(2)} + \frac{1}{50}$$

$$= \frac{\cancel{200} - 75 + 100 y^{(2)} + 6}{300}$$

$$= \frac{131 - 100 y^{(2)}}{300}$$

EXERCISE 13-3

Q1:

$$S = (1 - Y^{(1)}) \times (1 - I)$$

$$= 1 - Y^{(1)} - I + I \times Y^{(1)}$$

$$Y^{(1)} \times I = \begin{cases} 1 & : (1,1) \\ 0 & : \text{otherwise} \end{cases}$$

$$I \times (Y^{(1)} - 0.5)_+ = \begin{cases} 1/2 & : (1,1) \\ 0 & : \text{otherwise} \end{cases}$$

$$\text{Hence, } Y^{(1)} \times I = 2 \times I \times (Y^{(1)} - 0.5)_+$$

$$\text{or } Y^{(1)} \times I = 2 \times Y^{(3)}$$

$$\text{Hence, } S = 1 - Y^{(1)} - Y^{(2)} + 2 \times Y^{(3)}$$

$$\text{or } S = \underline{\gamma} \cdot \underline{Y} \text{ with } \underline{\gamma} = (1, -1, -1, 2)$$

$$\frac{\Theta^{MV}}{S} = \underline{\gamma} = (1, -1, -1, 2)$$

REMARK: It can be proven that the market in Ex 13-3 is complete, see

Q2:

$$\underline{\theta}[S] = \frac{\Theta^{MV}}{S} \cdot \underline{y} + \pi [S - \frac{\Theta^{MV}}{S} \cdot \underline{Y}]$$

$$= \frac{\Theta^{MV}}{S} \cdot \underline{y} + \pi [0]$$

$$= (1, -1, -1, 2) \cdot (1, y^{(1)}, y^{(2)}, y^{(3)}) + 0$$

$$= 1 - y^{(1)} - y^{(2)} + 2y^{(3)}$$

$$= 1 - \frac{1}{2} - \frac{2}{3} + 2 \times \frac{1}{6}$$

$$= \frac{1}{6}$$

Q1: $\underline{\Theta}_S^{MV}$ follows from:

$$\begin{cases} \text{Var}^P[Y^{(1)}] \times \Theta_S^{(1)} + \text{Cov}^P[Y^{(1)}, I] \times \Theta_S^{(2)} = \text{Cov}^P[Y^{(1)}, S] \\ \text{Cov}^P[I, Y^{(1)}] \times \Theta_S^{(1)} + \text{Var}^P[I] \times \Theta_S^{(2)} = \text{Cov}^P[I, S] \\ \Theta_S^{(3)} = e^{-2} \left(E^P[S] - E^P[Y^{(1)}] \times \Theta_S^{(1)} - E^P[I] \times \Theta_S^{(2)} \right) \end{cases}$$

$Y^{(1)} \perp (I, S)$

$$\Leftrightarrow \begin{cases} \text{Var}^P[Y^{(1)}] \times \Theta_S^{(1)} + 0 \times \Theta_S^{(2)} = 0 \\ 0 \times \Theta_S^{(1)} + \text{Var}^P[I] \times \Theta_S^{(2)} = \text{Cov}^P[I, S] \\ \Theta_S^{(3)} = \dots \end{cases}$$

$$\Rightarrow \begin{cases} \Theta_S^{(1)} = 0 \\ \Theta_S^{(2)} = \text{Cov}^P[I, S] / \text{Var}^P[I] \\ \Theta_S^{(3)} = e^{-2} \left(E^P[S] - E^P[I] \times \frac{\text{Cov}^P[I, S]}{\text{Var}^P[I]} \right) \end{cases}$$

$\rightarrow$ * no investment in stocks
(because of independence)

* $\text{Cov}^P[I, S] \uparrow \Rightarrow$ $\begin{cases} \text{more investment in index} \\ \text{less in zero-coupon bonds} \end{cases}$

* If $\text{Cov}^P[I, S] = 0 \Rightarrow \underline{\Theta}_S = (e^{-2} E^P[S], 0, 0)$

Q2: • $\text{Cov}^P[I, S] = \text{Cov}^P[S + (I - S), S]$

$(I - S)$ and S are independent

The I_i are P-c.i.o.l.

$$\begin{aligned}
 &= \text{Var}^P[S] + \text{Cov}^P[I - S, S] \\
 &= \text{Var}^P[S] \\
 &= N^{\text{ins}} \times \text{Var}^P\left[\frac{I}{1}\right] \\
 &= N^{\text{ins}} \times \mu \times (1 - \mu)
 \end{aligned}$$

• $\text{Var}^P[I] = N^{\text{nat}} \times \mu \times (1 - \mu)$

• $E^P[S] = N^{\text{ins}} \times \mu$

• $E^P[I] = N^{\text{nat}} \times \mu$

$$\begin{cases}
 \theta_S^{(0)} = e^{-2} \left(N^{\text{ins}} \times \mu - N^{\text{nat}} \times \mu \frac{N^{\text{ins}} \times \mu \times (1 - \mu)}{N^{\text{nat}} \times \mu \times (1 - \mu)} \right) = 0 \\
 \theta_S^{(1)} = 0 \\
 \theta_S^{(2)} = \frac{N^{\text{ins}} \times \mu \times (1 - \mu)}{N^{\text{nat}} \times \mu \times (1 - \mu)} = \frac{N^{\text{ins}}}{N^{\text{nat}}}
 \end{cases}$$

Exercise 14

Q3: $\delta[S] = \frac{\theta^{MV}}{S} \cdot y + \pi \left[S - \frac{\theta^{MV}}{S} \cdot y \right]$

$\theta^{MV}_S = (0, 0, \frac{N^{ins}}{N^{not}})$

$$= \frac{N^{ins}}{N^{not}} \times y^{(2)} + \pi \left[S - \frac{N^{ins}}{N^{not}} \times I \right]$$

$$= \frac{N^{ins}}{N^{not}} \times y^{(2)} + d \times e^{-2} \times \delta^P \left[S - \frac{N^{ins}}{N^{not}} \times I \right]$$

~~$\frac{N^{ins}}{N^{not}} \times y^{(2)}$~~

• $\text{Var}^P \left[S - \frac{N^{ins}}{N^{not}} \times I \right]$

$$= \text{Var}^P \left[S \times \left(1 - \frac{N^{ins}}{N^{not}} \right) - \frac{N^{ins}}{N^{not}} (I - S) \right]$$

$(S, I-S)$ are independent

$$= \text{Var}^P \left[S \times \left(1 - \frac{N^{ins}}{N^{not}} \right) \right] + \text{Var}^P \left[\frac{N^{ins}}{N^{not}} (I - S) \right]$$

$$= \left(\frac{N^{not} - N^{ins}}{N^{not}} \right)^2 \text{Var}^P[S] + \left(\frac{N^{ins}}{N^{not}} \right)^2 \text{Var}^P[I - S]$$

$$= \left(\frac{N^{not} - N^{ins}}{N^{not}} \right)^2 N^{ins} \times p \times (1-p)$$

$$+ \left(\frac{N^{ins}}{N^{not}} \right)^2 (N^{not} - N^{ins}) \times p \times (1-p)$$

$$= (N^{not} - N^{ins}) \times N^{ins} \times p \times (1-p) \left\{ \frac{N^{not} - N^{ins}}{(N^{not})^2} + \frac{N^{ins}}{(N^{not})^2} \right\}$$

$$= (N^{not} - N^{ins}) \times \frac{N^{ins}}{N^{not}} \times p \times (1-p)$$

• $\delta[S] = \frac{N^{ins}}{N^{not}} \times y^{(2)} + d \times e^{-2} \sqrt{\frac{N^{ins}}{N^{not}} \times (N^{not} - N^{ins}) \times p \times (1-p)}$

~~Exercise 14~~

- $\frac{S[S]}{N^{ins}} =$ premium per insured

$$= \frac{y^{(2)}}{N^{not}} + dxe^{-2} \sqrt{\frac{N^{not} - N^{ins}}{N^{not} \times N^{ins}} \times p \times (1-p)}$$

$$= \frac{y^{(2)}}{N^{not}} + dxe^{-2} \sqrt{\left(\frac{1}{N^{ins}} - \frac{1}{N^{not}}\right) \times p \times (1-p)}$$

- $\frac{S[S]}{N^{ins}}$ is a decreasing function of N^{ins} .

- $\lim_{N^{ins} \rightarrow N^{not}} \frac{S[S]}{N^{ins}} = \frac{y^{(2)}}{N^{not}}$

Hence, $N^{ins} = N^{not} \Rightarrow S[S] = y^{(2)}$

$\hookrightarrow$ in this case, $S = y^{(2)}$

EXERCISE 15

Q1: (1) conditional standard deviation principle

- π_Y is normalised:

$$\pi_Y[0] = e^{-2} \left(E^P[0|Y] + d \sigma^P[0|Y] \right) = 0$$
- π_Y is translation invariant:

$$\begin{aligned} \pi_Y[S+a] &= e^{-2} \left(E^P[S+a|Y] + d \sigma^P[S+a|Y] \right) \\ &= e^{-2} \left(E^P[S|Y] + a + d \sigma^P[S|Y] \right) \\ &= \pi_Y[S] + e^{-2} a \end{aligned}$$

• We have proven already that π_Y is a conditional valuation

• π_Y is MAC:

$$\begin{aligned} \pi_Y[S+S^q] &= e^{-2} \left\{ E^P[S+S^q|Y] + d \sigma^P[S+S^q|Y] \right\} \\ \text{sh} = \underline{Y} \cdot \underline{Y} &\rightarrow e^{-2} \left\{ E^P[S|Y] + S^q + d \sigma^P[S|Y] \right\} \\ &= \pi_Y[S] + e^{-2} \times \underline{Y} \cdot \underline{Y} \end{aligned}$$

• π_Y is MAC:

$$\begin{aligned} \pi_Y[S^\perp] &= e^{-2} \left\{ E^P[S^\perp|Y] + d e^{-2} \sigma^P[S^\perp|Y] \right\} \\ &= e^{-2} \left\{ E^P[S^\perp] + d e^{-2} \sigma^P[S^\perp] \right\} = \pi[S^\perp] \end{aligned}$$

with $\pi[S] = e^{-2} \left\{ E^P[S] + d e^{-2} \sigma^P[S] \right\}$, $\forall S \in C$
 $\hookrightarrow$ underlying actuarial valuation

Q1: (2) conditional cost-of-capital principle:

- π_Y is normalized:

$$\pi_Y [0] = e^{-2} \left\{ E^P [0 | Y] + i \left(\text{Var}_\pi^P [0 | Y] - E^P [0 | Y] \right) \right\} \\ = 0$$

- π_Y is translation invariant:

$$\pi_Y [S+a] = \left\{ e^{-2} E^P [S+a | Y] + i \left(\text{Var}^P [S+a | Y] - E^P [S+a | Y] \right) \right\} \\ = \pi_Y [S] + e^{-2} \cdot a$$

- π_Y is MAC:

$$\pi_Y [S+S^Q] = e^{-2} \left\{ E^P [S+S^Q | Y] + i \left(\text{Var}_\pi^P [S+S^Q | Y] - E^P [S+S^Q | Y] \right) \right\} \\ = e^{-2} \left\{ E^P [S | Y] + S^Q + i \left(\text{Var}_\pi^P [S | Y] + \cancel{S^Q} - E^P [S | Y] + \cancel{S^Q} \right) \right\} \\ \stackrel{S^Q = Y \cdot Y}{=} \pi_Y [S] + e^{-2} Y \cdot Y$$

- π_Y is MOC:

$$\pi_Y [S^\perp] = e^{-2} \left\{ E^P [S^\perp] + i \left(\text{Var}_\pi^P [S^\perp] - E^P [S^\perp] \right) \right\} \\ = \pi [S^\perp]$$

where $\pi [S] = e^{-2} \left\{ E^P [S] + i \left(\text{Var}_\pi^P [S] - E^P [S] \right) \right\}$
 $\hookrightarrow$ underlying actuarial valuation $\forall S \in C$

Q2: (1) conditional SD principle:

$$\begin{aligned}
 \pi_Y [f(Y) \times S^\perp] &= e^{-2} \left\{ E^P [f(Y) \times S^\perp | Y] + \alpha \sigma^P [f(Y) \times S^\perp | Y] \right\} \\
 f(Y) \geq 0 \quad \downarrow &= f(Y) \times e^{-2} \left\{ E^P [S^\perp] + \alpha \sigma^P [S^\perp] \right\} \\
 &= f(Y) \times \pi [S^\perp] \\
 &\text{with } \pi [S] = e^{-2} \left\{ E^P [S] + \alpha e^{-2} \sigma^P [S] \right\}, \forall S \in C
 \end{aligned}$$

(2) conditional CoC principle:

$$\begin{aligned}
 \pi_Y [f(Y) \times S^\perp] &= e^{-2} \left\{ E^P [f(Y) \times S^\perp | Y] + i \left(\text{Var}_\pi^P [f(Y) \times S^\perp | Y] - E^P [f(Y) \times S^\perp | Y] \right) \right\} \\
 &= f(Y) \times \left\{ e^{-2} E^P [S^\perp] + i \left(\text{Var}_\pi^P [S^\perp] - E^P [f(Y)] \right) \right\} \\
 &= f(Y) \times \pi [S^\perp]
 \end{aligned}$$

with ~~Var~~ $\pi =$

$$\pi [S] = e^{-2} \left\{ E^P [S] + i \left(\text{Var}_\pi^P [S] - E^P [S] \right) \right\} \quad \forall S \in C$$

Two-step valuations

► Theorem:

ρ is a **TS valuation** $\Leftrightarrow \rho$ is a **fair valuation**

↳ PROOF: SEE NEXT PAGE

► Exercise 16:

- Consider the MVHB valuation ρ :

$$\rho[S] = \theta_S^{MV} \cdot \mathbf{y} + \pi[S - \theta_S^{MV} \cdot \mathbf{Y}] \quad \text{for any } S \in \mathcal{C}$$

- Q: Show that ρ can be expressed as a TS valuation:

$$\rho[S] = \mathbb{E}^Q[\pi_{\mathbf{Y}}[S]] \quad \text{for any } S \in \mathcal{C}$$

with

$$\pi_{\mathbf{Y}}[S] = \left(\theta_S^{MV} + (\pi[S - \theta_S \cdot \mathbf{Y}], 0, \dots, 0) \right) \cdot \mathbf{Y}$$

PROOF OF THEOREM:

(a) $\mathcal{P}$ is a TS valuation $\Rightarrow \mathcal{P}$ is a fair valuation

PROOF: . Consider the TS valuation

$$\mathcal{P}[S] = E^{\mathbb{Q}}[\pi_Y(S)] \quad , \quad \forall S \in C^q$$

. $\mathcal{P}$ is a MAC valuation:

$$\begin{aligned} \text{Indeed, } \mathcal{P}[S + S^q] &= E^{\mathbb{Q}}[\pi_Y(S + S^q)] \\ &\stackrel{\pi_Y \text{ is MAC}}{=} E^{\mathbb{Q}}[\pi_Y(S) + e^{-2} \cdot \underline{Y} \cdot \underline{Y}] \quad \begin{matrix} S^q = \underline{Y} \cdot \underline{Y} \end{matrix} \\ &= E^{\mathbb{Q}}[\pi_Y(S)] + e^{-2} E^{\mathbb{Q}}[\underline{Y} \cdot \underline{Y}] \\ &= \mathcal{P}[S] + \underline{Y} \cdot \underline{Y} \end{aligned}$$

. $\mathcal{P}$ is a MOC valuation:

$$\mathcal{P}[S^{\perp}] = E^{\mathbb{Q}}[\pi_Y(S^{\perp})] \stackrel{\pi_Y \text{ is MOC}}{=} E^{\mathbb{Q}}[\pi[S^{\perp}]] \stackrel{\text{underlying act. val.}}{=} \pi[S^{\perp}]$$

. Conclusion: $\mathcal{P}$ is a fair valuation $\square$

(b) $\mathcal{P}$ is a fair valuation $\Rightarrow \mathcal{P}$ is a TS valuation

PROOF: . Consider a fair valuation $\mathcal{P}$.

. $\mathcal{P}$ can be expressed as

$$\mathcal{P}[S] = \frac{\partial f}{\partial S} \cdot \underline{Y} \stackrel{\text{see Theorem}}{=} e^{-2} E^{\mathbb{Q}}\left[\frac{\partial f}{\partial S} \cdot \underline{Y}\right]$$

$\hookrightarrow$ fair hedger

$$= E^{\mathbb{Q}}[\pi_Y[S]] \text{ with } \pi_Y[S] = \frac{\partial f}{\partial S} \cdot \underline{Y}$$

. Conclusion: $\mathcal{P}$ is a TS valuation $\square$

EXERCISE 16

→ MOC valuation

$$\underline{Q} : \mathcal{S}[S] = \underline{\Theta}_S^{MV} \cdot \underline{Y} + \pi [S - \underline{\Theta}_S^{MV} \cdot \underline{Y}]$$

↳ fair hedger

$$\begin{aligned} &= e^{-2} E^{\mathbb{Q}} [\underline{\Theta}_S^{MV} \cdot \underline{Y}] \\ &\quad + e^{-2} E^{\mathbb{Q}} [\left(\pi [S - \underline{\Theta}_S^{MV} \cdot \underline{Y}], 0, \dots, 0 \right) \cdot \underline{Y}] \\ &= e^{-2} E^{\mathbb{Q}} \left[\left[\underline{\Theta}_S^{MV} + \left(\pi [S - \underline{\Theta}_S^{MV} \cdot \underline{Y}], 0, \dots, 0 \right) \right] \cdot \underline{Y} \right] \\ &= e^{-2} E^{\mathbb{Q}} [\underline{\mu}_S \cdot \underline{Y}] \end{aligned}$$

↳ fair hedger (see Lemma)

$$= E^{\mathbb{Q}} [e^{-2} \underline{\mu}_S \cdot \underline{Y}]$$

- We have that $e^{-2} \underline{\mu}_S \cdot \underline{Y}$ is a fair conditional valuation.
- Hence, $\mathcal{S}$ is TS valuation:

$$\mathcal{S}[S] = E^{\mathbb{Q}} [\pi_Y[S]]$$

with fair conditional valuation π_Y defined by

$$\pi_Y[S] = \left[\underline{\Theta}_S^{MV} + \left(\pi [S - \underline{\Theta}_S^{MV} \cdot \underline{Y}], 0, \dots, 0 \right) \right] \cdot \underline{Y}$$

EXERCISE 17

Q: • Premium = $g[S^d \times S^u]$

$$= E^Q[\pi_y[S^d \times S^u]]$$

$$= E^Q[S^d] \times \pi[S^u]$$

• L = loss at time 1

= claim payments - invested premiums

$$= S^d \times S^u - \frac{E^Q[S^d] \times \pi[S^u]}{e^{-r} E^Q[S^d]} \times S^d$$

units bought
at time 0

$$S^d > 0 \Rightarrow e^{-r} E^Q[S^d] > 0$$

$$= S^d \times \{S^u - e^r \pi[S^u]\}$$

• $\boxed{P[L \leq 0]} = \frac{P[\cancel{S^d} \times \{S^u - e^r \pi[S^u]\} \leq 0]}{= \boxed{P[S^u \leq e^r \pi[S^u]]}}$

$L \rightarrow$ depends on π and r

- does not depend on financial market performance!

EXERCISE 18

Q1: • underlying actuarial valuation of TSS valuation:

$$\pi[S] = e^{-2} \left(E^P[S] + d \sigma^P[S] \right), \quad \forall S \in \mathbb{C}$$

$$\begin{aligned} \boxed{\frac{S[S^Q \times S^L]}{B-S \text{ for } TS \text{ valuation}}} &= \frac{E^Q[S^Q] \times \pi[S^L]}{e^{-2} E^Q[S^Q] \times (E^P[S^L] + d \sigma^P[S^L])} \\ &= e^{-2} E^Q[S^Q] \times (E^P[S^L] + d \sigma^P[S^L]) \end{aligned}$$

Q2: • $E^P[S^L] = N \times E^P[X_1^L]$

• $\text{Var}^P[S^L] = N \times \text{Var}^P[X_1^L]$

$$\boxed{\frac{1}{N} S[S^Q \times S^L] = e^{-2} E^Q[S^Q] \times \left(E^P[X_1^L] + \frac{d}{\sqrt{N}} \sigma^P[X_1^L] \right)}$$

Q3: $\boxed{P[L \leq 0]} = P\left[S^Q \times S^L \leq \frac{S[S^Q \times S^L]}{e^{-2} E^Q[S^Q]} \times S^Q\right]$

Exercise 17

$$= P[S^L \leq e^{-2} \pi[S^L]]$$

$$= P[S^L \leq E^P[S^L] + d \times \sigma^P[S^L]]$$

$$= \boxed{P\left[\frac{S^L - E^P[S^L]}{\sigma^P[S^L]} \leq d\right]}$$

REMARK: $P[L \leq 0] \approx \Phi[d] = \begin{cases} 84,13\% & : d=1 \\ 84,42\% & : d=2 \\ 88,87\% & : d=3 \end{cases}$

normal approximation

EXERCISE 18

Q1: $S^L = \max(Y^{(1)}, K) = Y^{(1)} + Y^{(2)}$

Ex. 18-Q2

$$\frac{1}{N} \mathbb{E}[S^L \times S^L] = e^{-2} \mathbb{E}^Q[S^L] \times \left\{ \mathbb{E}^P[X_1^L] + \frac{\alpha}{\sqrt{N}} \sigma^P[X_1^L] \right\}$$

$$= (Y^{(1)} + Y^{(2)}) \times \left(\mu + \frac{\alpha}{\sqrt{N}} \sqrt{\mu(1-\mu)} \right)$$

Q2: $\mathbb{P}[L \leq 0] \stackrel{\text{Ex. 18-Q3}}{=} \mathbb{P}\left[\frac{S^L - \mathbb{E}^P[S^L]}{\sigma^P[S^L]} \leq \alpha\right] = \Phi(d)$

Q3: $K = 0 \Rightarrow S^L = \max(Y^{(1)}, 0) = Y^{(1)}$

$\mathbb{E}[S^L \times S^L] = \mathbb{E}[Y^{(1)} \times S^L]$

Ex 18-Q1

$= Y^{(1)} \times (\mathbb{E}^P[S^L] + \alpha \sigma^P[S^L])$

$\mathbb{P}[L \leq 0] = \mathbb{P}\left[Y^{(1)} \times S^L \leq \frac{\mathbb{E}[Y^{(1)} \times S^L]}{Y^{(1)}} \times Y^{(1)}\right]$

$= \mathbb{P}\left[Y^{(1)} \times S^L \leq \frac{Y^{(1)} \times (\mathbb{E}^P[S^L] + \alpha \sigma^P[S^L])}{Y^{(1)}} \times Y^{(1)}\right]$

$= \mathbb{P}\left[Y^{(1)} \times X \leq \alpha \times Y^{(1)}\right] \text{ with } X = \frac{S^L - \mathbb{E}^P[S^L]}{\sigma^P[S^L]}$

$= \mathbb{P}[Y^{(1)} \times X \leq \alpha \times Y^{(1)} | Y^{(1)} = 0] \times \mathbb{P}[Y^{(1)} = 0]$

$+ \mathbb{P}[Y^{(1)} \times X \leq \alpha \times Y^{(1)} | Y^{(1)} > 0] \times \mathbb{P}[Y^{(1)} > 0]$

$= \mathbb{P}[Y^{(1)} = 0] + \mathbb{P}[X \leq \alpha | Y^{(1)} > 0] \times \mathbb{P}[Y^{(1)} > 0]$

$= \mathbb{P}[Y^{(1)} = 0] + \mathbb{P}\left[\frac{S^L - \mathbb{E}^P[S^L]}{\sigma^P[S^L]} \leq \alpha\right] \times \mathbb{P}[Y^{(1)} > 0]$

$= \mathbb{P}[Y^{(1)} = 0] + \Phi[d] \times \mathbb{P}[Y^{(1)} > 0]$

EXERCISE 20

Q1: • $E^P[S|Y] = E^P[f(Y) \times S_N^L | Y] = f(Y) \times E^P[S_N^L]$

• $Var^P[S|Y] = Var^P[f(Y) \times S_N^L | Y]$

$$= (f(Y))^2 \times Var^P[S_N^L]$$

$$= (f(Y))^2 \times \left\{ E^P[Var^P[S_N^L | Z]] + Var^P[E^P[S_N^L | Z]] \right\}$$

$$= (f(Y))^2 \times \left\{ E^P[N \times Var^P[X_1 | Z]] + Var^P[N \times E^P[X_1 | Z]] \right\}$$

$$= (f(Y))^2 \times \left\{ N \times E^P[Var[X_1 | Z]] + N^2 \times Var^P[E^P[X_1 | Z]] \right\}$$

$$= (f(Y))^2 \times \{ N \times A + N^2 \times B \}$$

• $S[f(Y) \times S_N^L] = \cancel{e^{-2} E^Q[f(Y) \times S_N^L]}$

$$= e^{-2} E^Q[E^P[S|Y]] + \alpha \sigma^P[S|Y]$$

$$= e^{-2} E^Q \left[f(Y) \times E^P[S_N^L] + \alpha \times f(Y) \sqrt{N \times A + N^2 \times B} \right]$$

$$= e^{-2} E^Q[f(Y)] \times \left\{ E^P[S_N^L] + \alpha \sqrt{N \times A + N^2 \times B} \right\}$$

$$\frac{S[f(Y) \times S_N^L]}{N} = e^{-2} E^Q[f(Y)] \times \left\{ E^P[X_1^*] + \alpha \sqrt{\frac{A}{N} + B} \right\}$$

Q2:

• From Exercise 3', we find:

$$\frac{8[S^h \times S_N^L]}{N} = Y^h + Y_N^d + Y^2$$

with:

$$\begin{cases} \bullet E^Q[Y^h] = E^Q[S^h] \times E^P[X_1] \\ \bullet E^P[(Y^h)^2] = E^P[(S^h)^2] \times (E^P[X_1])^2 \\ \bullet \text{Var}^P[Y_N^d] = \frac{A}{N} \times E^P[(S^h)^2] \\ \bullet \text{Var}^P[Y^2] = B \times E^P[(S^h)^2] \end{cases}$$

$$\begin{aligned} \bullet \frac{8[S^h \times S_N^L]}{N} &= e^{-2} E^Q[S^h] \times \left\{ E^P[X_1] + \alpha \sqrt{\frac{A}{N} + B} \right\} \\ &= e^{-2} \left\{ E^Q[Y^h] + E^Q[S^h] \times \alpha \sqrt{\frac{\text{Var}^P[Y_N^d] + \text{Var}^P[Y^2]}{E^P[(S^h)^2]}} \right\} \end{aligned}$$

$$= e^{-2} E^Q[Y^h] \left\{ 1 + \frac{\alpha \cancel{E^P[X_1]} \times \sqrt{\frac{\text{Var}^P[Y_N^d] + \text{Var}^P[Y^2]}{E^P[(S^h)^2]}}}{\cancel{E^P[X_1]}} \right\}$$

$$= e^{-2} E^Q[\cancel{Y^h}] \left\{ 1 + \alpha \sqrt{\frac{\text{Var}^P[\cancel{Y_N^d}] + \text{Var}^P[\cancel{Y^2}]}{E^P[(\cancel{Y^h})^2]}} \right\}$$

EXERCISE 21-1

Q1: • $f(Y) = f(1, Y^{(1)}) = \begin{cases} f(1, 0) & : Y^{(1)} = 0 \\ f(1, 1) & : Y^{(1)} = 1 \end{cases}$

• Hedge for $f(Y)$:

$$\underline{V} \cdot \underline{Y} = f(Y)$$

$$\Leftrightarrow V^{(0)} \times 1 + Y^{(1)} \times Y^{(1)} = f(Y)$$

$$\Leftrightarrow \begin{cases} V^{(0)} + Y^{(1)} \times 0 = f(1, 0) & : Y^{(1)} = 0 \\ V^{(0)} + Y^{(1)} \times 1 = f(1, 1) & : Y^{(1)} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} V^{(0)} = f(1, 0) \\ Y^{(1)} = f(1, 1) - f(1, 0) \end{cases}$$

Hence, $f(Y) = f(1, 0) \times Y^{(0)} + (f(1, 1) - f(1, 0)) \times Y^{(1)}$

Q2: • TS hedge $\underline{\theta}_S^{TS}$ of $\pi_Y[S]$:

$$\underline{\theta}_S^{TS} \cdot \underline{Y} = \pi_Y[S]$$

$$\Leftrightarrow \theta_S^{(0)} \times 1 + \theta_S^{(1)} \times Y^{(1)} = \pi_Y[(1-I)(1-Y^{(1)})]$$

$$\Leftrightarrow \begin{cases} \theta_S^{(0)} + \theta_S^{(1)} \times 0 = \pi_Y[1-I] & : Y^{(1)} = 0 \\ \theta_S^{(0)} + \theta_S^{(1)} \times 1 = \pi_Y[0] & : Y^{(1)} = 1 \end{cases}$$

$\in \mathbb{C}^L$, see Ex. 13-1-Q1

$$\Leftrightarrow \begin{cases} \theta_S^{(0)} = \pi[1-I] \\ \theta_S^{(1)} = -\pi[1-I] \end{cases} \Leftrightarrow \underline{\theta}_S^{TS} = (\pi[1-I]; -\pi[1-I])$$

$$\begin{aligned}
 \bullet \quad \pi_Y[S] &= \theta_S^{TS} \cdot Y \\
 &= (\pi[1-I]; -\pi[1-I]) \cdot (1, Y^{(1)}) \\
 &= \pi[1-I] \times (1 - Y^{(1)})
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad S[S] &= E^Q[\pi_Y[S]] \\
 &= E^Q[\pi[1-I] \times (1 - Y^{(1)})] \\
 &= \pi[1-I] \times E^Q[1 - Y^{(1)}] \\
 &= \pi[1-I] \times (1 - E^Q[Y^{(1)}]) \\
 &= \pi[1-I] \times (1 - \frac{1}{2}) \\
 &= \frac{1}{2} \pi[1-I]
 \end{aligned}$$

$\hookrightarrow$ We do not need to choose Q because any $f(X)$ is hedgeable. Hence, $\pi_Y[S]$ is hedgeable and $S[S]$ is the price of the hedge

Exercise 21-1

Q3 → • $S[S] = e^{-2} E^Q \left[E^P[S|Y] + i \left(\text{Var}_P^P[S|Y] - E^P[S|Y] \right) \right]$

with underlying actuarial valuation:

$$\pi[X] = e^{-2} \left(E^P[X] + i \left(\text{Var}_P^P[X] - E^P[X] \right) \right) \quad \forall X \in C^+$$

• $S[S] = S[(1-I) \cdot (1-Y^{(1)})]$

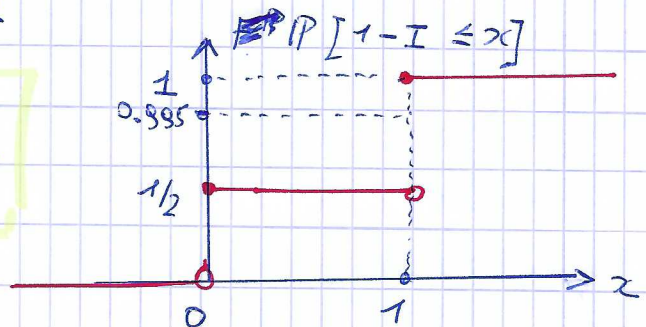
Q2 $= \frac{1}{2} \pi[1-I]$

$$= \frac{1}{2} \left\{ E^P[1-I] + 0.06 \left(\text{Var}_{0.995}^P[1-I] - E^P[1-I] \right) \right\}$$

• $1-I = \begin{cases} 0 & : 1.1 = \frac{1}{2} \\ 1 & : 1.0 = \frac{1}{2} \end{cases}$

$$\Rightarrow E^P[1-I] = \frac{1}{2}$$

• $\text{Var}_{0.995}^P[1-I] = 1$



$$S[S] = \frac{1}{2} \left\{ \frac{1}{2} + 0.06 \left(1 - \frac{1}{2} \right) \right\} = \frac{53}{200}$$

EXERCISE 21-2

Q1: $Q = (q_{00}, q_{01}, q_{10}, q_{11})$ is EMM

$$\Leftrightarrow q_{00} > 0, q_{01} > 0, q_{10} > 0, q_{11} > 0$$

$$\text{such that } \begin{cases} E^Q[Y^{(0)}] = 1 \\ E^Q[Y^{(1)}] = 1/2 \\ E^Q[Y^{(2)}] = 2/3 \end{cases}$$

$$\Leftrightarrow q_{00} > 0, q_{10} > 0, q_{01} > 0, q_{11} > 0$$

$$\text{such that } \begin{cases} q_{00} + q_{01} + q_{10} + q_{11} = 1 \\ 0 \times q_{00} + 1 \times q_{01} = 1/2 \\ 0 \times q_{00} + 1 \times q_{01} = 2/3 \end{cases}$$

$$\Leftrightarrow q_{00} > 0, \dots, q_{11} > 0$$

$$\text{such that } \begin{cases} q_{00} + q_{10} = 1 \\ q_{01} = 1/2 \\ q_{11} = 2/3 \end{cases}$$

$$\Leftrightarrow q_{00} > 0, q_{01} > 0, q_{10} > 0, q_{11} > 0$$

$$\text{such that } \begin{cases} q_{00} = 1/2 \\ q_{10} = 1/2 \\ q_{01} = 1/3 \\ q_{11} = 2/3 \end{cases}$$

Exercise 21-2

$$\Leftrightarrow q_{00} > 0, q_{01} > 0, q_{10} > 0, q_{11} > 0$$

$$\text{such that } \begin{cases} q_{00} = q \\ q_{01} = q_{0\cdot} - q_{00} = \frac{1}{2} - q \\ q_{10} = q_{\cdot 0} - q_{00} = \frac{1}{3} - q \\ q_{11} = q_{1\cdot} - q_{10} = \frac{2}{3} - \left(\frac{1}{3} - q\right) = \frac{1}{3} + q \end{cases}$$

$$\Leftrightarrow \begin{cases} q_{00} = q \\ q_{01} = \frac{1}{2} - q \\ q_{10} = \frac{1}{3} - q \\ q_{11} = \frac{1}{3} + q \end{cases} \quad \text{for some } q \in \left(0, \frac{1}{3}\right)$$

Q2: $S = (1-I) \times (1-Y'') = f(Y) \Rightarrow S \in C^1$

$\hookrightarrow S$ is a derivative

$E^Q[S] = E^Q[(1-I) \times (1-Y'')]$

$= q_{00} \Rightarrow S \notin C^1$

$\hookrightarrow S$ is not hedgeable
bc. $E^Q[S]$ depends on Q

Q3: From the B-S formula for TS valuations we have

$$S[f(Y)] = e^{-2} E^Q[f(Y)] \quad (*)$$

(provided $\pi_Y[f(Y)] = e^{-2} f(Y)$)

This result holds for the TS CoC valuation
 $\hookrightarrow$ if $f(Y) \geq 0$

Hence, $S[(1-I)(1-Y'')]$

$= E^Q[(1-I) \times (1-Y'')]$

$= q_{00} \in (0, \frac{1}{3})$

$\hookrightarrow$ depends on choice of EMM

Exercise 21-2

Q4

- In this market, we have from Ex 21-1-Q1 that $\mathbb{Q}$ is an EMM

$$\Leftrightarrow q_{0,\cdot} = \frac{1}{2}, \quad q_{1,\cdot} = \frac{1}{2}, \quad q_{\cdot,0} = \frac{1}{3}, \quad q_{\cdot,1} = \frac{2}{3}$$

$\hookrightarrow$ Only the coeff's of $y^{(1)}$ and I are fixed, no condition on dependency structure.

- Let $\mathbb{Q}^\perp$ be the EMM where $y^{(1)}$ and I are $\mathbb{Q}$ -independent.

- Then $q_{\infty}^\perp = q_{0,\cdot}^\perp \times q_{\cdot,0}^\perp = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

- In this case, $\boxed{p[S]} = q_{\infty}^\perp = \boxed{\frac{1}{6}}$

EXERCISE 21-3

Q1: $\mathbb{Q}$ is EMM $\Leftrightarrow \begin{cases} q_0 > 0, q_{01} > 0, q_{10} > 0, q_{11} > 0 \\ E^{\mathbb{Q}}[Y^{(m)}] = Y^{(m)}; m=0,1,2,3 \end{cases}$

$\Leftrightarrow \begin{cases} q_0 = q \\ q_{10} = 1/3 - q \\ q_{01} = 1/2 - q \\ q_{11} = 1/6 + q \\ E^{\mathbb{Q}}[Y^{(3)}] = 1 \end{cases} \text{ for some } q \in (0, 1/3)$

$\Leftrightarrow \begin{cases} q_0 = q \\ q_{10} = 1/3 - q \\ q_{01} = 1/2 - q \\ q_{11} = 1/6 + q \\ q_{11} \times \frac{1}{2} = \frac{1}{6} \end{cases} \text{ for some } q \in (0, 1/3)$

$\Leftrightarrow \begin{cases} q_0 = 1/6 \\ q_{10} = 1/6 \\ q_{01} = 1/2 - q = 1/3 \\ q_{11} = 1/3 \end{cases}$

• There is only 1 EMM

• Hence, the market is complete in $(\Omega, \mathcal{F}, \mathbb{P})$

Exercise 21-3

Q2: • $\underline{y} \cdot \underline{y} = (1 - I) \times (1 - Y^{(1)})$

$$\Leftrightarrow y^{(0)} \times 1 + y^{(1)} \times y^{(1)} + y^{(2)} \times y^{(2)} + y^{(3)} \times y^{(3)} = (1 - I) \times (1 - y^{(1)})$$

$$\Leftrightarrow \begin{cases} y^{(0)} = 1 \\ y^{(0)} + y^{(2)} = 0 \\ y^{(0)} + y^{(1)} = 0 \\ y^{(0)} + y^{(1)} + y^{(2)} + \frac{y^{(3)}}{2} = 0 \end{cases} \begin{matrix} : (0, 0) \\ : (0, 1) \\ : (-1, 0) \\ : (-1, 1) \end{matrix}$$

$$\Leftrightarrow \begin{cases} y^{(0)} = 1 \\ y^{(1)} = -1 \\ y^{(2)} = -1 \\ y^{(3)} = 2 \end{cases}$$

• Hence, $(1 - I) \times (1 - Y^{(1)}) = \underline{y} \cdot \underline{y}$ with

$$\underline{y} = (1, -1, -1, 2)$$

Q3: • For any hedgeable claim, we have

$$S[S^Q] = e^{-\frac{1}{2}} E^{\mathbb{Q}}[S^Q]$$

$$\bullet \underline{S[(1 - I) \times (1 - Y^{(1)})]}$$

$$= E^{\mathbb{Q}}[(1, -1, -1, 2) \cdot \underline{y}]$$

$$= E^{\mathbb{Q}}[1 - y^{(1)} - y^{(2)} + 2 y^{(3)}]$$

$$= 1 - \frac{1}{2} - \frac{2}{3} + 2 \times \frac{1}{6}$$

$$= 1 - \frac{1}{2} - \frac{2}{3} + \frac{2}{6}$$

$$= \frac{1}{6}$$

EXERCISE 22

Q1: $E^P[S|I] = \frac{N^{ins}}{N^{not}} I$

$Var^P[S|I] = \frac{N^{ins} \times (N^{not} - N^{ins})}{N^{not} \times (N^{not} - 1)} \times I \times \left(1 - \frac{I}{N^{not}}\right)$

↳ For a proof of these 2 expressions, see end of this exercise

$S[S] = e^{-\lambda} E^Q \left[E^P[S|Y] + \lambda \sigma^P[S|Y] \right]$

$\cdot Y = (e^{\lambda}, \mathbb{I}_{Y^{(1)}}, I)$

$\cdot S \perp Y^{(1)} \text{ (under } P)$

$= e^{-\lambda} E^Q \left[E^P[S|I] + \lambda \sigma^P[S|I] \right]$

~~$= e^{-\lambda} E^Q \left[E^P[S|I] \right]$~~

$= e^{-\lambda} E^Q \left[\frac{N^{ins}}{N^{not}} I \right]$

$+ \lambda e^{-\lambda} E^Q \left[\sqrt{\frac{N^{ins} \times (N^{not} - N^{ins})}{N^{not} \times (N^{not} - 1)}} \times I \times \left(1 - \frac{I}{N^{not}}\right) \right]$

$\frac{S[S]}{N^{ins}} = \frac{Y^{(2)}}{N^{not}} + \lambda e^{-\lambda} E^Q \left[\sqrt{\left(\frac{1}{N^{ins}} - \frac{1}{N^{not}}\right) \times \frac{I}{N^{not} - 1} \times \left(1 - \frac{I}{N^{not}}\right)} \right]$

Q2:

$$Q \text{ is an EMM} \Leftrightarrow \begin{cases} Q \sim P \\ e^{-\lambda} E^Q [Y^{(m)}] = f^{(m)}; m=0,1,2 \end{cases}$$

$$\Leftrightarrow \begin{cases} Q \sim P \\ \sum_{x_1, x_2, x_3} Q[Y^{(1)} = x_1, I = x_2, S = x_3] = 1 \\ e^{-\lambda} \sum_{\substack{x_1 \in A \\ N \text{ not}}} Q[Y^{(1)} = x_1] = f^{(1)} \\ \sum_{x_2=0} x_2 \times Q[I = x_2] = f^{(2)} \end{cases}$$

Q3: • $S[S] = \dots + d e^{-\lambda} E^Q [\sqrt{aI + bI^2}]$

for $a = \dots$ and $b = \dots$

• $\sqrt{aI + bI^2}$ is NOT hedgeable

$\Rightarrow S[S]$ depends on choice of Q .

Q4:

• Suppose all I_i are $\mathbb{Q}$ -i.i.d.

• Then:
$$e^{-\lambda} E^{\mathbb{Q}} [Y^{(2)}] = \lambda$$

$$\Leftrightarrow \sum_{i=1}^{N^{\text{tot}}} E^{\mathbb{Q}} [I_i] = e^{\lambda} \lambda$$

$$\Leftrightarrow E^{\mathbb{Q}} [I_i] = \frac{e^{\lambda} \lambda}{N^{\text{tot}}}$$

$$\Leftrightarrow \mathbb{Q} [I_i = 1] = \frac{e^{\lambda} \lambda}{N^{\text{tot}}}$$

REMARK: we do not
the $\mathbb{Q}$ -independence
assumption, identical
distribution under
 $\mathbb{Q}$ is sufficient

~~• $Y^{(2)} \sim \text{Bin}(N^{\text{tot}}, \mathbb{Q} [I_i = 1])$~~

• $Y^{(2)} \sim \text{Bin}(N^{\text{tot}}, \mathbb{Q} [I_i = 1])$

$$\Rightarrow \mathbb{Q} [Y^{(2)} = x_2] = \binom{N^{\text{tot}}}{x_2} \left(\frac{e^{\lambda} \lambda}{N^{\text{tot}}} \right)^{x_2} \left(1 - \frac{e^{\lambda} \lambda}{N^{\text{tot}}} \right)^{N^{\text{tot}} - x_2}$$

$\downarrow$
 $(x_2 = 0, 1, \dots, N^{\text{tot}})$

$\hookrightarrow$ Here, we need $\mathbb{Q}$ -independence
assumption

• $S \sim \text{Bin}(N^{\text{tot}}, \mathbb{Q} [I_i = 1])$

$\Rightarrow \mathbb{Q} [S = s] = \dots$

Q5:

- $S[\xi]$ is uniquely determined because the colf of $\gamma^{(2)}$ is fixed
- Notice however that $\mathbb{Q}$ is not uniquely determined. E.g., the $\mathbb{Q}$ -distribution of $\gamma^{(2)}$ is not fixed!

NOTATIONS:

~~Exercice~~

Exercice 22

- $m = N^{(w)}$
- $n = N^{\text{tot}}$
- $V^P = \text{Var}^P$

We want to prove that

$$\mathbb{E}^P[S | I] = \frac{m}{n} I.$$

$$V^P[S | I] = \frac{m(n-m)}{n(n-1)} I \left(1 - \frac{1}{n} I\right).$$

Proof. By symmetry,

$$\begin{aligned} I &= \mathbb{E}^P[I | I] \\ &= \mathbb{E}^P[I_1 + I_2 + \dots + I_n | I] \\ &= n \mathbb{E}^P[I_1 | I]. \end{aligned}$$

So, we have

$$\mathbb{E}^P[I_1 | I] = \frac{I}{n},$$

and

$$\mathbb{E}^P[S | I] = m \mathbb{E}^P[I_1 | I] = \frac{m}{n} I.$$

Again using symmetry

$$\begin{aligned} I^2 &= \mathbb{E}^P[I^2 | I] \\ &= n \mathbb{E}^P[I_1^2 | I] + n(n-1) \mathbb{E}^P[I_1 I_2 | I] \\ &= I + n(n-1) \mathbb{E}^P[I_1 I_2 | I]. \end{aligned}$$

Hence, we have

$$\mathbb{E}^P[I_1 I_2 | I] = \frac{1}{n(n-1)} I(I-1).$$

So,

$$\begin{aligned} V^P[S | I] &= \mathbb{E}^P[S^2 | I] - (\mathbb{E}^P[S | I])^2 \\ &= \mathbb{E}^P[(I_1 + I_2 + \dots + I_m)^2 | I] - \frac{m^2}{n^2} I^2 \\ &= m \mathbb{E}^P[I_1^2 | I] + m(m-1) \mathbb{E}^P[I_1 I_2 | I] - \frac{m^2}{n^2} I^2 \\ &= \frac{m}{n} I + \frac{m(m-1)}{n(n-1)} I(I-1) - \frac{m^2}{n^2} I^2 \\ &= \frac{m}{n} I \left(1 + \frac{m-1}{n-1} (I-1) - \frac{m}{n} I\right) \\ &= \frac{m}{n} I \left(\frac{n-m}{n-1} - \frac{n-m}{n(n-1)} I\right) \\ &= \frac{m(n-m)}{n(n-1)} I \left(1 - \frac{1}{n} I\right). \end{aligned}$$

EXERCISE 23

Q1: PROOF:

(1) $\mathcal{P}$ is a valuation:

$$\bullet \mathcal{P}[0] = \underline{\Theta}_0^{MV} \cdot \gamma + d e^{-2} E^Q [\sigma^P[0|Y]]$$

↓

normalised

$$= \underline{0} \cdot \gamma + d e^{-2} E^Q [0] = 0$$

$$\bullet \mathcal{P}[S+a] = \underline{\Theta}_{S+a}^{MV} \cdot \gamma + d e^{-2} E^Q [\sigma^P[S+a|Y]]$$

↓

translation
invariant

$$= \left(\underline{\Theta}_S^{MV} + (e^{-2}a, 0, \dots, 0) \right) \cdot \gamma + d e^{-2} E^Q [\sigma^P[S|Y]]$$

$$= \mathcal{P}[S] + e^{-2} \cdot a$$

(2) $\mathcal{P}$ is a MAC valuation:

$$\mathcal{P}[S+S^A] = \underline{\Theta}_{S+S^A}^{MV} \cdot \gamma + d e^{-2} E^Q [\sigma^P[S+S^A|Y]]$$

↓

$S^A = \mu \cdot \gamma$

$$= \left(\underline{\Theta}_S^{MV} + \mu \right) \cdot \gamma + d e^{-2} E^Q [\sigma^P[S|Y]]$$

$$= \mathcal{P}[S] + \mu \cdot \gamma$$

(3) $\mathcal{P}$ is a MOC valuation:

$$\mathcal{P}[S^\perp] = \underline{\Theta}_{S^\perp}^{MV} \cdot \gamma + d e^{-2} E^Q [\sigma^P[S^\perp|Y]]$$

$S^\perp \in \mathcal{C}^\perp$

$$= (e^{-2} E^P[S^\perp], 0, \dots, 0) \cdot \gamma + d e^{-2} \sigma^P[S^\perp]$$

$$= e^{-2} \left\{ E^P[S^\perp] + d \sigma^P[S^\perp] \right\}$$

$$= e^{-2} \pi[S^\perp]$$

with $\pi[S] = e^{-2} \left\{ E^P[S] + d \sigma^P[S] \right\}$, $\forall S \in \mathcal{C}$
 $\hookrightarrow$ actuarial valuation

Q2: $\underline{Q[S]} = \underline{\theta}_S^{MV} \cdot \underline{f} + d e^{-2} E^Q [\sigma^P [S | Y]]$

$$= \underline{\theta}_S^{MV} \cdot \underline{f} + e^{-2} E^P [S - \underline{\theta}_S^{MV} \cdot \underline{Y}] + d e^{-2} E^Q [\sigma^P [S - \underline{\theta}_S^{MV} \cdot \underline{Y} | Y]]$$

$$= \underline{\theta}_S^{MV} \cdot \underline{f} + \pi [S - \underline{\theta}_S^{MV} \cdot \underline{Y}]$$

with $\pi [X] = e^{-2} E^P [X] + d e^{-2} E^Q [\sigma^P [X | Y]]$
 $\hookrightarrow$ This is a ~~continuous~~ valuation

• Still to prove: π is a MOC valuation

$$\pi [S^\perp] = e^{-2} E^P [S^\perp] + d e^{-2} \sigma^P [S^\perp], \quad S^\perp \in C^\perp$$

$\hookrightarrow$ Hence, in $C^\perp$, π reduces to an actuarial valuation $\Rightarrow \pi$ is MOC valuation

• Conclusion:

$\underline{Q}$ is a MVHB valuation with underlying MOC valuation π as defined above.

Q3: $S[S]$

$$\begin{aligned}
 S[S] &= \frac{\Theta^{MV}}{S} \cdot Y + \alpha e^{-2} E^Q[\sigma^P[S|Y]] \\
 &= e^{-2} E^Q\left[\frac{\Theta^{MV}}{S} \cdot Y\right] + \alpha e^{-2} E^Q[\sigma^P[S|Y]] \\
 &= E^Q\left[e^{-2}\left(\frac{\Theta^{MV}}{S} \cdot Y + \alpha \sigma^P[S|Y]\right)\right] \\
 &= E^Q[\pi_Y[S]]
 \end{aligned}$$

with $\pi_Y[S] = e^{-2}\left(\frac{\Theta^{MV}}{S} \cdot Y + \alpha \sigma^P[S|Y]\right)$

• Still to prove: π_Y is a fair cond. valuation

(1) π_Y is a conditional valuation:

$$\hookrightarrow \pi_Y[0] = 0 \text{ and } \pi_Y[S + e] = \pi_Y[S] + e^{-2} \alpha$$

(2) π_Y is a MAC conditional valuation

$$\begin{aligned}
 \hookrightarrow \pi_Y[S + S^q] &= e^{-2}\left(\frac{\Theta^{MV}}{S + S^q} \cdot Y + \alpha \sigma^P[S + S^q | Y]\right) \\
 S^q &= \mu \cdot Y \quad \downarrow \\
 &= e^{-2}\left(\frac{\Theta^{MV}}{S} \cdot Y + \mu \cdot Y + \alpha \sigma^P[S|Y]\right) \\
 &= \pi_Y[S] + e^{-2} \mu \cdot Y
 \end{aligned}$$

(3) π_Y is a MOC conditional valuation:

$$\begin{aligned}
 \hookrightarrow \pi_Y[S^\perp] &= e^{-2}\left(\frac{\Theta^{MV}}{S^\perp} \cdot Y + \alpha \sigma^P[S^\perp | Y]\right) \\
 &= e^{-2}\left(E^P[S^\perp] + \alpha \sigma^P[S^\perp]\right) \\
 &= \pi[S^\perp]
 \end{aligned}$$

with $\pi[X] = e^{-2}\left(E^P[S] + \alpha \sigma^P[S]\right)$, $X \in C$
 $\hookrightarrow$ actuarial valuation

Q4:

$$\begin{aligned}
 \bullet \quad g[S^q \times S^q] &= \Theta_{S^q \times S^q}^{MV} \cdot \gamma + d e^{-2} E^Q [\sigma^P[S^q \times S^q | \gamma]] \\
 &\quad \text{with } S^q = \mu \cdot \gamma \quad \downarrow \\
 &= E^P[S^q] \times \mu \cdot \gamma + d e^{-2} E^Q[S^q \times \sigma^P[S^q]] \\
 &= E^P[S^q] \times \mu \cdot \gamma + d e^{-2} E^Q[S^q] \times \sigma^P[S^q] \\
 &= e^{-2} E^Q[S^q] \times \{E^P[S^q] + d \sigma^P[S^q]\}
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad g^{TS}[S^q \times S^q] &= e^{-2} E^Q[E^P[S^q \times S^q | \gamma]] \\
 &\quad + d e^{-2} E^Q[\sigma^P[S^q \times S^q | \gamma]] \\
 &= e^{-2} E^Q[S^q \times E^P[S^q]] \\
 &\quad + d e^{-2} E^Q[S^q \times \sigma^P[S^q]] \\
 &= e^{-2} E^Q[S^q] \times \{E^P[S^q] + d \sigma^P[S^q]\}
 \end{aligned}$$

Conclusion: $\boxed{g[S^q \times S^q] = g^{TS}[S^q \times S^q]}$

Q5:

$$\bullet \quad \mathcal{S}[S] = \mathcal{S}^{TS}[S], \quad \forall S \in \mathcal{C}$$

$$\Leftrightarrow \underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}} + e^{-2} E^{\mathcal{Q}} [\sigma^P[S|Y]]$$

$$= e^{-2} E^{\mathcal{Q}} [E^P[S|Y]] + e^{-2} E^{\mathcal{Q}} [\sigma^P[S|Y]], \quad \forall S \in \mathcal{C}$$

$$\Leftrightarrow \underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}} = e^{-2} E^{\mathcal{Q}} [E^P[S|Y]], \quad \forall S \in \mathcal{C}$$

$\bullet$ Suppose that $E^P[S|Y] \notin \mathcal{C}^A$

Then : $\bullet$ $\underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}}$ is independent of choice of $\mathcal{Q}$
 $\bullet$ $e^{-2} E^{\mathcal{Q}} [E^P[S|Y]]$ depends on choice of $\mathcal{Q}$

$\bullet$ Conclusion: In general, $\mathcal{S}[S] \neq \mathcal{S}^{TS}[S]$

Q6: $\bullet$ Suppose that any $f(Y)$ is measurable.
 Then, for any S we have

$$E^P[S|Y] \in \mathcal{C}^A$$

$$\bullet \text{ Hence, } E^P[S|Y] = \frac{\underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}}}{E^P[S|Y]}$$

Exercise 7 $\downarrow$

$$= \underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}}$$

$$\bullet \text{ Hence, } e^{-2} E^{\mathcal{Q}} [E^P[S|Y]]$$

$$= e^{-2} E^{\mathcal{Q}} [\underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}}] = \underline{\underline{\mathcal{Q}}}_S^{MV} \cdot \underline{\underline{y}}, \quad \forall S \in \mathcal{C}$$

$\bullet$ Conclusion: $\mathcal{S}[S] = \mathcal{S}^{TS}[S], \quad \forall S \in \mathcal{C}$

$\hookrightarrow$ in case any $f(Y)$ is measurable