Financial valuation

Arbitrage-free pricing, pandemics and longevity[1]

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How to determine the value of things?

- "Res tantum valet quantum vendi potest." (Latin quote)
  A thing is worth only what someone else will pay for it.
- Question to be answered in this chapter:
  What would be possible current prices for a future insurance-linked claim if it was introduced today in an arbitrage-free market of traded assets?

Main references

- **On the (in-)dependence between financial and actuarial risks**.
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Agenda

- 1. Insurance securitization.
- 2. The financial-actuarial world.
- 3. Equivalent martingale measures.
- 4. A special finite-state single period world.
- 5. Example 1 - Pandemic risk.
- 6. Example 2 - Longevity risk.
- 7. Some more exercises.
- 8. Some conclusions and final remarks.

1. Insurance securitization

Some definitions

- **Insurance risk**
  = risk to which an insurer is exposed, due to selling insurance policies.
- **Insurance-linked securities (ILS)**
  = traded financial securities with payoffs which are contingent on insurance risk.
- **Examples of ILS**:
  - Catastrophe bonds.
  - Longevity bonds.
  - (Survivor swaps).
- ILS are available only to sophisticated investors.
- In this chapter we will answer the following **question**:
  What is an appropriate/acceptable market price for an ILS when it is introduced in an arbitrage-free market?

Insurance securitization

Some definitions

- **De-risking**
  = transfer of insurance risk from the insurer to other parties.
- **Classical risk transfer**
  = de-risking via (re-)insurance.
- **Alternative risk transfer (ART)**
  = de-risking via techniques different from (re-)insurance.
- **Insurance securitization**:
  = transfer of insurance risk to capital markets, which takes place if the insurer (partially) hedges his insurance risk with the help of an ILS.

Insurance securitization

Some definitions

- **Long position**:
  You are long risk X if you **gain from an increase** of X.
  - Buying a stock is setting up a long position in that stock.
  - The pharma industry is long longevity[2].

- **Short position**:
  You are short risk X if you **gain from a decrease** of X.
  - Buying a put option is setting up a long position in the put, but a short position in the underlying stock.
  - A pension insurer is short longevity.
  - An earthquake insurer is short earthquake risk.

- In an **insurance securitization** context, the insurer (partially) hedges his short position in insurance risk by combining it with a long position in the same (or a positively correlated) risk.

[2] Longevity risk refers to the chance that actual survival rates exceed assumed survival rates.

Insurance securitization

Catastrophe bonds

- **Examples of catastrophic risks**:
  - Natural disasters (flood, hurricane, windstorm, hailstorm, earthquake).
  - Man-made disasters (explosion, terroristic attack, climate change).
  - Epidemic, pandemic.
- **Definition**: A **catastrophe bond** (CAT bond) is a bond of which the payment of coupons and/or principal is reduced in case a pre-defined catastrophic event (such as an earthquake) occurs. The level of the reduction depends on a 'trigger' mechanism.
- The issuer (buyer) of a CAT bond takes a long (short) position in the underlying catastrophic risk.

Insurance securitization

Catastrophe bonds

- The individual risks in a catastrophic insurance portfolio are often highly positive dependent, implying **systematic risk**. This means that the traditional insurance technique, based on the **law of large numbers**, is not appropriate.
- An **earthquake insurer** can partially hedge his **short position in earthquake risk** by taking a *long position* in earthquake risk by issuing a CAT bonds (via a SPV, see further).
- CAT bonds are bought by ***investors*** for their attractive returns and as a diversifying asset (low dependence between returns on CAT bonds and returns of other asset classes).
- **Types of trigger mechanisms for CAT bonds**:
  - The insurer's actual losses (avoids basis risk, but may cause *moral hazard*).
  - An 'industry-wide loss index' or a 'parametric trigger' (avoids moral hazard, but may cause *basis risk)*.

Insurance securitization

CAT bond transaction structure

- **Who issues a CAT bond?**
  - In practice, a CAT bond is not issued by an insurer.
  - Instead, the insurer establishes a company, called a Special Purpose Vehicle (SPV) to issue the CAT bond.
  - The SPV is based on Cayman Islands, Bermuda, Ireland, ...
  - This construction reduces counterparty risk.
- **Relation investors - SPV**:
  - Initially, investors buy the CAT bond from the SPV.
  - Later, the SPV pays the (eventually reduced) coupons and principle to the investor.
- **Relation SPV - insurer**:
  - The insurer (also called ceding company or sponsor) enters into a reinsurance contract with the SPV.
  - Initially, the insurer pays the reinsurance premium to the SPV.
  - Later, if a covered catastrophe has occured, the SPV pays the insurer the reduced part of the coupons and principle according to the terms of the reinsurance contract.

Insurance securitization

CAT bonds - historical note

- CAT bonds were first issued in the mid 1990's.
- The CAT bond market emerged in the aftermath of significant catastrophe losses in first half of 1990's:
  - Hurricane Andrew in 1992.
  - Northridge earthquake in 1994.
- **Early successful emissions**:
  - USAA / Residential Re (June 1997)
  - Swiss Re (July 1997)
  - Tokio Marine 1 Fire / Parametric Re (December 1997).
- Since then, the market of CAT bonds has been growing fast.
- CAT bonds are commonly (not always) offered with a maturity of 1 to 5 years.

Insurance securitization

Catastrophic mortality bonds

- **Insurer's catastrophic mortality risk**: The risk that the insurer suffers financially because of a catastrophic mortality (much higher than what he assumed) in his life insurance portfolio.
- **Definition**: A **catastrophic mortality bond** (CATM bond) is a CAT bond of which the payment of coupons and/or principal is reduced in case of a pre-defined catastrophic mortality in a given population.
- **VITA I**:
  - First CATM bond, issued by Swiss Re.
  - A principal-at-risk bond, term: 12/2003 - 01/2007.
  - Designed to securitize exposure of Swiss Re to certain catastrophic mortality events (severe outbreak of influenza, terroristic attack, natural catastrophe) during lifetime of bond.
- Since then, the market of short term CATM bonds is growing.

Insurance securitization

Pandemic bonds

- An **epidemic** is the rapid spread of an infectious disease to a large number of people in a given population within a short period of time.
- A **pandemic** is an epidemic that has spread across a large region, for instance multiple continents, or worldwide.
- **Examples of pandemics**:
  - Black death (1346-1353): $75-200$ million deaths.
  - Spanish flu: (1918-1922): $17-100$ million deaths.
  - Hong Kong flu: (1968-1969): 1 million deaths.
  - HIV/AIDS: (1981-...): > 35 million deaths.
  - Covid-19: (2019-2020): 817 351 (on August 25, 2020[3]).

[3] See www.worldometers.info for up-to-date info.

Insurance securitization

Pandemic bonds

- **Definition**: A **pandemic bond** is CATM bond of which the payment of coupons and/or principal is reduced in case a pre-defined pandemic event occurs.
- The issuer (buyer) of a pandemic bond takes a long (short) position in the underlying pandemic risk.
- **Anonymous CATM bond investor** (2017):

"If there will be one day such a severe worldwide pandemic that one of the bonds I bought will be triggered, there will be more important things to look after than an investment portfolio."

Insurance securitization

Pandemic Emergency Financing Facility (PEF)

- The **PEF** is a financing mechanism provided by the World Bank, intended to assist the world's poorest countries to fight deadly, cross-border pandemic outbreaks.
- As part of this mechanism, $320m **pandemic bonds** were issued July 2017, which mature July 2020 (extendable 1 yr).
- These bonds were issued in **two classes**:
  - **Class A** only applies to pandemic flu and coronavirus, and is subject to a highter threshold of deaths before the bond is triggered.
  - **Class B** has a higher risk for the investor.
- **Parties involved**:
  - **The World Bank** is the issuer of the pandemic bonds.
  - **Investors** buy these pandemic bonds. In return, they receive coupons as well as the principal at maturity.
  - However, when a qualifying pandemic occurs, affected **developing countries** receive part of the principal (rather than being returned to the investors).

Insurance securitization

Pandemic Emergency Financing Facility (PEF)

- **Covered diseases**:
  - Pandemic flu.
  - Filovirus.
  - Coronavirus.
  - Rift Valley fever, Lassa fever, Crimean-Congo hemorrhagic fever.
- The **trigger mechanism** depends on:
  - the number of IBDR or IDA countries affected,
  - the number of cases in each of those countries,
  - the number of deaths,
  - the growth rate of the cases.
- Although the PEF was triggered in 2020 by the COVID-19 outbreak, it came under heavy **criticism**:
  - Coupons paid to investors (about 13% interest) are too high.
  - The triggers are too stringent and too complex.
  - Money was not eligeable to be released until 12 weeks after the start of the outbreak (while early action is essential).

Intermezzo: Pandemics and Black Swans

- **Definition**: A **Black Swan**[4] is an event with the following properties:
  - It is a rare event: very low probability that it occurs.
  - It has an extreme impact: consequences are huge.
  - It is unpredictable: nothing in the past can convincingly point to its possibility.
- **Examples of Black Swans**:
  - The personal computer.
  - The rise of the internet.
  - Harry Potter.
  - The Indian Ocean tsunami of 2004.
- Winning the lottery is not a Black Swan.
- Whether an event is a Black or a White Swan, may depend on the observer (9/11, the turkey problem).

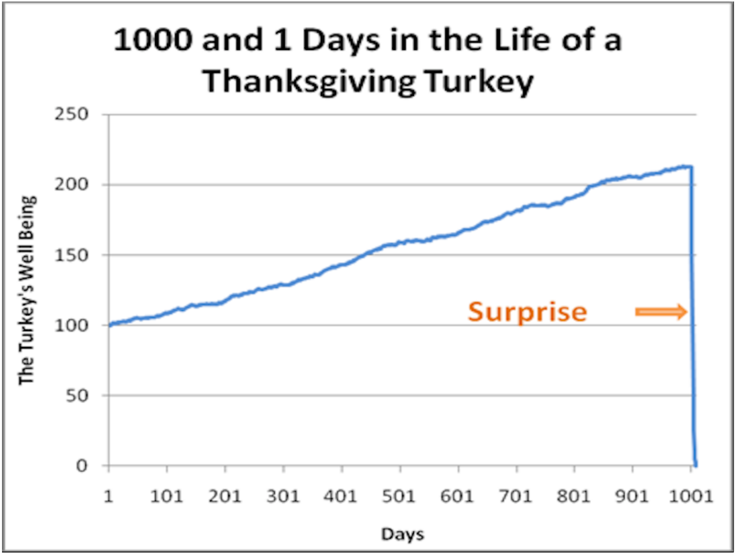
[4] The Black Swan - The impact of the highly improbable, Nassim Nicholas Taleb (2007).

Intermezzo: Pandemics and Black Swans

The Turkey problem

Intermezzo: Pandemics and Black Swans

The Turkey problem



Intermezzo: Pandemics and Black Swans

- **Nassim Taleb** (The Black Swan, 2007):
  - "As we travel more on this planet, epidemics will be more accute - we will have a germ population dominated by a few numbers, and the succesful killer will spread vastly more effectively."
  - "**I see the risk of a very strange acute virus spreading throughout the planet**."
- **Bill Gates** (TED talk, 2015):
  - "The world is simply not prepared to deal with a disease - an especially virulent flu, for example - that infects large numbers of people very quickly."
  - "**Of all the things that could kill 10 million people or more, by far the most likely is an epidemic**."

Intermezzo: Pandemics and Black Swans

- **Sebastian Farquhar** (Global Priorities Project, 2016):
  - "Pandemics have wiped out millions in previous centuries."
  - "We recommend that the WHO, nation states, and other bodies should increase their planning for especially **bad global pandemics, which might kill very large numbers of people**."
  - "Some of these threats seem unlikely, and they probably will not hit us tomorrow or the day after. But it only takes one to change the world we live in forever.
- **Nassim Taleb** (2020):
  - "**It (Covid-19) was not a Black Swan**, it was a White Swan. I am so irritated people would say it is a Black Swan."
  - "And there's definitely no excuse for governments not to be prepared for something like this."

Intermezzo: Pandemics and Black Swans

Figure: Nassim Taleb visiting KU Leuven, December 2017.

Insurance securitization

Longevity bonds

- **Insurer's longevity risk**: The risk that an insurer suffers financially because of high survival probabilities (much higher than what he assumed) in his life annuity portfolio.
- **Definition**: A **longevity bond** is a bond of which coupons and/or principal are increasing functions of the number of survivors in a pre-defined population.
- The issuer (buyer) of a longevity bond takes a short (long) position in the underlying longevity risk.
- A **pension insurer** can partially hedge his **short position in longevity** by taking a long position in longevity risk by buying a longevity bond.
- A **pharma company** (or health care company) can partially hedge its **long position in longevity** by taking a *short position* in longevity by issuing longevity bonds.

Insurance securitization

Longevity bonds

- Longevity bonds are also bought by **investors** (e.g. hedge funds) because of the high yield and for diversification reasons (low stochastic dependence between longevity and yields of financial securities).
- **Historical note**:
  - Maiden attempt: On November 2004, EIB / BNP Paribas announced the issue of the first longevity bond.
  - The maturity was 25 years.
  - Annual coupons were based on realized mortality rates of English and Welsh males aged 65 in 2003.
  - The bond was only partially subscribed and was later withdrawn.
  - Neither issuers nor investors have so far fully embraced longevity bonds.

2. The financial-actuarial world

- We will investigate the pricing mechanism of insurance-linked claims in arbitrage-free markets, in a single period framework.
- **The financial-actuarial world** : $(\Omega, \mathcal{G}, \mathbb{P})$
  - Time $0 =$ now.
  - $\Omega =$ set of all possible states of the world at time 1.
  - $\mathcal{G} = \sigma-$ algebra of all events that may (or may not) occur.
  - $\mathbb{P} =$ physical probability measure.
- **Claims**:
  - A (contingent) claim is a r.v. defined on $(\Omega, \mathcal{G})$.
  - Examples:
    - Time-1 value of a traded stock.
    - Insurance portfolio liability due at time 1.
  - The linear space of all claims in which we are interested is denoted by $\mathcal{C}$.
    - We identify claims which are equal a.s.
    - We assume that $\mathcal{C} \equiv L^2$ (or different if stated explicitely).

- $(\Omega, \mathcal{G}, \mathbb{P})$ is home to a **market of n + 1 traded assets**.
- These assets are denoted by $0, 1, ..., n$.
- **Price process of asset** m:
  - Current price:

$$\boxed{y^{(m)} > 0}$$

  - Price at time 1:

$$\boxed{Y^{(m)} \in \mathcal{C}}$$

- **Notations**:

$$\mathbf{y} = \left(y^{(0)}, y^{(1)}, \ldots, y^{(n)}\right)$$

and

$$\mathbf{Y} = \left(Y^{(0)}, Y^{(1)}, \ldots, Y^{(n)}\right)$$

The financial-actuarial world

The market of traded assets

- **Risk-free zero coupon bond**: (asset 0)
  - Current price: $y^{(0)} = 1$.
  - Payoff at time 1:

$$\boxed{Y^{(0)} = e^r}$$

- **Risky asset m**: $(m = 1, 2, \ldots, n)$
  - Current price: $y^{(m)} > 0$.
  - Price at time 1:

$$\boxed{Y^{(m)} \geq 0 : \text{non-deterministic}}$$

- **Assumption 1**: The $n+1$ assets can be bought or sold in any quantity in a deep, liquid, transparent and frictionless market.

- **Assumption 2**: The assets are *non-redundant*:
  - For any real number vector $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$ one has that
  $$\boxed{\boldsymbol{\theta} \cdot \mathbf{Y} = 0 \Rightarrow \boldsymbol{\theta} = (0, 0, \ldots, 0)}$$
  - Remark: (in-)equalities between r.v.'s have to be understood in the a.s. sense.

- **Definition**:
  A **trading strategy** is a vector $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$, with $\theta^{(m)}$, $m = 0, 1, \ldots, n$, the number of units invested in asset m at time 0.
- The linear space of all trading strategies is denoted by Θ.
- **Value of trading strategy $\boldsymbol{\theta}$**:
  - Time-0 value:
  $$\boldsymbol{\theta} \cdot \mathbf{y} = \sum_{m=0}^{n} \theta^{(m)} \ y^{(m)}$$
  - Time-1 value:
  $$\boldsymbol{\theta} \cdot \mathbf{Y} = \sum_{m=0}^{n} \theta^{(m)} \ Y^{(m)}$$

- **Definition**:
  Trading strategy $\boldsymbol{\theta} \in \Theta$ is an **arbitrage opportunity** in case the following conditions are fulfilled:
  - cost of $\boldsymbol{\theta}$ at time 0 is 0:
    $$\boxed{\boldsymbol{\theta}.\mathbf{y} = 0}$$
  - profit of $\boldsymbol{\theta}$ at time 1 is non-negative:
    $$\boxed{\mathbb{P}\left[\boldsymbol{\theta}.\mathbf{Y} \geq 0\right] = 1}$$
  - profit of $\boldsymbol{\theta}$ at time 1 is positive with positive probability:
    $$\boxed{\mathbb{P}\left[\boldsymbol{\theta}.\mathbf{Y} > 0\right] > 0}$$
- **Definition**:
  The market of traded assets is an **arbitrage-free market** in case it allows no arbitrage opportunities.

- **Definition**:
  Claim S is a **hedgeable claim** in case there exists an investment strategy $\boldsymbol{\theta}$ such that

$$\boxed{\mathbb{P}\left[S = \boldsymbol{\theta}.\mathbf{Y}\right] = 1}$$

  - $\boldsymbol{\theta}$ is called the *hedge* or the *replicating portfolio* of S.

- **Definition**:
  The market of traded assets is a **complete market** in case any claim can be hedged.

3. Equivalent martingale measures

- **Definition**:
  $\mathbb{Q}$ is an **equivalent martingale measure** in case the following conditions are satisfied:
  - $\mathbb{Q}$ is a probability measure defined on $(\Omega, \mathcal{G})$.
  - $\mathbb{Q}$ and $\mathbb{P}$ are equivalent:

$$\mathbb{Q}[A] = 0 \text{ if and only if } \mathbb{P}[A] = 0, \quad \text{for all } A \in \mathcal{G}.$$

  - $\mathbb{Q}$ is a martingale measure:
    The current price of any traded asset can be expressed as

$$y^{(m)} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}\right], \quad \text{for } m = 0, 1, ..., n.$$

- **Convention**: Hereafter, $\mathbb{Q}$ always has to be understood as a probability measure on $(\Omega, \mathcal{G})$.
- **Notation**:
  Equivalence of $\mathbb{Q}$ and $\mathbb{P}$ is denoted by $\mathbb{Q} \sim \mathbb{P}$.

Equivalent martingale measures

The fundamental theorems of asset pricing

- **First fundamental theorem of asset pricing**:
  The market is **arbitrage-free** if and only if there exists *at least one* equivalent martingale measure $\mathbb{Q}$.
- **Second fundamental theorem of asset pricing**:
  The arbitrage-free market is **complete** if and only if there exists a *unique* equivalent martingale measure $\mathbb{Q}$.

- **Assumption 3**: We always assume that **the market of traded assets is arbitrage-free**.
- However, we do not assume that the arbitrage-free market is complete.

Equivalent martingale measures

Physical probabilty measure vs. equivalent martingale measures

- **Physical probability measure $\mathbb{P}$:**
  - Also called: real-world probability measure.
  - Used for risk and portfolio management, e.g.

$$\boxed{\mathbb{P}\left[\boldsymbol{\theta}.\mathbf{Y} \geq \boldsymbol{\theta}.\mathbf{y}\right] = ?}$$

  - $\mathbb{P}$ is 'chosen' by the actuary.
- **Equivalent martingale measure $\mathbb{Q}$:**
  - Also called: risk-neutral measure or pricing measure.
  - Used for expressing prices of traded assets:

$$\boxed{y^{(m)} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}\right]}$$

  - The set of feasible $\mathbb{Q}$'s is 'chosen' by the arbitrage-free market.

Equivalent martingale measures

- **Definition**: Consider an EMM $\mathbb{Q}$ and the mapping $\rho^{\mathbb{Q}}$, with

$$\boxed{\rho^{\mathbb{Q}}[S] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}[S]} \quad \text{for any } S \in \mathcal{C}$$

  - $\rho^{\mathbb{Q}}$ is called a **financial valuation** (or a risk-neutral valuation).
  - $\rho^{\mathbb{Q}}[S]$ is called an arbitrage-free (time-0) price of S.

- **Arbitrage-free price of hedgeable claims**:
  - Consider a hedgeable claim $S = \boldsymbol{\theta}.\mathbf{Y}$.
  - The unique arbitrage-free price of S is given by $\boldsymbol{\theta}.\mathbf{y}$.
  - This time-0 price can be expressed as

$$\boxed{\boldsymbol{\theta}.\mathbf{y} = \rho^{\mathbb{Q}}[S]} \quad \text{for any EMM } \mathbb{Q}.$$

- **Arbitrage-free prices of non-hedgeable claims**:
  - Consider a non-hedgeable claim S.
  - The set of possible arbitrage-free prices of S is given by

$$\boxed{\left\{\rho^{\mathbb{Q}}[S] \mid \mathbb{Q} \text{ is an EMM}\right\}}$$

Equivalent martingale measures

Independence between financial and actuarial claims

- Consider a financial claim (e.g. time-1 value $Y^{(1)}$ of stock 1) and an actuarial claim (e.g. number of survivors S at time 1 from a closed group set up at time 0).
- $\mathbb{P}$-independence between the financial and the actuarial claim is often a **reasonable** assumption.
- $\mathbb{Q}$-independence is a **convenient** assumption.
- **Questions**:
  - Does $\mathbb{P}$-independence implies $\mathbb{Q}$-independence?
  - What is the intuitive meaning of $\mathbb{Q}$-independence?
  - Does $\mathbb{P}$-independence implies the existence of an EMM $\mathbb{Q}$ such that the financial and actuarial claims are $\mathbb{Q}$-independent?
  - Is there any relation between $\mathbb{P}$- and $\mathbb{Q}$-dependency structures?

4. A special finite-state single period world

- Hereafter, we will often consider a special finite-state single period world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$.
- Here, the **universe Ω** is given by :

$$\boxed{\Omega = \{(\varphi_i, \alpha_j) \mid i = 1, ..., I \text{ and } j = 1, ..., J\}}$$

  - with $\left(\varphi_i, \alpha_j\right)$ = state of the world at time 1 (= end of period).
    - φ_i = *financial* substate.
    - α_j = *actuarial* substate.
  - We distinguish a finite number I of different financial substates and a finite number J of different actuarial substates.
- The **σ - algebra 2^{Ω}** is the set of all subsets of Ω.
- The **physical probability measure $\mathbb{P}$** follows from

$$\boxed{\mathbb{P}\left[(\varphi_i, \alpha_j)\right] = p_{ij} \geq 0} \qquad \text{for any } i \text{ and } j$$

- Any **EMM $\mathbb{Q}$** is characterized by

$$\boxed{\mathbb{Q}\left[(\varphi_i, \alpha_j)\right] = q_{ij} \geq 0} \qquad \text{for any } i \text{ and } j$$

A special finite-state single period world

- The **financial projections** of $\mathbb{P}$ and $\mathbb{Q}$:

$$p_{i\cdot} = \sum_{j=1}^{J} p_{ij} \qquad \text{for } i = 1, 2, \ldots, I$$

and

$$q_{i\cdot} = \sum_{j=1}^{J} q_{ij} \qquad \text{for } i = 1, 2, \ldots, I$$

- The **actuarial projections** of $\mathbb{P}$ and $\mathbb{Q}$:

$$p_{\cdot j} = \sum_{i=1}^{I} p_{ij} \qquad \text{for } j = 1, 2, \ldots, J$$

and

$$q_{\cdot j} = \sum_{i=1}^{I} q_{ij} \qquad \text{for } j = 1, 2, \ldots, J$$

A special finite-state single period world

- Assume (only on this slide) that all probabilities related to the different states-of-the-world are positive:

$$\boxed{p_{ij} > 0} \qquad \text{for all } i = 1, 2, \ldots I \text{ and } j = 1, 2, \ldots J$$

- Under this assumption, one has that:
  - For any $A \in 2^{\Omega}$,

$$\boxed{\mathbb{P}\left[A\right] = 1 \Leftrightarrow A = \Omega}$$

  - For any probability measure $\mathbb{Q}$ on $\left(\Omega, 2^{\Omega}\right)$,

$$\boxed{\mathbb{Q} \sim \mathbb{P} \Leftrightarrow q_{ij} > 0} \qquad \text{for all } i \text{ and } j$$

A special finite-state single period world

Exercise 1:

- Consider the special finite-state world $(\Omega, \mathcal{G}, \mathbb{P})$ with the market of traded assets, as described above.
- **Q1**: Show that if there exists at least one EMM $\mathbb{Q}$, then the market is **arbitrage** $-$ **free**.
- For any possible state (φ_i, α_j), the **Arrow-Debreu security** is defined by:

$$S_{ij}(\omega) = \begin{cases} 1 & \text{if } \omega = (\varphi_i, \alpha_j) \\ 0 & \text{elsewhere} \end{cases}$$

- **Q2**: Show that if the arbitrage-free market is **complete**, then there exists a unique EMM $\mathbb{Q}$.
    - Hint: Assume that there exist two EMM's, and express the time-0 prices of all Arrow-Debreu securities in terms of these EMM's.

5. Example 1 - Pandemic risk

A financial-actuarial world with 2 traded assets

- Consider a world, which is home to a traded zero-coupon bond, a traded stock and a non-traded pandemic index.
- **Traded zero-coupon bond**:
  - Current price: $y^{(0)} = 1$.
  - Price at time 1: $Y^{(0)} = 1$.
- **Traded stock**:
  - Current price: $y^{(1)} = 100$.
  - Price at time 1: $Y^{(1)}$ is either 50 or 150.
- **Non-traded pandemic index**:

$$\mathcal{I} = \begin{cases} 0 & : \text{no pandemic breakout in } (0,1) \\ 1 & : \text{pandemic breakout in } (0,1) \end{cases}$$

Example 1 (cont'd)

A financial-actuarial world with 2 traded assets

- We model this world with the following **probability space**:

$$\boxed{\left(\Omega, 2^{\Omega}, \mathbb{P}\right)}$$

- **Universe**:

$$\boxed{\Omega = \left\{(50, 0), (50, 1), (150, 0), (150, 1)\right\}}$$

with each couple a possible outcome of $\left(Y^{(1)}, \mathcal{I}\right)$.

- **Real-world probabilities**:

$$\begin{cases} \mathbb{P}[50, 0] & = & p_{50,0} > 0 \\ \mathbb{P}[50, 1] & = & p_{50,1} > 0 \\ \mathbb{P}[150, 0] & = & p_{150,0} > 0 \\ \mathbb{P}[150, 1] & = & p_{150,1} > 0 \end{cases}$$

$\rightarrow$ $Y^{(1)}$ and $\mathcal{I}$ are assumed to be $\mathbb{P}$-dependent .
$\rightarrow$ '$\mathbb{P}$-dependence' means 'non-independence under $\mathbb{P}$'.

Example 1 (cont'd)

A financial-actuarial world with 2 traded assets

- **No-arbitrage condition**:
  There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\boxed{\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[Y^{(0)}\right] = 1 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(1)}\right] = 100 \end{cases}} \tag{1}$$

- **Rewriting the no-arbitrage condition**:
  There exist positive $q_{50,0}$, $q_{50,1}$, $q_{150,0}$ and $q_{150,1}$ satisfying

$$\boxed{\begin{cases} q_{50\cdot} = 0.5 \\ q_{150\cdot} = 0.5 \end{cases}}$$

Example 1 (cont'd)

A financial-actuarial world with 2 traded assets

- **Two particular EMM's**: $\mathbb{Q}^*$ and $\mathbb{Q}^\perp$:

$$\begin{cases} q^*_{50,0} = 0.3 \\ q^*_{150,0} = 0.4 \\ q^*_{50,1} = 0.2 \\ q^*_{150,1} = 0.1 \end{cases} \quad \text{and} \quad \begin{cases} q^\perp_{50,0} = q^*_{50\cdot} \times q^*_{\cdot 0} = 0.35 \\ q^\perp_{150,0} = q^*_{150\cdot} \times q^*_{\cdot 0} = 0.35 \\ q^\perp_{50,1} = q^*_{50\cdot} \times q^*_{\cdot 1} = 0.15 \\ q^\perp_{150,1} = q^*_{150\cdot} \times q^*_{\cdot 1} = 0.15 \end{cases}$$

- **Conclusions**:
  - The market of traded assets is arbitrage-free but incomplete.
  - The $\mathbb{P}$-dependence of $Y^{(1)}$ and $\mathcal{I}$ does not necessarily imply $\mathbb{Q}$-dependence of $Y^{(1)}$ and $\mathcal{I}$ under any EMM $\mathbb{Q}$:
    - $Y^{(1)}$ and $\mathcal{I}$ are $\mathbb{Q}^\perp$-independent.
    - $Y^{(1)}$ and $\mathcal{I}$ are $\mathbb{Q}^*$-dependent.

Example 1 (cont'd)

Exercise 2:

- Consider the market with 2 traded assets of Example 1.
- **Q1**: Show that the set of EMM's is given by the set of all $\mathbb{Q}$ specified by

$$\begin{cases} q_{50,0} = \frac{1}{2} - q \\ q_{50,1} = q \\ q_{150,0} = \frac{1}{2} - q' \\ q_{150,0} = q' \end{cases}$$

  for some $0 < q < \frac{1}{2}$ and $0 < q' < \frac{1}{2}$.
- **Q2**: Determine the hedge and the no-arbitrage price of

$$\left(100 - Y^{(1)}\right)_+$$

- **Q3**: Determine the set of admissable no-arbitrage prices of the pandemic index $\mathcal{I}$ and of

$$\left(100 - Y^{(1)}\right)_+ \times \mathcal{I}$$

Example 1' - Pandemic risk

A financial-actuarial worlds with 3 traded assets

- Consider a world which is home to a traded zero-coupon bond, a traded stock and a traded pandemic bond.
- **Traded zero-coupon bond**:
  - Current price: $y^{(0)} = 1$.
  - Price at time 1: $Y^{(0)} = 1$.
- **Traded stock**:
  - Current price: $y^{(1)} = 100$.
  - Price at time 1: $Y^{(1)}$ is either 50 or 150.
- **Traded zero-coupon pandemic bond**.
  - Current price: $y^{(2)} = 70$.
  - Payoff at time 1:

$$\boxed{Y^{(2)} = 100 \times (1 - \mathcal{I})} = \begin{cases} 100 & : \text{if no pandemic breaks out} \\ 0 & : \text{if a pandemic breaks out} \end{cases}$$

Example 1' (cont'd)

A financial-actuarial world with 3 traded assets

- We model this world with the following **probability space**:

$$\boxed{\left(\Omega, 2^{\Omega}, \mathbb{P}\right)}$$

- **Universe**:

$$\boxed{\Omega = \left\{(50, 0), (50, 1), (150, 0), (150, 1)\right\}}$$

with each couple a possible outcome of $\left(Y^{(1)}, \mathcal{I}\right)$.

- **Real-world probabilities**:

$$\begin{cases} \mathbb{P}\left[50, 0\right] & = & p_{50,0} > 0 \\ \mathbb{P}\left[50, 1\right] & = & p_{50,1} > 0 \\ \mathbb{P}\left[150, 0\right] & = & p_{150,0} > 0 \\ \mathbb{P}\left[150, 1\right] & = & p_{150,1} > 0 \end{cases}$$

$\rightarrow$ $Y^{(1)}$ and $\mathcal{I}$ are assumed to be $\mathbb{P}$-dependent.

Example 1' (cont'd)

A financial-actuarial world with 3 traded assets

- **No-arbitrage condition**:
  There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\boxed{\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[Y^{(0)}\right] = 1 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(1)}\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(2)}\right] = 70 \end{cases}} \qquad (2)$$

- **Rewriting the no-arbitrage condition**:
  There exist positive $q_{50,0}$, $q_{50,1}$, $q_{150,0}$ and $q_{150,1}$ satisfying

$$\boxed{\begin{cases} q_{50\cdot} = 0.5 \\ q_{150\cdot} = 0.5 \\ q_{\cdot 0} = 0.7 \\ q_{\cdot 1} = 0.3 \end{cases}}$$

Example 1' (cont'd)

A financial-actuarial world with 3 traded assets

- The set of EMM's is given by the set of all $\mathbb{Q}$ specified by

$$\begin{cases} q_{50,0} = 0.5 - q \\ q_{50,1} = q \\ q_{150,0} = 0.2 + q \\ q_{150,1} = 0.3 - q \end{cases}$$

  for some $0 < q < 0.3$.
- **Two particular EMM's**:
  $\mathbb{Q}^*$ and $\mathbb{Q}^\perp$ (defined above).
- **Conclusions**:
  - The market of traded assets is arbitrage-free but incomplete.
  - $Y^{(1)}$ and $\mathcal{I}$ are $\mathbb{Q}^*$-dependent.
  - $\mathbb{Q}^\perp$ is the unique EMM under which $Y^{(1)}$ and $\mathcal{I}$ are independent.

Example 1' (cont'd)

A financial-actuarial world with 3 financial assets

Exercise 3:

- Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ described in Example 1'.
- This world is home to the market with the following traded assets, defined above:
  - Asset 0: **Zero-coupon bond**.
  - Asset 1: **Stock**.
  - Asset 2: **Zero-coupon pandemic bond**.
- **Q1**: Show that the payoff $\left(100 - Y^{(1)}\right)_{+} \times \mathcal{I}$, which is due at time 1, cannot be replicated in this market.
- **Q2**: Determine the set of admissable arbitrage-free prices of

$$\left(100 - Y^{(1)}\right)_{+} \times \mathcal{I}$$

Example 1" - Pandemic risk

A financial-actuarial world with 4 traded assets

- Consider a world, which is home to a traded zero-coupon bond, a traded stock, a traded pandemic bond and a traded combined security.
- Traded **zero-coupon bond**: see Example 1'.
- Traded **stock**: see Example 1'.
- Traded **zero-coupon pandemic bond**: see Example 1'.
- Traded **combined security**:
  - Current price: $y^{(3)}$.
  - Payoff at time 1:

$$Y^{(3)} = \left(100 - Y^{(1)}\right)_+ \times \mathcal{I}$$

Example 1" (cont'd)

A financial-actuarial world with 4 traded assets

- We model this world with the following **probability space**:

$$\boxed{\left(\Omega, 2^{\Omega}, \mathbb{P}\right)}$$

- **Universe**:

$$\boxed{\Omega = \left\{\left(50, 0\right), \left(50, 1\right), \left(150, 0\right), \left(150, 1\right)\right\}}$$

with each couple a possible outcome of $\left(Y^{(1)}, \mathcal{I}\right)$.

- **Real-world probabilities**:

$$\begin{cases} \mathbb{P}\left[50, 0\right] & = & p_{50,0} > 0 \\ \mathbb{P}\left[50, 1\right] & = & p_{50,1} > 0 \\ \mathbb{P}\left[150, 0\right] & = & p_{150,0} > 0 \\ \mathbb{P}\left[150, 1\right] & = & p_{150,1} > 0 \end{cases}$$

$\rightarrow$ $Y^{(1)}$ and $\mathcal{I}$ are assumed to be $\mathbb{P}$-dependent.

Example 1" (cont'd)

A financial-actuarial world with 4 traded assets.

- **No-arbitrage condition**:
  There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\boxed{\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[Y^{(0)}\right] = 1 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(1)}\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(2)}\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(3)}\right] = y^{(3)} \end{cases}} \tag{3}$$

Example 1" (cont'd)

A financial-actuarial world with 4 traded assets

- **Rewriting the no-arbitrage condition**:
  There exist positive $q_{50,0}$, $q_{50,1}$, $q_{150,0}$ and $q_{150,1}$ satisfying

$$\boxed{\begin{cases} q_{50,0} = \frac{25-y^{(3)}}{50} \\ q_{150,0} = \frac{10+y^{(3)}}{50} \\ q_{50,1} = \frac{y^{(3)}}{50} \\ q_{150,1} = \frac{15-y^{(3)}}{50} \end{cases}}$$

  - The market is is arbitrage-free and complete if and only if $y^{(3)} \in (0, 15)$.
  - $Y^{(1)}$ and $\mathcal{I}$ are independent under the unique EMM $\mathbb{Q}$ if and only if $y^{(3)} = 7.5$.
- **Conclusion**:
  - In an arbitrage-free and complete market, it may happen that the pricing measure $\mathbb{Q}$ does not maintain the dependency which holds between claims under $\mathbb{P}$.

Example 1" (cont'd)

A financial-actuarial world with 4 traded assets

Exercise 4:

- Consider the financial-actuarial world of Example 1".
- This world is home to the market with the following traded assets, defined above:
  - Asset 0: **Zero-coupon bond**.
  - Asset 1: **Stock**.
  - Asset 2: **Pandemic bond**.
  - Asset 3: **Combined security** with $y^{(3)} = 15$.
- **Q**: Show that the following time-0 investment strategy gives rise to an arbitrage-opportunity:
  - buy 100 zero-coupon bonds (asset 0),
  - sell 1 pandemic bond (asset 2),
  - sell 2 shares of the combined security (asset 3).

Example 1" (cont'd)

A financial-actuarial world with 4 traded assets

Exercise 5:

- Consider the financial-actuarial world of Example 1".
- This world is home to the market with the following traded assets, defined above:
  - Asset 0: **Zero-coupon bond**.
  - Asset 1: **Stock**.
  - Asset 2: **Pandemic bond**.
  - Asset 3: **Combined security** with $y^{(3)} \in (0, 15)$.
- **Q**: Determine the replicating porfolio and the time-0 price of the time-1 payoff $100 \times \left[Y^{(1)}\right]^{\mathcal{I}}$.

6. Example 2 - Longevity risk

A financial-actuarial world with 4 traded assets

- Consider a world which is home to a 'zero-coupon bond', a 'barometer of the economy' and a 'longevity index'.
- **Zero-coupon bond**: $y^{(0)} = 1$ and $Y^{(0)} = 1$.
- **Barometer of the economy**:

$$\text{Economy at time 1 is} \begin{cases} \textbf{B}\text{ooming} \\ \text{in } \textbf{M}\text{oderate growth} \\ \text{in } \textbf{R}\text{ecession} \end{cases}$$

- **Longevity index**: Consider a longevity index for a given population:

$$\mathcal{I} = \begin{cases} 0 & : \text{'few' survive at time 1} \\ 1 & : \text{'many' survive at time 1} \end{cases}$$

- Further, in this world, we observe a longevity bond and a combined security with payoffs which are functions of the barometer and the longevity index, see further.

Example 2 (cont'd)

A financial-actuarial world with 4 traded assets

- We model this financial-actuarial world in the following **probability space**:

$$\boxed{\left(\Omega, 2^{\Omega}, \mathbb{P}\right)}$$

- **Universe**:

$$\Omega = \left\{ \left(B, 0\right), \left(M, 0\right), \left(R, 0\right), \left(B, 1\right), \left(M, 1\right), \left(R, 1\right) \right\}$$

  - Each element of Ω is a possible time-1 outcome of the random couple ('barometer', 'survival index').
- **Real-world probabilities**:
  - Each possible state-of-the-world has a positive probability.
  - 'Barometer' and 'survival index' are assumed to be $\mathbb{P}$-independent.
  - Hence,

$$\mathbb{P}\left[\left(B, 0\right)\right] = p_{B,0} = p_{B\cdot} \times p_{\cdot 0} > 0$$

Example 2 (cont'd)

A financial-actuarial world with 4 traded assets

- Traded asset 0: **Zero-coupon bond**.
- Traded asset 1: **Stock**.
  - Current price: $y^{(1)} = 50$.
  - Payoff at time 1:

$$Y^{(1)} = \begin{cases} 100, & \text{if } B \\ 0, & \text{otherwise} \end{cases}$$

- Traded asset 2: Zero-coupon **longevity bond**.
  - Current price: $y^{(2)} = 70$.
  - Payoff at time 1:

$$Y^{(2)} = 100 \times \mathcal{I}$$

- Traded asset 3: **Combined security**.
  - Current price: $y^{(3)}$.
  - Payoff at time 1:

$$Y^{(3)} = Y^{(1)} \times (1 - \mathcal{I})$$

Example 2 (cont'd)

A financial-actuarial world with 4 traded assets

- **No-arbitrage condition**:
  There exists a $\mathbb{Q} \sim \mathbb{P}$ satisfying

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[Y^{(0)}\right] = 1 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(1)}\right] = 50 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(2)}\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[Y^{(3)}\right] = y^{(3)} \end{cases} \tag{4}$$

Example 2 (cont'd)

A financial-actuarial world with 4 traded assets

- **Rewriting the no-arbitrage condition**:
  There exist positive $q_{B,0}, q_{B,1}, \dots, q_{R,1}$ satisfying

$$\begin{cases} q_{B,0} = \frac{y^{(3)}}{100} \\ q_{B,1} = \frac{50-y^{(3)}}{100} \\ q_{M,0} + q_{R,0} = \frac{30-y^{(3)}}{100} \\ q_{M,1} + q_{R,1} = \frac{20+y^{(3)}}{100} \end{cases}$$

- The market is arbitrage-free and incomplete, provided $y^{(3)} \in (0, 30)$.
- There exists a EMM $\mathbb{Q}$ such that the 'barometer' and the 'longevity index' are $\mathbb{Q}$-independent if and only if $y^{(3)} = 15$.

Example 2 (cont'd)

A financial-actuarial world with 4 traded assets

- **Conclusion**: In an arbitrage-free and incomplete market it may be impossible to find a pricing measure $\mathbb{Q}$ that maintains the $\mathbb{P}$ - independence property which holds between a 'barometer' and a 'longevity index'.
- **Remark**:
  - Combining the financial substates M and R of Example 2 in a single financial substate (M, R) leads to an arbitrage-free and complete market, provided $y^{(3)} \in (0, 30)$.
  - In that sense, the incompleteness of the market in Example 2 is *artificial*.
  - In Exercise 6, we will consider a *less artificial* incomplete market where it is impossible to find a pricing measure $\mathbb{Q}$ that maintains the $\mathbb{P}$ - independence property that holds between a 'barometer' and a 'longevity index'.

Example 2 (cont'd)

A financial-actuarial world with 5 traded assets

Exercise 6:

- Let us consider the financial-actuarial world $(\Omega, 2^{\Omega}, \mathbb{P})$ of Example 2.
- This world is home to the market with the traded assets 0 (**zero-coupon bond**), 1 (**stock**), 2 (**longevity bond**), 3 (**combined security)** as defined in Example 2, with $y^{(3)} \in (0, 30)$.
- Furthermore, there is a second traded **combined security** 4:
  - Current price: $y^{(4)} = 14$.
  - Payoff at time 1:

$$\boxed{Y^{(4)} = 100 \times 1_{(R,1)}}$$

- **Q1**: Show that this market of 5 traded assets is arbitrage-free, but incomplete.
- **Q2**: Show that there exists an EMM $\mathbb{Q}$ under which 'barometer of the economy' and 'longevity index' are $\mathbb{Q}$-independent if and only if $y^{(3)} = 15$.

Example 2 (cont'd)

A financial-actuarial world with 5 traded assets

Exercise 7:

- Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ considered in Exercise 6.
- This world is home to the market with the following traded assets, defined above:
  - Asset 0: **Zero-coupon bond**.
  - Asset 1: **Stock**.
  - Asset 2: **Longevity bond**.
  - Asset 3: **Combined security** with $y^{(3)} = 60$.
  - Asset 4: **Combined security** with $y^{(4)} = 14$.
- **Q**: Determine an arbitrage-opportunity in this market.

7. Some more exercises

Exercise 8: **CAT bonds**

- Consider a world which is home to a traded zero-coupon bond, a non-traded earthquake index and a traded CAT bond.
- **Traded zero-coupon bond**:
  - Time-0 price: $y^{(0)} = 1$, time-1 value: $Y(0) = 1$.
- **Non-traded earthquake index**, representing the magnitude of earthquake loss over period $(0, 1)$:

$$\mathcal{I} = \begin{cases} 0 \\ 50 \\ 100 \end{cases}$$

- **Traded CAT bond**:
  - Time-0 price: $y^{(1)} = 90$.
  - Time-1 value:

$$Y^{(1)} = \begin{cases} 100, & \text{if } \mathcal{I} < 100 \\ 0, & \text{if } \mathcal{I} = 100 \end{cases}$$

Some more exercises

Exercise 8 (cont'd):

- We model this world with the following **probability space**:

$$\boxed{(\Omega, 2^\Omega, \mathbb{P})}$$

- **Universe**:

$$\Omega = \{0, 50, 100\}$$

  with each element a possible outcome of earthquake index $\mathcal{I}$.
- **Real-world probabilities**:

$$\mathbb{P}[0] = p_0 > 0 \qquad \mathbb{P}[50] = p_{50} > 0 \qquad \mathbb{P}[100] = p_{100} > 0$$

- **Q1**: Show that in this world, $\mathbb{Q}$ is an EMM if and only if

$$\boxed{\begin{cases} q_0 = q \\ q_{50} = 0.9 - q \\ q_{100} = 0.1 \end{cases}}$$

  for some q in the interval $(0,\ 0.9)$.

Some more exercises

Exercise 8 (cont'd):

- Suppose now that a second cat bond (asset 2) is introduced in the market.
- **Second traded CAT bond**:
  - Time-0 price: $y^{(2)}$.
  - Time-1 value:

$$Y^{(2)} = \begin{cases} 100 & \text{if } \mathcal{I} = 0 \\ 60 & \text{if } \mathcal{I} = 50 \\ 20 & \text{if } \mathcal{I} = 100 \end{cases}$$

- **Q2**: Show that $y^{(2)}$ is a possible non-arbitrage time-0 price of $Y^{(2)}$ if and only if

$$56 < y^{(2)} < 92$$

- **Q3**: Suppose that $y^{(2)} = 40$. Determine $\theta^{(0)}$ such that the trading strating $\boldsymbol{\theta} = \left(\theta^{(0)}, -2, 1\right)$ is an arbitrage opportunity.

Some more exercises

Exercise 8 (cont'd):

- In the remainder of this exercise, we consider the arbitrage-free market with traded claims $Y^{(0)}$, $Y^{(1)}$ and $Y^{(2)}$.
- **Q4**: Show that this arbitrage-free market is complete with EMM given by

$$\boxed{\begin{cases} q_0 = \frac{y^{(2)}-56}{40} \\ q_{50} = \frac{92-y^{(2)}}{40} \\ q_{100} = 0.1 \end{cases}}$$

- **Q5**: Determine the hedge and the no-arbitrage price of the earthquake index $\mathcal{I}$.

Some more exercises

Exercise 9: **Longevity and mortality**

- Consider a world which is home to:
  - a non-traded survival index,
  - a traded zero-coupon bond,
  - a traded longevity bond,
  - a traded mortality-linked security.
- **Non-traded survival index**:
  This index represents the number of persons of a closed group who survive from time 0 until 1:

$$\mathcal{I} = \begin{cases} 0 : & \text{much less than expected number of people survive} \\ 1 : & \text{around the expected number of people survive} \\ 2 : & \text{much more than expected number of people survive} \end{cases}$$

Some more exercises

Exercise 9: (cont'd)

- Traded asset 0: **Zero-coupon bond**:

$$\boxed{y^{(0)} = Y(0) = 1}$$

- Traded asset 1: **Longevity bond**:

  Time-0 price $y^{(1)}$ and

$$\boxed{Y^{(1)} = \begin{cases} 100, & \text{if } \mathcal{I} = 0, 1 \\ 150, & \text{if } \mathcal{I} = 2 \end{cases}}$$

- Traded asset 2: **Mortality-linked security**:

  Time-0 price $y^{(2)}$ and:

$$\boxed{Y^{(2)} = \begin{cases} 150, & \text{if } \mathcal{I} = 0 \\ 100, & \text{if } \mathcal{I} = 1, 2 \end{cases}}$$

Some more exercises

Exercise 9: (cont'd)

- We model this world with the following **probability space**:

$$\boxed{\left(\Omega, 2^{\Omega}, \mathbb{P}\right)}$$

- **Universe**:

$$\Omega = \{0, 1, 2\}$$

  where any element of Ω is a possible time-1 outcome of the survival index $\mathcal{I}$.

- **Real-world probabilities**:

$$\mathbb{P}[0] = p_0 > 0 \qquad \mathbb{P}[1] = p_1 > 0 \qquad \mathbb{P}[2] = p_2 > 0$$

Some more exercises

Exercise 9: (cont'd)

- **Q1**: Show that the market is arbitrage-free if and only if

$$\boxed{100 < y^{(1)} < 250 - y^{(2)} < 150}$$

- **Q2**: Show that the arbitrage-free market is complete and determine the unique EMM.
- **Q3**: In this arbitrage-free and complete market, determine the hedge and the no-arbitrage price of the survival index $\mathcal{I}$.
- **Q4**: Suppose now that

$$\boxed{100 < y^{(1)} = 250 - y^{(2)} < 150}$$

Determine $\theta^{(0)}$ such that the trading strating

$$\boxed{\boldsymbol{\theta} = \left(\theta^{(0)}, -1, -1\right)}$$

is an arbitrage opportunity.

8. Some conclusions and final remarks

- **Arbitrage-free pricing of insurance-linked claims**:
  - The typical unhedgeability of such claims involves an inherent uncertainty about the choice of the EMM $\mathbb{Q}$.
  - We considered CAT bonds, CATM bonds, pandemic bonds and longevity bonds.
  - We illustrated their pricing by some theoretical and simple but insightful examples.
- **Independence of financial and actuarial claims**:
  - Such claims may be $\mathbb{P}$-independent, but this is definitely not always the case!
  - Assuming $\mathbb{Q}$-independence between such claims (motivated by their $\mathbb{P}$-independence) is convenient but meaningless.
  - Postulating a $\mathbb{Q}$-measure with $\mathbb{Q}$-independence between such claims (motivated by their $\mathbb{P}$-independence) and calibrating the model to observed market prices may lead to inconsistencies.

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