Fair valuation of insurance liabilities

Combining 'financial market'-consistency and 'actuarial model'-consistency in a single-period framework[1]

Jan Dhaene

KU Leuven, Belgium

How to determine the value of things?

- "Res tantum valet quantum vendi potest." (Latin quote)
  A thing is worth only what someone else will pay for it.
- "What is a cynic? ... A man who knows the price of everything and the value of nothing."
  Oscar Wilde, Lady Windermere's Fan (1893).
- "The greatest of all gifts is the power to estimate things at their true worth."
  François De La Rochefoucauld (1613-1680).
- Question to be answered in this chapter:
  For what price would another party be willing to take over a future insurance liability, taking into account hedging opportunities in the financial market?

Main reference

- **Fair valuation of insurance liabilities: merging actuarial judgement and market-consistency.**
  J. Dhaene, B. Stassen, K. Barigou, D. Linders, Z. Chen (2017).
  Insurance : Mathematics & Economics, 76, 14-27.

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On the fair valuation of insurance liabilities: merging market-consistency and actuarial considerations
（保険負債の公正価値評価について：市場整合性と保険数理的考察）

平成３０年１０月２９日（月）～１１月２日（金）
京都大学理学研究科３号館１２７大会議室
会場への交通については、理学研究科数学教室　ウェブページをご参照ください。
https://www.math.kyoto-u.ac.jp/ja/overview/access

対象者

アクチュアリーサイエンスに興味のある学部生、大学院生。日本アクチュアリー会の会員(事前申込不要。他大学、理学部・理学研究科以外の学生の参加も可。)

言語

英語

Prof. Jan Dhaene

KU Leuven,Belgium
Faculty of Business and Economics
Head of the Research Centre Insurance
(Actuarial Research Group)
ベルギーアクチュアリー会（IA|BE）会員

スケジュール

10月29日(月)	14:45	～	16:15	Introduction & Modelling the financial-actuarial world
10月30日(火)	10:30	～	12:00	Fair valuation
10月31日(水)	10:30	～	12:00	Hedging
11月 1日(木)	10:30	～	12:00	Hedge-Based valuation
11月 2日(金)	10:30	～	12:00	Two-Step valuation
	14:00	～	15:30	Discussion Session

講義概要

本セミナーでは、2016年1月に導入された欧州保険資本規制（Solvency II）で要請されている保険負債の公正価値評価（fair valuation）について1期間モデルを用いた考察を行う。公正価値を市場整合的（market-consistent）かつ保険数理的（actuarial）な評価として定義し、ヘッジを用いた評価（hedge-based valuation）では、まず保険負債に対する最良ヘッジを設定し、残りの部分を保険数理的手法で評価する。また2段階評価(two-step valuation) の手法を紹介し、これらがいずれも同一となることを導く。Discussion Session では、学生の様々な質問に対して、講師がアクチュアリーや大学での経験をふまえたアドバイスを行う。

講師はアクチュアリー教育における幅広い知識と経験を有しており、著書（共著）に「Modern Actuarial Risk Theory-Using R」「Actuarial Theory for Dependent Risks-Measures, Orders and Models」などがある。また保険数理・保険分野の複数のジャーナルの編集委員を務めている。

主催：京都大学大学院理学研究科数学教室
協賛：公益社団法人日本アクチュアリー会（IAJ）

問い合わせ:〒606-8502　京都市左京区北白川追分町
京都大学大学院理学研究科　数学事務室
E-mail:jimushitsu@math.kyoto-u.ac.jp
Tel:075-753-3700

University of Kyoto, Department of Mathematics,
Kyoto, Japan, October - November 2018.

Agenda

- 1. Introduction.
- 2. The financial-actuarial world.
- 3. Valuations.
- 4. Hedgers.
- 5. Linking valuations and hedgers.
- 6. Hedge-based valuations.
- 7. Two-step valuations.
- 8. Conclusions.
- 9. References.
- 10. Appendix.

1. Introduction

Different types of valuations

- **Valuation according to the financial quant:**
  - A *financial valuation* is based on the principle of **no-arbitrage**:
    - Fundamental Theorem of Asset Pricing.
    - Claim S due at time 1:

$$\text{Time-0 value of } S = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S\right]$$

    - Set of feasible $\mathbb{Q}$'s follows from observed **financial market prices**.
- **Valuation according to the traditional actuary:**
  - An *actuarial valuation* is based on principle of **diversification**:
    - Law of Large Numbers.
    - Claim S due at time 1:

$$\text{Time-0 value of } S = e^{-r}\ \left(\mathbb{E}^{\mathbb{P}}\left[S\right] + \text{RM}\left[S\right]\right)$$

    - Expectation $\mathbb{E}^{\mathbb{P}}\left[S\right]$ and risk margin $\text{RM}[S]$ follow from an **actuarial model** set up by the actuary.

Introduction

Different types of valuations

- **Valuation according to Solvency II**[2]:
  - *If (part of) the cash flows of an insurance liability can be replicated, then the value of the (part of the) cash flows is determined on the basis of the* **market value** *of these financial instruments.*
  - *Otherwise, the value is equal to the sum of the* **best estimate**[3] *and a* **risk margin**[4].
- We will define 5 types of valuations: *financial, actuarial, market-consistent, model-consistent* and *fair* valuations.
- Results can be applied in a *reserving* and in a *pricing* context.

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[2] Solvency II Directive 2009/138/EC, Article 77.

[3] *Best Estimate*: The probability-weighted average, also referred to as the mean (Solvency II Glossary).

[4] *Risk Margin*: The value of the deviation risk of the actual outcome compared with the best estimate, expressed in terms of a defined risk measure (Solvency II Glossary).

Introduction

What question does each valuation answer?

- **Financial quant**: *What is a correct price of claim S when traded in an arbitrage-free market* **?**
  - **Hedgeable claim**: Value is equal to the financial market price of the underlying hedge.
  - Non-hedgeable claim: Inherent uncertainty about how S would be priced.
- **Traditional actuary**: *For what price is another party willling to take over liability S, ignoring the financial market, except for the riskfree bank account?*
  - **Orthogonal claim**: A valuation based on an actuarial model is appropriate.
  - Non-orthogonal claim: Ignorance of existence of the financial market.
- **Modern actuary**: *For what price is another party willing to take over liability S, taking into account hedging opportunities in the financial market?*

2. The financial-actuarial world

- We investigate the valuation of insurance liabilities in a single period financial-actuarial framework.
- **The financial-actuarial world** : $(\Omega, \mathcal{G}, \mathbb{P})$
  - Time $0 =$ now.
  - $\Omega =$ set of all possible states of the world at time 1.
  - $\mathcal{G} = \sigma-$ algebra of all events that may (or may not) occur.
  - $\mathbb{P} =$ physical probability measure.
- **Contingent claims**:
  - A (contingent) claim is a r.v. defined on $(\Omega, \mathcal{G})$.
  - Examples:
    - Time-1 price of a traded asset,
    - Insurance liability due at time 1.
  - The linear space of all claims in which we are interested is denoted by $\mathcal{C}$.
    - We identify claims which are equal a.s.
    - We assume that $\mathcal{C} \equiv L^2$ (or different if stated explicitely).

The financial-actuarial world

- $(\Omega, \mathcal{G}, \mathbb{P})$ is home to a **market of n + 1 traded assets**.
- These assets are denoted by $0, 1, ..., n$.
- **Price process of asset** m:
  - Current price:
  $$\boxed{y^{(m)} > 0}$$
  - Price at time 1:
  $$\boxed{Y^{(m)} \in \mathcal{C}}$$
- **Notations**:
$$\mathbf{y} = \left(y^{(0)}, y^{(1)}, \ldots, y^{(n)}\right)$$
and
$$\mathbf{Y} = \left(Y^{(0)}, Y^{(1)}, \ldots, Y^{(n)}\right)$$

The financial-actuarial world

- **Risk-free zero coupon bond**: (asset 0)
  - Current price: $y^{(0)} = 1$.
  - Payoff at time 1:

$$\boxed{Y^{(0)} = e^r}$$

- **Risky asset** m: $(m = 1, 2, \ldots, n)$
  - Current price: $y^{(m)} > 0$.
  - Price at time 1:

$$\boxed{Y^{(m)} \geq 0 : \text{non-deterministic}}$$

The financial-actuarial world

- **Trading strategies**:
  - A trading strategy is a vector $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$, with $\theta^{(m)}$, $m = 0, 1, \ldots, n$, the number of units invested in asset m at time 0.
  - The linear space of all trading strategies is denoted by Θ.
- **Value of trading strategy $\boldsymbol{\theta}$**:
  - Time-0 value:

$$\boldsymbol{\theta} \cdot \mathbf{y} = \sum_{m=0}^{n} \theta^{(m)} \ y^{(m)}$$

  - Time-1 value:

$$\boldsymbol{\theta} \cdot \mathbf{Y} = \sum_{m=0}^{n} \theta^{(m)} \ Y^{(m)}$$

The financial-actuarial world

- **Assumption 1**: The $n+1$ assets can be bought or sold in any quantity in a deep, liquid, transparent and frictionless market.

- **Assumption 2**: The assets are *non-redundant*:
  - For any trading strategy $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$ one has that

$$\boxed{\boldsymbol{\theta} \cdot \mathbf{Y} = 0 \Rightarrow \boldsymbol{\theta} = (0, 0, \ldots, 0)}$$

  - Remark: (in-)equalities between r.v.'s have to be understood in the a.s. sense.

- **Assumption 3**: The market is *arbitrage-free*: There exists no investment strategy $\boldsymbol{\theta} \in \Theta$ such that

$$\boxed{\boldsymbol{\theta} \cdot \mathbf{y} = 0, \quad \mathbb{P}\left[\boldsymbol{\theta} \cdot \mathbf{Y} \geq 0\right] = 1 \quad \text{and} \quad \mathbb{P}\left[\boldsymbol{\theta} \cdot \mathbf{Y} > 0\right] > 0}$$

The financial-actuarial world

- **Equivalent martingale measures**:
  $\mathbb{Q}$ is an *equivalent martingale measure* (or a risk-neutral measure) if :
  - $\mathbb{Q}$ is a probability measure defined on $(\Omega, \mathcal{G})$.
  - $\mathbb{Q}$ and $\mathbb{P}$ are equivalent:

$$\mathbb{Q}[A] = 0 \text{ if and only if } \mathbb{P}[A] = 0, \quad \text{for all } A \in \mathcal{G}.$$

  - The current price of any traded asset can be expressed as

$$y^{(m)} = e^{-r} \, \mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}\right], \quad \text{for } m = 0, 1, ..., n.$$

- **Fundamental theorem of asset pricing**:
  - The no-arbitrage condition is equivalent to the existence of at least one equivalent martingale measure.

- **Definition** :
  $S^h \in \mathcal{C}$ is a **hedgeable claim** if there exists a trading strategy $\boldsymbol{\nu} = \left(\nu^{(0)}, \ldots, \nu^{(n)}\right)$ such that

$$\boxed{S^h = \boldsymbol{\nu} \cdot \mathbf{Y}}$$

- Time-0 price of a hedgeable claim:
  - Suppose that $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$.
  - Let $\mathbb{Q}$ be an EMM.
  - Current price:

$$\boxed{\text{Time-0 price of } S^h = \boldsymbol{\nu} \cdot \mathbf{y} = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[S^h\right]}$$

- The linear space of all hedgeable claims is denoted by $\mathcal{C}^h$.

The financial-actuarial world

Hedgeable claims

Exercise 1:

- Consider the hedgeable claim S^h.
- **Q**: Show that the hedge of S^h is uniquely determined.

- **Definition** :
  $S^{\perp} \in \mathcal{C}$ is an **orthogonal claim** if it is $\mathbb{P}$-independent of the traded claims $Y^{(1)}, \ldots, Y^{(n)}$:

$$\boxed{S^{\perp} \perp \left(Y^{(1)}, Y^{(2)}, \ldots Y^{(n)}\right)}$$

- The linear space of all orthogonal claims is denoted by $\mathcal{C}^{\perp}$.
- **Remark**: The risk-free claims $a \in \mathbb{R}$ are the only claims which are both hedgeable and orthogonal.

Exercise 2: **The Cost-of-Capital principle**.

- Consider a portfolio of N insurance policies with respective claims $X_1, X_2, \ldots, X_N \in \mathcal{C}$.
- Assumptions:
  - Under $\mathbb{P}$, the X_i are i.i.d. with mean μ and variance $\sigma^2 > 0$.
  - Each $X_i \perp \left(Y^{(1)}, Y^{(2)}, \ldots Y^{(n)}\right)$.
  - N is sufficiently large such that $\sum_{i=1}^{N} X_i$ is (approximately) normal distributed.
- The **insurance portfolio liability** is given by

$$\boxed{S^{\perp} = \sum_{i=1}^{N} X_i}$$

Exercise 2: **The Cost-of-Capital principle** (cont'd).

- Suppose that the value $\rho\left[S^{\perp}\right]$ of $S^{\perp}$ is determined by :

$$\boxed{\rho\left[S^{\perp}\right]=e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]+\mathrm{RM}\left[S^{\perp}\right]\right)}$$

  - $\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ is the *best estimate* of $S^{\perp}$.
  - $\mathrm{RM}\left[S^{\perp}\right]$ is the *risk margin*, determined according to the *cost-of-capital* approach:

$$\mathrm{RM}\left[S^{\perp}\right]=i\left(\mathrm{VaR}_{p}^{\mathbb{P}}\left[S^{\perp}\right]-\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right)$$

    for given probability level p and perunage i.

- **Q**: Determine $\rho\left[S^{\perp}\right]$ and show that the value per policy $\rho\left[S^{\perp}\right]/N$ is a decreasing function of N.

The financial-actuarial world

Hybrid claims

- **Definition** :
  Claim S is a **hybrid claim** if it is neither hedgeable nor orthogonal:
  $$\boxed{S \in \mathcal{C} \setminus \left(\mathcal{C}^h \cup \mathcal{C}^\perp\right)}$$
- **Examples**:
  - Sum of a hedgeable and an orthogonal claim:
    $$S = S^h + S^\perp$$
  - Product of a hedgeable and an orthogonal claim:
    $$S = S^h \times S^\perp$$
- **Remarks**:
  - Insurance regulations allow different approaches for valuating hybrid claims.
  - Insurance securitization may lead to hybrid liabilities for insurers.

Exercise 3: **Decomposing insurance liabilities**.

- Consider a **portfolio of N insurance contracts**, with payoff of contract i at time 1 given by $S^h \times X_i$ for any i.
- Assumptions:
  - $S^h \in \mathcal{C}^h$.
  - $X_1, X_2, \ldots, X_N$ are $\mathbb{P}-$i.i.d. orthogonal claims.
- **Insurance portfolio liability**:

$$\boxed{S^h \times S_N^\perp}$$

with $S_N^\perp = \sum_{i=1}^N X_i \in \mathcal{C}^\perp$.

Exercise 3 (cont'd):

- The **insurance portfolio liability per policy** can be decomposed into a hedgable and a diversifiable hybrid claim:

$$S^h \times \frac{S_N^{\perp}}{N} = Y^h + Y_N^d$$

- **Hedgeable claim**:

$$Y^h = S^h \times \mathbb{E}^{\mathbb{P}}\left[X_1\right]$$

- **Diversifiable hybrid claim**:

$$Y_N^d = S^h \times \left(\frac{S_N^{\perp}}{N} - \mathbb{E}^{\mathbb{P}}\left[X_1\right]\right)$$

- **Q1**: Show that the variance of the **diversifiable claim** is given by

$$\mathrm{Var}^{\mathbb{P}}\left[Y_N^d\right] = \frac{1}{N} \times \mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}\left[X_1\right]$$

Exercise 3 (cont'd):

- A sequence of r.v.'s $X_1, X_2, X_3 \dots$ **converges in probability** to a r.v. X, notation $X_N \xrightarrow{P} X$, if for all $\epsilon > 0$ one has that

$$\lim_{N\to\infty} \mathbb{P}\left[|X_N - X| > \epsilon\right] = 0$$

- **Q2**: Use *Chebyshev's inequality* to show that $Y_1^d, Y_2^d, Y_{3,}^d \dots$ converges in probability to zero:

$$\boxed{Y_N^d \xrightarrow{P} 0}$$

- This convergence result can also be stated as follows:

$$\boxed{S^h \times \frac{S_N^\perp}{N} \xrightarrow{P} S^h \times \mathbb{E}^{\mathbb{P}}\left[X_1\right]}$$

- Special case: The (weak) **Law of Large Numbers**:

$$\boxed{\frac{S_N^\perp}{N} \xrightarrow{P} \mathbb{E}^{\mathbb{P}}\left[X_1\right]}$$

Exercise 3 (cont'd):

- In the remainder of this exercise, assume that

$$X_i = \begin{cases} 0 & : \text{ insured } i \text{ dies before time } 1 \\ 1 & : \text{ insured } i \text{ is alive at time } 1 \end{cases}$$

  with

$$\mathbb{P}\left[X_i = 1\right] = p$$

- **Q3**: Derive an expression for $\text{Var}^{\mathbb{P}}\left[Y_N^d\right]$.

Exercise 3': **Decomposing insurance liabilities**.

- Consider a **portfolio of N insurance contracts**, with payoff of contract i at time 1 given by $S^h \times X_i$ for any i.
- Assumptions:
  - $S^h \in \mathcal{C}^h$ and any $X_i \in \mathcal{C}$.
  - There exists a r.v. $Z \in \mathcal{C}^\perp$ with support $A \subseteq \mathbb{R}$, such that for any $z \in A$, one has that $(X_1 \mid Z = z), \ldots, (X_N \mid Z = z)$ are $\mathbb{P}-$i.i.d. orthogonal claims.
- **Q1**: Show that $X_1, X_2, \ldots, X_N \in \mathcal{C}^\perp$.
- **Insurance portfolio liability**:

$$\boxed{S^h \times S_N^\perp}$$

with $S_N^\perp = \sum_{i=1}^N X_i \in \mathcal{C}^\perp$.

Exercise 3' (cont'd):

- The **insurance portfolio liability per policy** can be decomposed into a hedgable claim, a diversifiable hybrid claim and a residual hybrid claim:

$$S^h \times \frac{S_N^{\perp}}{N} = Y^h + Y_N^d + Y^r$$

- **Hedgeable claim**:

$$Y^h = S^h \times \mathbb{E}^{\mathbb{P}}\left[X_1\right]$$

- **Diversifiable hybrid claim**:

$$Y_N^d = S^h \times \left(\frac{S_N^{\perp}}{N} - \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]\right)$$

- **Residual hybrid claim**:

$$Y^r = S^h \times \left(\mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right] - \mathbb{E}^{\mathbb{P}}\left[X_1\right]\right)$$

Exercise 3' (cont'd):

- **Q2**: Show that the variance of the **diversifiable claim** is given by

$$\boxed{\mathrm{Var}^{\mathbb{P}}\left[Y_N^d\right] = \frac{1}{N} \times \mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right] \times \mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}^{\mathbb{P}}\left[X_1 \mid Z\right]\right]}$$

- **Q3**: Show that the variance of the **residual claim** is given by

$$\boxed{\mathrm{Var}^{\mathbb{P}}\left[Y^r\right] = \mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]\right]}$$

- **Q4**: Show that the diversifiable claim and the residual claim are uncorrelated:

$$\boxed{\mathrm{Covar}^{\mathbb{P}}\left[Y_N^d,\, Y^r\right] = 0}$$

Exercise 3' (cont'd):

- **Q5**: Use *Chebyshev's inequality* to show that $Y_1^d, Y_2^d, Y_{3,}^d \ldots$ converges in probability to zero:

$$\boxed{Y_N^d \xrightarrow{\mathrm{P}} 0}$$

- This convergence result can also be stated as follows:

$$\boxed{S^h \times \frac{S_N^{\perp}}{N} \xrightarrow{\mathrm{P}} S^h \times \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]}$$

- Special case: **Conditional** (weak) **Law of Large Numbers**:

$$\boxed{\frac{S_N^{\perp}}{N} \xrightarrow{\mathrm{P}} \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]}$$

Exercise 3' (cont'd):

- In the remainder of this exercise, assume that

$$X_i = \begin{cases} 0 & : \text{insured } i \text{ dies before time 1} \\ 1 & : \text{insured } i \text{ is alive at time 1} \end{cases}$$

with

$$\mathbb{P}\left[X_i = 1 \mid Z = z\right] \overset{\text{not.}}{=} p\left(z\right), \qquad \text{for any } z \in A$$

and

$$\mathbb{P}\left[X_i = 1\right] = \mathbb{E}^{\mathbb{P}}\left[p(Z)\right] \overset{\text{not.}}{=} p$$

- Z describes *systematic survival risk* (= population-wide variability of survival).

Exercise 3' (cont'd):

- **Q6**: Show that

$$\boxed{\text{covar}^{\mathbb{P}}\left[X_1, X_2\right] = \text{Var}^{\mathbb{P}}\left[p\left(Z\right)\right]}$$

- **Q7**: Show that

$$\boxed{\mathbb{P}\left[X_2 = 1 \mid X_1 = 1\right] = p + \frac{\text{Var}^{\mathbb{P}}\left[p(Z)\right]}{p}}$$

- **Q8**: Derive an expression for $\text{Var}^{\mathbb{P}}\left[Y_N^d\right]$ and $\text{Var}^{\mathbb{P}}\left[Y^r\right]$.

3. Valuations

- **Definition**: A **valuation** is a mapping $\rho : \mathcal{C} \to \mathbb{R}$, attaching a real number to any claim S:

$$\boxed{S \to \rho\left[S\right]}$$

  such that
  - ρ is **normalized**:

$$\boxed{\rho\left[0\right] = 0}$$

  - ρ is **translation invariant**:

$$\boxed{\rho\left[S + a\right] = \rho\left[S\right] + e^{-r} a} \quad \text{for any } S \in \mathcal{C} \text{ and } a \in \mathbb{R}$$

- **Interpretation**: $\rho\left[S\right]$ is the time-0 value of insurance claim S.

- As we identify r.v.'s which are equal in the a.s. sense, we have that

$$\boxed{\mathbb{P}\left[X = Y\right] = 1 \Rightarrow \rho\left[X\right] = \rho\left[Y\right]}$$

Valuations

- For any $X, Y \in \mathcal{C}$, the notation $X \overset{\mathbb{P}}{=} Y$ is used for

$$\mathbb{P}\left[X \leq x\right] = \mathbb{P}\left[Y \leq x\right], \text{ for all } x \in \mathbb{R}$$

- Let $\mathcal{C}'$ be a linear subspace of linear space $\mathcal{C}$ and consider the mapping $\rho : \mathcal{C}' \to \mathbb{R}$.
- **Properties that ρ may or may not satisfy**:
  - *$\mathbb{P}$-law invariance*:

$$\boxed{\rho\left[X\right] = \rho\left[Y\right]} \quad \text{for any } X, Y \in \mathcal{C}' \text{ with } X \overset{\mathbb{P}}{=} Y$$

  - *Positive homogeneity*: $\boxed{\rho\left[aX\right] = a\,\rho\left[X\right]}$ for any scalar $a > 0$ and any $X \in \mathcal{C}'$.
  - *Subadditivity*:

$$\boxed{\rho\left[X + Y\right] \leq \rho\left[X\right] + \rho\left[Y\right]} \quad \text{for any } X, Y \in \mathcal{C}'$$

Valuations

Two well-known types of valuations

- **Definition**: A **financial valuation** is a valuation $\rho : \mathcal{C} \to \mathbb{R}$ such that

$$\boxed{\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S\right]} \quad \text{for any } S \in \mathcal{C}$$

  where $\mathbb{Q}$ is an equivalent martingale measure (EMM).

- **Definition**: An **actuarial valuation** is a valuation $\rho : \mathcal{C} \to \mathbb{R}$ such that

$$\boxed{\rho\left[S\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[S\right] + \text{RM}\left[S\right]\right)} \text{ for any } S \in \mathcal{C}$$

  where the mapping $\text{RM} : \mathcal{C} \to \mathbb{R}$ is $\mathbb{P}$-law invariant.

- **Properties of the risk margin**:

$$\boxed{\text{RM}\left[0\right]=0 \quad \text{and} \quad \text{RM}\left[S+a\right]=\text{RM}\left[S\right]}$$

- **The definition of an actuarial valuation**:
  - is based on the **subjective choice** of the properties that $\text{RM}:\mathcal{C}\rightarrow\mathbb{R}$ has to satisfy.
  - Here, we assume $\mathbb{P}$-law invariance of this mapping RM.
  - **Other possible choices**:
    - RM is $\mathbb{P}$-law invariant and subadditive in $\mathcal{C}$.
    - $\text{RM}[S]=2\ \sigma_S^{\mathbb{P}}$, for any $\mathcal{S}$ in $\mathcal{C}$.
  - The equivalence results that we will derive hereafter remain to hold for any subjective choice of the properties that the mapping $\text{RM}:\mathcal{C}\rightarrow\mathbb{R}$ is supposed to satisfy.

Valuations

Two examples of actuarial valuations

- **The CoC principle**:

$$\boxed{\rho\left[S\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[S\right] + i\ \left(\text{VaR}_{p}^{\mathbb{P}}\left[S\right] - \mathbb{E}^{\mathbb{P}}\left[S\right]\right)\right)}$$

with $0 < i < 1, 0 < p < 1$.

- **The Standard Deviation (SD) principle**:

$$\boxed{\rho\left[S\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[S\right] + \alpha\ \sigma^{\mathbb{P}}\left[S\right]\right)}$$

with $\alpha \geq 0$.

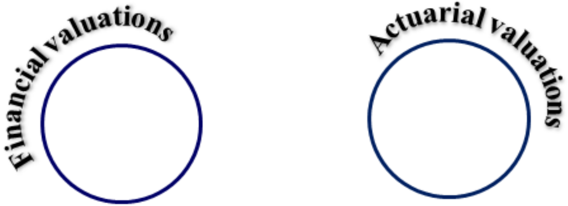
Valuations

Valuating different types of claims

- **Hedgeable claims**: A **financial valuation** is appropriate.
- **Orthogonal claims**: An **actuarial valuation** is appropriate.
- **Hybrid claims**:
  - In general, the classes of financial and actuarial valuations are disjunct.
  - Neither a financial nor an actuarial valuation is appropriate for valuating hybrid claims.
  - We will enlarge the class of financial valuations to the class of **market-consistent valuations,** such that hedgeable parts of claims are still valuated with a financial valuation.
  - We will enlarge the class of actuarial valuations to the class of **model-consistent valuations**, such that orthogonal claims are still valuated with an actuarial valuation.
  - We propose to valuate a hybrid claim by a **fair valuation**, which is defined as a valuation which is in the intersection of the classes of market-consistent and model-consistent valuations.

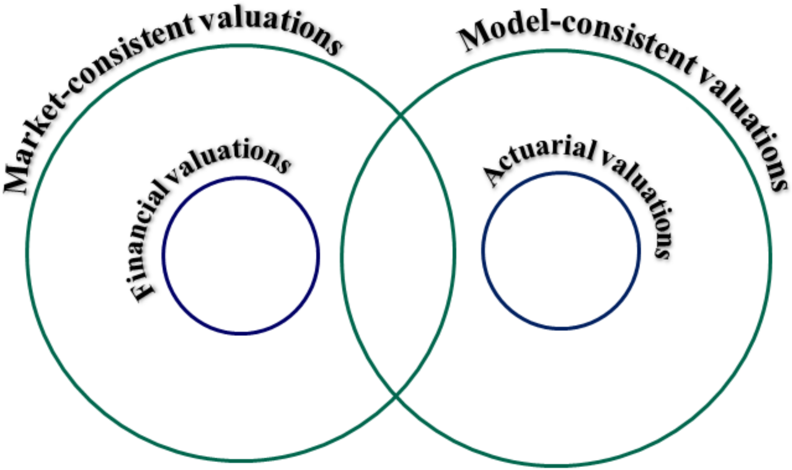
Valuations

Valuating different types of claims



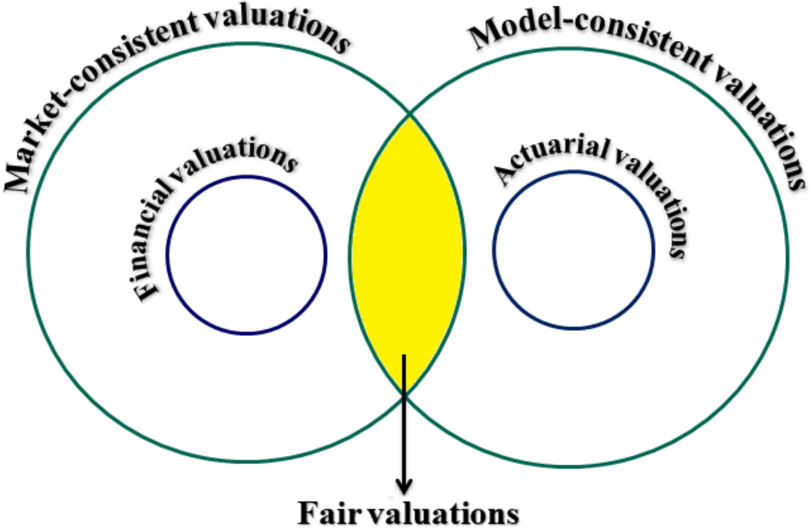
Valuations

Valuating different types of claims



Valuations

Valuating different types of claims



Valuations

Market-consistency

- **Definition**: A valuation ρ is **market-consistent**[5] if any hedgeable part of a claim is **marked-to-market**[6]:

$$\boxed{\rho\left[S+S^h\right]=\rho\left[S\right]+\boldsymbol{\nu}\cdot\mathbf{y}} \text{ for any } S\in\mathcal{C} \text{ and any } S^h=\boldsymbol{\nu}\cdot\mathbf{Y}\in\mathcal{C}^h$$

- **Remarks**:
  - Market-consistency means that the value of any hedgeable part of a claim is determined by the **financial market**, i.e. it is based on observed financial market prices.
  - Market–consistency is an an extension of the notion of translation invariance.
  - Any financial valuation is market-consistent.

[5] Cont (2006), Kupper et al. (2008), Malamud et al. (2008), Artzner & Eisele (2010), Stadje & Pelsser (2014).

[6] *Mark-to-market* is the practice of valuing ... using current market prices (Solvency II Glossary).

Valuations

Model-consistency

- **Definition**: A valuation $\rho : \mathcal{C} \to \mathbb{R}$ is **model-consistent** if any orthogonal claim is **marked-to-model**[7]:

$$\boxed{\rho\left[S^{\perp}\right] = \pi\left[S^{\perp}\right]} \text{ for all } S^{\perp} \in \mathcal{C}^{\perp}$$

  for a given actuarial valuation π.

- **Remarks**:
  - Model-consistency means that the value of any orthogonal claim is determined by an **actuarial model**, i.e. it is based on an actuarial valuation.
  - π is called the underlying actuarial valuation of ρ.
  - Any actuarial valuation is model-consistent.

[7] *Mark-to-model* is the practice of valuing ... based on modeling (Solvency II Glossary).

Valuations

- **Definition**: A valuation ρ is a **fair valuation**[8] if it is both market-consistent and model-consistent:
  - **Mark-to-market for any hedgeable part of a claim**:
    For any claim S and any hedgeable claim $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$, one has
    $$\boxed{\rho\left[S + S^h\right] = \rho\left[S\right] + \boldsymbol{\nu} \cdot \mathbf{y}}$$
  - **Mark-to-model for any orthogonal claim**:
    For any orthogonal claim $S^{\perp}$, one has
    $$\boxed{\rho\left[S^{\perp}\right] = \pi\left[S^{\perp}\right]}$$
    for a given actuarial valuation π.
- We consider the *generic meaning* of fair valuation, and not a particular meaning that is given to it by a particular regulation.

[8] *Fair Value* is the amount for which ... a liability could be settled between knowledgeable, willing parties in an arm's length transaction. This is similar to the concept of Market Value, but the Fair Value may be a mark-to-model price if no actual market price for the ... liability exists (Solvency II Glossary).

Valuations

Exercise 4:

- Consider an EMM $\mathbb{Q}$ and define the following **valuations**:
  - Valuation 1:

$$\boxed{\rho_1[S] = e^{-r}\,\mathbb{E}^{\mathbb{Q}}[S]} \quad \text{for any } S \in \mathcal{C}$$

  - Valuation 2:

$$\boxed{\rho_2[S] = e^{-r}\,\mathbb{E}^{\mathbb{P}}[S]} \quad \text{for any } S \in \mathcal{C}$$

  - Valuation 3:

$$\boxed{\rho_3[S] = e^{-r}\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]\right]} \quad \text{for any } S \in \mathcal{C}$$

- **Q**: Verify whether these valuations are market-consistent, model-consistent and/or fair.

Valuations

- **Definition**: A valuation ρ is **strong model-consistent** if any orthogonal part of a claim is marked-to-model:

$$\boxed{\rho\left[S+S^{\perp}\right]=\rho\left[S\right]+\pi\left[S^{\perp}\right]} \text{ for any } S\in\mathcal{C} \text{ and any } S^{\perp}\in\mathcal{C}^{\perp}$$

  where π is a given actuarial valuation.
- Strong model-consistency implies model-consistency.
- A strong model-consistent valuation is *additive for orthogonal claims*:

$$\boxed{\rho\left[X^{\perp}+Y^{\perp}\right]=\rho\left[X^{\perp}\right]+\rho\left[Y^{\perp}\right]}$$

  for any $X^{\perp}$ and $Y^{\perp}\in\mathcal{C}^{\perp}$.
- Conclusion: Strong model-consistency is not appropriate for fair valuation as it ignores diversification benefits from pooling $\mathbb{P}$ - i.i.d. orthogonal claims .

Valuations

- **Fair valuation of $S^h + S^\perp$:**

$$\rho\left[S^h + S^\perp\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] + \pi\left[S^\perp\right]$$

where π is the underlying actuarial valuation of ρ.

- **Fair valuation of $S^h \times S^\perp$:**
  - **General case**:
    - Solvency regulations do not specify the hedgeable part of $S^h \times S^\perp$.
  - **Special case: The Brennan-Schwartz**[9] formula:
    - Assumption: The insurer is risk-neutral towards the orthogonal risk:

$$S^\perp = \mathbb{E}^{\mathbb{P}}\left[S^\perp\right]$$

    - Fair Valuation of $S^h \times S^\perp$:

$$\rho\left[S^h \times S^\perp\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \mathbb{E}^{\mathbb{P}}\left[S^\perp\right]$$

[9] Brennan and Schwartz (1976, 1979a,b)

4. Hedgers

- **Definition**: A **hedger** is a function $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ which maps any claim S into a trading strategy:

$$\boxed{S \to \boldsymbol{\theta}_S = \left(\theta_S^{(0)}, \theta_S^{(1)}, \ldots, \theta_S^{(n)}\right)}$$

  such that

  - $\boldsymbol{\theta}$ is **normalized**:

$$\boxed{\boldsymbol{\theta}_0 = (0, 0, \ldots, 0)}$$

  - $\boldsymbol{\theta}$ is **translation invariant**:

$$\boxed{\boldsymbol{\theta}_{S+a} = \boldsymbol{\theta}_S + (e^{-r}a, 0, \ldots, 0)} \quad \text{for any } S \in \mathcal{C} \text{ and } a \in \mathbb{R}$$

Hedgers

- **Remarks**:
  - The mapping $\boldsymbol{\theta} : \mathcal{C} \longrightarrow \Theta$ is called a *hedger*.
  - The trading strategy $\boldsymbol{\theta}_S$ is called a *hedge* for S.
  - $\boldsymbol{\theta}_S$ may be a partial or a perfect hedge for S.
- **Time-0 value of the hedge $\boldsymbol{\theta}_S$**:

$$\boldsymbol{\theta}_S \cdot \mathbf{y} = \sum_{m=0}^{n} \theta_S^{(m)} \; y^{(m)} = e^{-r} \; \mathbb{E}^{\mathbb{Q}} \left[\boldsymbol{\theta}_S \cdot \mathbf{Y} \right]$$

- **Time-1 value of the hedge $\boldsymbol{\theta}_S$**:

$$\boldsymbol{\theta}_S \cdot \mathbf{Y} = \sum_{m=0}^{n} \theta_S^{(m)} \; Y^{(m)}$$

Hedgers

Possible properties of hedgers:

- *Positive homogeneity*:

$$\boxed{\boldsymbol{\theta}_{a\,S} = a\,\boldsymbol{\theta}_S} \quad \text{for any scalar } a > 0 \text{ and any } S \in \mathcal{C}$$

- *Additivity*:

$$\boxed{\boldsymbol{\theta}_{S_1+S_2} = \boldsymbol{\theta}_{S_1} + \boldsymbol{\theta}_{S_2}} \quad \text{for any } S_1, S_2 \in \mathcal{C}$$

Hedgers

- **Definitions**:
  - $\boldsymbol{\theta}$ is a **market-consistent hedger** in case

$$\boxed{\boldsymbol{\theta}_{S+S^h} = \boldsymbol{\theta}_S + \boldsymbol{\nu}} \quad \text{for any } S \in \mathcal{C} \text{ and any } S^h = \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^h$$

  - $\boldsymbol{\theta}$ is a **model-consistent hedger** in case there exists an actuarial valuation π such that

$$\boxed{\boldsymbol{\theta}_{S^\perp} = \left(\pi\left[S^\perp\right], 0, \ldots, 0\right)} \quad \text{for any } S^\perp \in \mathcal{C}^\perp$$

  - $\boldsymbol{\theta}$ is a **fair hedger** in case it is *market-consistent* and *model-consistent*.
- The actuarial valuation π is called the **underlying actuarial valuation** of the model-consistent hedger $\boldsymbol{\theta}$.

Hedgers

Exercise 5: Consider a claim S, an orthogonal claim $S^{\perp}$, a hedgeable claim $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$ and a scalar a . Prove the following statements:

- **Q1**: For any hedger $\boldsymbol{\theta}$:

$$\boxed{\boldsymbol{\theta}_a = \left(e^{-r}\ a, 0, \ldots, 0\right)}$$

- **Q2**: For any market-consistent hedger $\boldsymbol{\theta}$:

$$\boxed{\boldsymbol{\theta}_{S^h} = \boldsymbol{\nu}}$$

- **Q3**: For any fair hedger $\boldsymbol{\theta}$ with underlying actuarial valuation given by π:

$$\boxed{\boldsymbol{\theta}_{S^{\perp}+S^h} = \left(\pi\left[S^{\perp}\right], 0, \ldots, 0\right) + \boldsymbol{\nu}}$$

- **Goal**: Find the hedger $\boldsymbol{\theta}$ such that any claim S is *as close as possible* to $\boldsymbol{\theta}_S \cdot \mathbf{Y}$.
- **Definition**:
  - Consider the strictly convex function $u \geq 0$ with $u(0) = 0$.
  - The **convex hedger** $\boldsymbol{\theta}^u$ is defined by

$$\boxed{\boldsymbol{\theta}_S^u = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[u\left(S - \boldsymbol{\mu}\cdot\mathbf{Y}\right)\right]} \quad \text{for any } S \in \mathcal{C}$$

- **Theorem**: The convex hedger $\boldsymbol{\theta}^u$ is a fair hedger with underlying actuarial valuation π^u given by

$$\boxed{\pi^u\left[S^{\perp}\right] = \arg\min_{s\in\mathbb{R}} \mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - e^{r}s\right)\right]} \quad \text{for any } S^{\perp} \in \mathcal{C}^{\perp}$$

- **Definition**: The **Mean-Variance hedge** of S is the trading strategy $\boldsymbol{\theta}_S^{MV}$ that minimizes the *expected quadratic hedging error*:

$$\boxed{\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[\left(S - \boldsymbol{\mu}\cdot\mathbf{Y}\right)^2\right]}$$

- The MV hedge is also called the **quadratic hedge**.
- The **MV hedger** of S is the function $\boldsymbol{\theta}^{MV} : \mathcal{C} \to \Theta$ which maps any claim S into its MV hedge:

$$\boxed{S \to \boldsymbol{\theta}_S^{MV}}$$

- **Corollary**: The MV hedger $\boldsymbol{\theta}^{MV}$ is a fair hedger with underlying actuarial valuation π^{MV} given by

$$\boxed{\pi^{MV}\left[S^{\perp}\right] = e^{-r}\,\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]} \quad \text{for any } S^{\perp} \in \mathcal{C}^{\perp}$$

Hedgers

The Mean-Variance hedger

- **Theorem**: The Mean-Variance hedge $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \ldots, \theta_S^{(n)}\right)$ of $S \in \mathcal{C}$ is uniquely determined from

$$\boxed{\sum_{m=0}^{n} \mathbb{E}^{\mathbb{P}}\left[Y^{(k)}.Y^{(m)}\right] \times \theta_S^{(m)} = \mathbb{E}^{\mathbb{P}}\left[S.Y^{(k)}\right]} \quad \text{for } k = 0, 1, ..., n$$

- **Exercise 6**:
  - **Q1**: Give a proof of the Theorem.
  - **Q2**: Show that $\boldsymbol{\theta}_S^{MV}$ can also be determined from

$$\boxed{\sum_{m=1}^{n} \text{cov}^{\mathbb{P}}\left[Y^{(k)}, Y^{(m)}\right] \times \theta_S^{(m)} = \text{cov}^{\mathbb{P}}\left[Y^{(k)}, S\right]} \quad \text{for } k = 1, ..., n$$

and

$$\boxed{\theta_S^{(0)} = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S] - \sum_{m=1}^{n} \mathbb{E}^{\mathbb{P}}\left[Y^{(m)}\right] \times \theta_S^{(m)}\right)}$$

- **Theorem**: **Properties of the MV hedger**.
  - Let $S, S_1, S_2 \in \mathcal{C}$, $S^h = \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^h$, $S^\perp \in \mathcal{C}^\perp$ and $a > 0$.
  - A claim and the time-1 value of its MV hedge are equal in expectation:

$$\boxed{\mathbb{E}^{\mathbb{P}}[S] = \mathbb{E}^{\mathbb{P}}\left[\boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]}$$

  - The MV hedger is additive:

$$\boxed{\boldsymbol{\theta}_{S_1+S_2}^{MV} = \boldsymbol{\theta}_{S_1}^{MV} + \boldsymbol{\theta}_{S_2}^{MV}}$$

  - The MV hedger is positive homogeneous:

$$\boxed{\boldsymbol{\theta}_{a\times S}^{MV} = a \times \boldsymbol{\theta}_S^{MV}}$$

  - The MV hedge of the product of a hedgeable and an orthogonal claim:

$$\boxed{\boldsymbol{\theta}_{S^h\times S^\perp}^{MV} = \boldsymbol{\nu} \times \mathbb{E}^{\mathbb{P}}\left[S^\perp\right]}$$

Hedgers

The Mean-Variance hedger

- **Theorem**: **Further properties of the MV hedger.**
  - Consider $S \in \mathcal{C}$, $S^{\perp} \in \mathcal{C}^{\perp}$ and a Borel-measurable function f.
  - The MV hedge of the product of a derivative and an orthogonal claim:

$$\boxed{\boldsymbol{\theta}^{MV}_{f(\mathbf{Y})\times S^{\perp}} = \boldsymbol{\theta}^{MV}_{f(\mathbf{Y})} \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]}$$

  - The MV hedge of a claim vs. the MV hedge of its conditional expectation:

$$\boxed{\boldsymbol{\theta}^{MV}_{S} = \boldsymbol{\theta}^{MV}_{\mathbb{E}^{\mathbb{P}}[S|\mathbf{Y}]}}$$

- **Exercise 7**: Give a proof for the properties of the MV hedger considered on the previous and the current slide.

Hedgers

Exercise 3": **Decomposing insurance portfolio liabilities.**

- Consider a **portfolio of N insurance contracts**, with payoff of contract i at time 1 given by

$$\boxed{S \times X_i} \qquad i = 1, 2, \ldots, N$$

- Assumptions:
  - S and all X_i are elements of $\mathcal{C}$.
  - There exists a r.v. $Z \in \mathcal{C}$ with support A, such that for any $z \in A$, one has that $(X_1 \mid Z = z), \ldots, (X_N \mid Z = z)$ are $\mathbb{P}-$ i.i.d. claims.

- **Insurance portfolio liability per policy**:

$$\boxed{S \times \frac{S_N}{N}}$$

with $S_N = \sum_{i=1}^{N} X_i$

Hedgers

Exercise 3" (cont'd):

- The **insurance portfolio liability per policy** can be decomposed into:

$$\boxed{S \times \frac{S_N}{N} = Y^h + Y^d_N + Y^r}$$

- In this decomposition formula,
  - Y^h is given by

$$\boxed{Y^h = \theta^{MV}_{S \times \frac{S_N}{N}} \cdot \mathbf{Y}}$$

  - Y^d_N is given by

$$\boxed{Y^d_N = S \times \left(\frac{S_N}{N} - \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]\right)}$$

  - Y^r is given by

$$\boxed{Y^r = S \times \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right] - \theta^{MV}_{S \times \frac{S_N}{N}} \cdot \mathbf{Y}}$$

Hedgers

Exercise 3" (cont'd):

- According to the Conditional LLN, we have that

$$\frac{S_N}{N} \xrightarrow{\mathrm{P}} \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]$$

- **Q1**: Show that

$$\boxed{Y_N^d \xrightarrow{\mathrm{P}} 0}$$

  - Hint: If $Y_N \xrightarrow{\mathrm{P}} Y$ and $Z_N \xrightarrow{\mathrm{P}} Z$, then $f\left(Y_N, Z_N\right) \xrightarrow{\mathrm{P}} f\left(Y, Z\right)$ for any continuous function f.
  - This convergence result can also be stated as

$$\boxed{S \times \frac{S_N}{N} \xrightarrow{\mathrm{P}} S \times \mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]}$$

- **Q2**: Give an interpretation of Y^h, Y_N^d and Y^r.

Hedgers

Exercise 3" (cont'd):

- **Q3**: Show that in case $S \equiv f(\mathbf{Y})$ and $S_N/N \in \mathcal{C}^{\perp}$, the **insurance portfolio liability per policy** can be expressed as

$$\boxed{f(\mathbf{Y}) \times \frac{S_N}{N} = Y^h + Y^d_N + Y^r_{act} + Y^r_{fin}}$$

with

$$Y^h = \boldsymbol{\theta}^{MV}_{f(\mathbf{Y})} \cdot \mathbf{Y} \times \mathbb{E}^{\mathbb{P}}[X_1]$$

$$Y^d_N = f(\mathbf{Y}) \times \left(\frac{S_N}{N} - \mathbb{E}^{\mathbb{P}}[X_1 \mid Z] \right)$$

$$Y^r_{act} = f(\mathbf{Y}) \times \left(\mathbb{E}^{\mathbb{P}}[X_1 \mid Z] - \mathbb{E}^{\mathbb{P}}[X_1] \right)$$

$$Y^r_{fin} = \left(f(\mathbf{Y}) - \boldsymbol{\theta}^{MV}_{f(\mathbf{Y})} \cdot \mathbf{Y} \right) \times \mathbb{E}^{\mathbb{P}}[X_1]$$

- **Q4**: Give an interpretation of each of the 4 terms in this decomposition.

5. Linking valuations and hedgers

Lemma:

- Consider a hedger $\boldsymbol{\theta}$ and a valuation ρ. Define the hedger $\boldsymbol{\mu}$ by

$$\boxed{\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + \left(\rho\left[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}\right], 0, \ldots, 0\right)} \quad \text{for any } S \in \mathcal{C}$$

- If $\boldsymbol{\theta}$ is a market-consistent hedger, then $\boldsymbol{\mu}$ is a **market-consistent hedger**.
- If $\boldsymbol{\theta}$ is a model-consistent hedger[10] and ρ is a model-consistent valuation with underlying actuarial valuation π, then $\boldsymbol{\mu}$ is a **model-consistent hedger** with underlying actuarial valuation π.

[10] This condition can be weakened to '$\boldsymbol{\theta}$ hedges any orthogonal claim by a zero coupon bond trading strategy'.

Linking valuations and hedgers

Lemma:

- Let $\boldsymbol{\theta}$ be a fair hedger[11].
- Let ρ be a model-consistent valuation with underlying actuarial valuation π.
- Then the hedger $\boldsymbol{\mu}$ defined by

$$\boxed{\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + \left(\rho\left[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}\right], 0, \ldots, 0\right)} \quad \text{for any } S \in \mathcal{C}$$

is a **fair hedger** with underlying actuarial valuation π.

[11] This condition can be weakened to '$\boldsymbol{\theta}$ is a market-consistent hedger which hedges any orthogonal claim by a zero coupon bond trading strategy'.

Linking valuations and hedgers

Theorem:

- Consider the valuation $\rho : \mathcal{C} \to \mathbb{R}$.
- ρ is a **market-consistent valuation** if and only if there exists a market-consistent hedger $\boldsymbol{\mu}^{mac}$ such that

$$\boxed{\rho\left[S\right] = \boldsymbol{\mu}_S^{mac} \cdot \mathbf{y}} \quad \text{for any } S \in \mathcal{C}$$

- ρ is a **model-consistent valuation** if and only if there exists a model-consistent hedger $\boldsymbol{\mu}^{moc}$ such that

$$\boxed{\rho\left[S\right] = \boldsymbol{\mu}_S^{moc} \cdot \mathbf{y}} \quad \text{for any } S \in \mathcal{C}$$

- ρ is a **fair valuation** if and only if there exists a fair hedger $\boldsymbol{\mu}^{f}$ such that

$$\boxed{\rho\left[S\right] = \boldsymbol{\mu}_S^{f} \cdot \mathbf{y}} \quad \text{for any } S \in \mathcal{C}$$

Linking valuations and hedgers

Remark: **Why we can't get rid of actuaries**

- The valuation ρ defined by

$$\boxed{\rho\left[S\right]=e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S\right]} \quad \text{for any } S\in\mathcal{C}$$

  is not a fair valuation (see Exercise 4).

- A fair valuation ρ can always be expressed as

$$\boxed{\rho\left[S\right]=e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_S^f\cdot\mathbf{Y}\right]} \quad \text{for any } S\in\mathcal{C}$$

  for some **fair hedger** $\boldsymbol{\mu}_S^f$, which implies that an actuarial valuation is involved.

6. Hedge-based valuations

- **Definition**:
  $\rho : \mathcal{C} \to \mathbb{R}$ is a **hedge-based valuation** if

$$\boxed{\rho\,[S] = \boldsymbol{\theta}_S \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}]} \quad \text{for any } S \in \mathcal{C}$$

  where $\boldsymbol{\theta}$ *is a fair hedger* and π is a model-consistent valuation.
- **Remark**: An important subclass of the class of hedge-based (HB) valuations arises if we require π to be an actuarial valuation.
- **Exercise 8**:
  - Consider a hedge-based valuation ρ.
  - **Q1**: Show that ρ is normalized and translation invariant, and hence, a valuation.
  - **Q2**: Show that

$$\boxed{\rho\left[S^h\right] = e^{-r}\,\mathbb{E}^{\mathbb{Q}}\left[S^h\right]} \quad \text{and} \quad \boxed{\rho\left[S^{\perp}\right] = \pi[S^{\perp}]}$$

Hedge-based valuations

- **Theorem**:
  - A **HB valuation** ρ is **positive homogeneous**:

    $$\boxed{\rho\left[a\ S\right] = a\ \rho\left[S\right]} \quad \text{for any } a > 0 \text{ and } S \in \mathcal{C}$$

    if its underlying $\boldsymbol{\theta}$ and π are positive homogeneous.
  - A **HB valuation** ρ is **subadditive**:

    $$\boxed{\rho\left[S_1 + S_2\right] \leq \rho\left[S_1\right] + \rho\left[S_2\right]} \quad \text{for any } S_1, S_2 \in \mathcal{C}$$

    if its underlying $\boldsymbol{\theta}$ is additive and π is subadditive.
- **Theorem**:

  $$\boxed{\rho \text{ is a } \textbf{HB valuation} \Leftrightarrow \rho \text{ is a } \textbf{fair valuation}}$$

- **Definition**:
  - Consider the strictly convex function $u \geq 0$ with $u(0) = 0$.
  - The valuation $\rho : \mathcal{C} \to \mathbb{R}$ defined by

$$\boxed{\rho\,[S] = \boldsymbol{\theta}^u_S \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}^u_S \cdot \mathbf{Y}]}$$

  with convex hedger $\boldsymbol{\theta}^u$ and model-consistent valuation π is called a **convex hedge-based valuation** (CHB valuation).

- **Corollary**:

Any CHB valuation is a fair valuation

Hedge-based valuations

Mean-variance hedge-based valuations

- **Definition**:
  - Consider the MV hedger $\boldsymbol{\theta}^{MV}$ and a model-consistent valuation π.
  - The valuation $\rho : \mathcal{C} \to \mathbb{R}$ defined by

$$\boxed{\rho\,[S] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}]}$$

  is a **mean-variance hedge-based valuation** (MVHB)[12].

- **Corollary**: **Properties of the MVHB valuation** ρ with model-consistent valuation π.
  - ρ is a fair valuation.
  - If π is positive homogeneous, then ρ is positive homogeneous.
  - If π is subadditive, then ρ is subadditive.

[12] MV hedging for valuating insurance liabilities is also considered in Tsanakas, Wütrich and Cerny (2013).

Hedge-based valuations

Mean-variance hedge-based valuations

- For the subclass of MVHB valuations where π is an actuarial valuation, we find :

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + e^{-r}\ \text{RM}\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]}$$

  - **MVHB valuation with CoC principle π**:

$$\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + e^{-r} i\ \text{VaR}_p^{\mathbb{P}}\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]$$

  - **MVHB valuation with SD principle π**:

$$\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \alpha e^{-r} \sigma^{\mathbb{P}}\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]$$

- **Exercise 9**:
  - Consider the following fair valuation:

$$\boxed{\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right]} \quad \text{for any } S \in \mathcal{C}$$

  - **Q**: Express ρ as a MVHB valuation.

Hedge-based valuations

Brennan-Schwartz formula for MVHB valuations

- Consider the **MVHB valuation** ρ defined by

$$\boxed{\rho\left[X\right] = \boldsymbol{\theta}_X^{MV} \cdot \mathbf{y} + \pi[X - \boldsymbol{\theta}_X^{MV} \cdot \mathbf{Y}]} \quad \text{for any } X \in \mathcal{C}$$

- Consider the **product liability S**:

$$\boxed{S = S^h \times S^{\perp}} \quad \text{with } S^h \in \mathcal{C}^h \text{ and } S^{\perp} \in \mathcal{C}^{\perp}$$

- **Brennan-Schwartz formula for MVHB valuations**:

$$\boxed{\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] + \pi\left[S^h \times \left(S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right)\right]}$$

Hedge-based valuations

Exercise 10: **Brennan-Schwartz formula**.

- Consider the **MVHB valuation** ρ defined by

$$\rho\left[X\right] = \boldsymbol{\theta}_X^{MV} \cdot \mathbf{y} + \pi[X - \boldsymbol{\theta}_X^{MV} \cdot \mathbf{Y}], \quad \text{for any } X \in \mathcal{C}$$

  with π the **standard deviation principle**:

$$\pi\left[X\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[X\right] + \alpha\, \sigma^{\mathbb{P}}\left[X\right]\right), \quad \text{for all } X \in \mathcal{C}$$

- Consider the **product liability S**:

$$\boxed{S = S^h \times S^{\perp}} \quad \text{with } S^h \in \mathcal{C}^h \text{ and } S^{\perp} \in \mathcal{C}^{\perp}$$

- **Q1**: Show that

$$\boxed{\rho\left[S\right] = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] + \alpha\, e^{-r}\, \sigma^{\mathbb{P}}\left[S^{\perp}\right] \sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right]}}$$

Hedge-based valuations

Exercise 10: (cont'd).

- Consider an insurance portfolio of N policies, where policy i pays $S^h \times X_i^\perp$.
  - Assume that $S^h \in \mathcal{C}^h$ and the $X_i^\perp$ are $\mathbb{P}-$i.i.d. elements of $\mathcal{C}^\perp$.
- The aggregate claims of the insurance portfolio is given by $S^h \times S^\perp$, with

$$\boxed{S^\perp = \sum_{i=1}^N X_i^\perp}$$

- Suppose that each policy is charged a premium $\frac{\rho\left[S^h \times S^\perp\right]}{N}$.
- **Q2**: Show that

$$\boxed{\frac{\rho\left[S^h \times S^\perp\right]}{N} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right]\ \mathbb{E}^{\mathbb{P}}\left[X_1^\perp\right] + \alpha e^{-r} \frac{\sigma^{\mathbb{P}}\left[X_1^\perp\right]}{\sqrt{N}} \sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right]}}$$

Exercise 10: (cont'd).

- **Suppose**:
  - $S^h > 0$.
  - Each insurance contract is charged a premium $\frac{\rho\left[S^h \times S^\perp\right]}{N}$.
  - These premiums are fully invested in units of S^h.
- **Q3**: Show that the probability that the time-1 value of the invested premiums exceeds the time-1 liability $S^h \times S^\perp$ is given by

$$\boxed{\mathbb{P}\left[\frac{S^\perp - \mathbb{E}^{\mathbb{P}}\left[S^\perp\right]}{\sigma^{\mathbb{P}}\left[S^\perp\right]} \leq \alpha \frac{\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right]}}{\mathbb{E}^{\mathbb{Q}}\left[S^h\right]}\right]}$$

Exercise 11: **Unit-linked insurance**.

- Consider a **portfolio of N insureds**, with

$$X_i^{\perp} = \begin{cases} 0 & : \text{insured } i \text{ dies before time 1} \\ 1 & : \text{insured } i \text{ is alive at time 1} \end{cases}$$

  - The orthogonal claims $X_i^{\perp}$ are i.i.d. with mean p (under $\mathbb{P}$).
  - Number of survivors at time 1:

$$\boxed{S_N^{\perp} = \sum_{i=1}^{N} X_i^{\perp}}$$

- Each insured i underwrites a **unit-linked contract** with time-1 payoff:

$$\boxed{\max\left(Y^{(1)}, K\right) \times X_i^{\perp}} \quad \text{with } K > 0$$

- Suppose that the traded assets are $Y^{(0)}$, $Y^{(1)}$ and $Y^{(2)} = \left(K - Y^{(1)}\right)_+$.

Exercise 11: (cont'd).

- **Unit-linked insurance portfolio liability**:

$$S^h \times S_N^{\perp} = \max\left(Y^{(1)}, K\right) \times \sum_{i=1}^{N} X_i^{\perp}$$

- **Q1**: Show that the MV hedge of $S^h \times S_N^{\perp}$ is $N \times p \times (0, 1, 1)$.
- Consider the **MVHB valuation** ρ defined by

$$\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}] \quad \text{for any } S \in \mathcal{C}$$

with π the standard deviation principle:

$$\pi\left[X\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[X\right] + \alpha\ \sigma^{\mathbb{P}}\left[X\right]\right), \quad \text{for any } X \in \mathcal{C}$$

- **Q2**: Show that the **MVHB value of the unit-linked liability** (per policy) is given by

$$\frac{\rho\left[S^h \times S_N^{\perp}\right]}{N} = \left(y^{(1)} + y^{(2)}\right) p + \alpha\ e^{-r}\sqrt{\frac{p(1-p)}{N}}\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right]}$$

Exercise 11: (cont'd).

- **Suppose**:
  - Each unit-linked contract is charged a premium $\frac{\rho\left[S^h \times S_N^{\perp}\right]}{N}$.
  - These premiums are fully invested in units of S^h.
- Assumption: $S_N^{\perp}$ is (approx.) normal distributed (under $\mathbb{P}$).
- **Q3**: Show that the probability that the time-1 value of the invested premiums exceeds the time-1 liability $S^h \times S^{\perp}$ is given by

$$\boxed{\Phi\left[\alpha e^{-r} \frac{\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right]}}{\left(y^{(1)}+y^{(2)}\right)}\right]}$$

Hedge-based valuations

Exercise 12: **Unit-linked insurance.**

- Consider a **portfolio of N insurance contracts**, with payoff of contract i at time 1 given by $S^h \times X_i$ for any i.
- Assumptions:
  - $S^h \in \mathcal{C}^h$.
  - Any $X_i \in \mathcal{C}$.
  - There exists a r.v. $Z \in \mathcal{C}^\perp$ with support A, such that for any outcome $z \in A$, one has that $(X_1 \mid Z = z), \ldots, (X_N \mid Z = z)$ are $\mathbb{P}-$i.i.d. orthogonal claims.

Hedge-based valuations

Exercise 12: (cont'd).

- **Unit-linked insurance portfolio liability**:

$$\boxed{S^h \times S_N^\perp}$$

with $S_N^\perp = \sum_{i=1}^N X_i \in \mathcal{C}^\perp$.

- Consider the **MVHB valuation** of $S^h \times S_N^\perp$:

$$\boxed{\rho\left[S^h \times S_N^\perp\right] = \boldsymbol{\theta}^{MV}_{S^h \times S_N^\perp} \cdot \mathbf{y} + \pi[S^h \times S_N^\perp - \boldsymbol{\theta}^{MV}_{S^h \times S_N^\perp} \cdot \mathbf{Y}]}$$

with π the standard deviation principle:

$$\pi\left[X\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[X\right] + \alpha\, \sigma^{\mathbb{P}}\left[X\right]\right), \quad \text{for any } S \in \mathcal{C}$$

Hedge-based valuations

Exercise 12: (cont'd).

- **Q1**: Show that the **MVHB value of the unit-linked liability** per policy is given by

$$\boxed{\frac{\rho\left[S^h \times S^{\perp}_N\right]}{N} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right]\ \mathbb{E}^{\mathbb{P}}\left[X_1\right] + \alpha e^{-r}\sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(S^h\right)^2\right]\left(\frac{A}{N} + B\right)}}$$

  with

$$A = \mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}^{\mathbb{P}}\left[X_1 \mid Z\right]\right] \quad \text{and} \quad B = \mathrm{Var}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]\right]$$

- **Q2**: Show that the MVHB value of the unit-linked liability per policy can also be expressed as follows:

$$\boxed{\frac{\rho\left[S^h \times S^{\perp}_N\right]}{N} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[Y^h\right] + \alpha e^{-r}\sqrt{\mathrm{Var}^{\mathbb{P}}\left[Y^d_N\right] + \mathrm{Var}^{\mathbb{P}}\left[Y^r\right]}}$$

  with Y^h, Y^d_N and Y^r the hedgeable, diversifiable and residual claim as defined in Exercise 3'.

Hedge-based valuations

Exercise 12: (cont'd):

- In the remainder of this exercise, assume that

$$X_i = \begin{cases} 0 & : \text{insured } i \text{ dies before time } 1 \\ 1 & : \text{insured } i \text{ is alive at time } 1 \end{cases}$$

with $\mathbb{P}\left[X_i = 1 \mid Z\right] = p\left(Z\right)$ and $\mathbb{P}\left[X_i = 1\right] = p$.

- **Q3**: Show that

$$\boxed{A = \mathbb{E}^{\mathbb{P}}\left[p\left(Z\right) \times \left(1 - p\left(Z\right)\right)\right]} \quad \text{and} \quad \boxed{B = \text{Var}^{\mathbb{P}}\left[p\left(Z\right)\right]}$$

Hedge-based valuations

Exercise 13-1:

Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ with

- **Universe**:

$$\Omega = \{(0,0), (0,1), (1,0)\ (1,1)\}$$

  - First component = price $Y^{(1)}$ of stock 1 at time 1.
  - Second component = value of survival index $\mathcal{I}$ at time 1.

- **Probabilities**:

$$\begin{cases} p_{0,0} & = 1/6 \\ p_{1,0} & = 2/6 \\ p_{0,1} & = 1/6 \\ p_{1,1} & = 2/6 \end{cases}$$

Hedge-based valuations

Exercise 13-1 (cont'd):

The financial-actuarial world $\left(\Omega, 2^\Omega, \mathbb{P}\right)$ is home to

- A *traded* **zero-coupon bond**:
  - Current price: $y^{(0)} = 1$
  - Price at time 1: $Y^{(0)} = 1$
- A *traded* **stock**:
  - Current price: $y^{(1)} = 1/2$
  - Price at time 1: $Y^{(1)}$ is either 0 or 1
- A *non-traded* survival index:

$$\mathcal{I} = \begin{cases} 0 & \text{if few people survive} \\ 1 & \text{if many people survive} \end{cases}$$

- A *non-traded* combined claim:

$$\boxed{S = \left(1 - Y^{(1)}\right) \times (1 - \mathcal{I})}$$

Hedge-based valuations

Exercise 13-1 (cont'd):

- **Q1**: Show that the survival index $\mathcal{I}$ is an orthogonal claim.
- **Q2**: Show that the MV hedge of $\mathcal{I}$ is given by

$$\boxed{\boldsymbol{\theta}_{\mathcal{I}}^{MV} = \left(\tfrac{1}{2}, 0\right)}$$

- **Q3**: Determine the MVHB value $\rho\left[\mathcal{I}\right]$ of $\mathcal{I}$:

$$\boxed{\rho\left[\mathcal{I}\right] = \boldsymbol{\theta}_{\mathcal{I}}^{MV} \cdot \mathbf{y} + \pi[\mathcal{I} - \boldsymbol{\theta}_{\mathcal{I}}^{MV} \cdot \mathbf{Y}]}$$

- **Q4**: Determine the numerical value of $\rho\left[\mathcal{I}\right]$ in case π is a cost-of-capital principle:

$$\boxed{\rho\left[\mathcal{I}\right] = \boldsymbol{\theta}_{\mathcal{I}}^{MV} \cdot \mathbf{y} + 0.06\ \mathrm{VaR}_{0.995}^{\mathbb{P}}\left[\mathcal{I} - \boldsymbol{\theta}_{\mathcal{I}}^{MV} \cdot \mathbf{Y}\right]}$$

Hedge-based valuations

Exercise 13-1 (cont'd):

- **Q5**: Show that the MV hedge of S is given by

$$\boxed{\boldsymbol{\theta}_S^{MV} = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)}$$

- **Q6**: Determine the MVHB value $\rho\left[S\right]$ of S :

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}]}$$

- **Q7**: Determine the numerical value of $\rho\left[S\right]$ in case π is a cost-of-capital principle:

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + 0.06\ \mathrm{VaR}_{0.995}^{\mathbb{P}}\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]}$$

Hedge-based valuations

Exercise 13-2:

Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ with

- **Universe**:

$$\Omega = \left\{ (0,0), (0,1), (1,0)\ (1,1) \right\}$$

  - First component = price $Y^{(1)}$ of stock 1 at time 1.
  - Second component = value of survival index $\mathcal{I}$ at time 1.

- **Probabilities**:

$$\begin{cases} p_{0,0} & = 1/6 \\ p_{1,0} & = 2/6 \\ p_{0,1} & = 1/6 \\ p_{1,1} & = 2/6 \end{cases}$$

Hedge-based valuations

Exercise 13-2 (cont'd):

The financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ is home to

- A *traded* **zero-coupon bond**:
  - Current price: $y^{(0)}(0) = 1$.
  - Price at time 1: $Y^{(0)} = 1$.
- A *traded* **stock**:
  - Current price: $y^{(1)}(0) = 1/2$.
  - Price at time 1: $Y^{(1)}$, which is either 0 or 1.
- A *traded* **survival index**:
  - Current price: $y^{(2)} = 2/3$.
  - Payoff at time 1: $Y^{(2)} = \mathcal{I}$.
- A *non-traded* combined claim:

$$\boxed{S = \left(1 - Y^{(1)}\right) \times (1 - \mathcal{I})}$$

Hedge-based valuations

Exercise 13-2 (cont'd):

- **Q1**: Show that the MV hedge of S is given by

$$\boxed{\boldsymbol{\theta}_S^{MV} = \left(\tfrac{2}{3}, -\tfrac{1}{2}, -\tfrac{1}{3}\right)}$$

- **Q2**: Determine the MVHB value $\rho\left[S\right]$ of S:

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}]}$$

- **Q3**: Determine the numerical value of $\rho\left[S\right]$ in case π is a cost-of-capital principle:

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + 0.06\ \mathrm{VaR}_{0.995}^{\mathbb{P}}\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]}$$

Hedge-based valuations

Exercise 13-2':

- Consider the setting of Example 13-2, except that the current price of the traded survival index is given by $\boxed{y^{(2)} \in (0,1)}$.
- Consider the non-traded claim $S = \left(1 - Y^{(1)}\right) \times (1 - \mathcal{I})$.
- The MVHB value of S is determined by

$$\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + 0.06\ \mathrm{VaR}^{\mathbb{P}}_{0.995}\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right]$$

- **Q**: Show that

$$\boxed{\rho\left[S\right] = \frac{131 - 100\ y^{(2)}}{300}}$$

Hedge-based valuations

Exercise 13-3:

Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ with

- **Universe**:

$$\Omega = \{(0,0),(0,1),(1,0)\,(1,1)\}$$

  - First component = price $Y^{(1)}$ of stock 1 at time 1.
  - Second component = value of survival index $\mathcal{I}$ at time 1.

- **Probabilities**:

$$\begin{cases} p_{0,0} & = 1/6 \\ p_{1,0} & = 2/6 \\ p_{0,1} & = 1/6 \\ p_{1,1} & = 2/6 \end{cases}$$

Hedge-based valuations

Exercise 13-3 (cont'd):

The financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ is home to

- A *traded* **zero-coupon bond**: see Exercise 11-2.
- A *traded* **stock**: see Exercise 11-2.
- A *traded* **survival index**: see Exercise 11-2.
- A *traded* **call option**:
  - Current price: $y^{(3)} = 1/6$.
  - Payoff at time 1:

$$Y^{(3)} = \mathcal{I} \times \left(Y^{(1)} - 0.5\right)_{+}$$

- A *non-traded* combined claim:

$$\boxed{S = \left(1 - Y^{(1)}\right) \times (1 - \mathcal{I})}$$

Hedge-based valuations

Exercise 13-3 (cont'd):

- **Q1**: Show that the MV hedge of S is given by

$$\boxed{\boldsymbol{\theta}_S^{MV} = (1, -1, -1, 2)}$$

- **Q2**: Determine the MVHB value $\rho\,[S]$ of S :

$$\boxed{\rho\,[S] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}]}$$

Hedge-based valuations

Exercise 14:

- Consider a **national population** of N^{nat} members:

$$I_i = \begin{cases} 0 & : \text{ member } i \text{ dies before time 1} \\ 1 & : \text{ otherwise} \end{cases}$$

  - **National survival index**:

$$\boxed{I = I_1 + I_2 + \ldots + I_{N^{\text{nat}}}}$$

- Consider an **insured population** of N^{ins} members:

$$J_i = \begin{cases} 0 & : \text{ insured } i \text{ dies before time 1} \\ 1 & : \text{ otherwise} \end{cases}$$

  - **Insurance benefit payments at time 1**:

$$\boxed{S = J_1 + J_2 + \ldots + J_{N^{\text{ins}}}}$$

- **Remark**: The insured population is not necessary a subset of the national population.

Hedge-based valuations

Exercise 14 (cont'd):

- There is a **financial market** consisting of 3 traded assets:
  - **Zero-coupon bond**:
    - Current price: $y^{(0)} = 1$
    - Price at time 1: $Y^{(0)} = e^r$
  - **Stock**:
    - Current price: $y^{(1)}$
    - Price at time 1: $Y^{(1)} \in \mathcal{A}$.
  - **National survival index**:
    - Current price: $y^{(2)}$
    - Price at time 1: $Y^{(2)} = I$.
- **Financial-actuarial world** $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$:

$$\boxed{\Omega = \left\{ \left(x_1, x_2, x_3 \right) \mid x_1 \in \mathcal{A};\ x_2 = 0, ..., N^{\text{nat}};\ x_3 = 0, ..., N^{\text{ins}} \right\}}$$

  - Ω is support of $\left(Y^{(1)}, I, S\right)$.

Exercise 14 (cont'd):

- **Assumption**: $Y^{(1)}$ and (I, S) are $\mathbb{P}$ - independent.

- **Q1**: Show that the MV hedge $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)},\ \theta_S^{(1)},\ \theta_S^{(2)}\right)$ of S is given by

$$\begin{cases} \theta_S^{(0)} = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S] - \mathbb{E}^{\mathbb{P}}[I]\ \frac{\text{cov}^{\mathbb{P}}[I,S]}{\text{var}^{\mathbb{P}}[I]}\right) \\ \theta_S^{(1)} = 0 \\ \theta_S^{(2)} = \frac{\text{cov}^{\mathbb{P}}[I,S]}{\text{var}^{\mathbb{P}}[I]} \end{cases}$$

Hedge-based valuations

Exercise 14 (cont'd):

- **Additonal assumptions**:
  - $N^{\text{ins}} \leq N^{\text{nat}}$ and $J_i = I_i$ for $i = 1, 2, \ldots, N^{\text{ins}}$.
  - All I_i are i.i.d. under $\mathbb{P}$ with $\mathbb{P}\left[I_i = 1\right] = p > 0$.

- **Q2**: Show that the MV hedge $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)},\ \theta_S^{(1)},\ \theta_S^{(2)}\right)$ of S is now given by

$$\boxed{\begin{cases} \theta^{(0)} = 0 \\ \theta^{(1)} = 0 \\ \theta^{(2)} = \frac{N^{\text{ins}}}{N^{\text{nat}}} \end{cases}}$$

Hedge-based valuations

Exercise 14 (cont'd):

- Consider the MVHB valuation

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}]}$$

with π the standard deviation principle:

$$\boxed{\pi\left[X\right] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[X\right] + \alpha\ \sigma^{\mathbb{P}}\left[X\right]\right)} \qquad \text{for any } X \in \mathcal{C}$$

- **Q3**: Show that $\frac{\rho[S]}{N^{\text{ins}}}$ is given by

$$\boxed{\frac{\rho[S]}{N^{\text{ins}}} = \frac{y^{(2)}}{N^{\text{nat}}} + e^{-r}\ \alpha\ \sqrt{\left(\frac{1}{N^{\text{ins}}} - \frac{1}{N^{\text{nat}}}\right) p(1-p)}}$$

7. Two-step valuations

- **Definition**:
  - Consider the vector of asset prices $\mathbf{Y}$ in $(\Omega, \mathcal{G}, \mathbb{P})$.
  - A **derivative** of $\mathbf{Y}$ is a r.v. of the form $f(\mathbf{Y})$, for some Borel measurable function f.

- **Equivalent definition**:
  - Let $\mathcal{F}^{\mathbf{Y}} \subseteq \mathcal{G}$ be the sigma-algebra generated by the asset prices $\mathbf{Y}$.
  - A **derivative** of $\mathbf{Y}$ is a r.v. on $\left(\Omega, \mathcal{F}^{\mathbf{Y}}\right)$.

- We denote the linear space of all derivatives of $\mathbf{Y}$ by $\mathcal{C}^{\mathbf{Y}}$.

- **Examples of derivatives**:
  - Conditional expectation: $f(\mathbf{Y}) = \mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]$
  - Conditional variance: $f(\mathbf{Y}) = \mathrm{Var}^{\mathbb{P}}[S \mid \mathbf{Y}]$
  - Time-1 value of trading strategy $\boldsymbol{\mu}$: $f(\mathbf{Y}) = \boldsymbol{\mu} \cdot \mathbf{Y}$

Two-step valuations

- **Definition**: A **conditional valuation** is a mapping $\pi_{\mathbf{Y}} : \mathcal{C} \to \mathcal{C}^{\mathbf{Y}}$ attaching a derivative of $\mathbf{Y}$ to any claim S:

$$\boxed{S \longrightarrow \pi_{\mathbf{Y}}[S]}$$

  such that

  - $\pi_{\mathbf{Y}}$ is **normalized**:

$$\boxed{\pi_{\mathbf{Y}}[0] = 0}$$

  - $\pi_{\mathbf{Y}}$ is **translation invariant**:

$$\boxed{\pi_{\mathbf{Y}}[S + a] = \pi_{\mathbf{Y}}[S] + e^{-r}\ a} \qquad \text{for any } S \in \mathcal{C} \text{ and } a \in \mathbb{R}$$

- **Examples of conditional valuations**:
  - $\pi_{\mathbf{Y}}[S] = e^{-r}\ \mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]$
  - $\pi_{\mathbf{Y}}[S] = e^{-r}\ \boldsymbol{\theta}_S \cdot \mathbf{Y}$, where $\boldsymbol{\theta}$ is a hedger

Two-step valuations

- **Definitions**:
  - Consider the **conditional valuation** $\pi_{\mathbf{Y}} : \mathcal{C} \to \mathcal{C}^{\mathbf{Y}}$.
  - $\pi_{\mathbf{Y}}$ is **market-consistent** if

$$\boxed{\pi_{\mathbf{Y}}\left[S + S^h\right] = \pi_{\mathbf{Y}}\left[S\right] + e^{-r}\ \boldsymbol{\nu} \cdot \mathbf{Y}} \quad \text{for any } S \in \mathcal{C} \text{ and } S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$$

  - $\pi_{\mathbf{Y}}$ is **model-consistent** if there exists an actuarial valuation π such that

$$\boxed{\pi_{\mathbf{Y}}\left[S^{\perp}\right] = \pi\left[S^{\perp}\right]} \qquad \text{for any } S^{\perp} \in \mathcal{C}^{\perp}$$

  - $\pi_{\mathbf{Y}}$ is **fair** if it is market-consistent and model-consistent.

- **Example of a fair conditional valuation**:

$$\boxed{\pi_{\mathbf{Y}}\left[S\right] = e^{-r}\ \boldsymbol{\theta}_S^f \cdot \mathbf{Y}} \qquad \text{where } \boldsymbol{\theta}^f \text{ is a fair hedger}$$

Two-step valuations

- For any market-consistent conditional valuation $\pi_{\mathbf{Y}}$, one has that

$$\boxed{\pi_{\mathbf{Y}}\left[S^h\right] = e^{-r}\ S^h} \text{ for any } S^h$$

- But there exist market-consistent conditional valuations $\pi_{\mathbf{Y}}$ and derivatives $f(\mathbf{Y})$ for which

$$\boxed{\pi_{\mathbf{Y}}\left[f(\mathbf{Y})\right] \neq e^{-r}\ f(\mathbf{Y})}$$

- <u>Example</u>:
  - Consider the market-consistent conditional valuation $\pi_{\mathbf{Y}}[S] = e^{-r}\boldsymbol{\theta}_S^{MV}\boldsymbol{\cdot}\mathbf{Y}$.
  - In case $f(\mathbf{Y}) \notin \mathcal{C}^h$, one has that

$$\pi_{\mathbf{Y}}\left[f(\mathbf{Y})\right] \neq e^{-r} f(\mathbf{Y})$$

Two-step valuations

Exercise 15:

- **Q1**: Show that the following mappings $S \to \pi_{\mathbf{Y}}[S]$, for any $S \in \mathcal{C}$, are fair conditional valuations:
  - **Conditional standard deviation principle**: $(\alpha \geq 0)$

$$\boxed{\pi_{\mathbf{Y}}[S] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + \alpha\, \sigma^{\mathbb{P}}[S \mid \mathbf{Y}]\right)}$$

  - **Conditional cost-of-capital principle**: $(i, p \in [0,1))$

$$\boxed{\pi_{\mathbf{Y}}[S] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + i\ \left(\mathrm{VaR}_p^{\mathbb{P}}[S \mid \mathbf{Y}] - \mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]\right)\right)}$$

- **Q2**: For both conditional valuations, show that

$$\boxed{\pi_{\mathbf{Y}}\left[f(\mathbf{Y}) \times S^{\perp}\right] = f(\mathbf{Y}) \times \pi\left[S^{\perp}\right]}$$

holds for any non-negative $f(\mathbf{Y}) \in \mathcal{C}^{\mathbf{Y}}$ and any $S^{\perp} \in \mathcal{C}^{\perp}$.

Two-step valuations

- **Definition**[13]: A mapping $\rho : \mathcal{C} \to \mathbb{R}$ is a **two-step valuation** (TS valuation) if there exists a fair conditional valuation $\pi_{\mathbf{Y}}$ and an EMM $\mathbb{Q}$ such that

$$\boxed{\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]} \qquad \text{for any } S \in \mathcal{C}$$

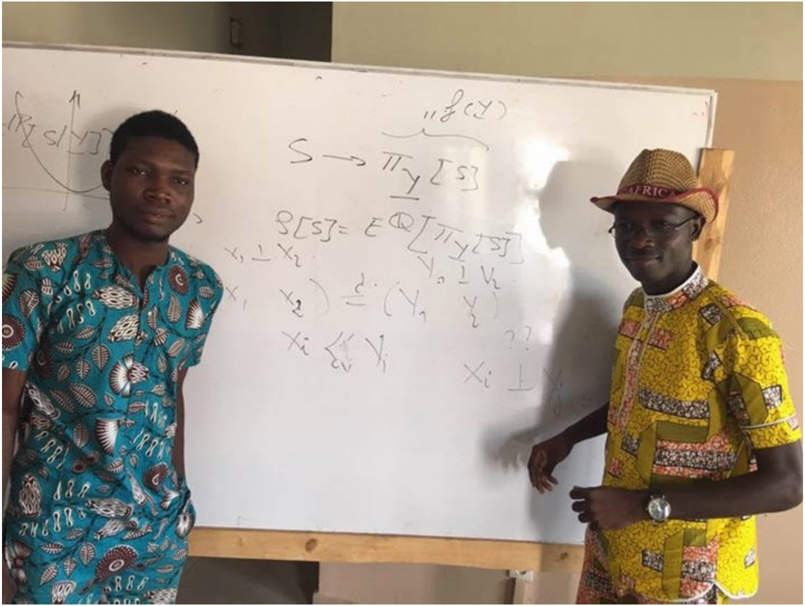
- **Examples**:
  - **TS standard deviation valuation**: (TSSD)

$$\boxed{\rho[S] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + \alpha\ \sigma^{\mathbb{P}}[S \mid \mathbf{Y}]\right]}$$

  - **Two-step CoC valuation**: (TSCoC)

$$\boxed{\rho[S] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + i\ \left(\mathrm{VaR}_p^{\mathbb{P}}[S \mid \mathbf{Y}] - \mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]\right)\right]}$$

[13] Pelsser & Stadje (2014) define sligthly different two-step valuations in a complete financial market setting.



Université d'Abomey-Calavi, Benin, May 2017.

Two-step valuations

- **Theorem**:

$$\boxed{\rho \text{ is a } \textbf{TS valuation} \Leftrightarrow \rho \text{ is a } \textbf{fair valuation}}$$

- **Exercise 16**:
  - Consider the MVHB valuation ρ:

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}]} \qquad \text{for any } S \in \mathcal{C}$$

  - **Q**: Show that ρ can be expressed as a TS valuation:

$$\boxed{\rho\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[S\right]\right]} \qquad \text{for any } S \in \mathcal{C}$$

with

$$\pi_{\mathbf{Y}}\left[S\right] = \left(\boldsymbol{\theta}_S^{MV} + \left(\pi\left[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}\right], 0, \ldots, 0\right)\right) \cdot \mathbf{Y}$$

Two-step valuations

Two-step valuations in a market where any derivative is hedgeable

- **Assumption** (only made on this slide!):
  - Any derivative $f(\mathbf{Y})$ is hedgeable.
  - Equivalently, the financial market of $(n+1)$ traded assets is complete in $\left(\Omega, \mathcal{F}^{\mathbf{Y}}, \mathbb{P}\right)$.
- **Definition**: The **two-step hedger** of the fair conditional valuation $\pi_{\mathbf{Y}}$ is the mapping $\boldsymbol{\theta}^{TS} : \mathcal{C} \rightarrow \Theta$ such that

$$\boxed{\boldsymbol{\theta}_S^{TS} \cdot \mathbf{Y} = e^r \ \pi_{\mathbf{Y}}[S]} \qquad \text{for any } S \in \mathcal{C}$$

- **Properties**:
  - $\boldsymbol{\theta}^{TS}$ is uniquely determined.
  - $\boldsymbol{\theta}^{TS}$ is a fair hedger.
- **Two-step values**:

$$\boxed{\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]] = \boldsymbol{\theta}_S^{TS} \cdot \mathbf{y}} \qquad \text{for any } S \in \mathcal{C}$$

Two-step valuations

The Brennan - Schwartz formula for TS valuations

- Consider the **TS valuation** ρ defined by

$$\boxed{\rho\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[S\right]\right]} \quad \text{for any } S \in \mathcal{C}$$

  with fair conditional valuation $\pi_{\mathbf{Y}}$ and underlying actuarial valuation π.

- **Brennan - Schwartz formula for TS valuations**:
  - Let $f(\mathbf{Y}) \in \mathcal{C}^{\mathbf{Y}}$ and $S^{\perp} \in \mathcal{C}^{\perp}$, such that

$$\pi_{\mathbf{Y}}\left[f(\mathbf{Y}) \times S^{\perp}\right] = f(\mathbf{Y}) \times \pi\left[S^{\perp}\right]$$

  - Then one has

$$\boxed{\rho\left[f(\mathbf{Y}) \times S^{\perp}\right] = \mathbb{E}^{\mathbb{Q}}\left[f(\mathbf{Y})\right] \times \pi\left[S^{\perp}\right]}$$

- This generalized B-S formula holds in particular for the TSSD valuation and the TSCoC valuation, provided $f(\mathbf{Y}) \geq 0$.

Two-step valuations

Exercise 17:

- Consider the **TS valuation** given by

$$\boxed{\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]} \qquad \text{for any } S \in \mathcal{C}$$

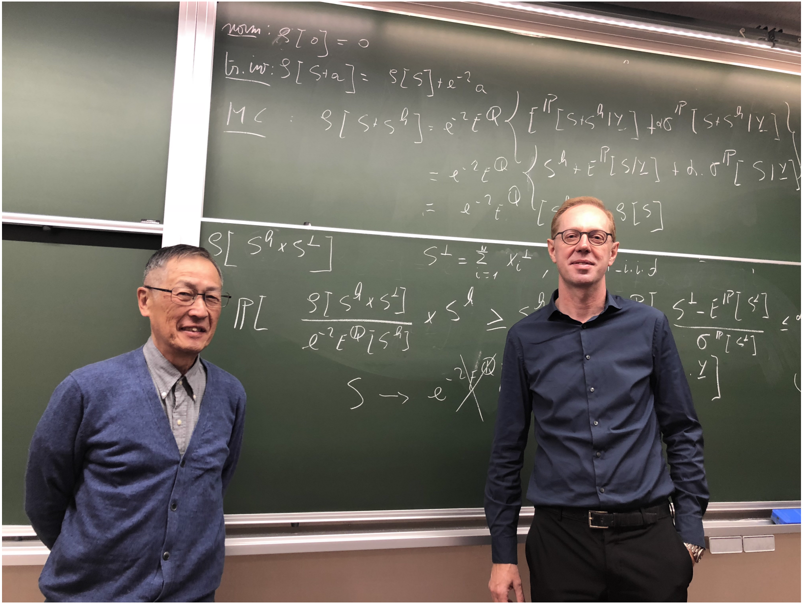
- Consider a unit-linked insurance portfolio liability $S^h \times S^{\perp}$:
  - Suppose that

$$S^h > 0 \qquad \text{and} \qquad \pi_{\mathbf{Y}}\left[S^h \times S^{\perp}\right] = S^h \times \pi\left[S^{\perp}\right]$$

  - The insurer charges a premium $\rho\left[S^h \times S^{\perp}\right]$.
  - The premium is fully invested in S^h.
- **Q**: Show that the probability that at time 1, the insurer will be able to pay $S^h \times S^{\perp}$ from the invested premium is given by

$$\boxed{\mathbb{P}\left[S^{\perp} \leq e^r\ \pi\left[S^{\perp}\right]\right]}$$

- This result holds in particular for the TSSD valuation and the TSCoC valuation.



Prof. Dr. Ichiro Shigekawa, head actuarial science section,
Kyoto University, November 2018.

Two-step valuations

Exercise 18: **Brennan-Schwartz formula**.

- Consider the TSSD valuation $\rho : \mathcal{C} \to \mathbb{R}$, defined by

$$\boxed{\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right] + \alpha\ \sigma^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right]} \quad \text{for any } S \in \mathcal{C}$$

- Consider the portfolio liability $S^h \times S^{\perp}$ with $0 < S^h \in \mathcal{C}^h$ and $S^{\perp} \in \mathcal{C}^{\perp}$.
- **Q1**: Show that

$$\boxed{\rho\left[S^h \times S^{\perp}\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] + \alpha\ \sigma^{\mathbb{P}}\left[S^{\perp}\right]\right)}$$

- **Q2**: Suppose that $S^{\perp} = X_1^{\perp} + \ldots + X_N^{\perp}$ with $X_1^{\perp}, \ldots, X_N^{\perp}$ being $\mathbb{P}$ - i.i.d. orthogonal claims. Show that

$$\boxed{\frac{1}{N}\ \rho\left[S^h \times S^{\perp}\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \left(\mathbb{E}^{\mathbb{P}}\left[X_1^{\perp}\right] + \frac{\alpha}{\sqrt{N}}\ \sigma^{\mathbb{P}}\left[X_1^{\perp}\right]\right)}$$

Exercise 18: (cont'd)

- **Suppose**:
  - Each unit-linked contract is charged a premium $\frac{\rho\left[S^h \times S^\perp\right]}{N}$.
  - These premiums are fully invested in units of S^h.
- **Q3**: Show that the probability that the time-1 value of the invested premiums exceeds the time-1 liability $S^h \times S^\perp$ is given by

$$\boxed{\mathbb{P}\left[\frac{S^\perp - \mathbb{E}^{\mathbb{P}}\left[S^\perp\right]}{\sigma^{\mathbb{P}}\left[S^\perp\right]} \leq \alpha\right]}$$

Two-step valuations

Exercise 19: **Unit-linked insurance**.

- Consider a **portfolio of N insureds**, with

$$X_i^{\perp} = \begin{cases} 0 & : \text{ insured } i \text{ dies before time 1} \\ 1 & : \text{ insured } i \text{ is alive at time 1} \end{cases}$$

  - The orthogonal claims $X_i^{\perp}$ are i.i.d. with mean p (under $\mathbb{P}$).
  - Number of survivors:

$$\boxed{S^{\perp} = \sum_{i=1}^{N} X_i^{\perp}}$$

- Each insured i has underwritten a **unit-linked contract** with payoff at time 1 given by

$$\boxed{\max\left(Y^{(1)}, K\right) \times X_i^{\perp}} \text{ with } K > 0$$

- Suppose that the traded assets are $Y^{(0)}$, $Y^{(1)}$ and $Y^{(2)} = \left(K - Y^{(1)}\right)_+$.

Two-step valuations

Exercise 19 (cont'd):

- **Unit-linked insurance portfolio liability**:

$$S^h \times S^\perp = \max\left(Y^{(1)}, K\right) \times \sum_{i=1}^N X_i^\perp$$

- Consider the **fair valuation** ρ defined by

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right] + \alpha\ \sigma^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right] \quad \text{for any } S \in \mathcal{C}$$

with $\alpha \geq 0$.

- **Q1**: Show that the **fair value of the unit-linked liability** is given by

$$\rho\left[S^h \times S^\perp\right] = \left(y^{(1)} + y^{(2)}\right)\left(Np + \sqrt{N}\alpha\sqrt{p(1-p)}\right)$$

Two-step valuations

Exercise 19 (cont'd):

- **Suppose**:
  - Each unit-linked contract is charged a premium $\frac{\rho\left[S^h \times S^\perp\right]}{N}$.
  - These premiums are fully invested in S^h.
- **Q2**: What is the probability that the insurer will be able to pay his time-1 liability?
  - Assumption: $S^\perp$ is (approx.) normal distributed (under $\mathbb{P}$).
- **Q3**: Determine the probability that the insurer will be able to pay his time-1 liability in case of a pure unit-linked contract ($K = 0$).

Two-step valuations

Exercise 20:

- Consider a **portfolio of N insurance contracts**, with payoff of contract i at time 1 given by

$$\boxed{f(\mathbf{Y}) \times X_i} \qquad i = 1, 2, \ldots, N$$

- Assumptions:
  - $0 \leq f(\mathbf{Y}) \in \mathcal{C}^{\mathbf{Y}}$ and any $X_i \in \mathcal{C}$.
  - There exists a r.v. $Z \in \mathcal{C}^{\perp}$ with support A, such that for any $z \in A$, one has that $(X_1 \mid Z = z), \ldots, (X_N \mid Z = z)$ are $\mathbb{P}-$ i.i.d. orthogonal claims.

- **Unit-linked insurance portfolio liability**:

$$\boxed{f(\mathbf{Y}) \times S_N^{\perp}}$$

with $S_N^{\perp} = \sum_{i=1}^{N} X_i \in \mathcal{C}^{\perp}$.

Two-step valuations

Exercise 20 (cont'd):

- Consider the following **TSSD valuation**:

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right] + \alpha\ \sigma^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right]$$

  with $\alpha \geq 0$.

- **Q1**: Show that the **TSSD value of the unit-linked liability** per policy is given by

$$\boxed{\frac{\rho\left[f\left(\mathbf{Y}\right)\times S_N^{\perp}\right]}{N} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[f\left(\mathbf{Y}\right)\right]\left(\mathbb{E}^{\mathbb{P}}\left[X_1\right] + \alpha\sqrt{\frac{A}{N} + B}\right)}$$

  with

$$A = \mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}^{\mathbb{P}}\left[X_1 \mid Z\right]\right]$$

  and

$$B = \mathrm{Var}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X_1 \mid Z\right]\right]$$

Two-step valuations

Exercise 20 (cont'd):

- In the remainder of this exercise, assume that $f(\mathbf{Y}) \equiv S^h \in \mathcal{C}^h$.
- Let Y^h, Y_N^d and Y^r the hedgeable, diversifiable and residual part of $\frac{S^h \times S_N^\perp}{N}$ as defined in Exercise 3'.
- **Q2**: Show that the TSSD value of the unit-linked liability per policy can then be expressed as follows:

$$\boxed{\frac{\rho\left[S^h \times S_N^\perp\right]}{N} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[Y^h\right] \times \left\{1 + \alpha\sqrt{\frac{\mathrm{Var}^{\mathbb{P}}\left[Y_N^d\right] + \mathrm{Var}^{\mathbb{P}}\left[Y^r\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(Y^h\right)^2\right]}}\right\}}$$

Two-step valuations

Exercise 21-1:
Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ with

- **Universe**:

$$\Omega = \left\{ (0,0)\,, (0,1)\,, (1,0)\,(1,1) \right\}$$

  - First component = price $Y^{(1)}$ of stock 1 at time 1.
  - Second component = value of survival index $\mathcal{I}$ at time 1.

- **Probabilities**:

$$\begin{cases} p_{0,0} & = 1/6 \\ p_{1,0} & = 2/6 \\ p_{0,1} & = 1/6 \\ p_{1,1} & = 2/6 \end{cases}$$

Two-step valuations

Exercise 21-1 (cont'd):

The financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ is home to

- A *traded* **zero-coupon bond**:
  - Current price: $y^{(0)} = 1$
  - Price at time 1: $Y^{(0)} = 1$
- A *traded* **stock**:
  - Current price: $y^{(1)} = 1/2$
  - Price at time 1: $Y^{(1)}$ is either 0 or 1
- A *non-traded* **survival index**:

$$\mathcal{I} = \begin{cases} 0 & \text{if few people survive} \\ 1 & \text{if many people survive} \end{cases}$$

- A *non-traded* **claim**:

$$\boxed{S = (1 - \mathcal{I}) \times \left(1 - Y^{(1)}\right)}$$

Two-step valuations

Exercise 21-1 (cont'd):

- **Q1**: Show that any derivative of $\mathbf{Y} = \left(Y^{(0)}, Y^{(1)}\right)$ is hedgeable.
- Consider the fair conditional valuation $\pi_{\mathbf{Y}}$ (with underlying actuarial valuation π) and the non-traded claim S:

$$\boxed{S = (1 - \mathcal{I}) \times \left(1 - Y^{(1)}\right)}$$

- **Q2**: Express both the **TS hedge** $\boldsymbol{\theta}_S^{TS}$ of $\pi_{\mathbf{Y}}[S]$ and the **TS value** $\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]$ of S as functions of $\pi[1 - \mathcal{I}]$.
- **Q3**: Determine the **TS CoC value** $\rho[S]$ of S, when $i = 0.06$ and $p = 0.995$.

Two-step valuations

Exercise 21-2:
Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ with

- **Universe**:

$$\Omega = \left\{(0,0), (0,1), (1,0)\ (1,1)\right\}$$

  - First component = price $Y^{(1)}$ of stock 1 at time 1.
  - Second component = value of survival index $\mathcal{I}$ at time 1.

- **Probabilities**:

$$\begin{cases} p_{0,0} & = 1/6 \\ p_{1,0} & = 2/6 \\ p_{0,1} & = 1/6 \\ p_{1,1} & = 2/6 \end{cases}$$

Two-step valuations

Exercise 21-2 (cont'd):

The financial-actuarial world $(\Omega, 2^{\Omega}, \mathbb{P})$ is home to

- A *traded* **zero-coupon bond**:
  - Current price: $y^{(0)} = 1$.
  - Price at time 1: $Y^{(0)} = 1$.
- A *traded* **stock**:
  - Current price: $y^{(1)} = 1/2$.
  - Price at time 1: $Y^{(1)}$, which is either 0 or 1.
- A *traded* **survival index**:
  - Current price: $y^{(2)} = 2/3$.
  - Payoff at time 1: $Y^{(2)} = \mathcal{I}$.
- A *non-traded* **claim**:

$$\boxed{S = (1 - \mathcal{I}) \times \left(1 - Y^{(1)}\right)}$$

Two-step valuations

Exercise 21-2 (cont'd):

- **Q1**: Show that $\mathbb{Q} = \left(q_{00}, q_{10}, q_{01}, q_{11}\right)$ is an EMM for the financial market in $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ if and only if

$$\boxed{q_{00} \in \left(0, \tfrac{1}{3}\right), \quad q_{10} = \tfrac{1}{3} - q_{00}, \quad q_{01} = \tfrac{1}{2} - q_{00}, \quad q_{11} = \tfrac{1}{6} + q_{00}}$$

- **Q2**: Show that $S = (1 - \mathcal{I}) \times \left(1 - Y^{(1)}\right)$ is a non-hedgeable derivative of $\mathbf{Y} = \left(Y^{(0)}, Y^{(1)}, Y^{(2)}\right)$.
- **Q3**: Show that the **TS CoC value** $\rho\left[S\right]$ of S is given by

$$\boxed{\rho\left[S\right] = q_{00}}$$

- **Q4**: Determine the TS CoC value of S in case $\mathcal{I}$ and $Y^{(1)}$ are independent under $\mathbb{Q}$.

Two-step valuations

Exercise 21-3:
Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ with

- **Universe**:

$$\Omega = \left\{(0,0), (0,1), (1,0)\, (1,1)\right\}$$

  - First component = price $Y^{(1)}$ of stock 1 at time 1.
  - Second component = value of survival index $\mathcal{I}$ at time 1.

- **Probabilities**:

$$\begin{cases} p_{0,0} & = 1/6 \\ p_{1,0} & = 2/6 \\ p_{0,1} & = 1/6 \\ p_{1,1} & = 2/6 \end{cases}$$

Two-step valuations

Exercise 21-3 (cont'd):
The financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ is home to

- A *traded* **zero-coupon bond**: see Exercise 15-2.
- A *traded* **stock**: see Exercise 15-2.
- A *traded* **survival index**: see Exercise 15-2.
- A *traded* **call option**:
  - Current price: $y^{(3)} = 1/6$.
  - Payoff at time 1:

$$Y^{(3)} = \mathcal{I} \times \left(Y^{(1)} - 0.5\right)_+$$

- A *non-traded* **claim**:

$$\boxed{S = (1 - \mathcal{I}) \times \left(1 - Y^{(1)}\right)}$$

Two-step valuations

Exercise 21-3 (cont'd):

- **Q1**: Show that the market is complete in $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$.
- **Q2**: Determine the hedge of $S = (1 - \mathcal{I}) \times \left(1 - Y^{(1)}\right)$.
- **Q3**: Show that any fair value of S is given by

$$\boxed{\rho\left[S\right] = \frac{1}{6}}$$

Two-step valuations

Exercise 22:

- Consider a **national population** of N^{nat} members:

$$I_i = \begin{cases} 0 & : \text{ member } i \text{ dies before time } 1 \\ 1 & : \text{ otherwise} \end{cases}$$

  - **National survival index**:

$$\boxed{I = I_1 + I_2 + \ldots + I_{N^{\text{nat}}}}$$

- Consider an **insured population** of N^{ins} members:

$$J_i = \begin{cases} 0 & : \text{ insured } i \text{ dies before time } 1 \\ 1 & : \text{ otherwise} \end{cases}$$

  - **Insurance claim at time 1**:

$$\boxed{S = J_1 + J_2 + \ldots + J_{N^{\text{ins}}}}$$

- **Assumption**:
  - $N^{\text{ins}} \leq N^{\text{nat}}$ and $J_i = I_i$ for $i = 1, 2, \ldots, N^{\text{ins}}$.

Two-step valuations

Exercise 22 (cont'd):

There is a **financial market** consisting of 3 traded assets:

- **Zero-coupon bond**:
  - Current price: $y^{(0)} = 1$
  - Price at time 1: $Y^{(0)} = e^r$
- **Stock**:
  - Current price: $y^{(1)}$
  - Price at time 1: $Y^{(1)} \in \mathcal{A}$.
- **National survival index**:
  - Current price: $y^{(2)}$
  - Price at time 1: $Y^{(2)} = I$.

Two-step valuations

Exercise 22 (cont'd):

- **Financial-actuarial world** $(\Omega, 2^{\Omega}, \mathbb{P})$:

$$\Omega = \left\{ (x_1, x_2, x_3) \mid x_1 \in \mathcal{A};\ x_2 = 0, 1..., N^{\text{nat}},\ x_3 = 0, 1..., N^{\text{ins}} \right\}$$

  - Ω is a support of $\left(Y^{(1)},\ I,\ S \right)$.

- **Assumption**: $Y^{(1)}$ and (I, S) are $\mathbb{P}$ - independent.

- Suppose that the fair value of the insurance claim S is determined by the *TSSD principle*:

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right] + \alpha\ \sigma^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right]$$

Two-step valuations

Exercise 22 (cont'd):

- **Q1**: Show that $\rho[S]$ is given by

$$\boxed{\frac{\rho[S]}{N^{\text{ins}}} = \frac{y^{(2)}}{N^{\text{nat}}} + \alpha\ e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\sqrt{\left(\frac{1}{N^{\text{ins}}} - \frac{1}{N^{\text{nat}}}\right) \times \frac{Y^{(2)}}{N^{\text{nat}}-1} \times \left(1 - \frac{Y^{(2)}}{N^{\text{nat}}}\right)}\ \right]}$$

- **Additional assumptions**:
  - $\mathcal{A}$ is a countable set.
  - All I_i are i.i.d. under $\mathbb{P}$ with $\mathbb{P}[I_i = 1] = p > 0$.
- **Q2**: Write down the set of equations which determines the set of EMM's for the financial market in $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$.
- **Q3**: Does $\rho[S]$ depend on the choice of $\mathbb{Q}$?
- Suppose that $\mathbb{Q}$ is an EMM under which all I_i are i.i.d.
  - **Q4**: Determine the $\mathbb{Q}$-distributions of I_i, I and S, respectively.
  - **Q5**: Is $\rho[S]$ uniquely determined in this case?

Two-step valuations

Exercise 23: **HB valuations vs. TS valuations**.

- Let $\alpha \geq 0$. Consider the mapping $\rho : \mathcal{C} \to \mathbb{R}$ defined by

$$\boxed{\rho\left[S\right] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \alpha\ e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\sigma^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right]} \quad \text{for any } S \in \mathcal{C}$$

- **Q1**: Show that ρ is a **fair valuation**.
- **Q2**: Show that ρ can be expressed as a **MVHB valuation** with model-consistent valuation π given by

$$\boxed{\pi\left[X\right] = e^{-r}\ \mathbb{E}^{\mathbb{P}}\left[X\right] + \alpha\ e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\sigma^{\mathbb{P}}\left[X \mid \mathbf{Y}\right]\right]} \quad \text{for any } X \in \mathcal{C}$$

- **Q3**: Show that ρ can be expressed as a **TS valuation** with underlying fair conditional valuation $\pi_{\mathbf{Y}}$ given by

$$\boxed{\pi_{\mathbf{Y}}\left[S\right] = e^{-r}\ \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y} + \alpha\ e^{-r}\ \sigma^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]} \quad \text{for any } S \in \mathcal{C}$$

Two-step valuations

Exercise 23 (cont'd):

- **Q4**: Let $0 \leq S^h \in \mathcal{C}^h$ and $S^\perp \in \mathcal{C}^\perp$. Show that

$$\boxed{\rho\left[S^h \times S^\perp\right] = \rho^{TS}\left[S^h \times S^\perp\right]}$$

where ρ^{TS} is the **TSSD valuation** with parameter α.

- **Q5**: Does the following statement holds:

$$\rho\left[S\right] = \rho^{TS}\left[S\right], \quad \text{for any } S \in \mathcal{C}?$$

- **Q6**: Does the statement above holds in case all derivatives of $\mathbf{Y}$ are hedgeable?

8. Conclusions

- We introduced the **fair valuation** of financial-actuarial liabilities in a single period setting:
  - We combined prices observed in the **financial market** with a valuation based on an **actuarial model**.
  - Both $\mathbb{P}$- and $\mathbb{Q}$-measures are involved.
- We proved the **equivalence** of the following statements :

1. ρ is a fair valuation.
2. There exists a fair hedger $\boldsymbol{\theta}^{\mathrm{f}}$, such that

$$\boxed{\rho\left[S\right] = e^{-r}\boldsymbol{\theta}_S^{\mathrm{f}} \cdot \mathbf{y}} \qquad \text{for any } S \in \mathcal{C}$$

3. ρ is a hedge-based valuation.
4. ρ is a two-step valuation.

- These equivalences hold for any **subjective choice** of the properties that an actuarial valuation has to satisfy.

Main reference and some generalizations

- **Static valuation of time-T claims, static hedging:**
  - Dhaene, Stassen, Barigou, Linders & Chen (2017).
    Fair valuation of insurance liabilities: merging actuarial judgement and market-consistency.
    Insurance : Mathematics & Economics, 76, 14-27.

- **Static valuation of time-T claims, dynamic hedging**:
  - Barigou & Dhaene (2019).
    Fair valuation of insurance liabilities via mean-variance hedging in a multi-period setting.
    Scandinavian Actuarial Journal, 2019(2), 163-187.

- **Dynamic valuation of time-T claims, dynamic hedging**:
  - Barigou, Chen & Dhaene (2019).
    Fair dynamic valuation of insurance liabilities: merging actuarial judgement with market- and time-consistency.
    Insurance : Mathematics & Economics, 88, 19-29.

Appendix

- Consider the **fair valuation** ρ^{f}:

$$\boxed{\rho^{\mathrm{f}}\left[S\right] = \boldsymbol{\theta}_S^{\mathrm{f}} \cdot \mathbf{y}}$$

- ρ^{f} can be expressed as a **HB valuation**:

$$\boxed{\rho^{\mathrm{f}}\left[S\right] = \boldsymbol{\theta}_S^{\mathrm{f}} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{\mathrm{f}} \cdot \mathbf{Y}]}$$

with

$$\pi\left[X\right] = \boldsymbol{\theta}_X^{\mathrm{f}} \cdot \mathbf{y}.$$

- ρ^{f} can be expressed as a **TS valuation**:

$$\boxed{\rho^{\mathrm{f}}\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[S\right]\right]}$$

with

$$\pi_{\mathbf{Y}}\left[S\right] = e^{-r}\ \boldsymbol{\theta}_S^{\mathrm{f}} \cdot \mathbf{Y}.$$

Appendix

- Consider the **HB valuation** ρ^{HB}:

$$\boxed{\rho^{\mathrm{HB}}\left[S\right] = \boldsymbol{\theta}_S \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}]}$$

- ρ^{HB} can be expressed as a **fair valuation**:

$$\boxed{\rho^{\mathrm{HB}}\left[S\right] = \boldsymbol{\theta}_S^f \cdot \mathbf{y}}$$

with

$$\boldsymbol{\theta}_S^f = \boldsymbol{\theta}_S + \left(\pi[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}], 0, \ldots, 0\right).$$

- ρ^{HB} can be expressed as a **TS valuation**:

$$\boxed{\rho^{\mathrm{HB}}\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[S\right]\right]}$$

with $\mathbb{Q}$ any EMM and

$$\pi_{\mathbf{Y}}\left[S\right] = e^{-r}\ \left(\boldsymbol{\theta}_S + \left(\pi[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}], 0, \ldots, 0\right)\right) \cdot \mathbf{Y}.$$

Appendix

- Consider the **TS valuation** ρ^{TS}:

$$\boxed{\rho^{\mathsf{TS}}[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]}$$

- ρ^{TS} can be expressed as a **fair valuation**:

$$\boxed{\rho^{\mathsf{TS}}[S] = \boldsymbol{\theta}_S^{\mathsf{f}} \cdot \mathbf{y}}$$

with

$$\boldsymbol{\theta}_S^{\mathsf{f}} = \boldsymbol{\theta}_S^{\mathsf{MV}} + \left(\mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[S - \boldsymbol{\theta}_S^{\mathsf{MV}} \cdot \mathbf{Y}\right]\right], 0, \ldots, 0\right).$$

- ρ^{TS} can be expressed as a **HB valuation**:

$$\boxed{\rho^{\mathsf{TS}}[S] = \boldsymbol{\theta}_S^{\mathsf{f}} \cdot \mathbf{y} + \pi[S - \boldsymbol{\theta}_S^{\mathsf{MV}} \cdot \mathbf{Y}]}$$

with $\boldsymbol{\theta}^{\mathsf{f}}$ defined above and

$$\pi[X] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[X]].$$

Contact

- jan.dhaene@kuleuven.be
- www.jandhaene.org
- **Actuarial Research Group, KU Leuven**
  Naamsestraat 69, B-3000 Leuven, Belgium