Life Insurance Mathematics

Survival models[1]

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[1]Based on Chapter 2 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).

2.1 Summary

- The future lifetime r.v.[2]
- Probabilities of death and survival.
- The force-of-mortality.
- Basic actuarial notations.
- The curtate future lifetime r.v.

[2] *Nothing is certain in life except death and taxes* - Benjamin Frankin.

2.2 The future lifetime random variable

- **Status (x)**:

$$(x) \overset{\text{not.}}{=} \text{a life aged } x, \qquad x \geq 0$$

- **Future lifetime** of (x):

$$T_x$$

  - Assumption: T_x is a *continuous* r.v. on $(0, +\infty)$.

- **Age-at-death** of (x):

$$x + T_x$$

- **Lifetime distribution** of (x):

$$F_x(t) = \mathbb{P}[T_x \leq t]$$

- **Survival function** of (x):

$$S_x(t) = 1 - F_x(t)$$

2.2 The future lifetime random variable

- "*The first person to live for a thousand years is possibly already alive and of those of us aged between 20-30, there will certainly be some who reach 130 years old. This will have an instantaneous and catastrophic effect on the world population*". Richard Seymour.

- About a century ago, the British monarch started sending anniversary messages to "current citizens of [the monarch's] realms or UK Overseas Territories" who reached the age of 100. In 1917, King George V sent a total of 24 celebratory messages to centenarians. By 1952 this had increased more than 10-fold to 255, and in 2016, it has exploded to nearly 60-fold to 14500.

2.2 The future lifetime random variable

- Consider a person (x) with
  - Current future lifetime: T_x.
  - Future lifetime at birth: T_0.
  - Future lifetime at age $y \geq x$, given survival until age y: T_y.
- **Assumption**:
  For any $y \geq x$ and $t \geq 0$, we assume that

$$\boxed{\mathbb{P}[T_y \leq t] = \mathbb{P}[T_0 \leq y + t \mid T_0 > y]} \tag{2.1}$$

- **Interpretation**: Starting from the cdf of T_0, the only additonal information used to determine survival probabilties at age x and beyond is survival or not.
- **Corollary**: For any $t, u \geq 0$, we have that

$$\boxed{\mathbb{P}[T_{x+t} \leq u] = \mathbb{P}[T_x \leq t + u \mid T_x > t]}$$

2.2 The future lifetime random variable

- Lifetime distributions F_x and F_0:

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)} \tag{2.2}$$

- Survival functions S_x and S_0:

$$S_0(x+t) = S_0(x)\ S_x(t) \tag{2.4}$$

- Survival functions S_{x+t} and S_x:

$$S_x(t+u) = S_x(t)\ S_{x+t}(u) \tag{2.5}$$

2.2 The future lifetime random variable

- Consider (x) with continuous future lifetime T_x.
- $S_x(t)$ is a **survival function** for (x) if and only if the following conditions are satisfied:
  - Condition 1:
  $$S_x(0) = 1$$
  - Condition 2:
  $$\lim_{t \to +\infty} S_x(t) = 0$$
  - Condition 3:
  $$S_x(t) \text{ is a non-increasing continuous function of } t$$

2.2 The future lifetime random variable

- For all **survival functions** $S_x(t)$ in this course, we make the following assumptions:
  - Assumption 1:
  $$\frac{d}{dt} S_x(t) \text{ exists for all } t > 0$$
  - Assumption 2:
  $$\lim_{t \to +\infty} t \, S_x(t) = 0$$
  - Assumption 3:
  $$\lim_{t \to +\infty} t^2 \, S_x(t) = 0$$
- Assumptions 2 and 3 ensure that the mean and the variance of the distribution of T_x exist.

2.2 The future lifetime random variable

Example 2.1

- Assume that

$$F_0(t) = 1 - \left(1 - \frac{t}{120}\right)^{\frac{1}{6}} \text{ for } 0 \leq t \leq 120$$

- Calculate the probability that
  - a newborn survives beyond age 30,
  - a life aged 30 dies before age 50,
  - a life aged 40 survives beyond age 65.

2.3 The force of mortality

- Consider a person with survival function at birth $\mathbb{P}[T_0 > t]$.
- The **force-of-mortality** at age x:

$$\boxed{\mu_x \stackrel{\text{def.}}{=} \lim_{h\to 0^+} \frac{\mathbb{P}[T_x \le h]}{h}}$$

- Other expression for μ_x:

$$\mu_x = \lim_{h\to 0^+} \frac{\mathbb{P}[T_0 \le x + h \mid T_0 > x]}{h} \tag{2.6}$$

- Intuitive interpretation:

$$\boxed{\mu_x \, dx \approx \mathbb{P}[T_0 \le x + dx \mid T_0 > x]} \tag{2.8}$$

2.3 The force of mortality

- μ_x in terms of S_0:

$$\boxed{\mu_x = -\frac{1}{S_0(x)}\frac{d}{dx}S_0(x)} \tag{2.9}$$

- The pdf of T_x:

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t)$$

- μ_x in terms of f_0 and S_0:

$$\boxed{\mu_x = \frac{f_0(x)}{S_0(x)}}$$

2.3 The force of mortality

- Suppose that x is fixed and t is variable.
- Expression for μ_{x+t}:

$$\boxed{\mu_{x+t} = \frac{f_x(t)}{S_x(t)}} \tag{2.10}$$

- Intuitive interpretation:

$$\boxed{\mu_{x+t}\,dt \approx \mathbb{P}[T_x \leq t + dt \mid T_x > t]}$$

- An expression for $S_x(t)$:

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s} ds\right) \tag{2.11}$$

2.3 The force of mortality

Example 2.2

- Suppose that

$$F_0(x) = \mathbb{P}[\,T_0 \le x] = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}} \qquad \text{for } 0 \le t \le 120$$

- Derive an expression for μ_x.

2.3 The force of mortality

2.3.1 Mortality laws

- **Gompertz' law of mortality**:

$$\boxed{\mu_x = Bc^x, \qquad x > 0}$$

  where B and c are constants such that $B > 0$ and $c > 1$.

- **Makeham's law of mortality**:

$$\boxed{\mu_x = A + Bc^x, \qquad x > 0}$$

  where A, B and c are constants such that A, $B > 0$ and $c > 1$.

- Both models often provide a good fit to mortality data over certain age ranges, particularly from middle age to early old age.

2.3 The force of mortality

2.3.1 Mortality laws

Example 2.3

- **Gompertz' law of mortality.**
  Derive an expression for $S_x(t)$.
  - Solution:

$$\boxed{S_x(t) = \exp\left\{-\frac{B}{\ln c} c^x \left(c^t - 1\right)\right\}}$$

- **Makeham's law of mortality.**
  Derive an expression for $S_x(t)$.
  - Solution:

$$\boxed{S_x(t) = \exp\left\{-At - \frac{B}{\ln c} c^x \left(c^t - 1\right)\right\}} \qquad (2.12)$$

  - Remark: This expression is often written as

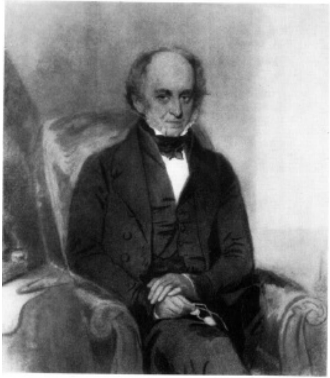
$$S_x(t) = s^t \; g^{c^x\left(c^t - 1\right)}$$

with $s = e^{-A}$ and $g = \exp\left(-B/\ln c\right)$.

2.3 The force of mortality

Benjamin Gompertz (1779 - 1865)

- Chief actuary of Alliance Assurance Company, England.
- Before Gompertz, demographers collected and compiled mortality data, but never much considered a formal 'law of mortality'.



BENJAMIN GOMPERTZ, 1779–1865

2.3 The force of mortality

Benjamin Gompertz (1779 - 1865)

- He published his law of mortality in: Gompertz, B. (1825), *Philosophical Transactions of the Royal Society of London 115: 513–585*.

XXIV. *On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of Life Contingencies. In a Letter to* Francis Baily, *Esq. F. R. S. &c. By* Benjamin Gompertz, *Esq. F. R. S.*

Read June 16, 1825.

Dear Sir,

The frequent opportunities I have had of receiving pleasure from your writings and conversation, have induced me to prefer offering to the Royal Society through your medium, this Paper on Life Contingencies, which forms part of a continuation of my original paper on the same subject, published among the valuable papers of the Society, as by passing through your hands it may receive the advantage of your judgment.

I am, Dear Sir, yours with esteem,

Benjamin Gompertz.

9th June 1825.

CHAPTER I.

Article 1. In continuation of Art. 2. of my paper on the valuation of life contingencies, published in the Philosophical Transactions of this learned Society, in which I observed the near agreement with a geometrical series for a short period of

2.3 The force of mortality

Benjamin Gompertz (1779 - 1865)

- Gompertz' law performs a fairly good fit from middle age (35) to old age (90).
- For very old ages , it overestimates death rates.
- It does not capture infant mortality and the accident hump.
- He was a Jewish religious man:
  "... *Neither profane history nor modern experience could contradict the possibility of the great age of the patriarchs of the scripture...*"

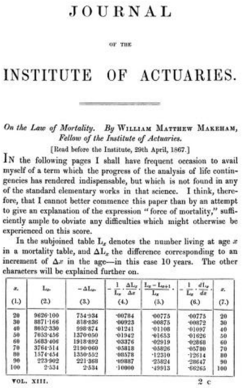
2.3 The force of mortality

William Makeham (1826 - 1891)

- Gompertz (1825):
  "*It is possible that death may be the consequence of two generally co-existing causes. The one is chance, without previous disposition to death or deterioration. The other is a deterioration, or an increased inability to withstand destruction...*"
- Makeham formalized and mathematized Gompertz' law of mortality.
- He published his law of mortality in Makeham, W.M. (1860). *"On the Law of Mortality and the Construction of Annuity Tables". J. Inst. Actuaries and Assur. Mag. 8: 301–310.*

2.3 The force of mortality

William Makeham (1826 - 1891)



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OF THE

INSTITUTE OF ACTUARIES.

On the Law of Mortality. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 29th April, 1867.]

IN the following pages I shall have frequent occasion to avail myself of a term which the progress of the analysis of life contingencies has rendered indispensable, but which is not found in any of the standard elementary works in that science. I think, therefore, that I cannot better commence this paper than by an attempt to give an explanation of the expression "force of mortality," sufficiently ample to obviate any difficulties which might otherwise be experienced on this score.

In the subjoined table L_x denotes the number living at age x in a mortality table, and ΔL_x the difference corresponding to an increment of Δx in the age—in this case 10 years. The other characters will be explained further on.

x. (1.)	L_x. (2.)	$-\Delta L_x$. (3.)	$-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}$. (4.)	$\frac{L_x - L_{x+1}}{L_x}$. (5.)	$-\frac{1}{L_x}\cdot\frac{dL_x}{dx}$. (6.)	x. (7.)
20	9626·100	754·934	·00784	·00775	·00775	20
30	8871·166	818·836	·00923	·00875	·00872	30
40	8052·330	998·874	·01241	·01108	·01097	40
50	7053·456	1370·050	·01942	·01653	·01626	50
60	5683·406	1918·892	·03376	·02919	·02868	60
70	3764·514	2190·060	·05818	·05826	·05780	70
80	1574·454	1350·552	·08578	·12310	·12614	80
90	223·902	221·368	·09887	·25824	·28647	90
100	2·534	2·534	·10000	·49913	·66265	100

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2.3 The force of mortality

Example 2.4

- Let

$$\mu_x = Bc^x, \qquad x > 0$$

- $B = 0.0003$ and $c = 1.07$
- Calculate $S_x(t)$ and $f_x(t)$ for $x = 20$, $x = 50$ and $x = 80$.

2.3 The force of mortality

Example 2.4

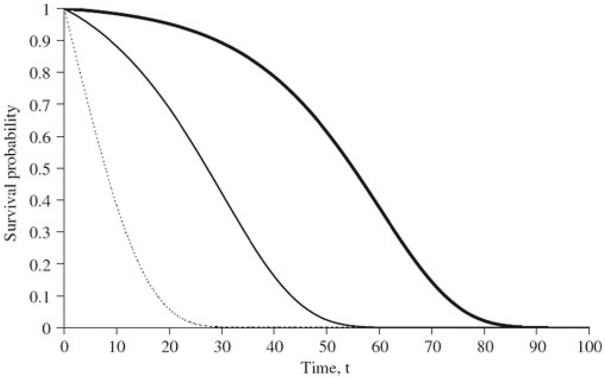

Figure 2.1 $S_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

2.3 The force of mortality

Example 2.4

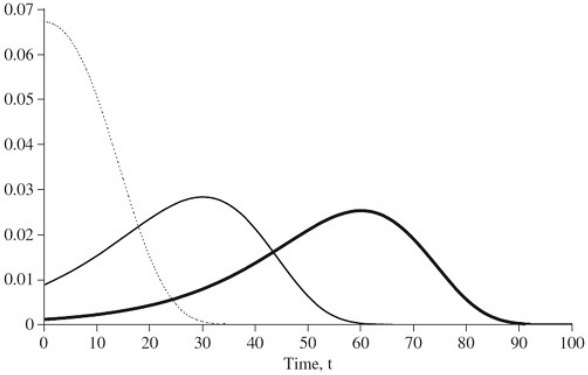

Figure 2.2 $f_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

2.4 Actuarial notation

- The **International Actuarial Association** (IAA) was founded in Brussels in 1895 , at the occasion of the 1st International Congress of Actuaries (ICA).
- At the 2nd ICA, which was held in London in 1898, standard **actuarial notation** was unanimously adopted.
- Actuarial notation is a shorthand method to allow actuaries to record mathematical formulas that deal with interest and mortality rates.

2.4 Actuarial notation

- **Survival rates**:

$$ {}_t p_x \overset{\text{not.}}{=} \mathbb{P}[T_x > t] = S_x(t) \tag{2.13} $$

- **Mortality rates**:

$$ {}_t q_x \overset{\text{not.}}{=} \mathbb{P}[T_x \leq t] = F_x(t) \tag{2.14} $$

- Deferred mortality rates:

$$ {}_{u|t} q_x \overset{\text{not.}}{=} \mathbb{P}[u < T_x \leq u + t] = S_x(u) - S_x(u + t) \tag{2.15} $$

- Simplified notations for 1 - year probabilities:

$$ p_x \overset{\text{not.}}{=} {}_1 p_x $$

$$ q_x \overset{\text{not.}}{=} {}_1 q_x $$

$$ {}_{u|} q_x \overset{\text{not.}}{=} {}_{u|1} q_x $$

2.4 Actuarial notation

Relations between survival and death rates

- Survival and mortality rate add to 1:

$$ {}_tp_x + {}_tq_x = 1 $$

- Survival rates at different ages:

$$ {}_{t+u}p_x = {}_tp_x \times {}_up_{x+t} \tag{2.16} $$

- Survival rates in terms of one-year survival rates:

$$ {}_np_x = p_x \times p_{x+1} \times \ldots \times p_{x+n-1} $$

- Deferred mortality rates:

$$ {}_{u|t}q_x = {}_up_x - {}_{u+t}p_x = {}_up_x \times {}_tq_{x+u} $$

2.4 Actuarial notation

Relations between forces-of-mortality and survival rates

- Force-of-mortality at age x:

$$\mu_x = \lim_{h \to 0^+} \frac{{}_hq_x}{h} = -\frac{1}{{}_xp_0} \frac{d}{dx} \, {}_xp_0 \tag{2.17}$$

- Force-of-mortality at age $x + t$:

$$\mu_{x+t} = -\frac{1}{{}_tp_x} \frac{d}{dt} {}_tp_x \tag{2.18}$$

- Density function of T_x:

$$f_x(t) = {}_tp_x \ \mu_{x+t} \tag{2.19}$$

- Survival rate in terms of forces-of-mortality:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds \right) \tag{2.20}$$

2.4 Actuarial notation

- Death rates and forces-of-mortality:

$$ {}_t q_x = \int_0^t {}_s p_x \; \mu_{x+s} ds \qquad (2.21) $$

- Graphical interpretation:

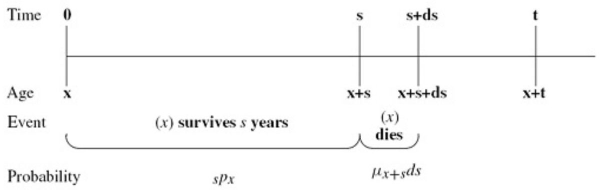

Figure 2.3 Time-line diagram for $_t q_x$

- Approximation:

$$ q_x \approx \mu_{x+\frac{1}{2}}, \qquad \text{when } q_x \text{ is small} $$

2.5 Mean and standard deviation of remaining lifetime

- **Complete expectation of life**:

$$\mathring{e}_x \stackrel{\text{def.}}{=} \mathbb{E}[T_x]$$

- Evaluating $\mathring{e}_x$:

$$\boxed{\mathring{e}_x = \int_0^\infty {}_t p_x \, dt} \tag{2.23}$$

- Second moment of T_x:

$$\mathbb{E}[T_x^2] = 2\int_0^\infty t \; {}_t p_x \, dt \tag{2.24}$$

- Variance of T_x:

$$V[T_x] := \mathbb{E}[T_x^2] - (\mathring{e}_x)^2$$

Example 2.6

- Let
$$F_0(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}} \text{ for } 0 \leq x \leq 120$$
- Calculate $\mathring{e}_x$ and $V\left[T_x\right]$ for $x = 30$ and for $x = 80$.

2.5 Mean and standard deviation of remaining lifetime

Table 2.1. *Values of* $\mathring{e}_x$, $SD[T_x]$ *and expected age at death for the Gompertz model with* $B = 0.0003$ *and* $c = 1.07$.

x	$\mathring{e}_x$	$\mathrm{SD}[T_x]$	$x + \mathring{e}_x$
0	71.938	18.074	71.938
10	62.223	17.579	72.223
20	52.703	16.857	72.703
30	43.492	15.841	73.492
40	34.252	14.477	74.752
50	26.691	12.746	76.691
60	19.550	10.693	79.550
70	13.555	8.449	83.555
80	8.848	6.224	88.848
90	5.433	4.246	95.433
100	3.152	2.682	103.152

2.5 Mean and standard deviation of remaining lifetime

- **Term expectation of life**:

$$\mathring{e}_{x:\overline{n}|} \stackrel{\text{def.}}{=} \mathbb{E}[\min(T_x, n)]$$

- Evaluating $\mathring{e}_{x:\overline{n}|}$:

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt$$

2.6 Curtate future lifetime

2.6.1 Curtate future lifetime and curtate expectation of life

- **Curtate future lifetime**:

$$K_x \overset{\text{def.}}{=} \lfloor T_x \rfloor$$

- Probability function of K_x:

$$\mathbb{P}\left[K_x = k\right] = {}_k p_x \; q_{x+k}, \qquad k = 0, 1, 2, \ldots$$

- **Curtate expectation of life**:

$$e_x \overset{\text{not.}}{=} \mathbb{E}\left[K_x\right]$$

- Evaluating e_x:

$$e_x = \sum_{k=1}^{\infty} {}_k p_x \tag{2.25}$$

- Second moment of K_x:

$$\mathbb{E}[K_x^2] = 2 \sum_{k=1}^{\infty} k \, {}_k p_x - e_x$$

- Numerical values:

Table 2.2. *Values of* e_x *and* $\mathring{e}_x$ *for Gompertz' law with* $B = 0.0003$ *and* $c = 1.07$.

x	e_x	$\mathring{e}_x$
0	71.438	71.938
10	61.723	62.223
20	52.203	52.703
30	42.992	43.492
40	34.252	34.752
50	26.192	26.691
60	19.052	19.550
70	13.058	13.555
80	8.354	8.848
90	4.944	5.433
100	2.673	3.152

- Approximation for the complete future lifetime:

$$\mathring{e}_x \approx e_x + \frac{1}{2} \tag{2.26}$$

2.7 Notes and further reading

- Other names for the force-of-mortality:
  - Survival analysis: *Hazard rate*.
  - Reliability theory: *Failure rate.*