Life Insurance Mathematics

Life insurance benefits[1]

Jan Dhaene

[1] Based on Chapter 4 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).

4.1 Summary

- Life Insurance contracts:
  - Whole life insurance.
  - Term insurance.
  - (Pure) endowment insurance.
- Actuarial valuation of life insurance benefits:
  - The actuarial value (or EPV) of a life insurance benefit cash flow.
  - Actuarial notation.
  - Continuous valuation via T_x.
  - Discrete valuation via K_x or $K_x^{(m)}$.

4.2 Introduction

- In previous chapters:
    - Models for future lifetimes.
- In coming chapters:
    - Valuating payments contingent on death or survival of policyholder or pension plan member,
    - by combining time value of money and survival models.
- Continuous time model:
    - Death benefits paid at moment of death.
    - Annuity benefits of € 1 per year paid in infinitesimal units of € dt in each interval $(t, t + dt)$.

- Discrete time model:
    - Death benefit paid at end of period.
    - Annuity benefits paid at beginning or end of period.

4.2 Introduction

"*The foundation of life insurance is the recognition of the value of a human life and the possibility of indemnification for the loss of that value*", C.F.C. Oviatt (1905).

"Protecting what matters in life."

4.2 Introduction

Why buying life insurance?

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Why buying life insurance?

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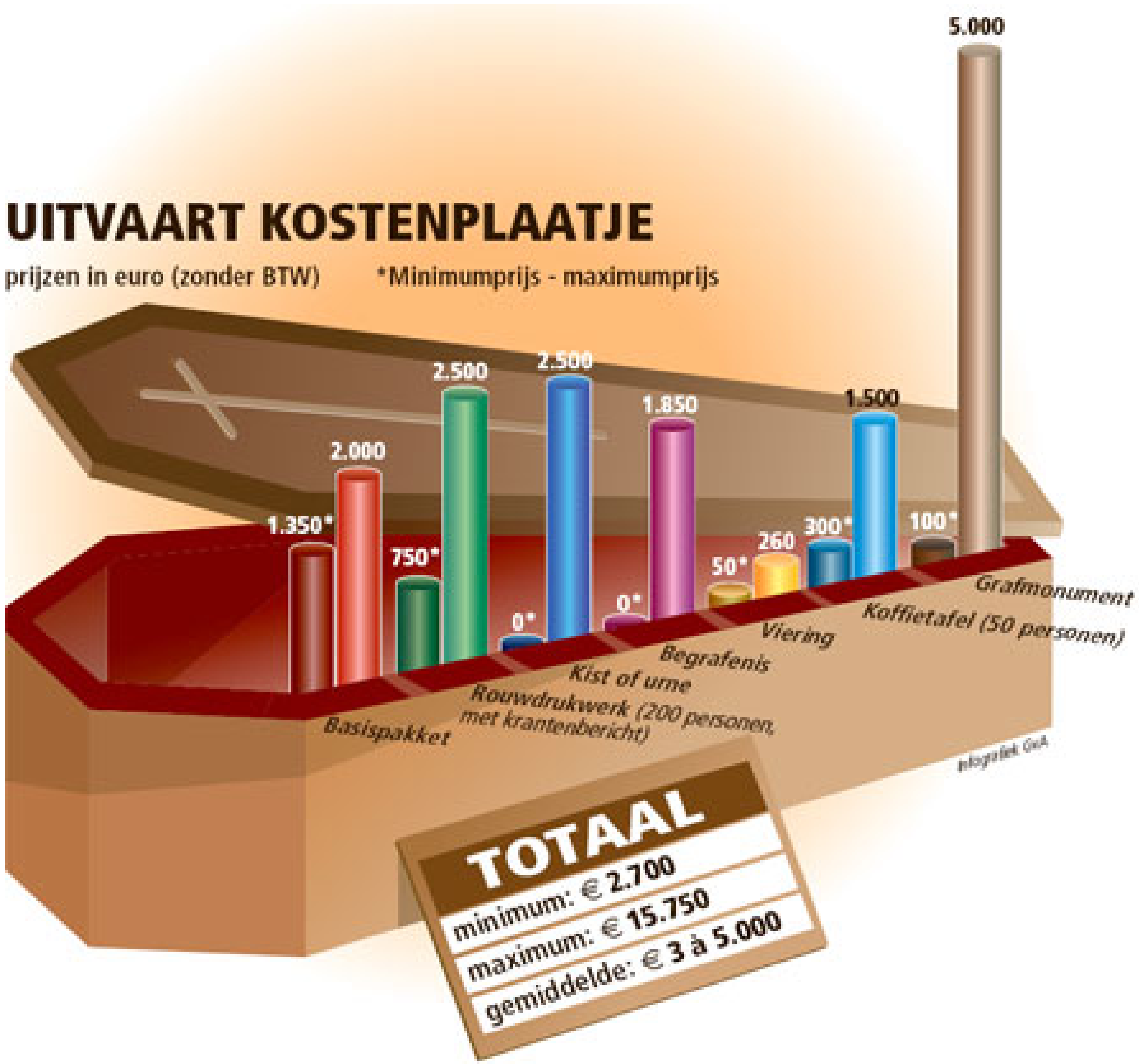
Why buying life insurance?



4.2 Introduction

Why buying life insurance?

4.3 Assumptions

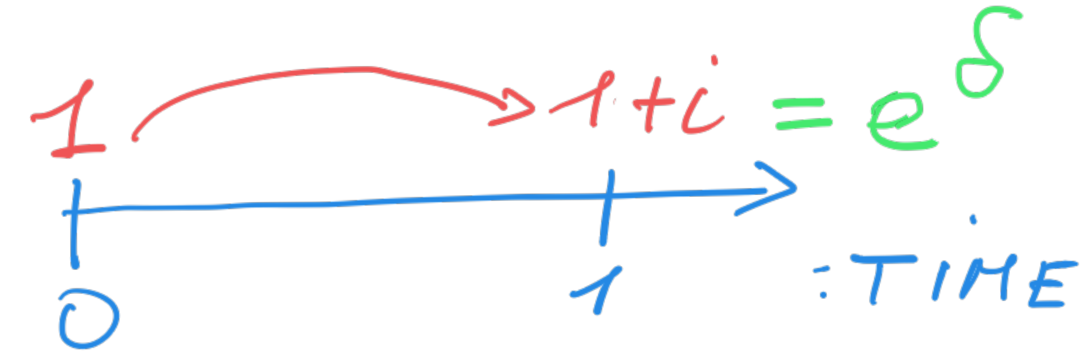
- Technical basis = a set of assumptions used for performing life insurance or pension calculations.
- Technical basis in this chapter (used in the examples):
  - The Standard Ultimate Survival Model:

$$\mu_x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$$

  - A constant interest.
  - These are (pedagogically) convenient assumptions.
- Conventions:
  - Time 0 = now.
  - Time unit is 1 year.

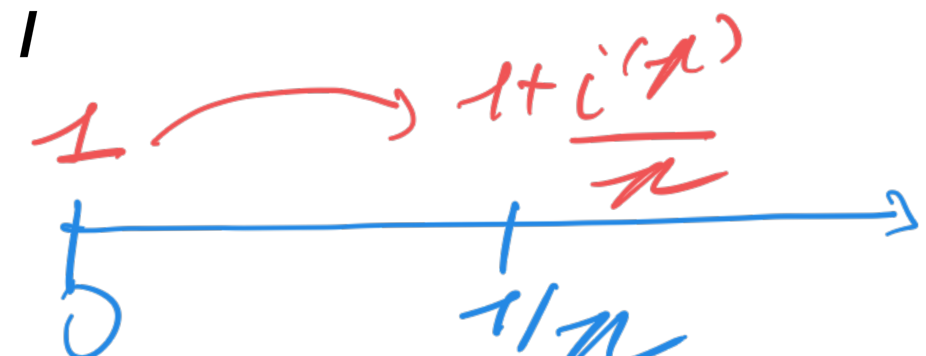
4.3 Assumptions

Some notions of financial algebra



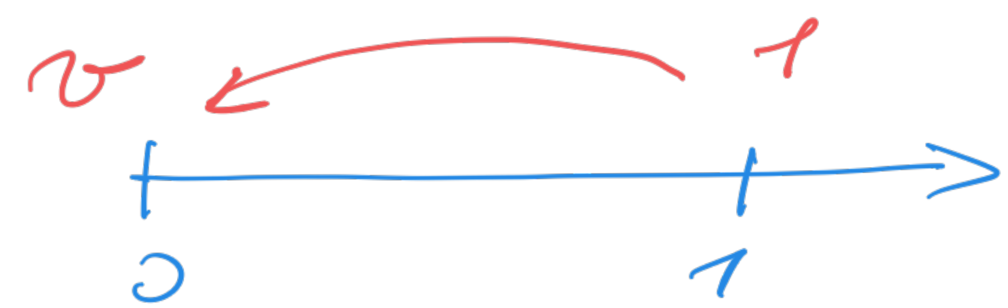
- i = annual rate of interest.
- $i^{(p)}$ = nominal interest (compounded p times per year):

$$\left(1+\frac{i^{(p)}}{p}\right)^{p}=1+i$$



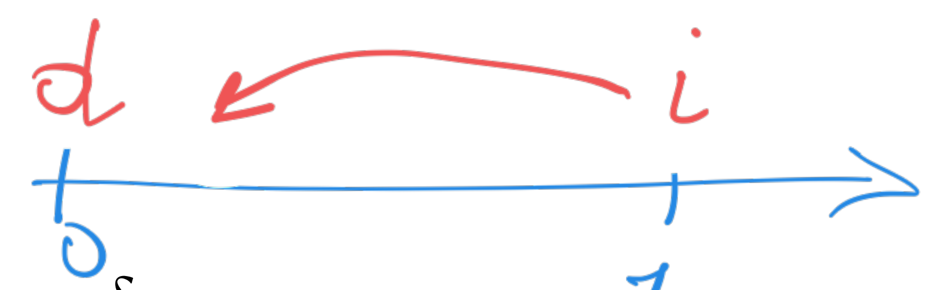
- δ = force of interest:

$$\delta=\ln\left(1+i\right)$$



- v = yearly discount factor:

$$v=\frac{1}{1+i}=e^{-\delta}$$

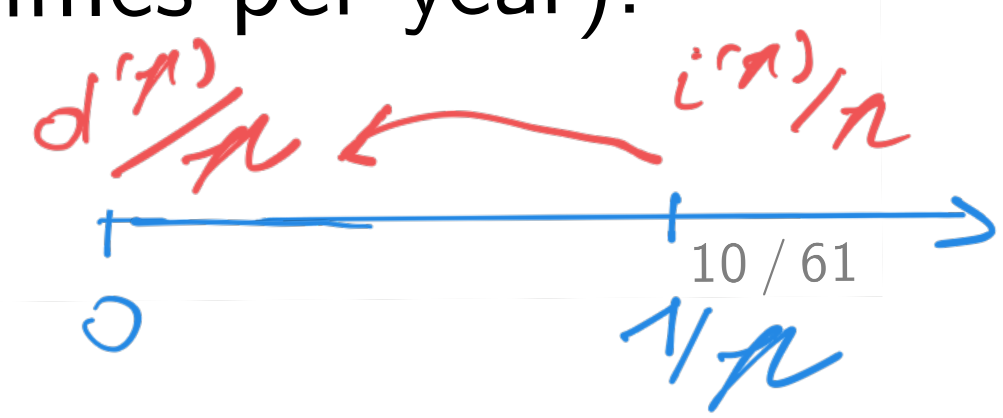


- d = discount rate per year:

$$d=1-v=i\ v=1-e^{-\delta}$$

- $d^{(p)}$ = nominal discount rate (compounded p times per year):

$$d^{(p)}=p\left(1-v^{\frac{1}{p}}\right)=i^{(p)}\ v^{1/p}$$

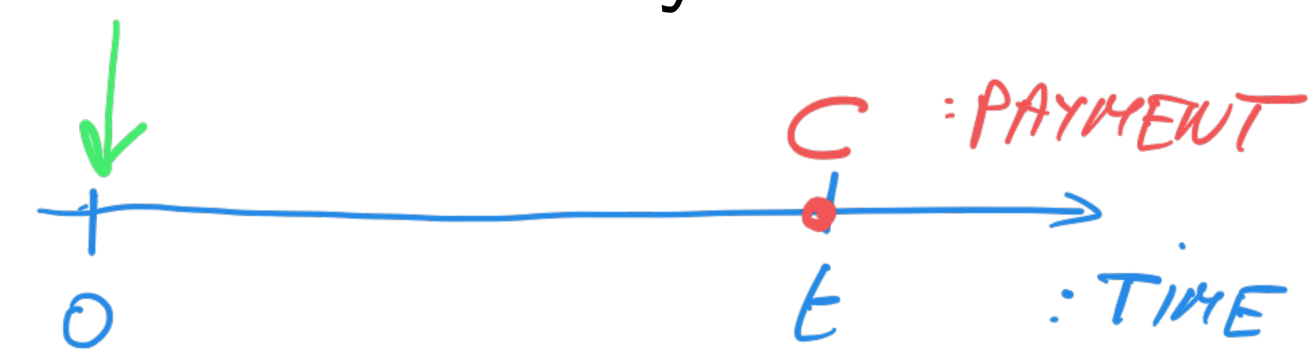


4.3 Assumptions

Cash flow notations

- The cash flow with payment c at time t is denoted by

$$(c, t)$$



- The cash flow $(\alpha c, t)$ is often denoted by

$$\alpha\,(c, t)$$

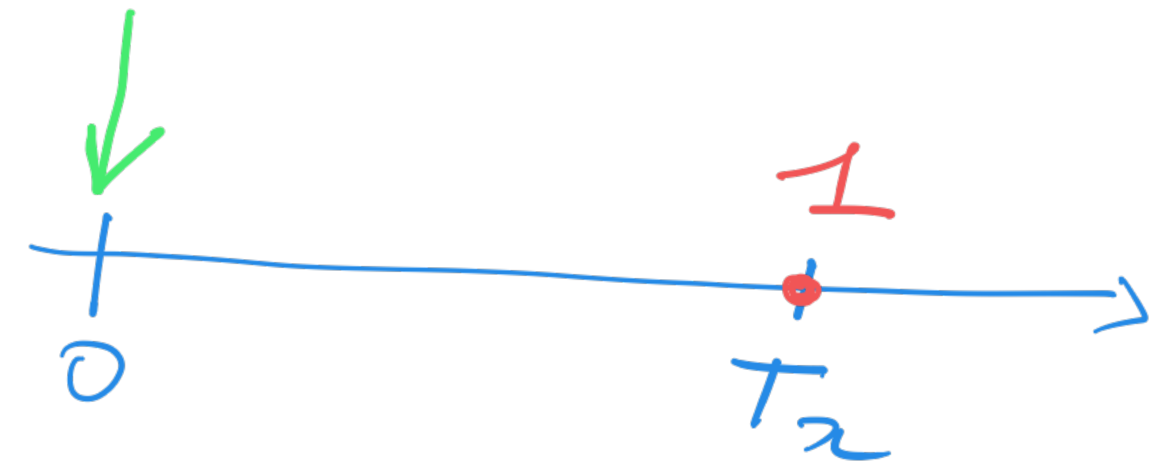
- In previous notations, c and t may be deterministic or random.

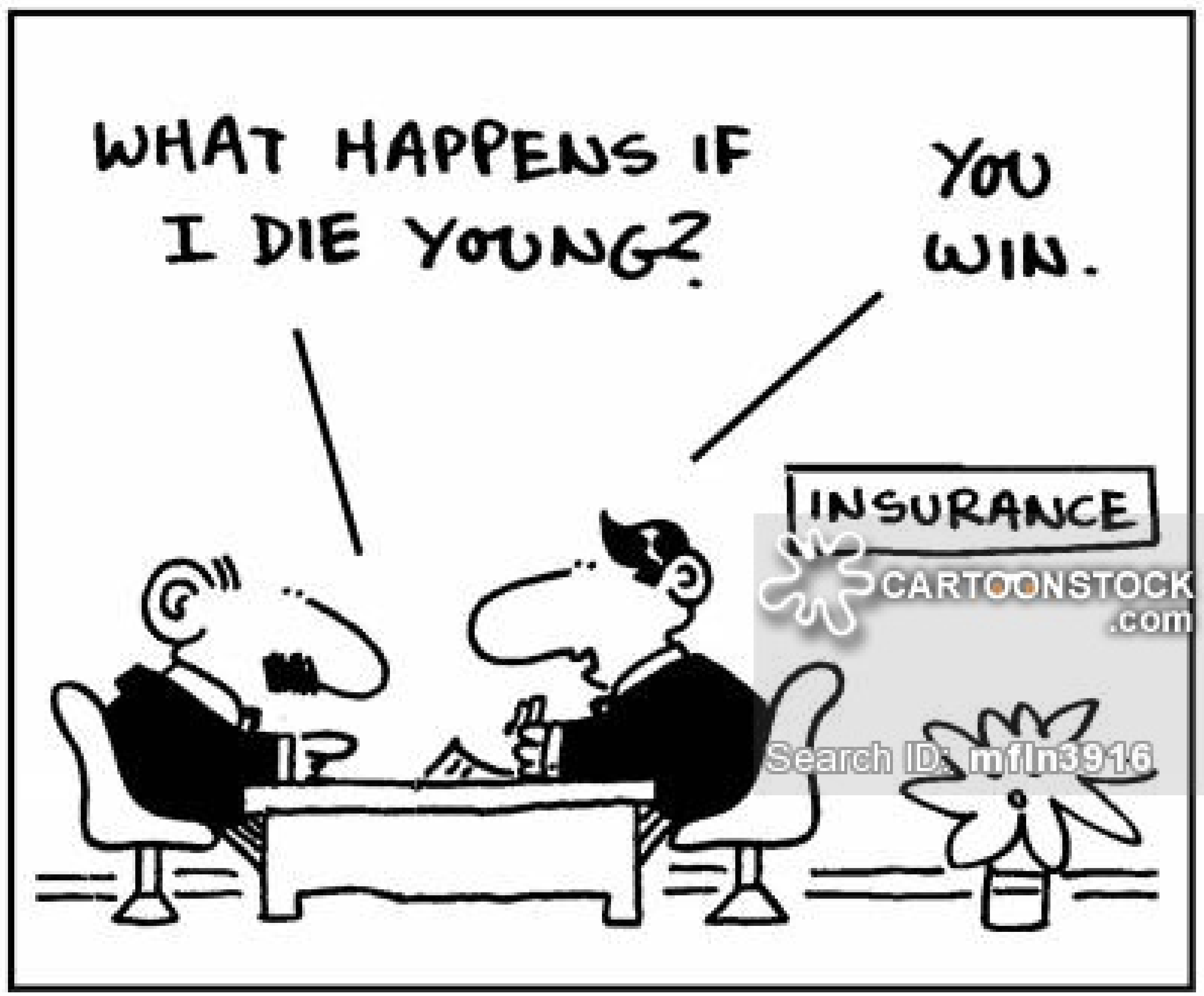
4.4 Valuation of insurance benefits

4.4.1 Whole life insurance: the continuous case

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at T_x.
- <u>Benefit cash flow</u>:

$$\boxed{(1,\, T_x)}$$





4.4 Valuation of insurance benefits

4.4.1 Whole life insurance: the continuous case

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at T_x.
- Benefit cash flow:

$$(1, T_x)$$

- Present value:

$$Z = v^{T_x} = e^{-\delta T_x}$$

- Actuarial value (or EPV):

$$\boxed{\overline{A}_x \overset{\text{not.}}{=} \mathbb{E}\left[e^{-\delta T_x}\right] = \int_0^\infty e^{-\delta t}\, {}_tp_x\, \mu_{x+t}\, dt} \qquad (4.1)$$

$$L_{>} = \int_0^\infty e^{-\delta t} \, f_x(t)\, dt = 5$$

4.4 Valuation of insurance benefits

4.4.1 Whole life insurance: the continuous case

$$\boxed{\bar{A}_x = \int_0^\infty e^{-\delta s}\, {}_sp_x\ \mu_{x+s}\ ds}$$

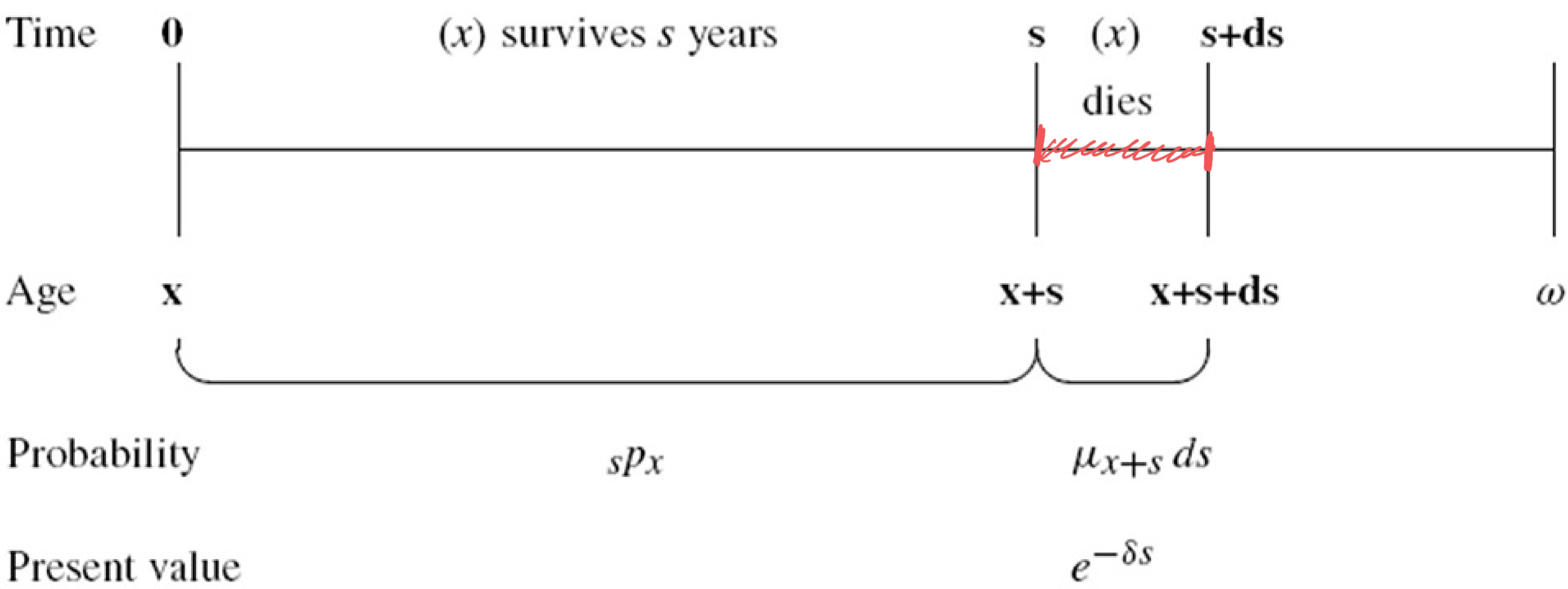


Figure 4.1 Time-line diagram for continuous whole life insurance.

4.4 Valuation of insurance benefits

4.4.1 Whole life insurance: the continuous case

- Cdf of $Z = e^{-\delta T_x}$:

$$\boxed{\mathbb{P}\left[Z \leq y\right] = {}_{u(y)}p_x}$$

with

$$u(y) = -\frac{\ln(y)}{\delta}$$

- Whole life insurance with payment of S at T_x:
  - Cash flow:

$$(S, T_x)$$

  - Actuarial value:

$$\mathbb{E}\left[S\ e^{-\delta T_x}\right] = S\ \overline{A}_x$$

4.4 Valuation of insurance benefits

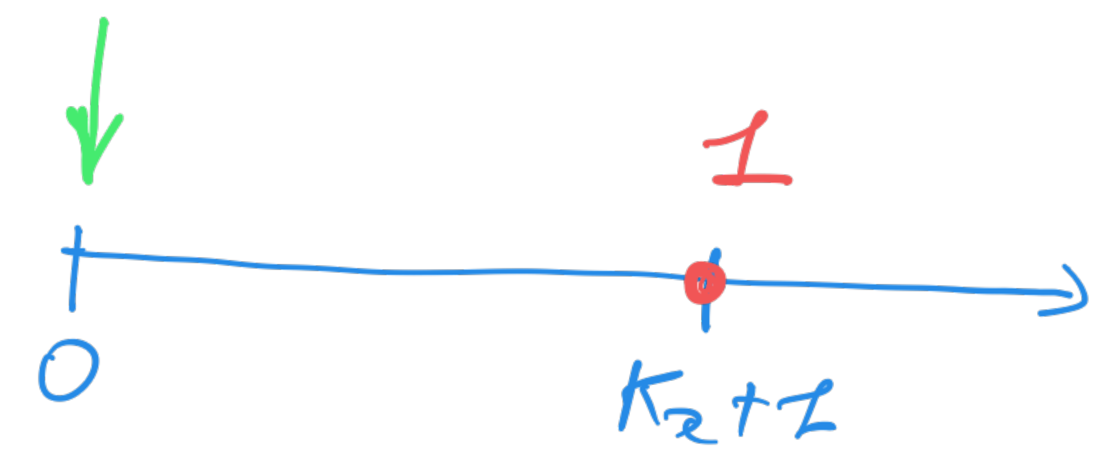
4.4.1 Whole life insurance: the continuous case

4.4 Valuation of insurance benefits

4.4.2 Whole life insurance: the annual case

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x + 1$.
- Benefit cash flow:

$$(1, K_x + 1)$$



- Present value:

$$Z = v^{K_x+1}$$

- Actuarial value:

$$\boxed{A_x \overset{\text{not.}}{=} \mathbb{E}[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} \; {}_{k|}q_x} \qquad (4.4)$$

4.4 Valuation of insurance benefits

4.4.2 Whole life insurance: the annual case

$$\boxed{A_x = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}q_x}$$

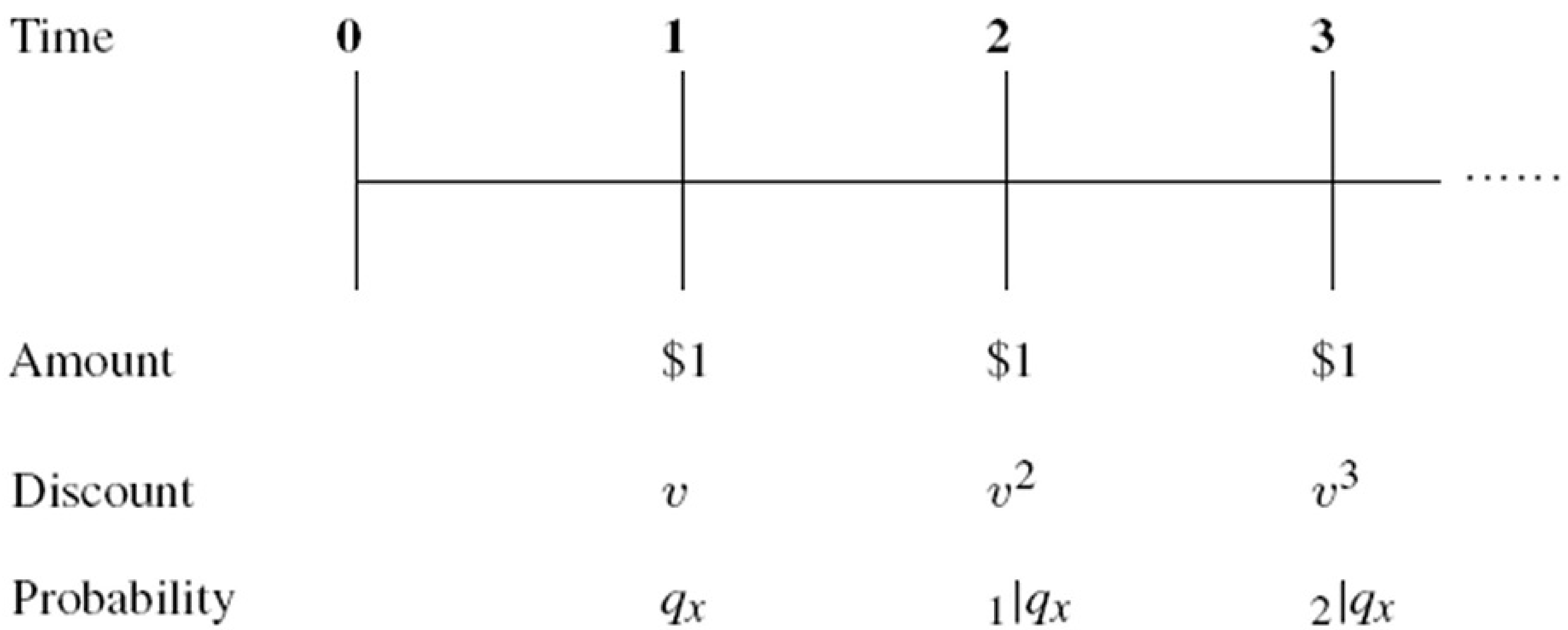


Figure 4.2 Time-line diagram for discrete whole life insurance.

4.4 Valuation of insurance benefits

4.4.2 Whole life insurance: the annual case

- **Single premium**:

$$\boxed{A_x = \sum_{k=0}^{\infty} v^{k+1} \; {}_{k|}q_x}$$

- Who got there first?
    - William Morgan (1779).
    - Richard Price (1783).
    - Francis Bailey (1810).

4.4 Valuation of insurance benefits

Some history

- Early forms of life insurance existed in Ancient Rome, where 'burial clubs' covered cost of members' funeral expenses and assisted survivors financially.
- Until 17th century, life insurance was often considered as 'profiting from one's death':
  - Opposition from the Church.
  - Illegal in many places.
- Modern life insurance policies were first established in the early 18th century.

4.4 Valuation of insurance benefits

Some history

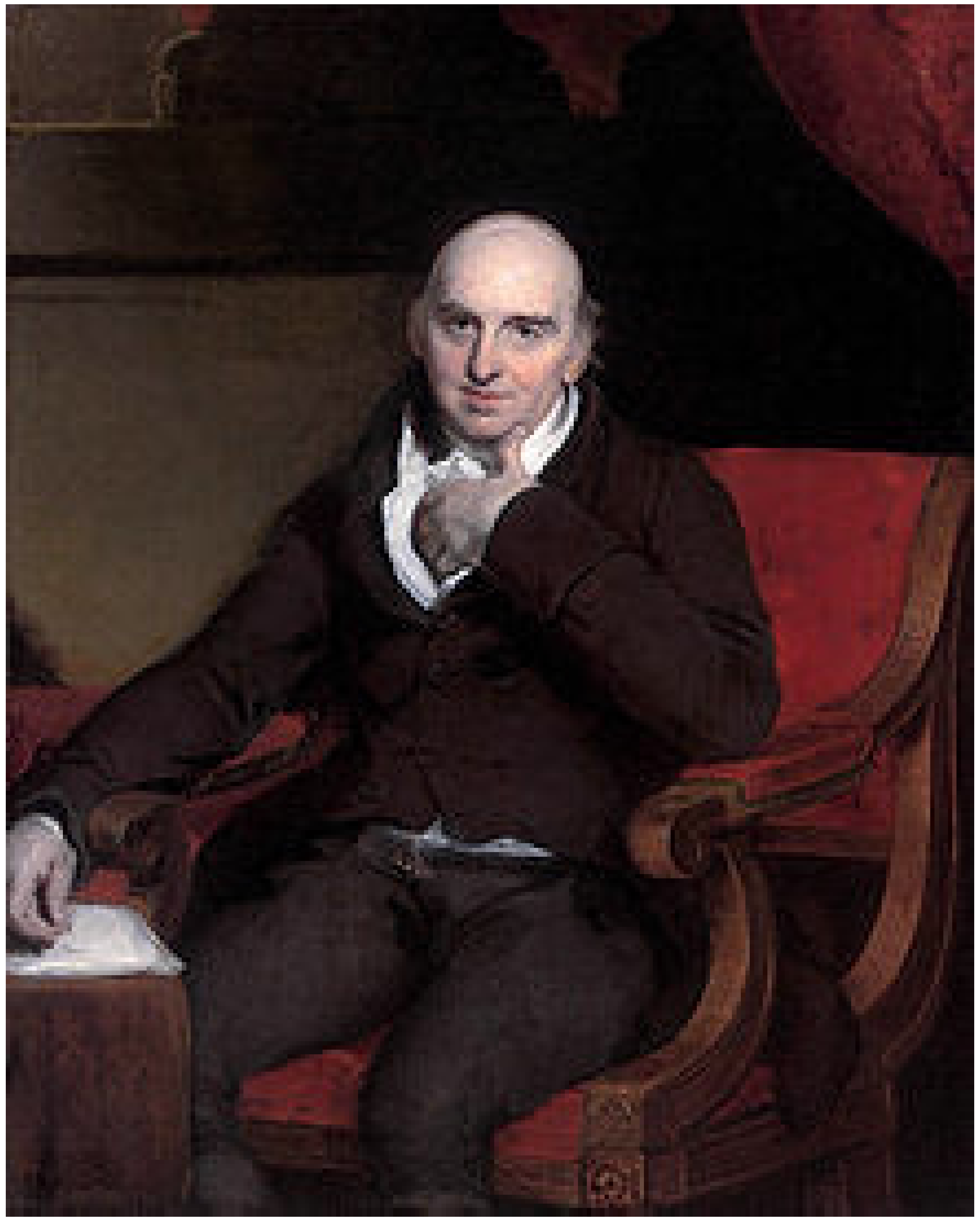
- Amicable Society for a Perpetual Assurance Office:
  - Founded in London, 1706.
  - First company offering life insurance.
- Society for Equittable Assurances on Lives and Survivorship:
  - Also known as '*Equittable Life*', founded in London, 1762.
  - World's oldest mutual life insurer.
  - Chief Executive Officer: **Edward Rowe Mores**, credited with being the first person to use the professional title *actuary* in relation to insurance.
  - In 1775, **William Morgan** was appointed as actuary. As he carried out the first actuarial valuation of liabilities in 1776, he is often considered as the first 'real' actuary.
  - 'Equittable Life' closed its doors to new business in 2000, due to huge unhedged biting guarantees (GAR's).

4.4 Valuation of insurance benefits

Some history

Edward Rowe Mores (1731-1778)

William Morgan (1750-1833)



4.4 Valuation of insurance benefits

4.4.3 Whole life insurance: the 1/m-thly case

- The floor function:

$$\lfloor r \rfloor = \text{ largest integer smaller than or equal to } r.$$

- The 1/m-thly curtate future lifetime of (x):

$$\boxed{K_x^{(m)} \overset{\text{not.}}{=} \frac{1}{m}\lfloor m T_x \rfloor} \qquad (4.7)$$

  - $K_x^{(m)}$ = future lifetime of (x) in years, rounded down to the lower 1/m-thly of the year.

- The pdf of $K_x^{(m)}$: For $k = 0, 1, 2, \ldots$,

$$\begin{aligned}
\mathbb{P}\left[K_x^{(m)} = \frac{k}{m}\right] &= \mathbb{P}\left[\frac{k}{m} \leq T_x < \frac{k+1}{m}\right] \\
&= {}_{\frac{k}{m}|\frac{1}{m}}q_x \\
&= {}_{\frac{k}{m}}p_x - {}_{\frac{k+1}{m}}p_x
\end{aligned}$$

- $K_x^{(m)} = \frac{1}{m} \lfloor m \times T_x \rfloor$

- Example:

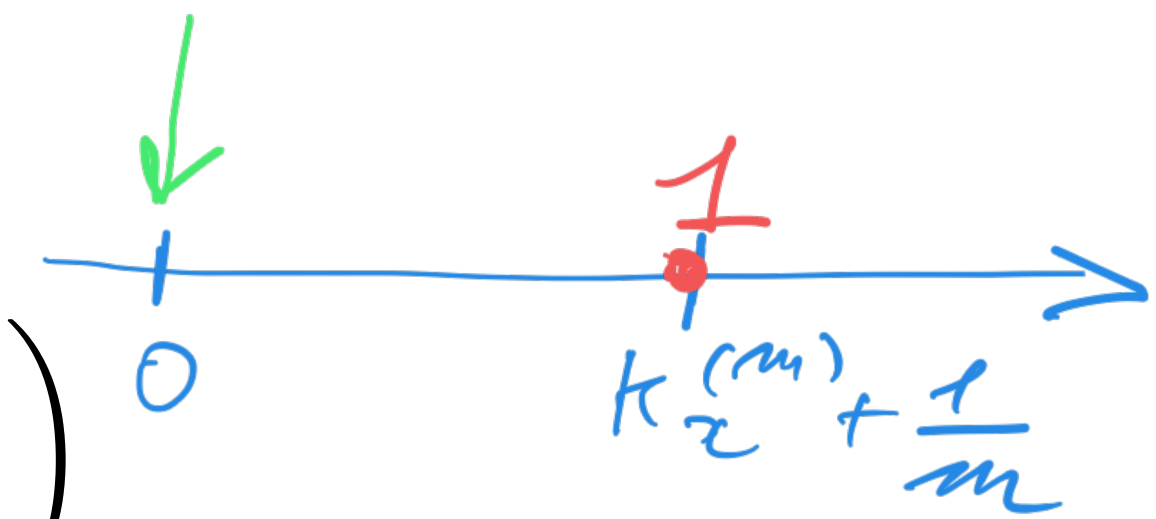
  - $T_x = 4.51$
  - $12 \times T_x = 12 \times 4.51 = 54.12$
  - $\lfloor 12 \times T_x \rfloor = 54$
  - $K_x^{(12)} = \frac{1}{12} \lfloor 12 \times T_x \rfloor = \frac{54}{12}$

    $= 4 + \frac{6}{12}$

4.4 Valuation of insurance benefits

4.4.3 Whole life insurance: the 1/m-thly case

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x^{(m)} + \frac{1}{m}$.
- Benefit cash flow:



$$\left(1, K_x^{(m)} + \frac{1}{m}\right)$$

- Present value:

$$Z = v^{K_x^{(m)} + \frac{1}{m}}$$

- Actuarial value:

$$\boxed{A_x^{(m)} \overset{\text{not.}}{=} \mathbb{E}\left[v^{K_x^{(m)} + \frac{1}{m}}\right] = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} \, {}_{\frac{k}{m}|\frac{1}{m}}q_x}$$

4.4 Valuation of insurance benefits

4.4.3 Whole life insurance: the 1/m-thly case

$$A_x^{(m)} = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x$$

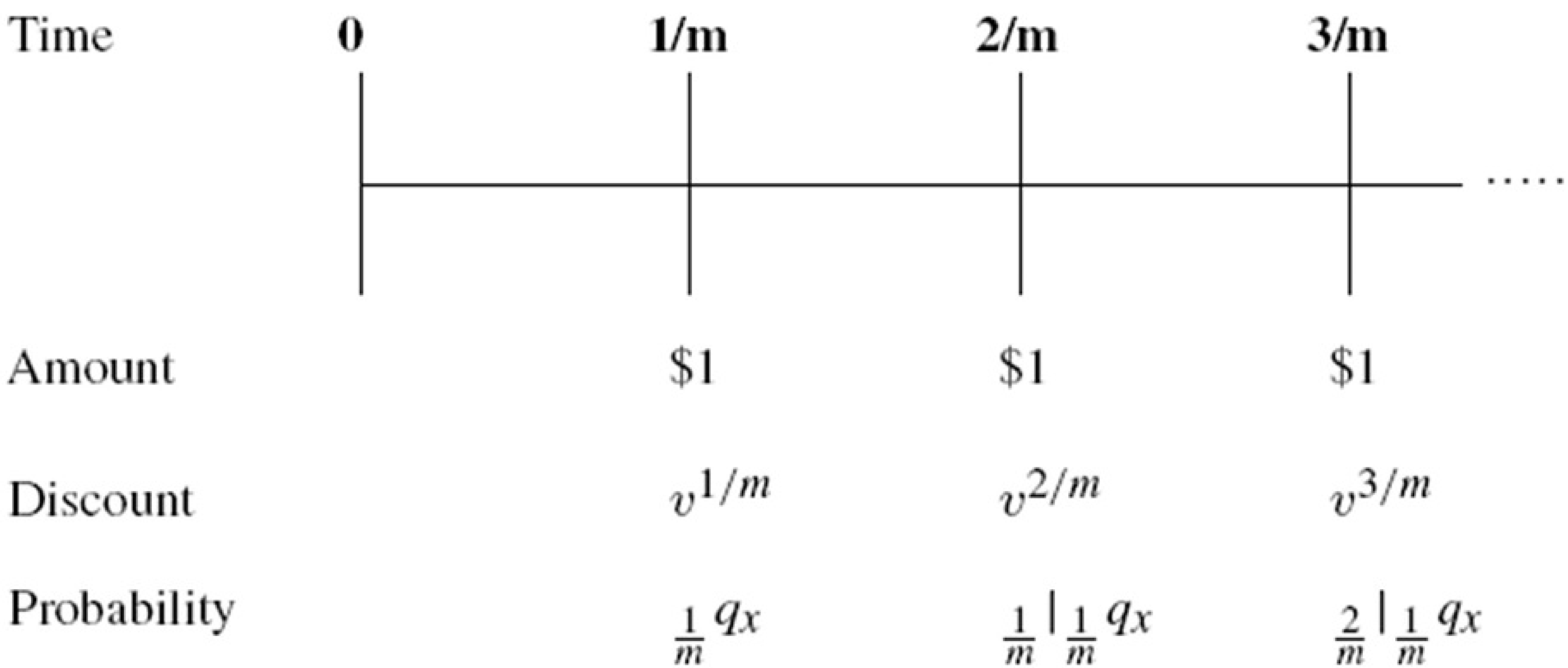


Figure 4.3 Time-line diagram for *m*thly whole life insurance.

4.4 Valuation of insurance benefits

4.4.4 Recursions

- Highest age ω of life table:

$$\omega \text{ is first age in life table such that } l_\omega = 0$$

$\to q_{\omega-1} = \frac{l_{\omega-1} - l_\omega}{l_{\omega-1}} = 1$

- **Backward recursion for the annual case**:
  - Starting value:

$$A_{\omega-1} = v$$

$= q_{\omega-1} \times v =$

  - Recursion: For $x = \omega - 2, \omega - 3, \ldots, x_0$,

$$\boxed{A_x = vq_x + vp_x A_{x+1}} \quad (4.8)$$

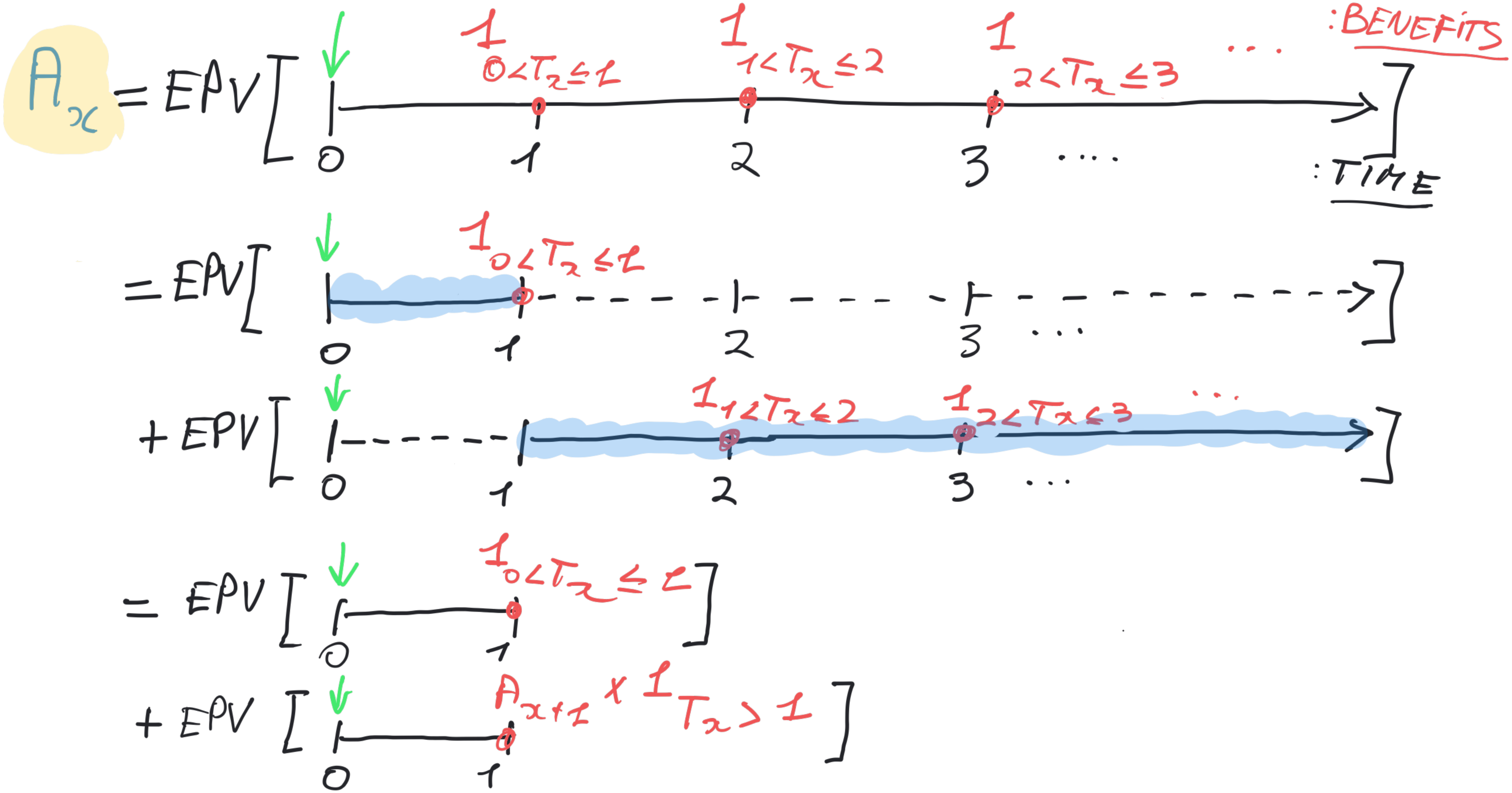
$\to$ PROOF: see book. $\to$ INTERPRETATION: see next slide.

- **Backward recursion for the 1/m-thly case**:
  - Starting value:

$$A^{(m)}_{\omega-\frac{1}{m}} = v^{\frac{1}{m}}$$

  - Recursion: For $x = \omega - \frac{2}{m}, \omega - \frac{3}{m}, \ldots,$

$$\boxed{A^{(m)}_x = v^{\frac{1}{m}} \, {}_{\frac{1}{m}}q_x + v^{\frac{1}{m}} \, {}_{\frac{1}{m}}p_x \, A^{(m)}_{x+\frac{1}{m}}}$$



$$= q_x \times v + p_x \times v \times A_{x+1} \qquad (4.8)$$

Example 4.1

- Technical basis:
  - Standard Ultimate Survival model (see Section 4.3).
  - Interest rate = 5%.
- Ultimate age: $\omega = 130$.
- Determine the values of A_x for $x = 20, 21, \ldots 100$.

4.4 Valuation of insurance benefits

4.4.4 Recursions

Example 4.1

Table 4.1. *Spreadsheet results for Example 4.1, for the calculation of A_x using the Standard Ultimate Survival Model.*

x	A_x	x	A_x	x	A_x
20	0.04922	47	0.16577	74	0.49215
21	0.05144	48	0.17330	75	0.50868
22	0.05378	49	0.18114	76	0.52536
23	0.05622	50	0.18931	77	0.54217
24	0.05879	51	0.19780	78	0.55906
25	0.06147	52	0.20664	79	0.57599
26	0.06429	53	0.21582	80	0.59293
27	0.06725	54	0.22535	81	0.60984
28	0.07034	55	0.23524	82	0.62666
29	0.07359	56	0.24550	83	0.64336
30	0.07698	57	0.25613	84	0.65990
31	0.08054	58	0.26714	85	0.67622
32	0.08427	59	0.27852	86	0.69229
33	0.08817	60	0.29028	87	0.70806
34	0.09226	61	0.30243	88	0.72349
35	0.09653	62	0.31495	89	0.73853
36	0.10101	63	0.32785	90	0.75317
37	0.10569	64	0.34113	91	0.76735
38	0.11059	65	0.35477	92	0.78104
39	0.11571	66	0.36878	93	0.79423
40	0.12106	67	0.38313	94	0.80688
41	0.12665	68	0.39783	95	0.81897
42	0.13249	69	0.41285	96	0.83049
43	0.13859	70	0.42818	97	0.84143
44	0.14496	71	0.44379	98	0.85177
45	0.15161	72	0.45968	99	0.86153
46	0.15854	73	0.47580	100	0.87068

Example 4.2

- Technical basis:
  - Standard Ultimate Survival model (see Section 4.3).
  - Interest rate = 5%.
- Ultimate age: $\omega = 130$.
- Determine the values of $A_x^{(12)}$ for x starting at age 20, in step of $1/12$.

4.4 Valuation of insurance benefits

4.4.4 Recursions

Example 4.2

Table 4.2. *Spreadsheet results for Example 4.2, for the calculation of $A_x^{(12)}$ using the Standard Ultimate Survival Model.*

x	${}_{\frac{1}{12}}p_x$	${}_{\frac{1}{12}}q_x$	$A_x^{(12)}$
20	0.999979	0.000021	0.05033
$20\frac{1}{12}$	0.999979	0.000021	0.05051
$20\frac{2}{12}$	0.999979	0.000021	0.05070
$20\frac{3}{12}$	0.999979	0.000021	0.05089
⋮	⋮	⋮	⋮
50	0.999904	0.000096	0.19357
$50\frac{1}{12}$	0.999903	0.000097	0.19429
⋮	⋮	⋮	⋮
$129\frac{10}{12}$	0.413955	0.586045	0.99427
$129\frac{11}{12}$			1

4.4 Valuation of insurance benefits

4.4.4 Recursions

Example 4.3

- For $x = 20, 40, 60, 80$ and 100, calculate the actuarial value of the following cash flows:
  - Continuous case:
  $$100\ 000 \times (1, T_x)$$
  - Monthly case:
  $$100\ 000 \times \left(1, K_x^{(12)} + \frac{1}{12}\right)$$
  - Annual case:
  $$100\ 000 \times (1, K_x + 1)$$

- Technical basis:
  - Standard Ultimate Survival model (see Section 4.3).
  - Interest rate = 5%.

4.4 Valuation of insurance benefits

4.4.4 Recursions

Example 4.3

Table 4.3. *Mean and standard deviation of the present value of a whole life insurance benefit of $100 000, for Example 4.3.*

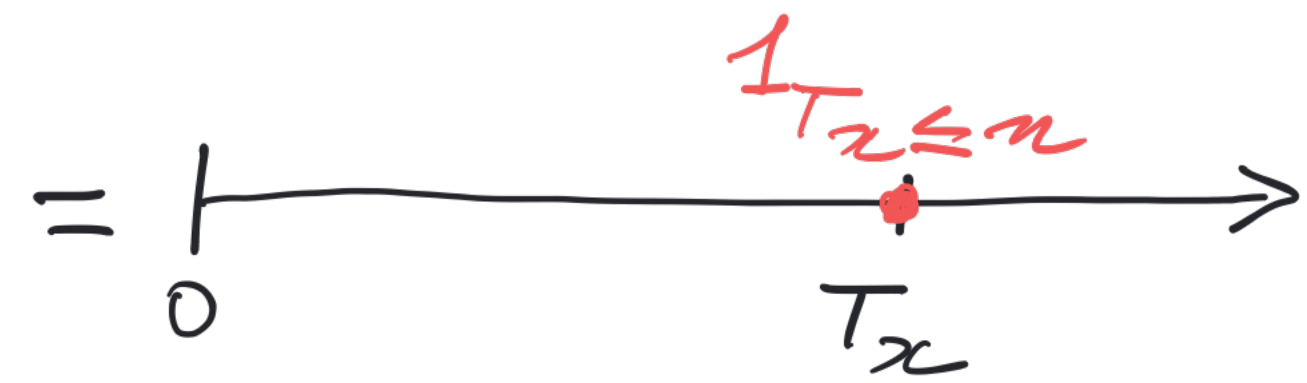
	Continuous (a)		Monthly (b)		Annual (c)	
Age, x	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
20	5 043	5 954	5 033	5 942	4 922	5 810
40	12 404	9 619	12 379	9 600	12 106	9 389
60	29 743	15 897	29 683	15 865	29 028	15 517
80	60 764	17 685	60 641	17 649	59 293	17 255
100	89 341	8 127	89 158	8 110	87 068	7 860

$\bar{A}_x > A_x^{(12)} > A_x$

Continuous case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $T_x \leq n$.
- <u>Benefit cash flow</u>:

$$\left(1_{\{T_x \leq n\}}, T_x\right) =$$



- <u>Present value</u>:

$$Z = e^{-\delta T_x} \, 1_{\{T_x \leq n\}}$$

- <u>Actuarial value</u>:

$$\boxed{\overline{A}^{\,1}_{x:\overline{n|}} \overset{\text{not.}}{=} \mathbb{E}[Z] = \int_0^n e^{-\delta t} \, {}_t p_x \; \mu_{x+t} \; dt} \qquad (4.9)$$

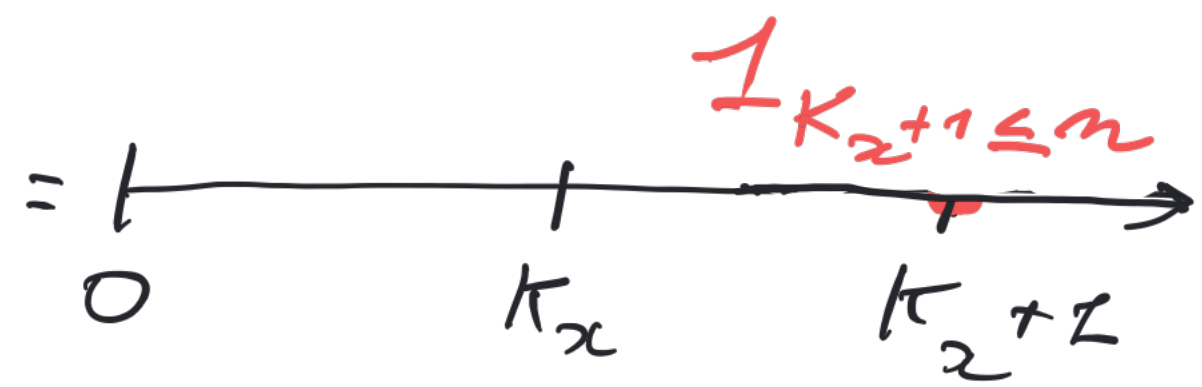
4.4 Valuation of insurance benefits

4.4.5 Term insurance

Annual case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x + 1$, provided $K_x + 1 \leq n$.
- Benefit cash flow:

$$\left(1_{\{K_x+1 \leq n\}},\ K_x + 1\right) =$$



- Present value:

$$Z = v^{K_x+1}\ 1_{\{K_x+1 \leq n\}}$$

- Actuarial value:

$$\boxed{A^{1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x} \qquad (4.10)$$

1/m-thly case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x^{(m)} + \frac{1}{m}$, provided $K_x^{(m)} + \frac{1}{m} \leq n$.
- Benefit cash flow:

$$\left(1_{\left\{K_x^{(m)}+\frac{1}{m}\leq n\right\}},\ K_x^{(m)} + \frac{1}{m}\right)$$

- Present value:

$$Z = v^{K_x^{(m)}+\frac{1}{m}}\, 1_{\left\{K_x^{(m)}+\frac{1}{m}\leq n\right\}}$$

- Actuarial value: $\mapsto k = m \times n - 1 \iff \frac{k+1}{m} = n$

$$A^{(m)\,1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \sum_{k=0}^{mn-1} v^{(k+1)/m}\ {}_{\frac{k}{m}|\frac{1}{m}}q_x \qquad (4.11)$$

4.4 Valuation of insurance benefits

4.4.5 Term insurance

Example 4.4

- For $x = 20, 40, 60$ and 80, calculate the following actuarial values:
  - Continuous case:
  $$\bar{A}^{\,1}_{x:\overline{10}|}$$
  - 1/4-thly case:
  $$A^{(4)\,1}_{\;\;x:\overline{10}|}$$
  - Annual case:
  $$A^{1}_{x:\overline{10}|}$$

- Technical basis:
  - Standard Ultimate Survival model (see Section 4.3).
  - Interest rate $= 5\%$.

Example 4.4

x	$\bar{A}^{\,1}_{x:\overline{10}\rceil}$ $>$	$A^{(4)\,1}_{x:\overline{10}\rceil}$ $>$	$A^{\,1}_{x:\overline{10}\rceil}$
20	0.00214	0.00213	0.00209
40	0.00587	0.00584	0.00573
60	0.04356	0.04329	0.04252
80	0.34550	0.34341	0.33722

4.4 Valuation of insurance benefits

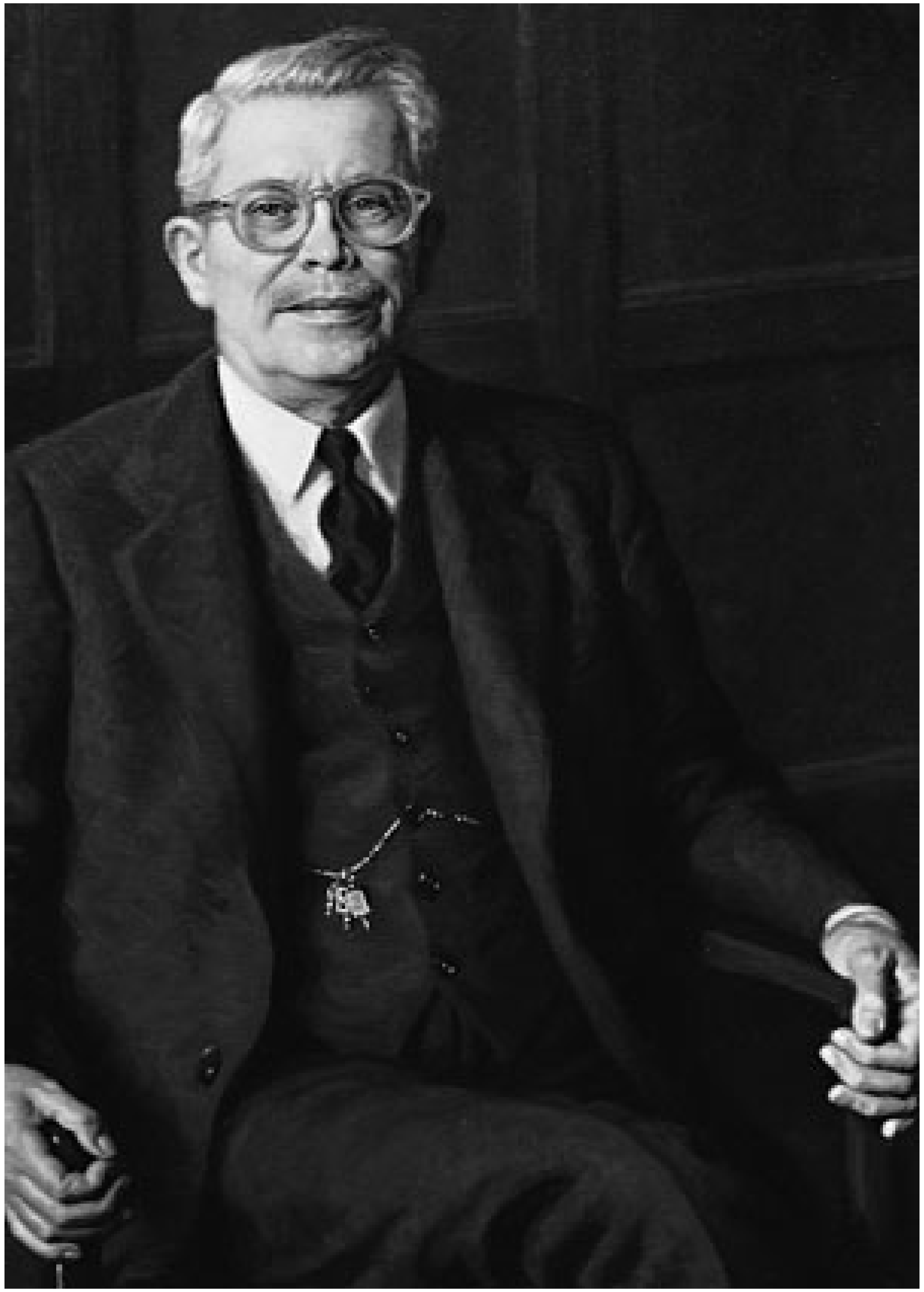
Samuel Huebner (1882 - 1964)

- Professor at Wharton School of Business.
- One of the first insurance economists.
- Author of 7 books and many articles on life insurance.
- The emperor of Japan awarded him an Order of the Sacred Treasure.
- In '*The Economics of Life Insurance*' (1927), he wrote:

> *Not to insure adequately through life insurance*
> *is to gamble with the greatest economic risk confronting man.*
> *If understood, the gamble is a particularly selfish one,*
> *since the blow, in the event the gamble is lost,*
> *falls upon an innocent household whose economic welfare*
> *should have been the family head's first consideration.*

4.4 Valuation of insurance benefits

Samuel Huebner (1882 - 1964)



- Consider a pure endowment insurance underwritten on (x) at time 0, with a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$\boxed{\left(1_{\{T_x>n\}},\ n\right)}$$

4.4 Valuation of insurance benefits

4.4.6 Pure endowment

- Consider a pure endowment insurance underwritten on (x) at time 0, with a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$\left(1_{\{T_x>n\}},\ n\right)$$

- Present value:

$$Z = v^n \, 1_{\{T_x>n\}}$$

- Actuarial value:

$$\boxed{{}_nE_x \overset{\text{not.}}{=} \mathbb{E}[Z] = v^n \, {}_np_x} \qquad (4.13)$$

- Alternate notation:

$$A_{x:\overset{\;1}{\overline{n|}}}$$

4.4 Valuation of insurance benefits

4.4.7 Endowment insurance

Continuous case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time T_x, provided $T_x \leq n$, and a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$(1, \min(T_x, n)) = \left(1_{\{T_x \leq n\}}, T_x\right) + \left(1_{\{T_x > n\}}, \ n\right)$$

- Present value:

$$Z = v^{\min(T_x, n)}$$

- Actuarial value:

$$\boxed{\overline{A}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \overline{A}^{\,1}_{x:\overline{n}|} + {}_nE_x} \tag{4.17}$$

4.4 Valuation of insurance benefits

4.4.7 Endowment insurance

Yearly case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time $K_x + 1$, provided $K_x + 1 \leq n$, and a payment of 1 at time n, provided $K_x + 1 > n$.
- Benefit cash flow:

$$\left(1, \min\left(K_x + 1, n\right)\right) = \left(1_{\{K_x+1\leq n\}},\ K_x + 1\right) + \left(1_{\{K_x+1>n\}},\ n\right)$$

- Present value:

$$Z = v^{\min(K_x+1,n)}$$

- Actuarial value:

$$\boxed{A_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = A^{1}_{x:\overline{n}|} + {}_{n}E_x} \qquad (4.19)$$

4.4 Valuation of insurance benefits

4.4.7 Endowment insurance

1/m-thly case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time $K_x^{(m)} + \frac{1}{m}$, provided $K_x^{(m)} + \frac{1}{m} \leq n$, and a payment of 1 at time n, provided $K_x^{(m)} + \frac{1}{m} > n$.
- Benefit cash flow:

$$\left(1, \min\left(K_x^{(m)} + \frac{1}{m}, n\right)\right)$$

- Present value:

$$Z = v^{\min\left(K_x^{(m)} + \frac{1}{m},\ n\right)}$$

- Actuarial value:

$$\boxed{A^{(m)}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = A^{(m)1}_{x:\overline{n}|} + {}_nE_x} \qquad (4.20)$$

Example 4.5

- For $x = 20, 40, 60$ and 80, calculate the following actuarial values:
  - Continuous case:
    $$\bar{A}_{x:\overline{10}|}$$
  - 1/4-thly case:
    $$A^{(4)}{}_{x:\overline{10}|}$$
  - Annual case:
    $$A_{x:\overline{10}|}$$
- Technical basis:
  - Standard Ultimate Survival model (see Section 4.3).
  - Interest rate $= 5\%$.

Example 4.5

Table 4.5. *EPVs of endowment insurance benefits.*

x	$\bar{A}_{x:\overline{10}\rceil}$	>	$A^{(4)}_{x:\overline{10}\rceil}$	>	$A_{x:\overline{10}\rceil}$
20	0.61438		0.61437		0.61433
40	0.61508		0.61504		0.61494
60	0.62220		0.62194		0.62116
80	0.68502		0.68292		0.67674

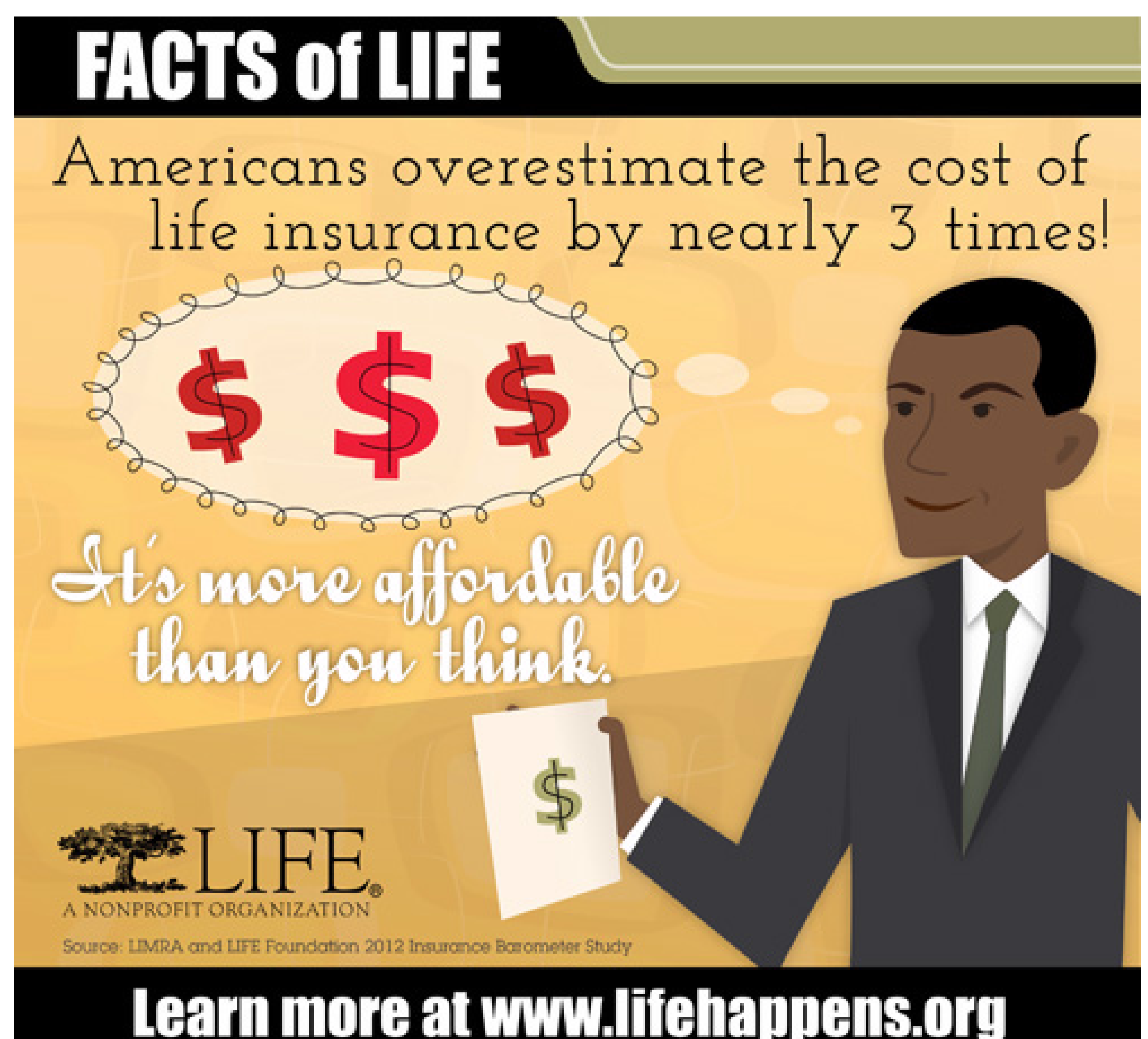
4.4 Valuation of insurance benefits

4.4.8 Deferred insurance

- Consider a deferred term insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $u < T_x \leq u + n$.
- Benefit cash flow:

$$\left(1_{\{u<T_x\leq u+n\}}, T_x\right)$$

- Price depends on x, u and n:



4.4 Valuation of insurance benefits

4.4.8 Deferred insurance

- Consider a deferred term insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $u < T_x \leq u + n$.
- Benefit cash flow:

$$\left(1_{\{u<T_x\leq u+n\}}, T_x\right)$$

- Present value:

$$Z = e^{-\delta T_x}\, 1_{\{u<T_x\leq u+n\}}$$

- Actuarial value:

$$\boxed{{}_{u|}\overline{A}^{\,1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \int_u^{u+n} e^{-\delta t}\, {}_t p_x\, \mu_{x+t}\, dt} \qquad (4.21)$$

4.4 Valuation of insurance benefits

4.4.8 Deferred insurance

- Deferred term insurance and actuarial discounting:

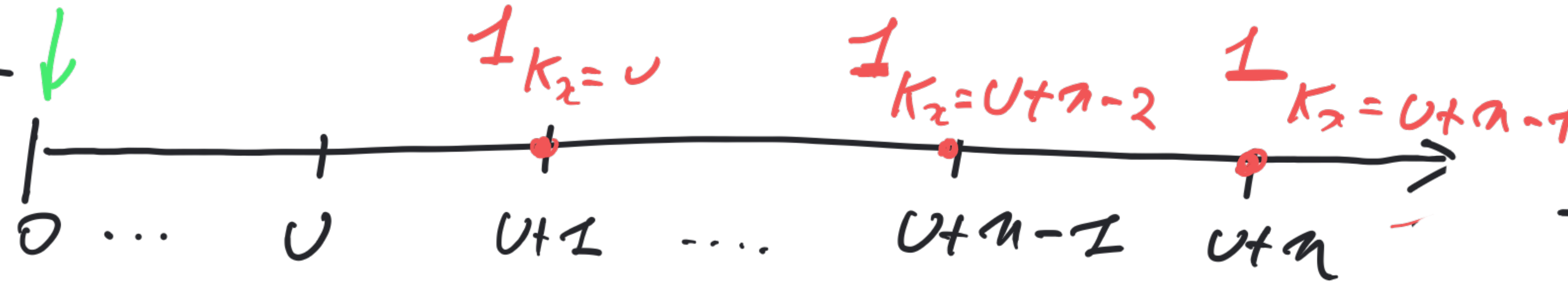
$$ {}_{u|}\bar{A}^{\,1}_{x:\overline{n}|} = {}_{u}E_x \times \bar{A}^{\,1}_{x+u:\overline{n}|} \tag{4.22} $$

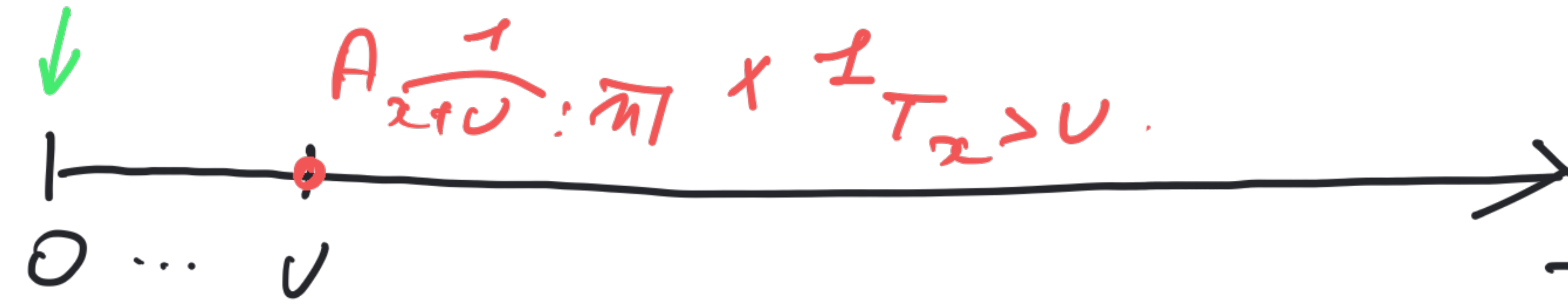
- Deferred vs. immediate insurance:

$$ {}_{u|}\bar{A}^{\,1}_{x:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{u+n}|} - \bar{A}^{\,1}_{x:\overline{u}|} \tag{4.23} $$

- Term insurance in terms of yearly insurances:

$$ \bar{A}^{\,1}_{x:\overline{n}|} = \sum_{r=0}^{n-1} {}_{r|}\bar{A}^{\,1}_{x:\overline{1}|} \tag{4.24} $$

• $_{u|}\bar{A}^{1}_{x:\overline{n}|} = EPV\Big[$  $\Big]$

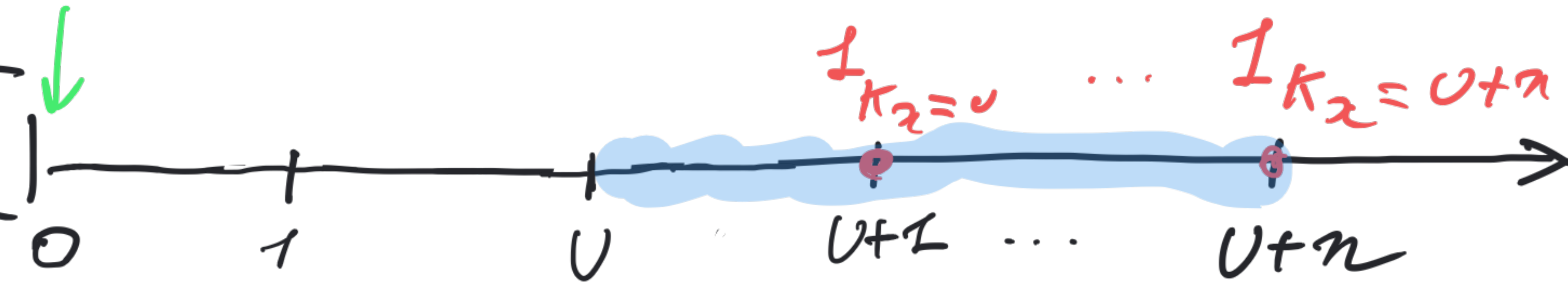
$= EPV\Big[$  $\Big]$

$$= {}_{u}p_{x} \times v^{u} \times A^{1}_{x+u:\overline{n}|}$$

$$= {}_{u}E_{x} \times \bar{A}^{1}_{x+u:\overline{n}|} \qquad (4.22')$$

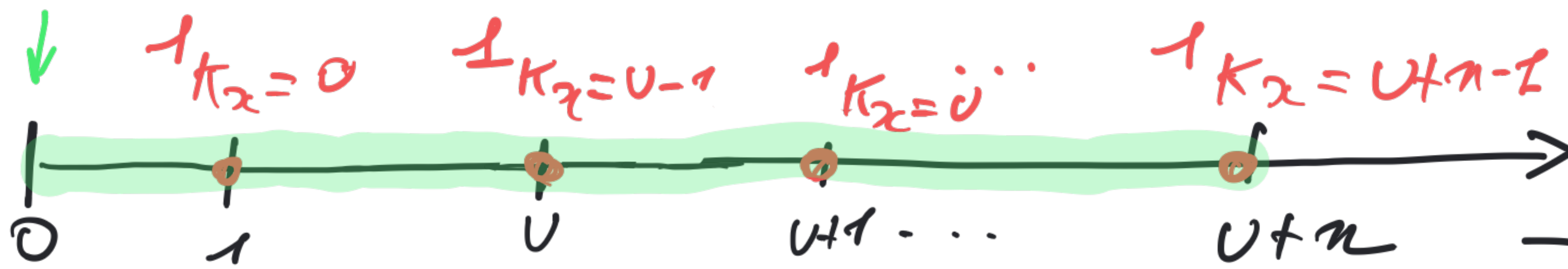
• $_{u}E_{x}$: actuarial discount factor over period $(0, u)$

$${}_{u|}A^{1}_{x:\overline{n}|} = EPV\Big[$$



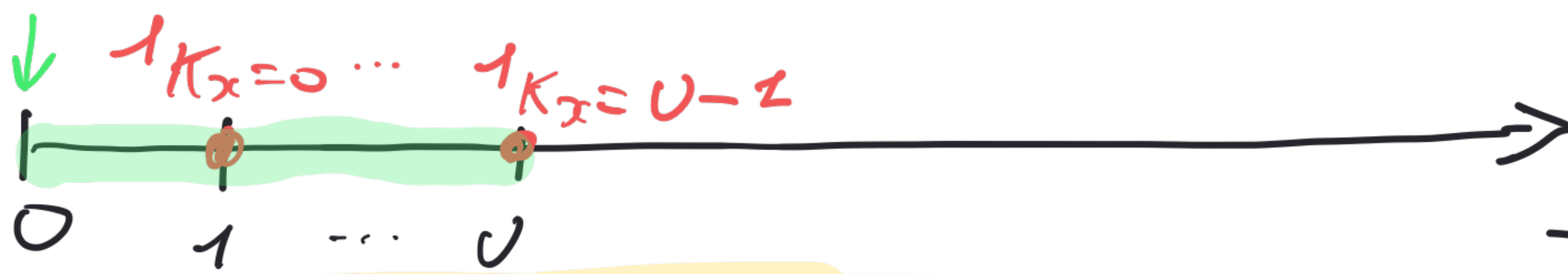
$$\Big]$$

$$= EPV\Big[$$



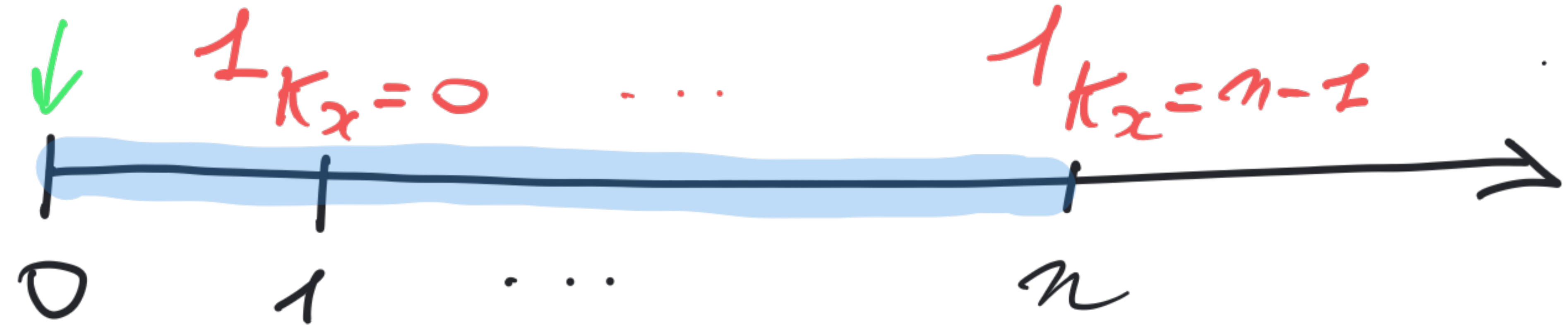
$$\Big]$$

$$- EPV\Big[$$



$$\Big]$$

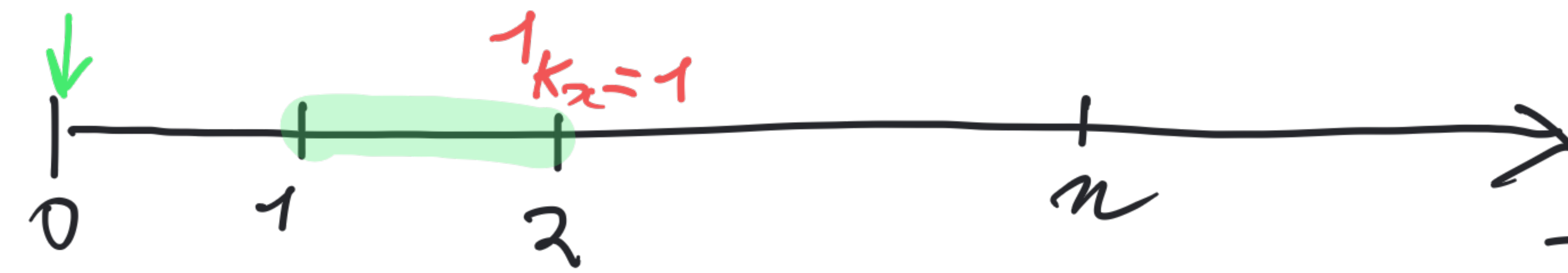
$$= A^{1}_{x:\overline{u+n}|} - A^{1}_{x:\overline{u}|} \qquad (4.23')$$

$A^{1}_{x:\overline{n|}} = EPV\Big[$ 

$\Big]$

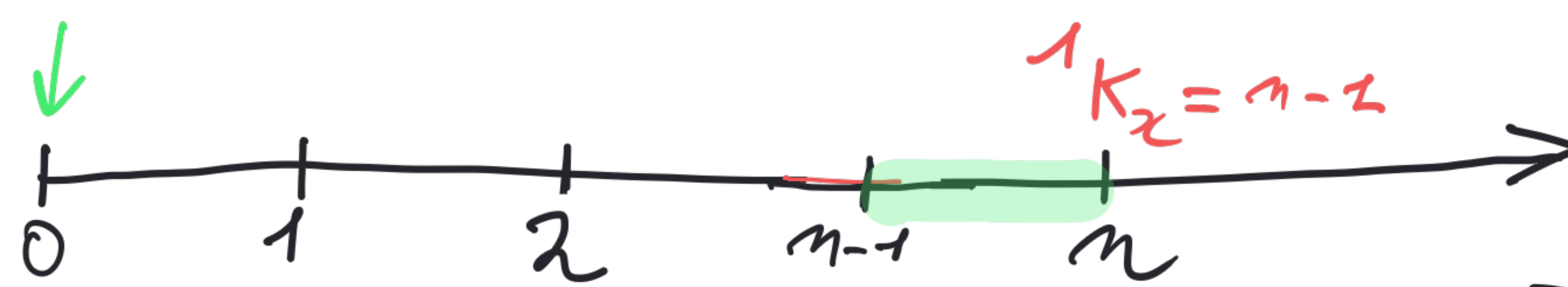
$= EPV\Big[$ 

$\Big]$

$+ EPV\Big[$ 

$\Big]$

$+ \dots$

$+ EPV\Big[$ 

$\Big]$

$$= {}_{0|}A^{1}_{x:\overline{1|}} + {}_{1|}A^{1}_{x:\overline{1|}} + \dots + {}_{n-1|}A^{1}_{x:\overline{1|}} \qquad (4.24)$$

4.4 Valuation of insurance benefits

4.4.8 Deferred insurance

Some more expressions

- Deferred term insurance in terms of yearly insurances:

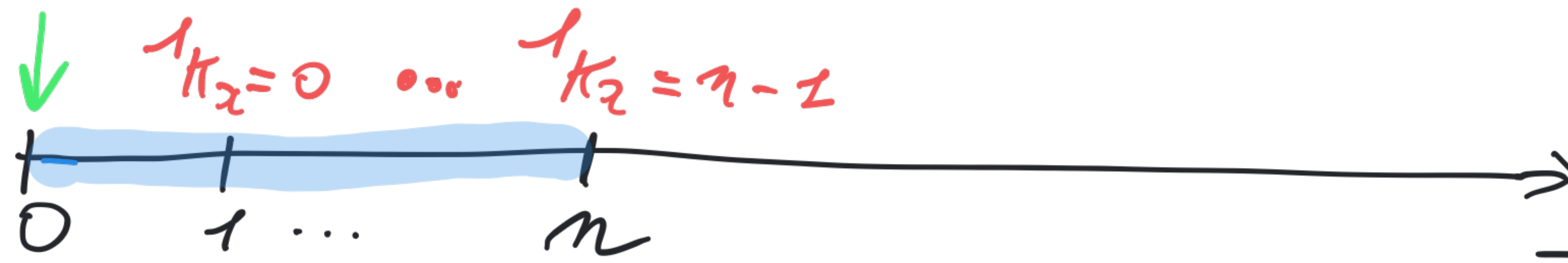
$$ {}_{u|}\bar{A}^{1}_{x:\overline{n}|} = \sum_{r=u}^{u+n-1} {}_{r|}\bar{A}^{1}_{x:\overline{1}|} $$

- Whole life insurance in terms of yearly insurances:

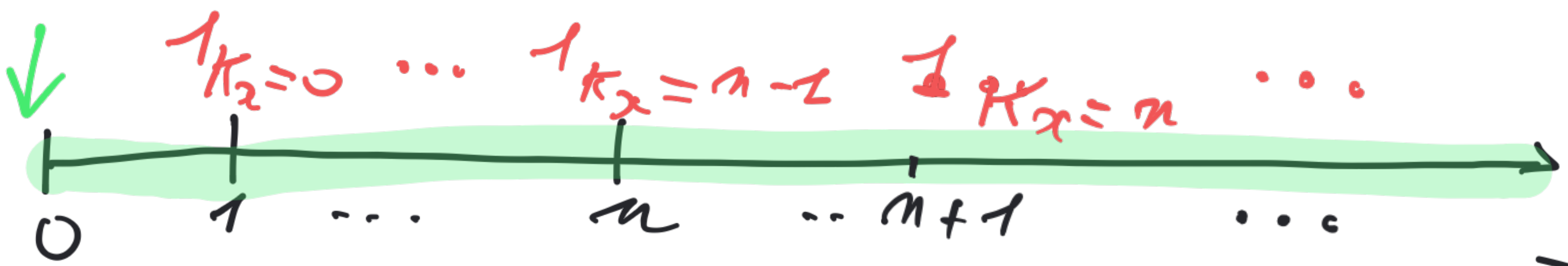
$$ \bar{A}_x = \sum_{r=0}^{\infty} {}_{r|}\bar{A}^{1}_{x:\overline{1}|} $$

- Term insurance in terms of whole life insurances:

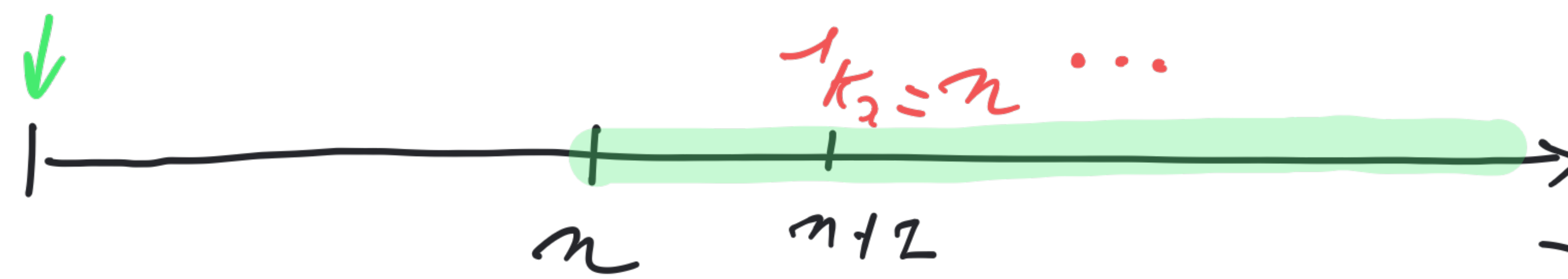
$$ \begin{aligned} A^{1}_{x:\overline{n}|} &= A_x - {}_{n|}A_x \\ &= A_x - {}_{n}E_x \times A_{x+n} \end{aligned} \qquad (4.25) $$

$$A^{1}_{x:\overline{n}|} = EPV\Big[$$


$$\Big]$$

$$= EPV\Big[$$


$$\Big]$$

$$- EPV\Big[$$


$$\Big]$$

$$= A_x - {}_{n|}A_x = A_x - {}_{n}E_x \times A_{x+n} \qquad (4\text{-}25)$$

4.5 Relating the continuous, discrete and 1/m-thly cases

- Suppose that there is only an integer life table available.
- How to determine $A_x^{(4)}$ and $\overline{A}_x$?
- The ratios $A_x^{(4)}/A_x$ and $\overline{A}_x/A_x$ are quite stable:

Table 4.6. *Ratios of $A_x^{(4)}$ to A_x and $\overline{A}_x$ to A_x, Standard Ultimate Survival Model.*

x	$A_x^{(4)}/A_x$	$\overline{A}_x/A_x$
20	1.0184	1.0246
40	1.0184	1.0246
60	1.0184	1.0246
80	1.0186	1.0248
100	1.0198	1.0261
120	1.0296	1.0368

- Hereafter, we will derive approximate values for these ratios.

4.5 Relating the continuous, discrete and 1/m-thly cases

4.5.1 Using the UDD assumption

- The UDD assumption: (y is an integer and $0 \leq s < 1$)
  - assumption:

$$_s q_y = s\ q_y \tag{3.7}$$

  - consequence:

$$q_y = {_s p_y}\ \mu_{y+s} \tag{3.11}$$

- Continuous case:

$$\boxed{\overline{A}_x \overset{\text{UDD}}{=} \frac{i}{\delta} A_x} \tag{4.26}$$

- $1/m$-thly case:

$$\boxed{A_x^{(m)} \overset{\text{UDD}}{=} \frac{i}{i^{(m)}} A_x} \tag{4.27}$$

- Endowment insurance:

$$\boxed{\overline{A}_{x:\overline{n|}} \overset{\text{UDD}}{=} \frac{i}{\delta} A^1_{x:\overline{n|}} + {_n E_x}} \tag{4.28}$$

4.5 Relating the continuous, discrete and 1/m-thly cases

4.5.2 Using the claims acceleration approach (read in book)

4.6 Variable insurance benefits

- Consider a life insurance underwritten on (x) at time 0, with a payment of T_x at time T_x.
- Benefit cash flow:

$$(T_x, T_x)$$

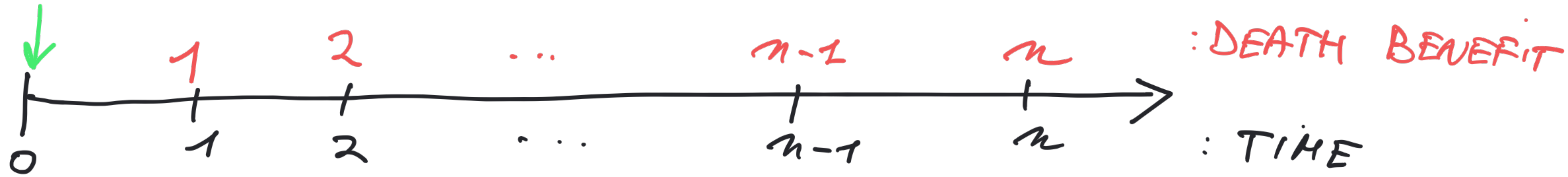
- Present value:

$$Z = T_x \, e^{-\delta T_x}$$

- Actuarial value:

$$\boxed{\left(\bar{I}\,\bar{A}\right)_x \overset{\text{not.}}{=} \mathbb{E}\left[Z\right] = \int_0^\infty t \, e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt} \qquad (4.32)$$

4.6 Variable insurance benefits



Example 4.6

- Consider a term insurance issued to (x) with benefit cash flow

$$\left((K_x + 1) \times 1_{\{K_x + 1 \leq n\}},\ K_x + 1 \right)$$

- Show that the EPV of this cash flow, notation $(IA)^{1}_{x:\overline{n}|}$, is given by

$$\sum_{k=0}^{n-1} v^{k+1} \, (k+1) \, {}_{k|}q_x$$

- Interprete the symbol $(IA)_x$.

4.6 Variable insurance benefits

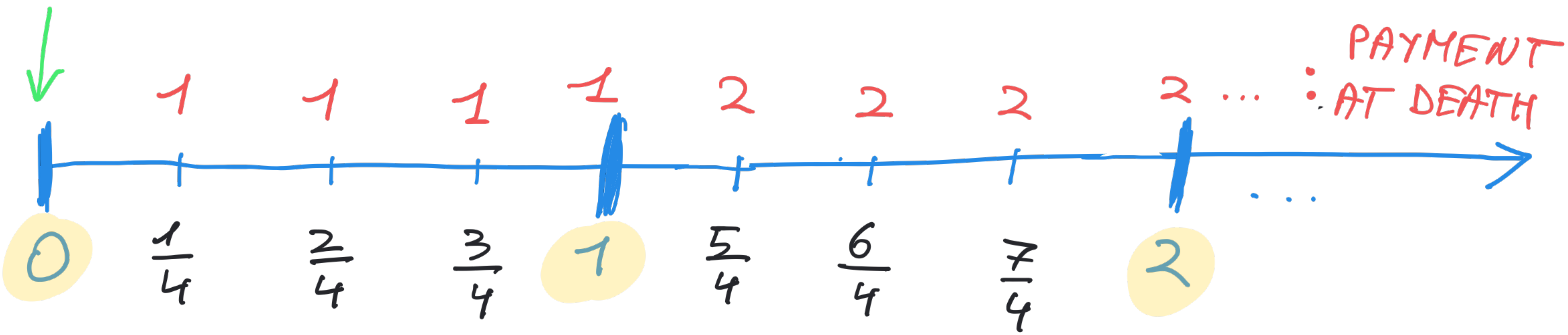
Example 4.7

- Consider an insurance issued to (x) with benefit cash flow

$$\left(K_x + 1,\ K_x^{(4)} + \frac{1}{4}\right)$$

- Show that the EPV of this cash flow, notation $\left(IA^{(4)}\right)_x$, is given by

$$\sum_{k=0}^{\infty} (k+1)\ {}_{k|}A^{(4)\,1}_{x:\overline{1|}}$$


1
1
1
1
2
2
2
2
PAYMENT AT DEATH
0
1/4
2/4
3/4
1
5/4
6/4
7/4
2

4.6 Variable insurance benefits

Example 4.8

- Consider a term insurance issued to (x) with benefit cash flow

$$\left((1+j)^{K_x} \times 1_{\{K_x+1 \leq n\}},\ K_x+1\right)$$

- Show that the EPV of this cash flow is given by

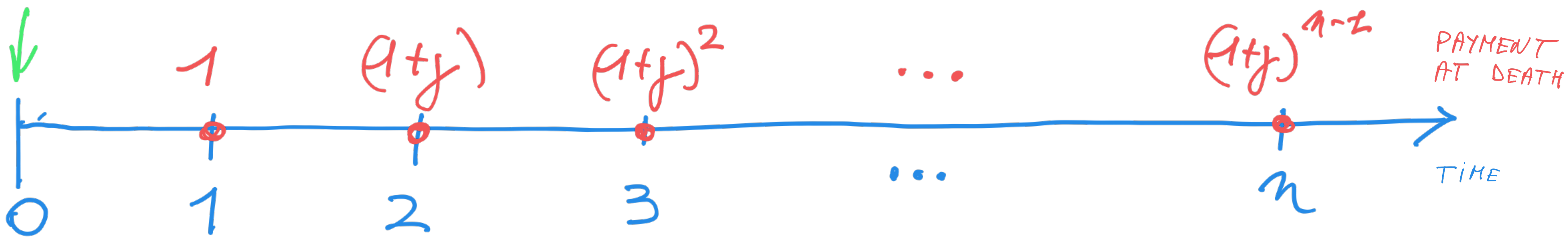
$$\frac{1}{1+j} A^{1}_{x:\overline{n}|\, i^*}$$

  - where

$$i^* = \frac{1+i}{1+j} - 1$$

  - and

$$A^{1}_{x:\overline{n}|\, i^*} = \sum_{k=0}^{n-1} \left(\frac{1}{1+i^*}\right)^{k+1} {}_{k|}q_x$$


1
(1+j)
(1+j)^2
...
(1+j)^n-1
PAYMENT AT DEATH
0
1
2
3
...
n
TIME

4.6 Variable insurance benefits

Example 4.9

- Consider a term insurance issued to (x) with benefit cash flow

$$\left((1+j)^{T_x} \times 1_{\{T_x \le n\}},\ \ T_x \right)$$

- Show that the EPV of this cash flow is given by $\bar{A}^{\,1}_{x:\overline{n|}\, i^*}$
  - where
$$i^* = \frac{1+i}{1+j} - 1$$
  - and
$$\bar{A}^{\,1}_{x:\overline{n|}\, i^*} = \int_0^n \left(\frac{1}{1+i^*} \right)^t {}_t p_x \mu_{x+t} dt$$

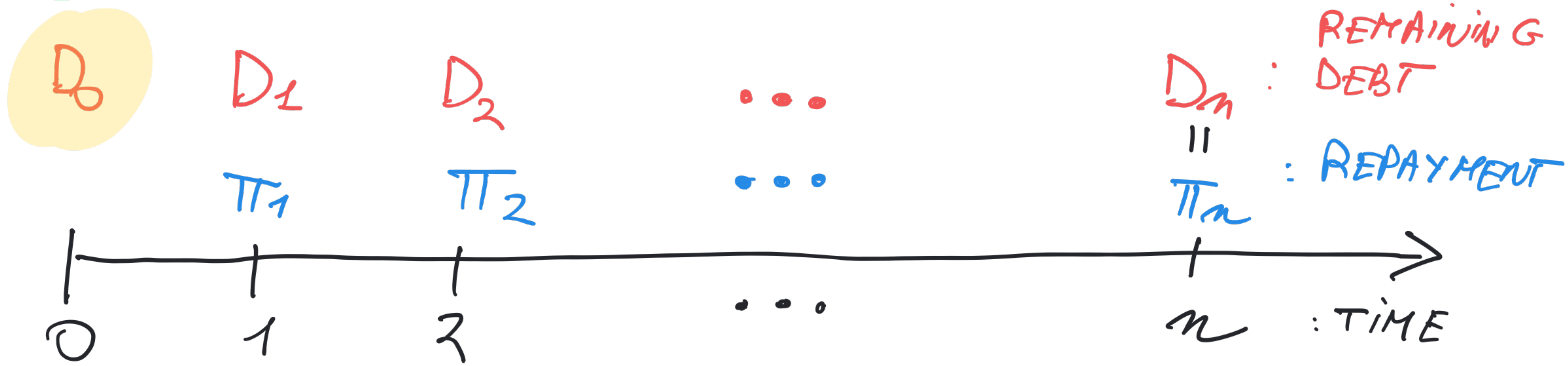
4.6 Variable insurance benefits

Term insurance linked to a mortgage loan

- Suppose (x) borrows the amount D_0 at time 0 at interest j.
- This debt is repaid by repayments π_k at times $1, 2, \ldots, n$.
  - Each π_k is the sum of *an interest payment* and *a reduction of the principal*.
- Let D_k be the remaining debt at time $k = 1, 2, \ldots, n$ (before payment of π_k).
  - Recursion:
    $$D_k = (D_{k-1} - \pi_{k-1})(1 + j)$$
  - Initial and terminal repayment conditions:
    $$\pi_0 = 0 \qquad \text{and} \qquad \pi_n = D_n$$
- Term insurance covering the remaining debt:

$$\text{Single premium} = \sum_{k=0}^{n-1} D_{k+1} \; {}_{k|}A^{1}_{x:\overline{1}|}$$

- **LOAN:**



- **TERM INSURANCE:**

4.7 Functions for select lives

- In this chapter, we developed results in terms of lives subject to ultimate mortality.
- All of the above development equally applies to lives subject to select mortality.
- Example:

$$\bar{A}_{[x]} = \int_0^\infty e^{-\delta t}\, {}_t p_{[x]}\, \mu_{[x]+t}\, dt$$

- Example:

$$A^{\;\;1}_{[x]+s:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_{[x]+s}$$

4.8 Notes and further reading

(read in book)