

# Life Insurance Mathematics

## Life annuities<sup>1</sup>

Jan Dhaene

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<sup>1</sup>Based on Chapter 5 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).

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# 5.1 Summary

- Life contingent annuities:
  - Whole life annuities vs. term annuities.
  - Annuities-due vs. immediate annuities.
  - Annuities payable yearly,  $1/m$ -thly or continuously.
- Actuarial valuation of life contingent annuities:
  - EPV of life annuity benefit cash flows.
  - Actuarial notation.
  - Continuous valuation via  $T_x$ .
  - Discrete valuation via  $K_x$  or  $K_x^{(m)}$ .

## 5.2 Introduction

- **Life annuity**: series of payments as long as a given person is alive on the payment dates.
- Payments:
  - at regular intervals,
  - (usually) of the same amount.
- Used for calculating:
  - pension benefits,
  - premiums,
  - policy values.



What is the relation between this grafitti painting (Brussels, January 2017) and life annuities?

## 5.2 Introduction

*'By providing financial protection against the major 18th and 19th century risk of dying too soon, life assurance became the biggest financial industry..., providing financial protection against the new risk of not dying soon enough may well become the next century's major and most profitable financial industry.'*  
(Peter Drucker, The Economist, 1999)



## 5.2 Introduction

Why buying a life annuity?

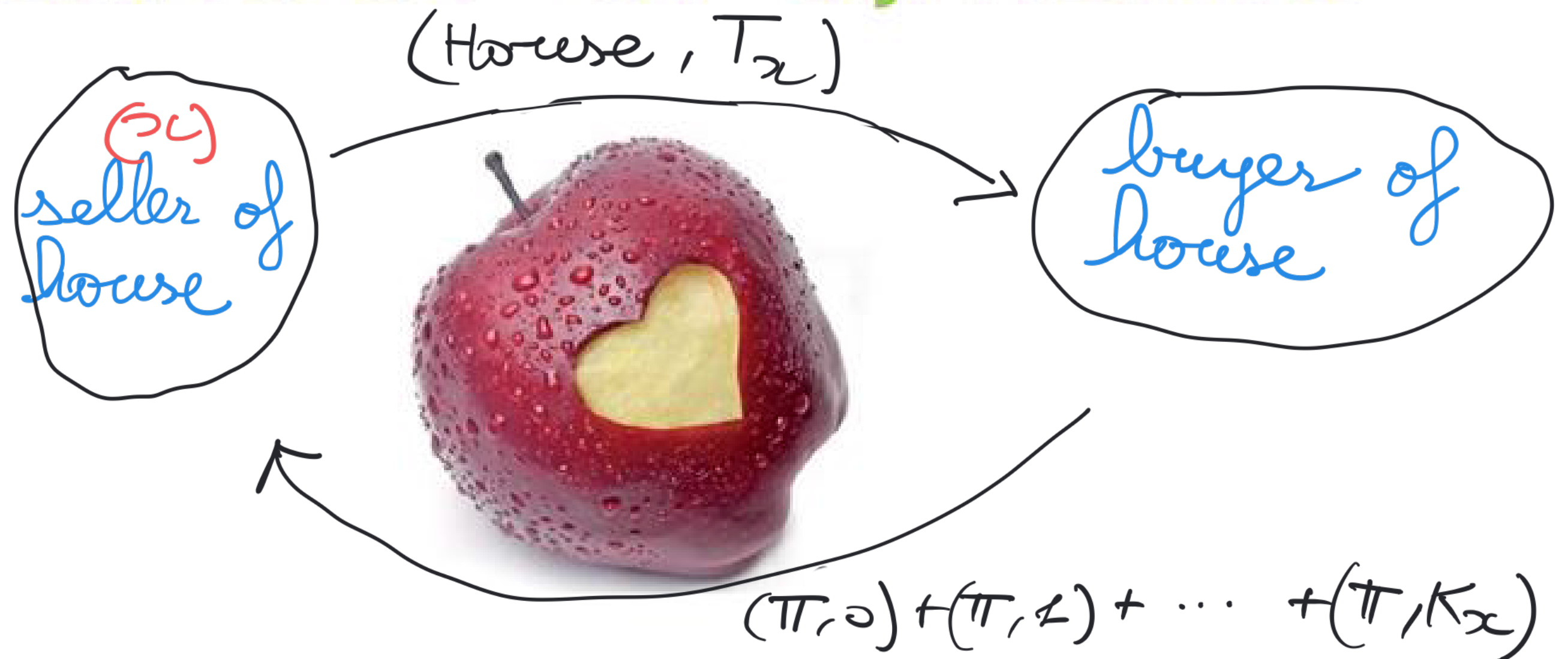




## 5.2 Introduction

Life annuity sale

# VERKOOOP OP LIJFRENTE



EEN APPELTJE VOOR DE DORST ?

## 5.2 Introduction

To buy or not to buy?

*LIFE INSURER*  
↓

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↑



# 5.2 Introduction

To buy or not to buy?

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|---------------------|----------|-----------------|---|---------|
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| .INX                | 1,639.00 | -17.78 (-1.07%) | - | -       |
| NDX                 | 3,078.99 | -43.68 (-1.40%) | - | -       |





## 5.3 Review of annuities-certain

### Cash flow notations

- The series of cash flows  $(c_k, k)$ ,  $k = m, \dots, n$ , is denoted by:

$$\boxed{\sum_{k=m}^n (c_k, k)}$$

- The continuous stream of payments  $c_\tau d\tau$  in any infinitesimal subinterval  $(\tau, \tau + d\tau)$  of  $(s, t)$ , is denoted by:

$$\boxed{\int_s^t (c_\tau d\tau, \tau)}$$

- Convention:

$$\sum_{k=m}^n (c_k, k) = (0, 0) \text{ if } m > n \text{ and } \int_s^t (c_\tau d\tau, \tau) = (0, 0) \text{ if } s > t$$

- In previous notations,  $m, n, c_k, c_\tau, s$  and  $t$  may be deterministic or random.
- Similar notations and conventions for series of cash flows with  $1/m$ -thly payments.

## 5.3 Review of annuities-certain

- Annuity-due:

$$PV \left[ \sum_{k=0}^{n-1} (1, k) \right] = \ddot{a}_{\overline{n}|} = 1 + v + \dots + v^{n-1} = \frac{1 - v^n}{d} \quad (5.1) \rightarrow$$

- Annuity-immediate:

$$PV \left[ \sum_{k=1}^n (1, k) \right] = a_{\overline{n}|} = v + \dots + v^n = \frac{1 - v^n}{i}$$

- Continuous annuity:

$$PV \left[ \int_0^n (dt, t) \right] = \bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - v^n}{\delta} \quad (5.2) \rightarrow$$

- Annuity-due with  $1/m$ -thly payments:

$$PV \left[ \sum_{k=0}^{m \cdot n - 1} \left( \frac{1}{m}, \frac{k}{m} \right) \right] = \ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m} \left( 1 + v^{\frac{1}{m}} + \dots + v^{n - \frac{1}{m}} \right) = \frac{1 - v^n}{d^{(m)}}$$

- Annuity-immediate with  $1/m$ -thly payments:

$$PV \left[ \sum_{k=1}^{m \cdot n} \left( \frac{1}{m}, \frac{k}{m} \right) \right] = a_{\overline{n}|}^{(m)} = \frac{1}{m} \left( v^{\frac{1}{m}} + \dots + v^{n - \frac{1}{m}} + v^n \right) = \frac{1 - v^n}{i^{(m)}}$$

$$\bullet \quad \ddot{a}_n = 1 + v + \dots + v^{n-1}$$

$$v \times \ddot{a}_n = v + v^2 + \dots + v^n$$

$$(1-v) \ddot{a}_n = 1 - v^n$$

$$\ddot{a}_n = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

(5.1)

$$\bullet \quad \bar{a}_n = \int_0^n v^t dt = \int_0^n e^{-\delta t} dt$$

$$= \frac{1}{\delta} \int_0^n d e^{-\delta t} = -\frac{1}{\delta} e^{-\delta t} \Big|_0^n$$

$$= -\frac{1}{\delta} (e^{-\delta n} - 1) = \frac{1-v^n}{\delta} \quad (5.2)$$

## 5.4 Annual life annuities

### 5.4.1 Whole life annuity-due (1)



- Consider an annuity underwritten to  $(x)$  at time 0. It pays 1 annually in advance as long as  $(x)$  is alive.
- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- A whole life annuity only 'dies' when the insured dies.



# 5.4 Annual life annuities

## 5.4.1 Whole life annuity-due (1)

- Consider an annuity underwritten to  $(x)$  at time 0. It pays 1 annually in advance as long as  $(x)$  is alive.
- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

(5.1) :  $1 + v + \dots + v^n = \ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}$

$$Y = 1 + v + \dots + v^{K_x} = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d}$$

- Actuarial value:

$$\ddot{a}_x \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}[v^{K_x+1}]}{d} = \frac{1 - A_x}{d} \quad (5.3)$$

## 5.4 Annual life annuities

### 5.4.1 Whole life annuity-due (2)

- Benefit cash flow:

$$\sum_{t=0}^{K_x} (1, t) = \sum_{t=0}^{\infty} (1_{\{T_x > t\}}, t)$$

$$1_{\{T_x > t\}} = \begin{cases} 0 & : T_x \leq t \\ 1 & : T_x > t \end{cases}$$

- Present value:

$$Y = \sum_{t=0}^{\infty} v^t 1_{\{T_x > t\}}$$

$$\begin{aligned} & \mathbb{E}[1_{\{T_x > t\}}] \\ &= {}_t p_x \end{aligned}$$

- Actuarial value:

$$\ddot{a}_x = \mathbb{E}[Y] = \sum_{t=0}^{\infty} v^t {}_t p_x \quad (5.5)$$

## 5.4 Annual life annuities

### 5.4.1 Whole life annuity-due (2)

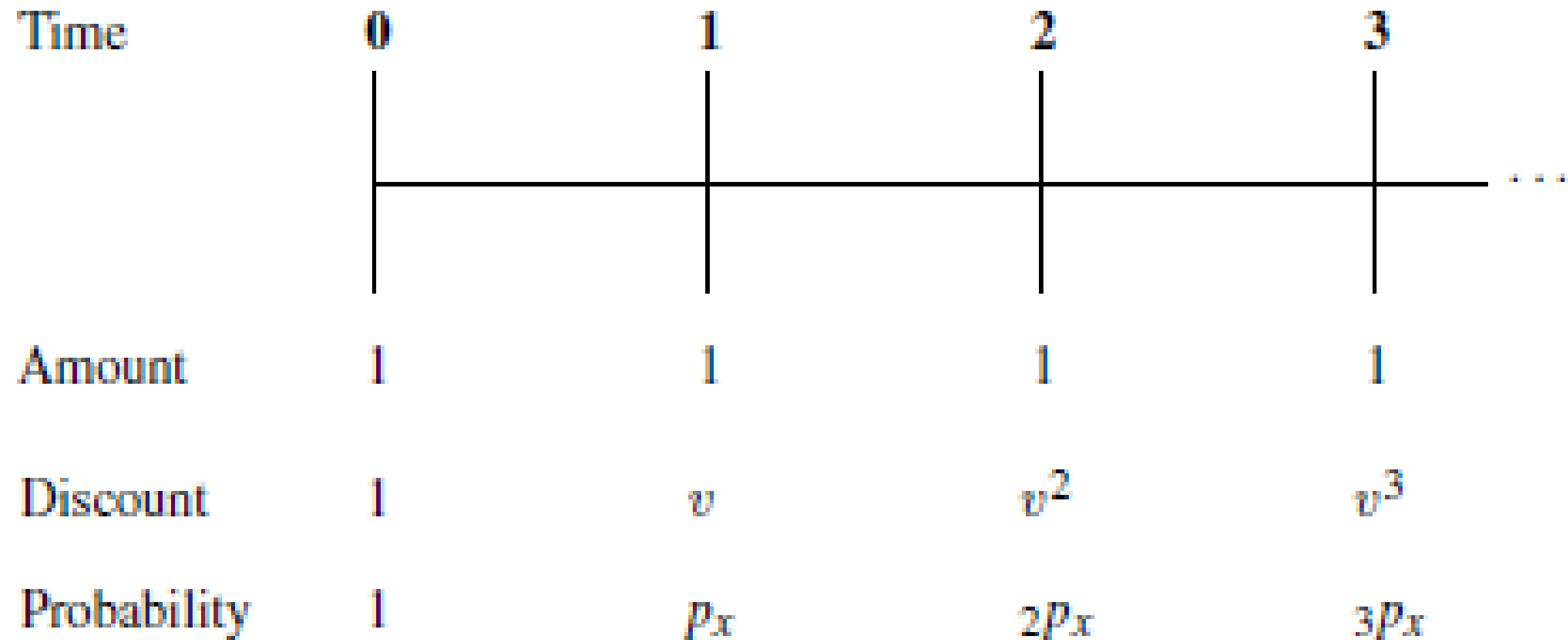


Figure 5.1 Time-line diagram for whole life annuity-due.



## 5.4 Annual life annuities

### 5.4.1 Whole life annuity-due (3)

- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = \ddot{a}_{\overline{K_x+1}|}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}[Y] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \times {}_k|q_x} \quad (5.6)$$



## 5.4 Annual life annuities

### Example 5.1

- Show algebraically that

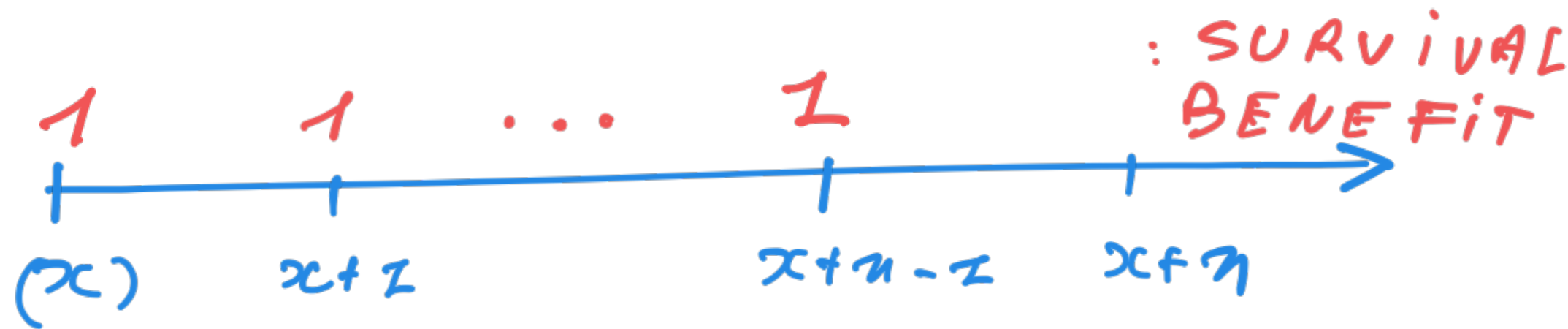
$$\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \times {}_k|q_x = \sum_{k=0}^{\infty} v^k {}_k p_x$$

- Proof:

$$\begin{aligned} \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \times {}_k|q_x &= \sum_{k=0}^{\infty} \left( \sum_{t=0}^k v^t \right) \times {}_k|q_x \\ &= \sum_{t=0}^{\infty} v^t \left( \sum_{k=t}^{\infty} {}_k|q_x \right) \\ &= \sum_{t=0}^{\infty} v^t {}_t p_x \end{aligned}$$

## 5.4 Annual life annuities

### 5.4.2 Term annuity-due (1)



- Consider an annuity underwritten to  $(x)$  at time 0. It pays 1 at times  $0, 1, \dots, n-1$ , provided  $(x)$  is alive.
- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

$$Y = 1 + v + \dots + v^{\min(K_x, n-1)} \stackrel{(5.1)}{=} \ddot{a}_{\min(K_x+1, n)} = \frac{1 - v^{\min(K_x+1, n)}}{d}$$

- Actuarial value:

$$\ddot{a}_{x:\overline{n}|} \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}[v^{\min(K_x+1, n)}]}{d} = \frac{1 - A_{x:\overline{n}|}}{d} \quad (5.7)$$

enrolment

$$1 = d \cdot \ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$$

## 5.4 Annual life annuities

### 5.4.2 Term annuity-due (2)

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t) = \sum_{t=0}^{n-1} (1_{\{T_x > t\}}, t)$$

- Present value:

$$Y = \sum_{t=0}^{n-1} v^t 1_{\{T_x > t\}}$$

- Actuarial value:

$$\ddot{a}_{x:\overline{n}|} \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \sum_{t=0}^{n-1} v^t {}_t p_x \quad (5.8)$$

## 5.4 Annual life annuities

### 5.4.2 Term annuity-due (2)

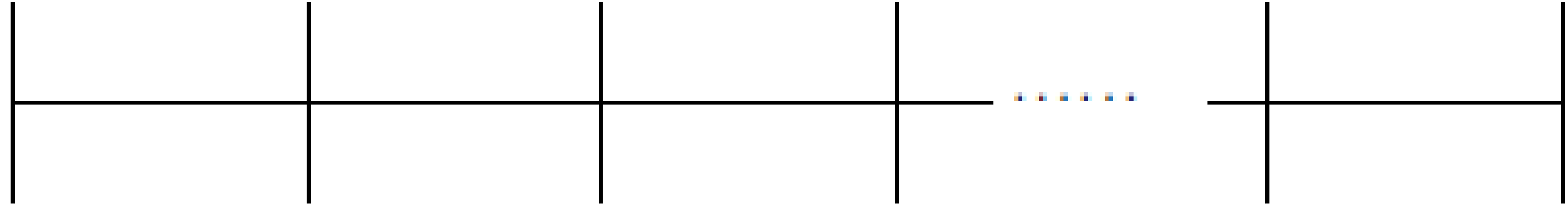
|             |   |       |        |        |       |           |   |
|-------------|---|-------|--------|--------|-------|-----------|---|
| Time        | 0   | 1     | 2      | 3      | ..... | n-1       | n |
|             |  |       |        |        |       |           |   |
| Amount      | 1   | 1     | 1      | 1      |       | 1         |   |
| Discount    | 1   | $v$   | $v^2$  | $v^3$  |       | $v^{n-1}$ |   |
| Probability | 1   | $p_x$ | $2p_x$ | $3p_x$ |       | $n-1p_x$  |   |

Figure 5.2 Time-line diagram for term life annuity-due.

## 5.4 Annual life annuities

### 5.4.2 Term annuity-due (3)

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

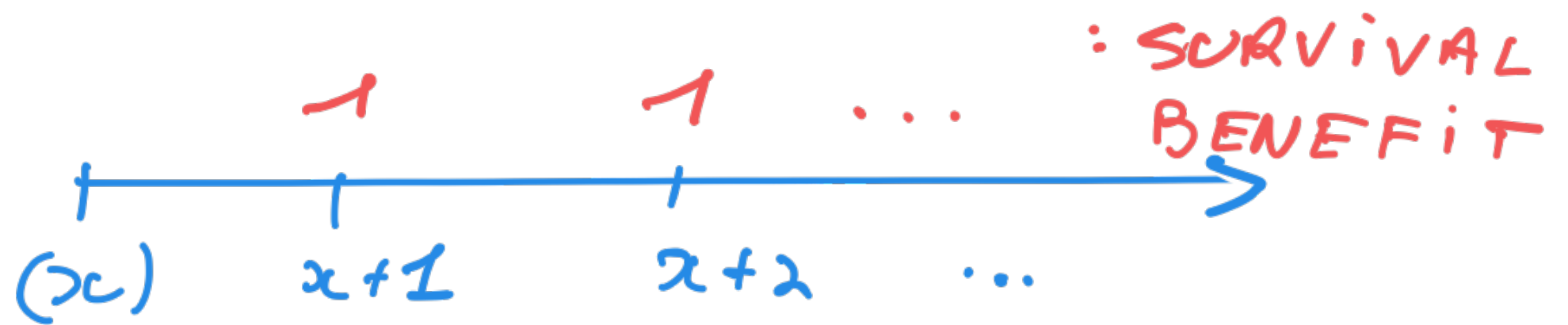
$$Y = \ddot{a}_{\overline{\min(K_x+1, n)}|}$$

- Actuarial value:

$$\ddot{a}_{x:\overline{n}|} = \mathbb{E}[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \times {}_k|q_x + {}_np_x \times \ddot{a}_{\overline{n}|}$$

## 5.4 Annual life annuities

### 5.4.3 Immediate life annuities



- Consider a **whole life immediate annuity** underwritten to  $(x)$  at time 0. It pays 1 annually in arrear, as long as  $(x)$  is alive.
- Benefit cash flow:

$$\sum_{t=1}^{K_x} (1, t) = \sum_{t=1}^{\infty} (1_{\{T_x > t\}}, t)$$

- Present value:

$$Y^* = \sum_{t=1}^{\infty} v^t 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{a_x \stackrel{\text{not.}}{=} \mathbb{E}[Y^*] = \ddot{a}_x - 1} \quad (5.9)$$

## 5.4 Annual life annuities

### 5.4.3 Immediate life annuities

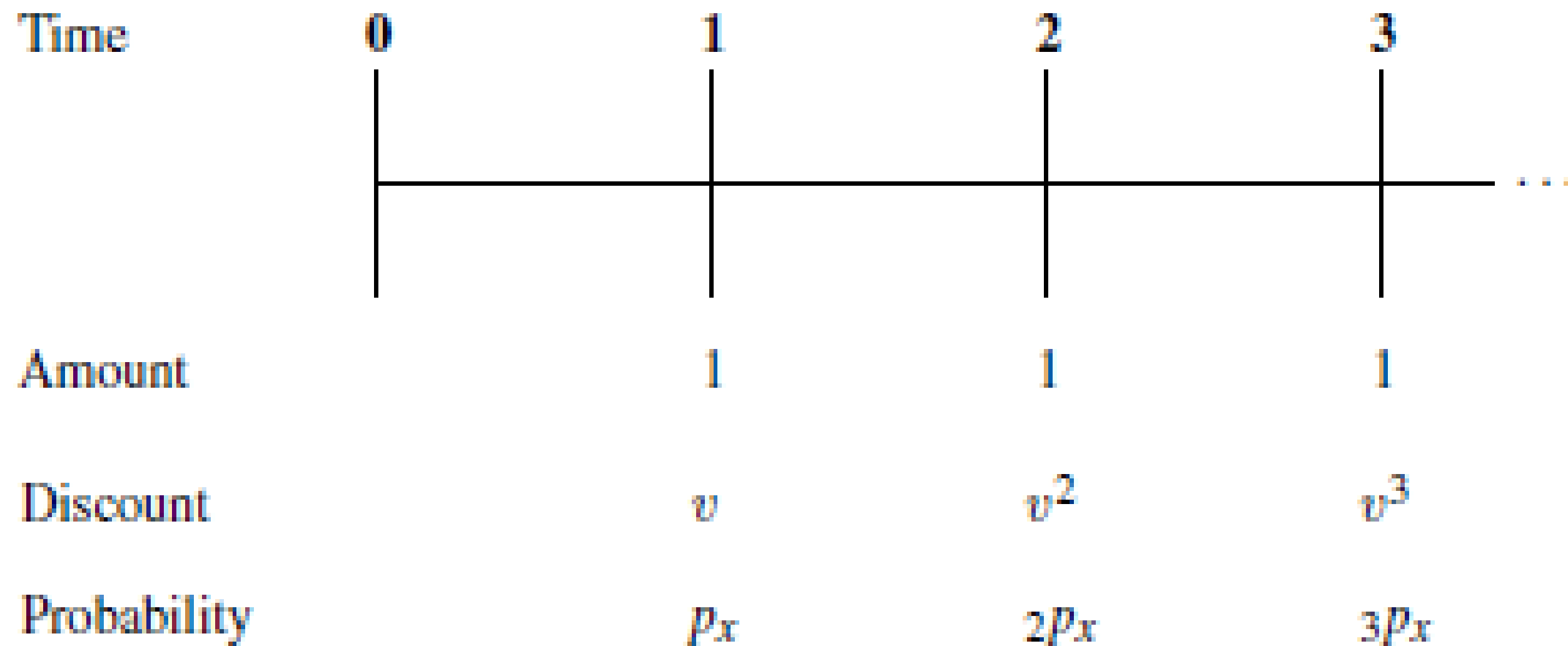


Figure 5.3 Time-line diagram for whole life immediate annuity.

## 5.4 Annual life annuities

### 5.4.3 Immediate life annuities



- Consider a **n-year term immediate annuity** underwritten to  $(x)$  at time 0. It pays 1 at times  $1, 2, \dots, n$ , provided  $(x)$  is alive.
- Benefit cash flow:

$$\sum_{t=1}^{\min(K_x, n)} (1, t) = \sum_{t=1}^n (1_{\{T_x > t\}}, t)$$

- Present value:

$$Y^* = \sum_{t=1}^n v^t 1_{\{T_x > t\}}$$

- Actuarial value:

$$a_{x:\overline{n}|} \stackrel{\text{not.}}{=} \mathbb{E}[Y^*] = \sum_{t=1}^n v^t {}_t p_x \quad (5.11)$$

- Relation:

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n {}_n p_x \quad (5.12)$$



## 5.4 Annual life annuities

### 5.4.4 Immediate life annuities

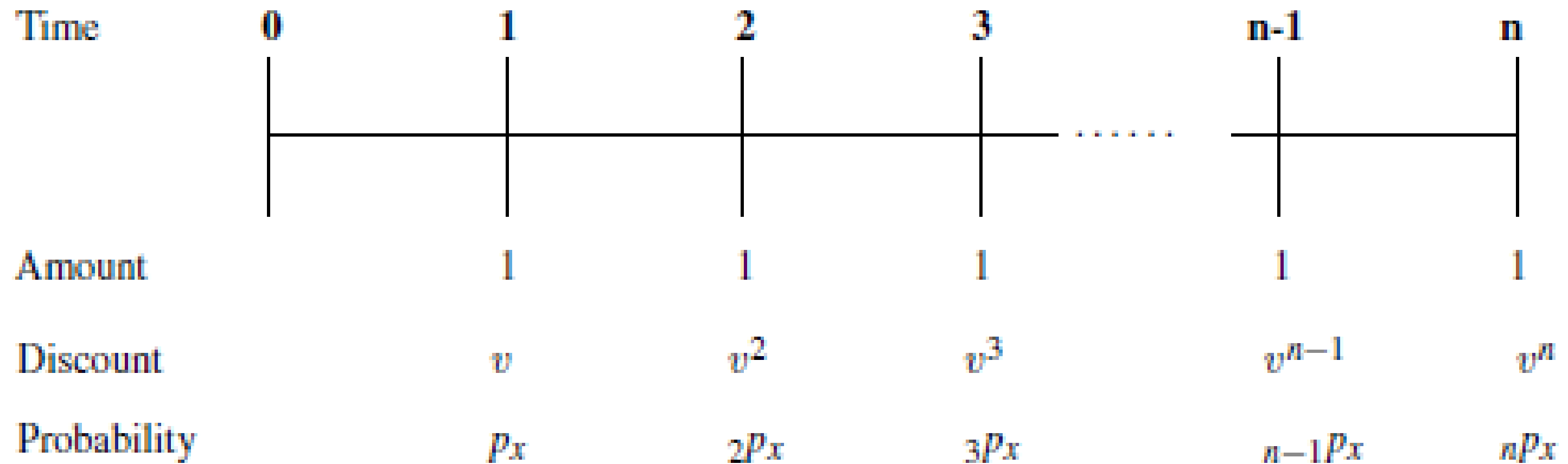


Figure 5.4 Time-line diagram for term life immediate annuity.

## 5.4 Annual life annuities

Some history

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x$$

Johan de Witt (1671)



Edmond Halley (1693).





## 5.4 Annual life annuities

### Some history

- In second half of 17th century, states and cities often raised money for public purposes by the sale of lifelong annuities to their residents. This led to huge *unfunded liabilities*.
- **Johan de Witt** (1625 - 1672):
  - First to use a (hypothetical) life table and interest rates to determine the value of a life annuity:
    - *Waardye van Lijfrenten naar Proportie van Los Renten* (1671).
  - At that time, he was prime minister of the State of Holland.
  - In 1672, he and his brother were cruelly lynched by Orangists.





## 5.4 Annual life annuities

### Some history

- **Edmond Halley** (1656 - 1742):
  - Was in 1690 asked by the Royal Society In London to estimate the value of the liabilities related to lifelong annuities.
  - In 1693, he published an article in 'Phylosophical Transactions of the Royal Society', in which
    - he displayed one of the first reliable lifetables based on demographic data,
    - and used these mortality rates and interest rates to determine the value of a life annuity.
  - Assisted and motivated Isaac Newton to publish his famous book 'Principles'.



## 5.5 Annuities payable continuously

### Whole life continuous annuity (1)

- Consider an annuity underwritten to  $(x)$  at time 0. It pays continuously at a rate of 1 per year as long as  $(x)$  is alive.
- Benefit cash flow:

$$\int_0^{T_x} (dt, t)$$

- Present value:

$$Y = \int_0^{T_x} v^t dt = \bar{a}_{\overline{T_x}|} = \frac{1 - v^{T_x}}{\delta} \quad (5.13)$$

(5.2) :  $\int_0^n v^t dt = \bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$

- Actuarial value:

$$\bar{a}_x \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}[v^{T_x}]}{\delta} = \frac{1 - \bar{A}_x}{\delta} \quad (5.14)$$

$$1 = \delta \cdot \bar{a}_x + \bar{A}_x$$

## 5.5 Annuities payable continuously

Whole life continuous annuity (2)

- Benefit cash flow:

$$\int_0^{T_x} (dt, t) = \int_0^{\infty} (1_{\{T_x > t\}} dt, t)$$

- Present value:

$$Y = \int_0^{\infty} e^{-\delta t} 1_{\{T_x > t\}} dt$$

- Actuarial value:

$$\bar{a}_x = \mathbb{E}[Y] = \int_0^{\infty} e^{-\delta t} {}_t p_x dt \quad (5.15)$$

## 5.5 Annuities payable continuously

### Whole life continuous annuity (2)

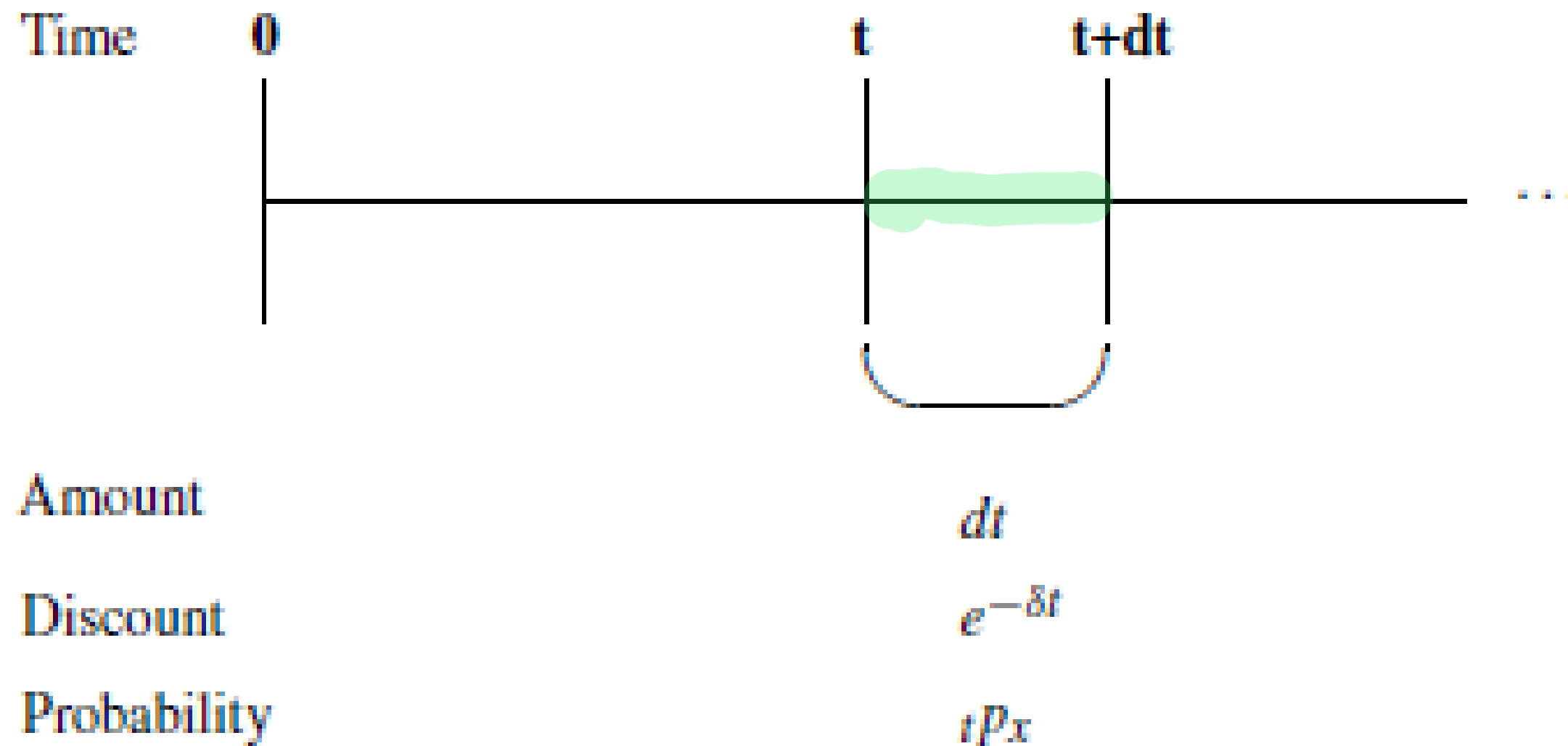


Figure 5.5 Time-line diagram for continuous whole life annuity.

## 5.5 Annuities payable continuously

Whole life continuous annuity (3)

- Benefit cash flow:

$$\int_0^{T_x} (dt, t)$$

- Present value:

$$Y = \int_0^{T_x} e^{-\delta t} dt = \bar{a}_{\overline{T_x}|}$$

- Actuarial value:

$$\bar{a}_x = \mathbb{E}[Y] = \int_0^{\infty} \bar{a}_{\overline{t}|} {}_t p_x \mu_{x+t} dt$$



## 5.5 Annuities payable continuously

### Term continuous annuity (1)

- Consider an annuity underwritten to  $(x)$  at time 0. It pays continuously at a rate of 1 per year, for a period of  $n$  years and provided  $(x)$  is alive.
- Benefit cash flow:

$$\int_0^{\min(T_x, n)} (dt, t)$$

- Present value:

$$Y = \int_0^{\min(T_x, n)} e^{-\delta t} dt = \bar{a}_{\min(T_x, n)} = \frac{1 - v^{\min(T_x, n)}}{\delta}$$

- Actuarial value:

$$\bar{a}_{x:\overline{n}|} \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}[v^{\min(T_x, n)}]}{\delta} = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta} \quad (5.16)$$

## 5.5 Annuities payable continuously

Term continuous annuity (2)

- Benefit cash flow:

$$\int_0^{\min(T_x, n)} (dt, t) = \int_0^n (1_{\{T_x > t\}} dt, t)$$

- Present value:

$$Y = \int_0^n e^{-\delta t} 1_{\{T_x > t\}} dt$$

- Actuarial value:

$$\boxed{\bar{a}_{x:\overline{n}|} = \mathbb{E}[Y] = \int_0^n e^{-\delta t} {}_t p_x dt} \quad (5.17)$$

## 5.5 Annuities payable continuously

Term continuous annuity (3)

- Benefit cash flow:

$$\int_0^{\min(T_x, n)} (dt, t)$$

- Present value:

$$Y = \int_0^{\min(T_x, n)} e^{-\delta t} dt = \bar{a}_{\overline{\min(T_x, n)}|}$$

- Actuarial value:

$$\bar{a}_{x:\overline{n}|} = \mathbb{E}[Y] = \int_0^n \bar{a}_{\overline{t}|} {}_t p_x \mu_{x+t} dt + \bar{a}_{\overline{n}|} \times {}_n p_x$$

## 5.6 Annuities payable 1/m-thly

### 5.6.1 Introduction

- Recall: Future lifetime of  $(x)$  in years, rounded down to the lower 1/m-th of the year:

$$K_x^{(m)} = \frac{1}{m} \lfloor mT_x \rfloor$$

- Recall: Annuity-due with 1/m-thly payments:

$$PV \left[ \sum_{k=0}^j \left( \frac{1}{m}, \frac{k}{m} \right) \right] = \ddot{a}_{\overline{(j+1)/m}|}^{(m)} = \frac{1}{m} \sum_{k=0}^j v^{k/m} = \frac{1 - v^{\frac{j+1}{m}}}{d^{(m)}} \quad (*)$$

## 5.6 Annuities payable 1/m-thly

### 5.6.2 Whole life annuities payable 1/m-thly (1)

- Consider an annuity underwritten to  $(x)$  at time 0. It pays an amount of 1 per year, payable in advance  $m$  times per year, throughout the lifetime of  $(x)$ .

- Benefit cash flow:

$$\sum_{k=0}^{m K_x^{(m)}} \left( \frac{1}{m}, \frac{k}{m} \right)$$

$$b = m \cdot K_x^{(m)} \Leftrightarrow \frac{b}{m} = K_x^{(m)}$$

- Present value:

$$Y = \frac{1}{m} \sum_{k=0}^{m K_x^{(m)}} v^{k/m} \stackrel{(*)}{=} \ddot{a}_{\overline{K_x^{(m)} + 1/m}|}^{(m)} = \frac{1 - v^{K_x^{(m)} + \frac{1}{m}}}{d^{(m)}}$$

$$(*) \text{ with } f = m \cdot K_x^{(m)}$$

- Actuarial value:

$$\ddot{a}_x^{(m)} \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}\left[v^{K_x^{(m)} + \frac{1}{m}}\right]}{d^{(m)}} = \frac{1 - A_x^{(m)}}{d^{(m)}} \quad (5.18)$$

## 5.6 Annuities payable 1/m-thly

### 5.6.2 Whole life annuities payable 1/m-thly (2)

- Actuarial value:

$$\ddot{a}_x^{(m)} = \frac{1}{m} \sum_{t=0}^{\infty} v^{t/m} {}_{t/m}p_x \quad (5.19)$$

- Annuity-immediate vs. annuity-due:

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} \quad (5.20)$$



## 5.6 Annuities payable 1/m-thly

### 5.6.2 Whole life annuities payable 1/m-thly (2)

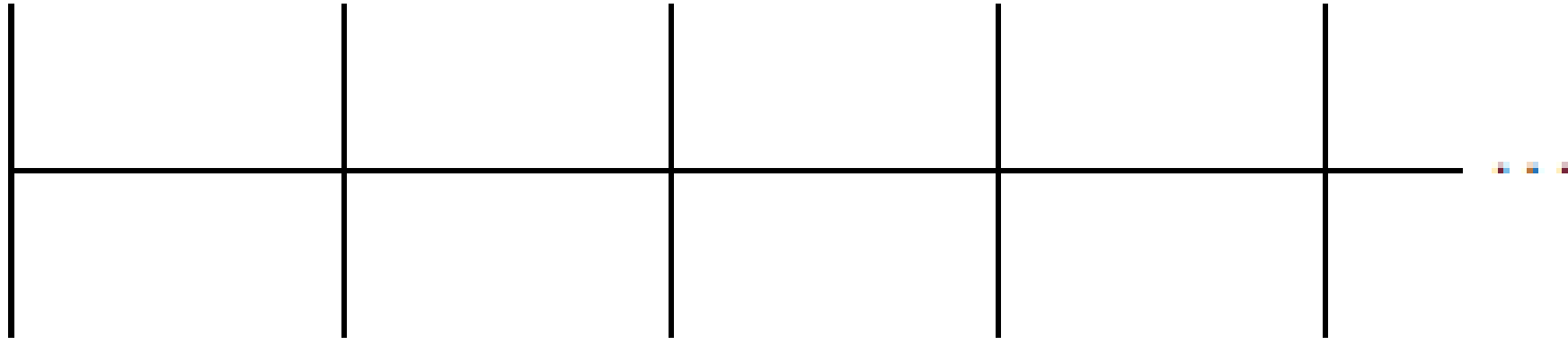
|             |  |                   |                   |                   |                   |     |
|-------------|--|-------------------|-------------------|-------------------|-------------------|-----|
| Time        | 0  | 1/m               | 2/m               | 3/m               | 4/m               | ... |
|             |  |                   |                   |                   |                   |     |
| Amount      | 1/m  | 1/m               | 1/m               | 1/m               | 1/m               |     |
| Discount    | 1  | $v^{1/m}$         | $v^{2/m}$         | $v^{3/m}$         | $v^{4/m}$         |     |
| Probability | 1  | $\frac{1}{m} P_x$ | $\frac{2}{m} P_x$ | $\frac{3}{m} P_x$ | $\frac{4}{m} P_x$ |     |

Figure 5.6 Time-line diagram for whole life 1/mthly annuity-due.

## 5.6 Annuities payable 1/m-thly

### 5.6.2 Whole life annuities payable 1/m-thly: example

$$\rightarrow E[\overline{T}_{90}] \approx 3$$

- In 1965, at age 90, Jeanne Calment sold her apartment by a life annuity sale.
- The buyer, André-François Raffray, was going to receive the apartment at the death of the seller:

$$\left( \text{apartment}, K_{90}^{(12)} + \frac{1}{12} \right)$$

- In return, Jeanne was going to receive a whole life annuity of 2500 fr. per month:

$$\sum_{k=0}^{12 K_{90}^{(12)}} \left( 2500, \frac{k}{12} \right)$$

- Jeann Calment died in 1997, aged 122.

## 5.6 Annuities payable 1/m-thly

### Term annuities payable 1/m-thly (1)

- Consider an annuity underwritten to  $(x)$  at time 0, paying 1 per year, payable in advance  $m$  times per year, throughout the lifetime of  $(x)$ , limited to a maximum of  $n$  years.

- Benefit cash flow:

$$\sum_{k=0}^{\min(m K_x^{(m)}, mn-1)} \left( \frac{1}{m}, \frac{k}{m} \right)$$

$\rightarrow D_2 = \min(m \cdot K_x^{(m)}, m \cdot n - 1)$   
 $\Leftrightarrow \frac{D_2}{m} = \min(K_x^{(m)}, n - \frac{1}{m})$

- Present value:

$$Y = \ddot{a}_{\min(K_x^{(m)} + 1/m, n)}^{(m)} = \frac{1 - v^{\min(K_x^{(m)} + 1/m, n)}}{d^{(m)}}$$

(\*) with  $j = \min(m \cdot K_x^{(m)}, m \cdot n - 1)$

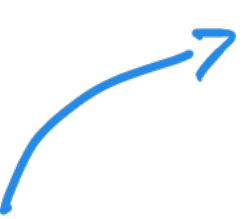
- Actuarial value:

$$\ddot{a}_{x:\overline{n}|}^{(m)} \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}\left[v^{\min(K_x^{(m)} + 1/m, n)}\right]}{d^{(m)}} = \frac{1 - A_{x:\overline{n}|}^{(m)}}{d^{(m)}} \quad (5.21)$$

## 5.6 Annuities payable 1/m-thly

Term annuities payable 1/m-thly (2)

- Actuarial value:


$$\ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1}{m} \sum_{r=0}^{mn-1} v^{r/m} {}_{\frac{r}{m}}p_x \quad (5.22)$$

- Annuity-immediate vs. annuity-due:

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m} (1 - v^n {}_np_x) \quad (5.23)$$

## 5.6 Annuities payable 1/m-thly

Term annuities payable 1/m-thly (2)

|             |     |                  |                  |                  |                  |       |                      |   |
|-------------|-----|------------------|------------------|------------------|------------------|-------|----------------------|---|
| Time        | 0   | 1/m              | 2/m              | 3/m              | 4/m              | ..... | n-1/m                | n |
| Amount      | 1/m | 1/m              | 1/m              | 1/m              | 1/m              |       | 1/m                  | 0 |
| Discount    | 1   | $v^{1/m}$        | $v^{2/m}$        | $v^{3/m}$        | $v^{4/m}$        |       | $v^{n-1/m}$          |   |
| Probability | 1   | $\frac{1}{m}p_x$ | $\frac{2}{m}p_x$ | $\frac{3}{m}p_x$ | $\frac{4}{m}p_x$ |       | $n - \frac{1}{m}p_x$ |   |

Figure 5.7 Time-line diagram for term life 1/mthly annuity-due.



## 5.7 Comparison of annuities by payment frequency

Table 5.1. Values of  $a_x$ ,  $a_x^{(4)}$ ,  $\bar{a}_x$ ,  $\ddot{a}_x^{(4)}$  and  $\ddot{a}_x$ .

| $x$ | $a_x$  | $a_x^{(4)}$ | $\bar{a}_x$ | $\ddot{a}_x^{(4)}$ | $\ddot{a}_x$ |
|-----|--------|-------------|-------------|--------------------|--------------|
| 20  | 18.966 | 19.338      | 19.462      | 19.588             | 19.966       |
| 40  | 17.458 | 17.829      | 17.954      | 18.079             | 18.458       |
| 60  | 13.904 | 14.275      | 14.400      | 14.525             | 14.904       |
| 80  | 7.548  | 7.917       | 8.042       | 8.167              | 8.548        |

- Technical basis:  
Standard Ultimate Survival Model and interest of 5%.
- Ordering:  
$$a_x < a_x^{(4)} < \bar{a}_x < \ddot{a}_x^{(4)} < \ddot{a}_x$$
- Reasons for this ordering:
  - Time value of money.
  - Payments only due upon survival.

## 5.7 Comparison of annuities by payment frequency

### Example 5.2

- Calculate values of

$$a_{x:\overline{10}|}, a_{x:\overline{10}|}^{(4)}, \ddot{a}_{x:\overline{10}|}, \ddot{a}_{x:\overline{10}|}^{(4)} \text{ and } \bar{a}_{x:\overline{10}|}^{(4)}$$

for  $x = 20, 40, 60$  and  $80$ .

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: 5%.

## 5.7 Comparison of annuities by payment frequency

### Example 5.2

Solution:

Table 5.2. Values of  $a_{x:\overline{10}|}$ ,  $a_{x:\overline{10}|}^{(4)}$ ,  $\bar{a}_{x:\overline{10}|}$ ,  $\ddot{a}_{x:\overline{10}|}^{(4)}$  and  $\ddot{a}_{x:\overline{10}|}$ .

| $x$ | $a_{x:\overline{10} }$ | $a_{x:\overline{10} }^{(4)}$ | $\bar{a}_{x:\overline{10} }$ | $\ddot{a}_{x:\overline{10} }^{(4)}$ | $\ddot{a}_{x:\overline{10} }$ |
|-----|------------------------|------------------------------|------------------------------|-------------------------------------|-------------------------------|
| 20  | 7.711                  | 7.855                        | 7.904                        | 7.952                               | 8.099                         |
| 40  | 7.696                  | 7.841                        | 7.889                        | 7.938                               | 8.086                         |
| 60  | 7.534                  | 7.691                        | 7.743                        | 7.796                               | 7.956                         |
| 80  | 6.128                  | 6.373                        | 6.456                        | 6.539                               | 6.789                         |

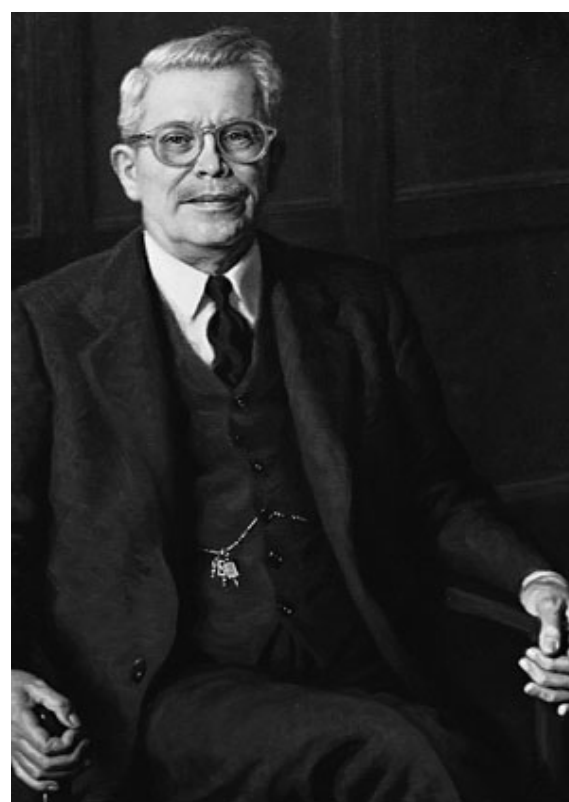
# Life annuities

Samuel Huebner (1882 - 1964)

- 'Annuitants are long lived. Freedom from financial worry and fear, and contentment with a double income, are conducive to longevity. ... I am inclined to believe that annuities serve in old age, much the same economic purpose that periodic medical examinations do during the working years of life.'
- 'Why exist on \$600, assuming 3% interest on \$20 000, and then live in fear, when \$1 600 may be obtained annually at age 65, through an annuity for all of life and minus all the fear?'

Live on interest:

$$20\,000 \times 3\% = 600$$

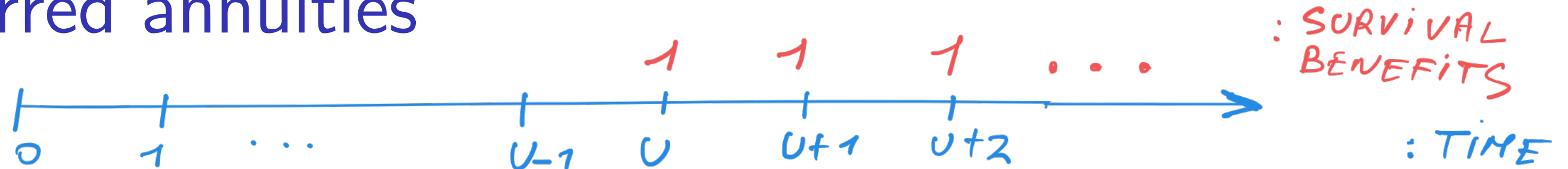


Buy life annuity

$$\text{. Suppose } \ddot{a}_{65} = 12.5$$

$$\text{. } 20\,000 = 1600 \times \ddot{a}_{65}$$

## 5.8 Deferred annuities



- Consider an annuity underwritten to  $(x)$  at time 0, with lifelong annual payments of 1 in advance, commencing at age  $x + u$  ( $u$  is a non-negative integer).
- Benefit cash flow:

$$\sum_{k=u}^{K_x} (1, k)$$

- Actuarial value:

$${}_u|\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|} \quad (5.25)$$

- Relation via actuarial discounting:

$${}_u|\ddot{a}_x = {}_uE_x \ddot{a}_{x+u} \quad (5.26) \rightarrow$$



$${}_v|\ddot{a}_x = EPV \left[ \begin{array}{c} \downarrow \\ \text{0} \quad 1 \quad \dots \quad v-1 \quad v \quad v+1 \quad \dots \\ \text{SURVIVAL BENEFITS} \\ \text{TIME} \end{array} \right]$$

$$= EPV \left[ \begin{array}{c} \downarrow \\ \text{0} \quad 1 \quad \dots \quad v-1 \quad v \\ \ddot{a}_{x+v} \times I_{T_x > v} \end{array} \right]$$

$$= {}_vE_x \times \ddot{a}_{x+v}$$

(5.26)

## 5.8 Deferred annuities

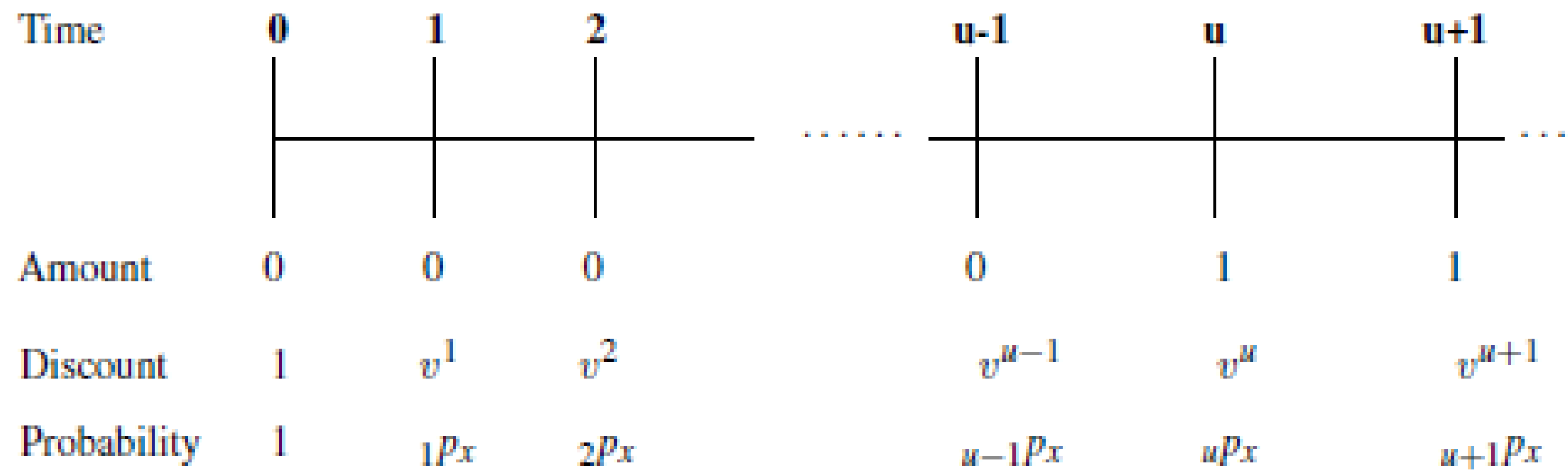


Figure 5.8 Time-line diagram for deferred annual annuity-due.

## 5.8 Deferred annuities

### Some more relations

- Deferred term immediate annuity:

$${}_u|a_{x:\overline{n}|} = {}_uE_x \times a_{x+u:\overline{n}|}$$

- Deferred annuity-due payable 1/m-thly:

$${}_u|\ddot{a}_x^{(m)} = {}_uE_x \times \ddot{a}_{x+u}^{(m)} \quad (5.27)$$

- Term annuity-due:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_nE_x \times \ddot{a}_{x+n} \quad (5.28)$$

- Term annuity-due payable 1/m-thly:

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_nE_x \times \ddot{a}_{x+n}^{(m)} \quad (5.29)$$

- Term-annuity with continuous payments:

$$\bar{a}_{x:\overline{n}|} = \sum_{u=0}^{n-1} {}_u|\bar{a}_{x:\overline{1}|} \quad (5.31)$$

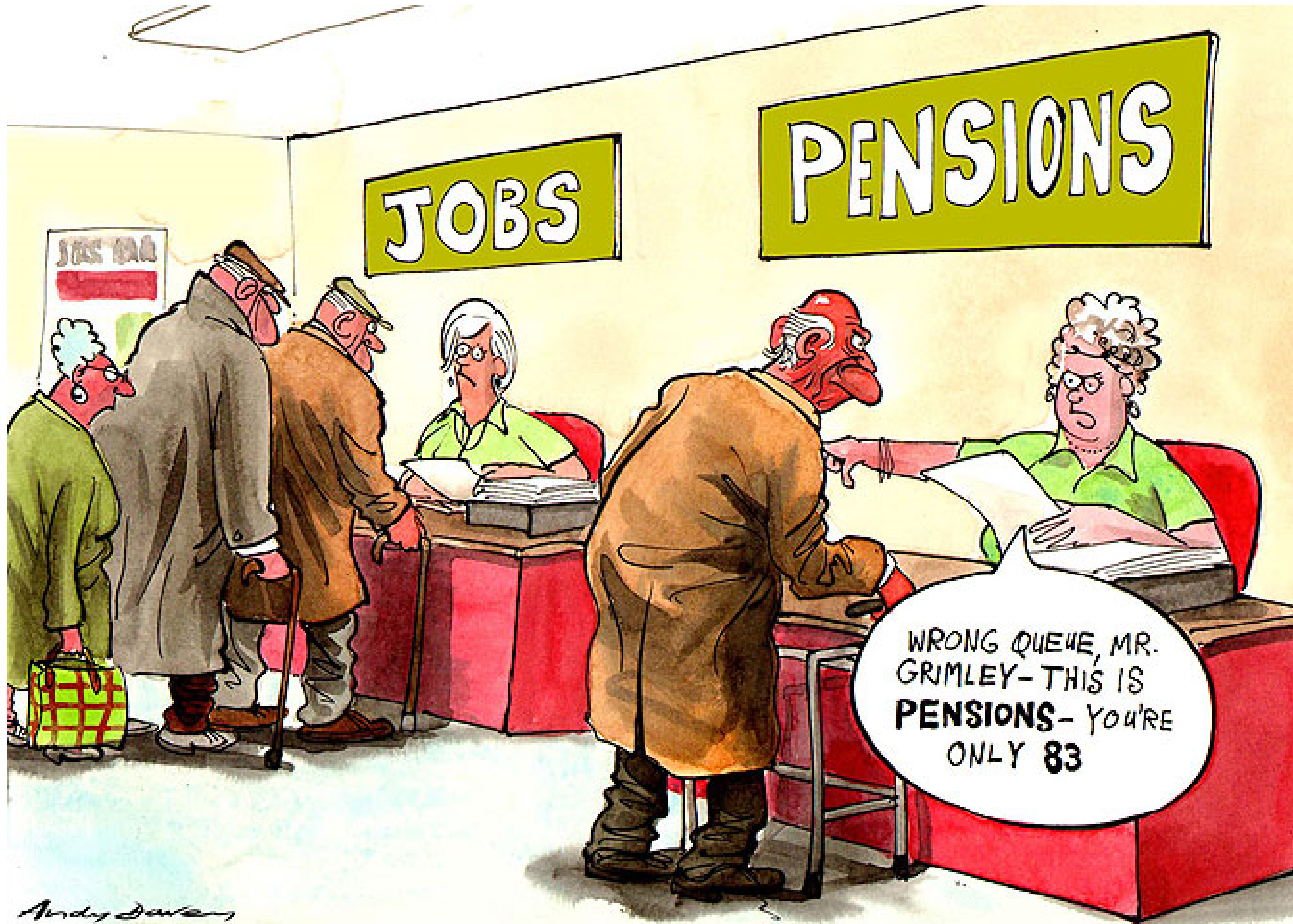
## 5.8 Deferred annuities

### Example 5.3

- Consider the following notations:
  - $Y_1 = \text{PV r.v. of a } u\text{-year deferred whole life annuity-due,}$
  - $Y_2 = \text{PV r.v. of a } u\text{-year term annuity-due,}$
  - $Y_3 = \text{PV r.v. of a whole life annuity-due.}$
- Show that
$$Y_3 = Y_1 + Y_2$$
- Assume annual payments.

## 5.8 Deferred annuities

Increasing retirement age



## 5.8 Deferred annuities

### Increasing retirement age

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: 3%.
- Pension reform: pension age is increased from 65 to 67.
- AV at age 65 of pension of 1 per year:
  - when pension starts at age 65:

$$\ddot{a}_{65} = 16.440$$

- when pension starts at age 67:

$${}_2|\ddot{a}_{65} = 14.474$$

- Relative decrease of pension liability for (65):

$${}_2|\ddot{a}_{65} = 88\% \times \ddot{a}_{65}$$

- In addition, (65) has to pay social security contributions between ages 65 and 67.



## 5.9 Guaranteed annuities

- Consider an annuity-due of 1 per year annually to  $(x)$ , which is guaranteed for a period of  $n$  years.
- Benefit cash flow:

$$\sum_{k=0}^{n-1} (1, k) + \sum_{k=n}^{K_x} (1, k)$$

- Present value:

$$Y = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{K_x} v^k$$

- Actuarial value:

$$\boxed{\ddot{a}_{\overline{x:n}|} \stackrel{\text{not.}}{=} \mathbb{E}[Y] = \ddot{a}_{\overline{n}|} + {}_nE_x \times \ddot{a}_{x+n}} \quad (5.32)$$

- Guaranteed annuity with monthly payments:

$$\ddot{a}_{\overline{x:n}|}^{(12)} \stackrel{\text{not.}}{=} \ddot{a}_{\overline{n}|}^{(12)} + {}_nE_x \times \ddot{a}_{x+n}^{(12)}$$

## 5.9 Guaranteed annuities

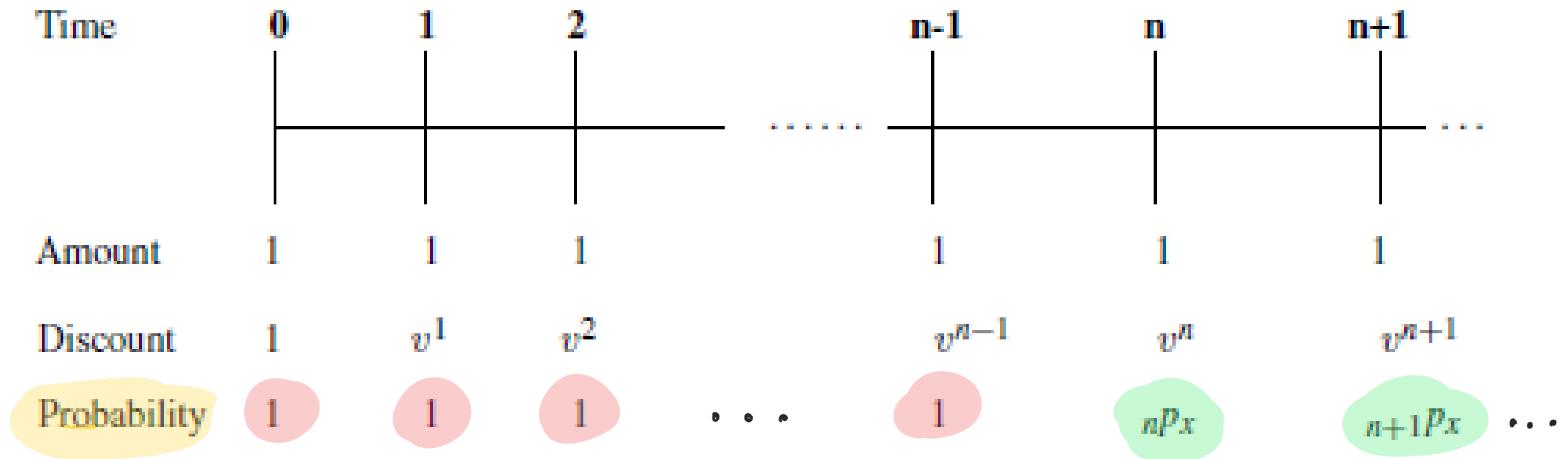


Figure 5.9 Time-line diagram for guaranteed annual annuity-due.

## 5.9 Guaranteed annuities

### Example 5.4

- A pension plan member is entitled to a pension with EPV given by

$$12\,000 \times \ddot{a}_{65}^{(12)}$$

- Alternatively, he can opt for a guaranteed annuity with EPV given by

$$12\,B \times \ddot{a}_{65:\overline{10}|}^{(12)}$$

- Determine  $B$ , such that both EPV's are equal.

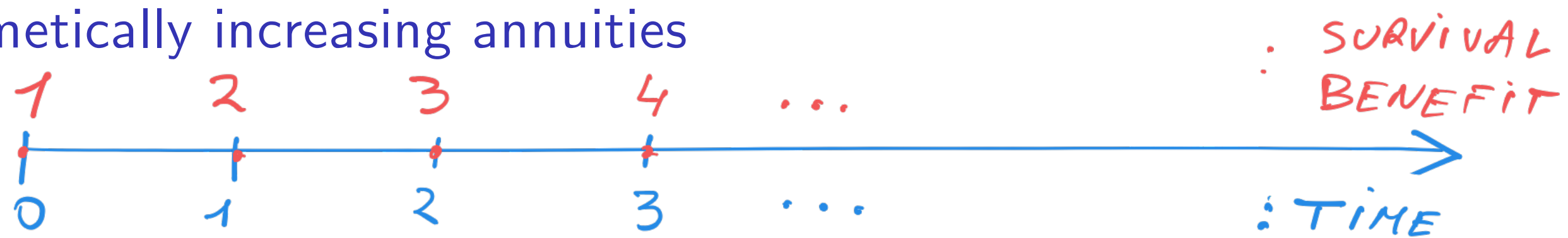
$B < 1000!$

- Technical basis:

- Mortality: Standard Ultimate Survival Model.
- Interest: 5%.

## 5.10 Increasing annuities

### 5.10.1 Arithmetically increasing annuities



- Consider an increasing annuity-due with a payment of  $t + 1$  at times  $t = 0, 1, 2, \dots$  provided  $(x)$  is alive at time  $t$ .
- Benefit cash flow:

$$\sum_{t=0}^{\infty} ((t + 1) \times 1_{\{T_x > t\}}, t)$$

- Actuarial value:

$$\boxed{(I\ddot{a})_x \stackrel{\text{not.}}{=} \sum_{t=0}^{\infty} v^t (t + 1) {}_t p_x} \quad (5.33)$$

## 5.10 Increasing annuities

### 5.10.1 Arithmetically increasing annuities

|             |   |        |        |        |        |     |
|-------------|---|--------|--------|--------|--------|-----|
| Time        | 0 | 1      | 2      | 3      | 4      | ... |
|             |   |        |        |        |        |     |
|             | — | —      | —      | —      | —      | ... |
| Amount      | 1 | 2      | 3      | 4      | 5      |     |
| Discount    | 1 | $v^1$  | $v^2$  | $v^3$  | $v^4$  |     |
| Probability | 1 | $1p_x$ | $2p_x$ | $3p_x$ | $4p_x$ |     |

Figure 5.10 Time-line diagram for arithmetically increasing annual annuity-due.

## 5.10 Increasing annuities

### 5.10.1 Arithmetically increasing annuities



- Consider an increasing annuity-due with a payment of  $t + 1$  at times  $t = 0, 1, 2, \dots, n - 1$ , provided  $(x)$  is alive at time  $t$ .
- Benefit cash flow:

$$\sum_{t=0}^{n-1} ((t + 1) \times 1_{\{T_x > t\}}, t)$$

- Actuarial value:

$$\boxed{(I\ddot{a})_{x:\overline{n}|} \stackrel{\text{not.}}{=} \sum_{t=0}^{n-1} v^t (t + 1) {}_t p_x} \quad (5.34)$$



# 5.10 Increasing annuities

## 5.10.1 Arithmetically increasing annuities

- Consider a **continuous annuity**, with a total payment equal to  $t$  in the  $t$ -th year,  $t = 1, 2, \dots, n$ , equally spread over the year, provided  $(x)$  is alive.

- Benefit cash flow:

$$\int_0^n (\lfloor t + 1 \rfloor \times 1_{\{T_x > t\}} dt, t) = \int_0^{\min(T_x, n)} (\lfloor t + 1 \rfloor x dt, t)$$

- Actuarial value:

$$(I\bar{a})_{x:\overline{n}|} \stackrel{\text{not.}}{=} \sum_{m=0}^{n-1} (m+1) {}_m|\bar{a}_{x:\overline{1}|}$$

## 5.10 Increasing annuities

### 5.10.1 Arithmetically increasing annuities

- Consider a **continuous annuity**, with a payment of  $t \, dt$  in the interval  $(t, t + dt)$ ,  $0 < t < n$ , provided  $(x)$  is alive.

- Benefit cash flow:

$$\int_0^n (t \times 1_{\{T_x > t\}} \, dt, \, t) = \int_0^{\min(T_x, n)} (t \cdot dt, \, t)$$

- Actuarial value:

$$\boxed{(\bar{I}\bar{a})_{x:\overline{n}|} \stackrel{\text{not.}}{=} \int_0^n t e^{-\delta t} {}_t p_x \, dt} \quad (5.35)$$

## 5.10 Increasing annuities

### 5.10.1 Arithmetically increasing annuities

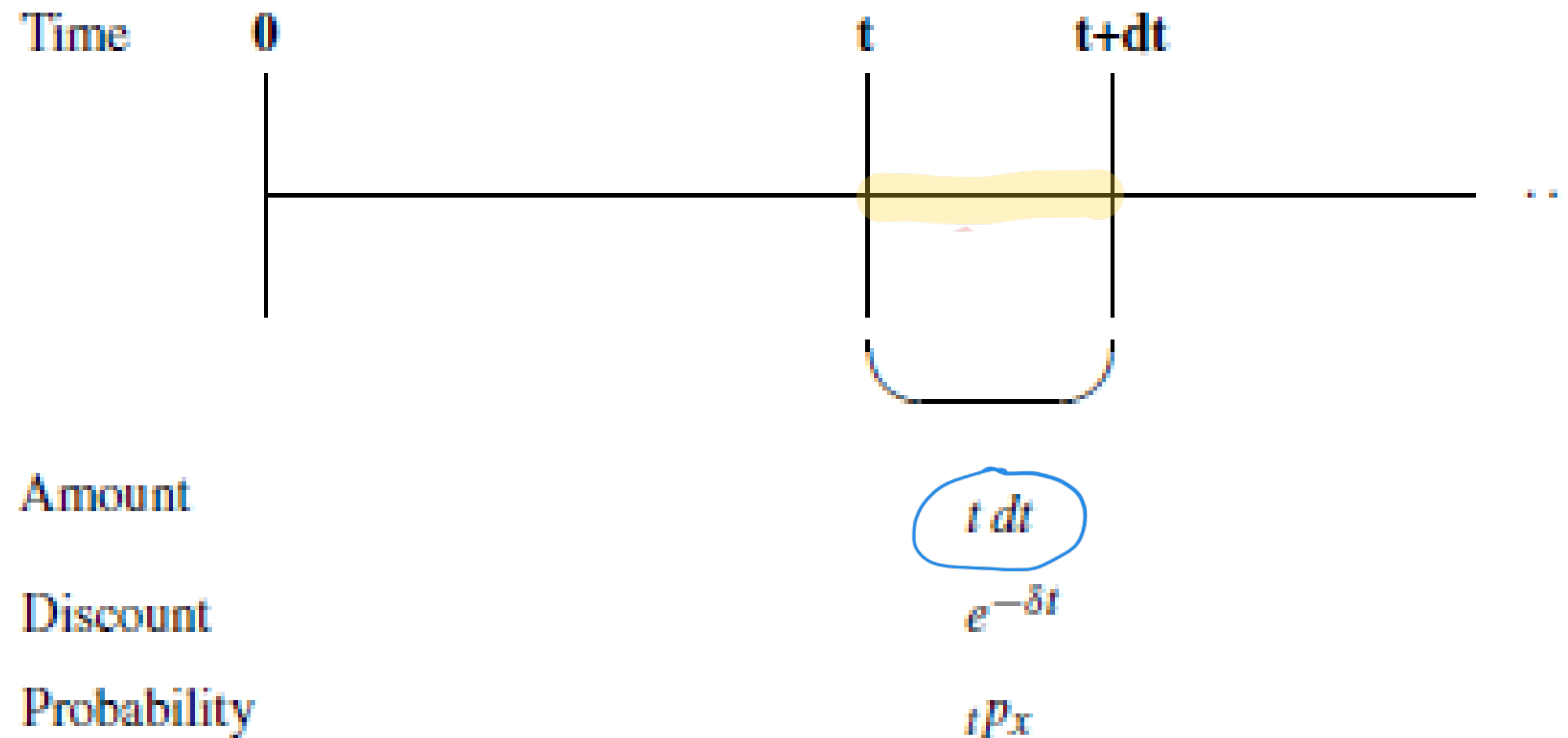


Figure 5.11 Time-line diagram for increasing continuous whole life annuity.

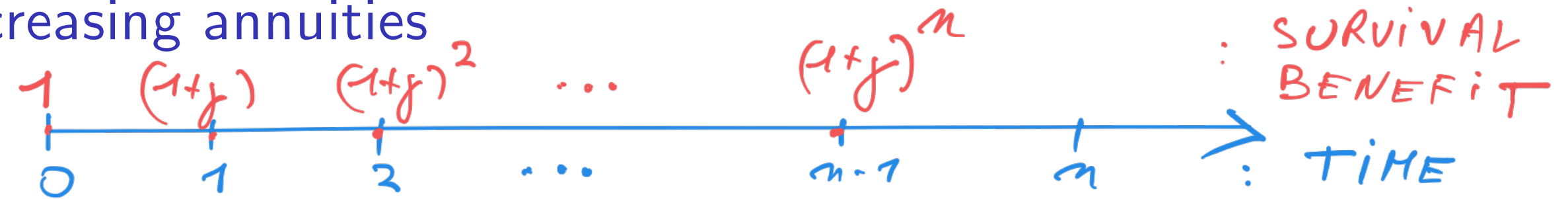
$$(\overline{I\overline{a}})_x$$

7

## 5.10 Increasing annuities

### 5.10.2 Geometrically increasing annuities

#### Example 5.5



- Consider an annuity-due with annual payments where the amount of the annuity is  $(1+j)^t$  at times  $t = 0, 1, 2, \dots, n-1$ , provided  $(x)$  is alive at that time.
- Benefit cash flow:

$$\sum_{t=0}^{n-1} \left( (1+j)^t \times 1_{\{T_x > t\}}, t \right)$$

- Show that the EPV of this cash flow is given by:

$$\sum_{t=0}^{n-1} (1+j)^t \times v^t \times {}_t p_x = \ddot{a}_{x:\overline{n}|} i^*$$

with

$$i^* = \frac{i - j}{1 + j}$$

## 5.10 Increasing annuities

### 5.10.2 Geometrically increasing annuities

#### Example 5.5

|             |       |           |           |           |           |       |
|-------------|-------|-----------|-----------|-----------|-----------|-------|
| Time        | 0     | 1         | 2         | 3         | 4         | ..... |
|             |       |           |           |           |           |       |
|             | ----- |           |           |           |           |       |
| Amount      | 1     | $(1+j)$   | $(1+j)^2$ | $(1+j)^3$ | $(1+j)^4$ |       |
| Discount    | 1     | $v^1$     | $v^2$     | $v^3$     | $v^4$     |       |
| Probability | 1     | ${}_1p_x$ | ${}_2p_x$ | ${}_3p_x$ | ${}_4p_x$ |       |

Figure 5.12 Time-line diagram for geometrically increasing annual annuity-due.

# 5.11 Evaluating annuity functions

## 5.11.1 Recursions

- Let  $\omega$  be the first integer age such that  $l_\omega = 0$ . Then,

$$q_{\omega-1} = 1$$

- Yearly annuity-due:

- Initial value:  $\ddot{a}_{\omega-1} = 1$ .
- Backward recursion: for  $x = \omega - 2, \omega - 3, \dots$ ,

$$\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$$

(5.36)  $\longrightarrow$

- $1/m$ -thly annuity-due:

- Initial value:  $\ddot{a}_{\omega-1/m}^{(m)} = \frac{1}{m}$ .
- Backward recursion: for  $x = \omega - \frac{2}{m}, \omega - \frac{3}{m}, \dots$ ,

$$\ddot{a}_x^{(m)} = \frac{1}{m} + v^{1/m} {}_{1/m}p_x \ddot{a}_{x+1/m}^{(m)}$$

(5.37)

$$\begin{aligned}
 \ddot{a}_x &= EPV \left[ \begin{array}{c} \text{SURVIVAL BENEFITS} \\ \downarrow \\ \begin{array}{ccccccc} 1 & & 1 & & 1 & & \dots \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & & 1 & & 2 & & \dots \end{array} \\ \text{TIME} \end{array} \right] \\
 &= EPV \left[ \begin{array}{c} \downarrow \\ \begin{array}{ccccccc} 1 & & & & & & \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & & 1 & & 2 & & \dots \end{array} \end{array} \right] \\
 &\quad + EPV \left[ \begin{array}{c} \downarrow \\ \begin{array}{ccccccc} & & 1 & & 1 & & \dots \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & & 1 & & 2 & & \dots \end{array} \end{array} \right] \\
 &= 1 + EPV \left[ \begin{array}{c} \downarrow \\ \begin{array}{c} \ddot{a}_{x+1} \times 1_{T_x \geq 1} \\ \vdots \\ 0 \quad 1 \end{array} \end{array} \right]
 \end{aligned}$$

$$= 1 + v \times p_x \times \ddot{a}_{x+1}$$

(5.36)



# 5.11 Evaluating annuity functions

## 5.11.2 Applying the UDD assumption

- How to evaluate the EPV of  $1/m$ -thly and continuous annuities, given only the EPVs of yearly annuities?
- Recall:

$$A_x^{(m)} \stackrel{\text{UDD}}{=} \frac{i}{i^{(m)}} A_x \quad \text{and} \quad \bar{A}_x \stackrel{\text{UDD}}{=} \frac{i}{\delta} A_x$$

and

$$\ddot{a}_x = \frac{1 - A_x}{d}, \quad \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}, \quad \bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$$

- Lifelong annuity-due with  $1/m$ -thly payments:

$$\ddot{a}_x^{(m)} \stackrel{\text{UDD}}{=} \alpha(m) \ddot{a}_x - \beta(m)$$

with

$$\alpha(m) = \frac{i d}{i^{(m)} d^{(m)}} \quad \text{and} \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} \quad (5.38)$$

$$\ddot{a}_x^{(m)} \quad (5.18) = \frac{1 - A_x^{(m)}}{d^{(m)}}$$

$$\stackrel{VDD}{=} \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}}$$

$$= \frac{i^{(m)} - i \cdot A_x}{i^{(m)} \cdot d^{(m)}}$$

$$(5.4) = \frac{i^{(m)} - i(1 - d \cdot \ddot{a}_x)}{i^{(m)} \cdot d^{(m)}}$$

$$= d^{(m)} \cdot \ddot{a}_x - \beta^{(m)}$$

with  $d^{(m)} = \frac{i \cdot d}{i^{(m)} \cdot d^{(m)}}$  and  $\beta^{(m)} = \frac{i - i^{(m)}}{i^{(m)} \cdot d^{(m)}}$

↑ independent of x ↑

# 5.11 Evaluating annuity functions

## 5.11.2 Applying the UDD assumption

- Limiting values:

$$\delta = \lim_{m \rightarrow \infty} i^{(m)} = \lim_{m \rightarrow \infty} d^{(m)}$$


- Lifelong annuity with continuous payments:

$$\bar{a}_x \stackrel{\text{UDD}}{=} \frac{i d}{\delta^2} \ddot{a}_x - \frac{i - \delta}{\delta^2}$$



- Term annuity-due with 1/*m*-thly payments:

$$\ddot{a}_{x:\overline{n}|}^{(m)} \stackrel{\text{UDD}}{=} \alpha(m) \ddot{a}_{x:\overline{n}|} - \beta(m) (1 - {}_nE_x)$$

(5.39) 

$$\bullet \quad \bar{a}_x = \lim_{m \rightarrow \infty} \ddot{a}_x^{(m)}$$

$$= \lim_{m \rightarrow \infty} (d(m) \ddot{a}_x - \beta(m))$$

$$= \frac{i \cdot d}{\delta^2} \ddot{a}_x - \frac{i - \delta}{\delta^2}$$

$$\bullet \quad \ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_nE_x \times \ddot{a}_{x+n}^{(m)}$$

$$= (d(m) \ddot{a}_x - \beta(m)) - {}_nE_x (d(m) \ddot{a}_{x+n} - \beta(m))$$

$$= d(m) \times (\ddot{a}_x - {}_nE_x \ddot{a}_{x+n}) - \beta(m) \cdot (1 - {}_nE_x)$$

$$= d(m) \cdot \ddot{a}_{x:\overline{n}|} - \beta(m) \times (1 - {}_nE_x)$$

# 5.11 Evaluating annuity functions

## 5.11.2 Applying the UDD assumption

- Approximations for  $\alpha(m)$  and  $\beta(m)$ : (see Eq. 5.18)

$$\alpha(m) \approx 1 \quad \text{and} \quad \beta(m) \approx \frac{m-1}{2m}$$

- Approximation for  $1/m$ -thly annuity:

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m} (1 - {}_nE_x) \quad (5.40)$$

- Approximation for continuous annuity:

$$\bar{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2} (1 - {}_nE_x)$$

$\lim_{m \rightarrow \infty}$

# 5.11 Evaluating annuity functions

## 5.11.3 Woolhouse's formula (read in book)

## 5.12 Numerical illustrations

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest:  $i = 0.1$

Table 5.3. Values of  $\ddot{a}_{x:\overline{10}|}^{(12)}$  for  $i = 0.1$ .

| $x$ | Exact  | UDD    | W2     | <del>W3</del> | <del>W3*</del> |
|-----|--------|--------|--------|---------------|----------------|
| 20  | 6.4655 | 6.4655 | 6.4704 | 6.4655        | 6.4655         |
| 30  | 6.4630 | 6.4630 | 6.4679 | 6.4630        | 6.4630         |
| 40  | 6.4550 | 6.4550 | 6.4599 | 6.4550        | 6.4550         |
| 50  | 6.4295 | 6.4294 | 6.4344 | 6.4295        | 6.4295         |
| 60  | 6.3485 | 6.3482 | 6.3535 | 6.3485        | 6.3485         |
| 70  | 6.0991 | 6.0982 | 6.1044 | 6.0990        | 6.0990         |
| 80  | 5.4003 | 5.3989 | 5.4073 | 5.4003        | 5.4003         |
| 90  | 3.8975 | 3.8997 | 3.9117 | 3.8975        | 3.8975         |
| 100 | 2.0497 | 2.0699 | 2.0842 | 2.0497        | 2.0496         |

- Exact:  $\ddot{a}_{x:\overline{10}|}^{(12)}$
- UDD:  $\ddot{a}_{x:\overline{10}|}^{(12)} \stackrel{\text{UDD}}{=} \alpha (12) \ddot{a}_{x:\overline{10}|} - \beta (12) (1 - {}_{10}E_x)$  (5.38)
- W2:  $\ddot{a}_{x:\overline{10}|}^{(12)} \approx \ddot{a}_{x:\overline{10}|} - \frac{11}{24} (1 - {}_{10}E_x)$  (5.40)



## 5.12 Numerical illustrations

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest:  $i = 0.05$

Table 5.4. Values of  $\ddot{a}_{x:\overline{25}|}^{(2)}$  for  $i = 0.05$ .

| $x$ | Exact   | UDD     | W2      | <del>W3</del> | <del>W3*</del> |
|-----|---------|---------|---------|---------------|----------------|
| 20  | 14.5770 | 14.5770 | 14.5792 | 14.5770       | 14.5770        |
| 30  | 14.5506 | 14.5505 | 14.5527 | 14.5506       | 14.5506        |
| 40  | 14.4663 | 14.4662 | 14.4684 | 14.4663       | 14.4663        |
| 50  | 14.2028 | 14.2024 | 14.2048 | 14.2028       | 14.2028        |
| 60  | 13.4275 | 13.4265 | 13.4295 | 13.4275       | 13.4275        |
| 70  | 11.5117 | 11.5104 | 11.5144 | 11.5117       | 11.5117        |
| 80  | 8.2889  | 8.2889  | 8.2938  | 8.2889        | 8.2889         |
| 90  | 4.9242  | 4.9281  | 4.9335  | 4.9242        | 4.9242         |
| 100 | 2.4425  | 2.4599  | 2.4656  | 2.4424        | 2.4424         |

- Exact:  $\ddot{a}_{x:\overline{25}|}^{(2)}$
- UDD:  $\ddot{a}_{x:\overline{25}|}^{(2)} \stackrel{\text{UDD}}{=} \alpha(2) \ddot{a}_{x:\overline{25}|} - \beta(2) (1 - {}_{25}E_x)$
- W2:  $\ddot{a}_{x:\overline{25}|}^{(2)} \approx \ddot{a}_{x:\overline{25}|} - \frac{1}{4} (1 - {}_{25}E_x)$

( 5.38 )

( 5.40 )

## 5.13 Functions for select lives

- Throughout this chapter we assumed an ultimate survival model.
- Results can easily be adapted to the case of a select survival model.
- **Continuous life annuities and endowment assurances:**

- Life annuity:

$$\bar{a}_{[x]+k:\overline{n}|} = \int_0^n e^{-\delta t} {}_t p_{[x]+k} dt$$

- Endowment insurance:

$$\bar{A}_{[x]+k:\overline{n}|} = \int_0^n e^{-\delta t} {}_t p_{[x]+k} \mu_{[x]+k+t} dt + {}_n E_{[x]+k}$$

- Relation:

$$\bar{a}_{[x]+k:\overline{n}|} = \frac{1 - \bar{A}_{[x]+k:\overline{n}|}}{\delta}$$

## 5.13 Functions for select lives

- Yearly and  $1/m$ -thly annuities:

$$\ddot{a}_{[x]+k} = \sum_{t=0}^{\infty} v^t {}_t p_{[x]+k} \quad \text{and} \quad \ddot{a}_{[x]+k}^{(m)} = \frac{1}{m} \sum_{t=0}^{\infty} v^{t/m} {}_{\frac{t}{m}} p_{[x]+k}$$

- Approximations:

$$\ddot{a}_{[x]+k}^{(m)} \stackrel{\text{UDD}}{=} \alpha(m) \ddot{a}_{[x]+k} - \beta(m) \approx \ddot{a}_{[x]+k} - \frac{m-1}{2m}$$

- Example 5.6:

- Technical basis:

- Mortality: Standard Select Survival Model.
- Interest:  $i = 0.05$ .

- Assumption:  $q_{131} = 1$

- Question: Produce a table showing values of  $\ddot{a}_{[x]}$ ,  $\ddot{a}_{[x]+1}$  and  $\ddot{a}_{x+2}$  for  $x = 20, 21, \dots, 80$ .

# 5.14 Notes and further reading

(read in book)