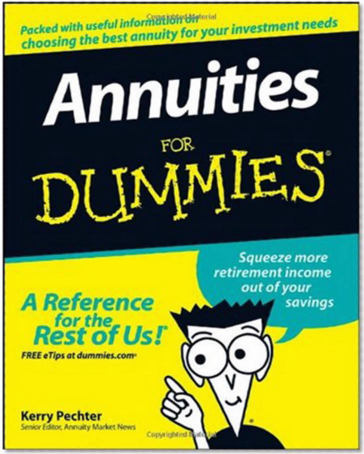
Life Insurance Mathematics

Life annuities[1]

Jan Dhaene

[1] Based on Chapter 5 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).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5.1 Summary

- Life contingent annuities:
  - Whole life annuities vs. term annuities.
  - Annuities-due vs. immediate annuities.
  - Annuities payable yearly, $1/m$-thly or continuously.
- Actuarial valuation of life contingent annuities:
  - EPV of life annuity benefit cash flows.
  - Actuarial notation.
  - Continuous valuation via T_x.
  - Discrete valuation via K_x or $K_x^{(m)}$.

5.2 Introduction

- **Life annuity**: series of payments as long as a given person is alive on the payment dates.
- Payments:
  - at regular intervals,
  - (usually) of the same amount.
- Used for calculating:
  - pension benefits,
  - premiums,
  - policy values.

What is the relation between this grafitti painting (Brussels, January 2017) and life annuities?

5.2 Introduction

'By providing financial protection against the major 18th and 19th century risk of dying too soon, life assurance became the biggest financial industry..., providing financial protection against the new risk of not dying soon enough may well become the next century's major and most profitable financial industry.'
(Peter Drucker, The Economist, 1999)

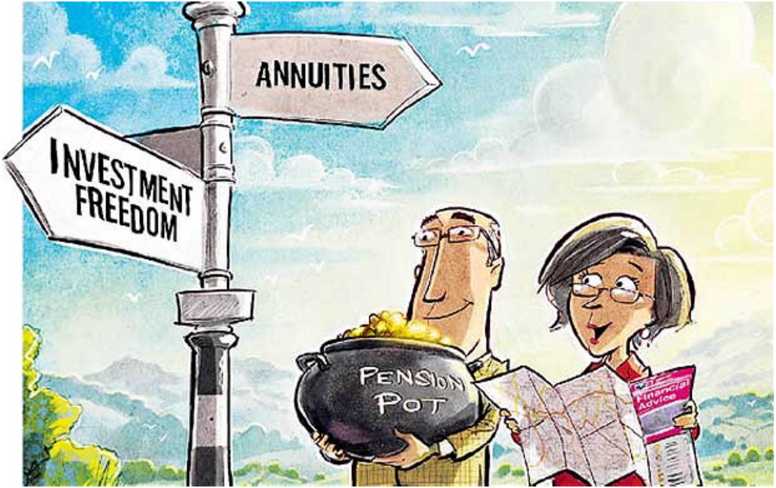
5.2 Introduction

Why buying a life annuity?

5.2 Introduction
Life annuity sale

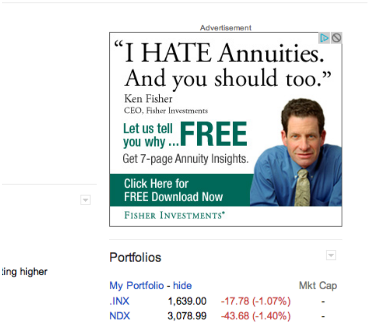
5.2 Introduction

To buy or not to buy?



5.2 Introduction

To buy or not to buy?



5.3 Review of annuities-certain

Cash flow notations

- The series of cash flows (c_k, k), $k = m, \ldots, n$, is denoted by:

$$\boxed{\sum_{k=m}^{n} (c_k, k)}$$

- The continuous stream of payments $c_\tau \, d\tau$ in any infinitesimal subinterval $(\tau, \tau + d\tau)$ of (s, t), is denoted by:

$$\boxed{\int_s^t (c_\tau d\tau,\ \tau)}$$

- Convention:

$$\sum_{k=m}^{n} (c_k, k) = (0, 0) \text{ if } m > n \text{ and } \int_s^t (c_\tau d\tau,\ \tau) = (0, 0) \text{ if } s > t$$

- In previous notations, m, n, c_k, c_τ, s and t may be deterministic or random.
- Similar notations and conventions for series of cash flows with $1/m$-thly payments.

5.3 Review of annuities-certain

- Annuity-due:

$$\ddot{a}_{\overline{n}|} = 1 + v + ... + v^{n-1} = \frac{1 - v^n}{d} \tag{5.1}$$

- Annuity-immediate:

$$a_{\overline{n}|} = v + ... + v^n = \frac{1 - v^n}{i}$$

- Continuous annuity:

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - v^n}{\delta} \tag{5.2}$$

- Annuity-due with $1/m$-thly payments:

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m}\left(1 + v^{\frac{1}{m}} + ... + v^{n-\frac{1}{m}}\right) = \frac{1 - v^n}{d^{(m)}}$$

- Annuity-immediate with $1/m$-thly payments:

$$a^{(m)}_{\overline{n}|} = \frac{1}{m}\left(v^{\frac{1}{m}} + ... + v^{n-\frac{1}{m}} + v^n\right) = \frac{1 - v^n}{i^{(m)}}$$

5.4 Annual life annuities

5.4.1 Whole life annuity-due (1)

- Consider an annuity underwritten to (x) at time 0. It pays 1 annually in advance as long as (x) is alive.
- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- A whole life annuity only 'dies' when the insured dies.

5.4 Annual life annuities

5.4.1 Whole life annuity-due (1)

- Consider an annuity underwritten to (x) at time 0. It pays 1 annually in advance as long as (x) is alive.
- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = 1 + v + \ldots + v^{K_x} = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d}$$

- Actuarial value:

$$\boxed{\ddot{a}_x \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{K_x+1}\right]}{d} = \frac{1-A_x}{d}} \qquad (5.3)$$

5.4 Annual life annuities

5.4.1 Whole life annuity-due (2)

- Benefit cash flow:

$$\sum_{t=0}^{K_x} (1,\ t) = \sum_{t=0}^{\infty} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y = \sum_{t=0}^{\infty} v^t \ 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}\left[Y\right] = \sum_{t=0}^{\infty} v^t \ {}_tp_x} \tag{5.5}$$

5.4 Annual life annuities

5.4.1 Whole life annuity-due (2)

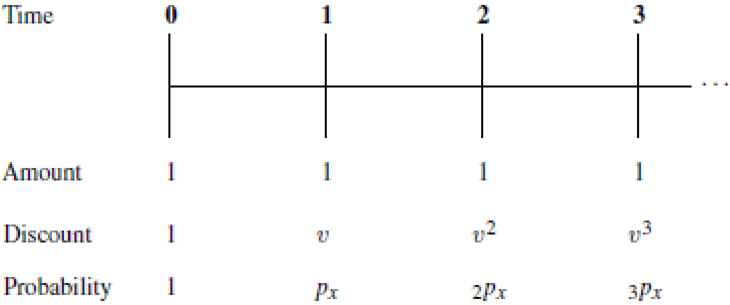


Figure 5.1 Time-line diagram for whole life annuity-due.

5.4 Annual life annuities

5.4.1 Whole life annuity-due (3)

- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = \ddot{a}_{\overline{K_x+1}|}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}\left[Y\right] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \times {}_{k|}q_x} \tag{5.6}$$

5.4 Annual life annuities

Example 5.1

- Show algebraically that

$$\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \times {}_{k|}q_x = \sum_{k=0}^{\infty} v^k \; {}_k p_x$$

- <u>Proof</u>:

$$\begin{aligned} \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \times {}_{k|}q_x &= \sum_{k=0}^{\infty} \left(\sum_{t=0}^{k} v^t \right) \times {}_{k|}q_x \\ &= \sum_{t=0}^{\infty} v^t \left(\sum_{k=t}^{\infty} {}_{k|}q_x \right) \\ &= \sum_{t=0}^{\infty} v^t \; {}_t p_x \end{aligned}$$

5.4 Annual life annuities

5.4.2 Term annuity-due (1)

- Consider an annuity underwritten to (x) at time 0. It pays 1 at times $0, 1, \ldots, n-1$, provided (x) is alive.
- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

$$Y = 1 + v + \ldots + v^{\min(K_x, n-1)} = \ddot{a}_{\overline{min(K_x+1,n)|}} = \frac{1 - v^{\min(K_x+1,n)}}{d}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{\min(K_x+1,n)}\right]}{d} = \frac{1-A_{x:\overline{n}|}}{d}} \qquad (5.7)$$

5.4 Annual life annuities

5.4.2 Term annuity-due (2)

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t) = \sum_{t=0}^{n-1} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y = \sum_{t=0}^{n-1} v^t \ 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \sum_{t=0}^{n-1} v^t \ {}_t p_x} \tag{5.8}$$

5.4 Annual life annuities

5.4.2 Term annuity-due (2)

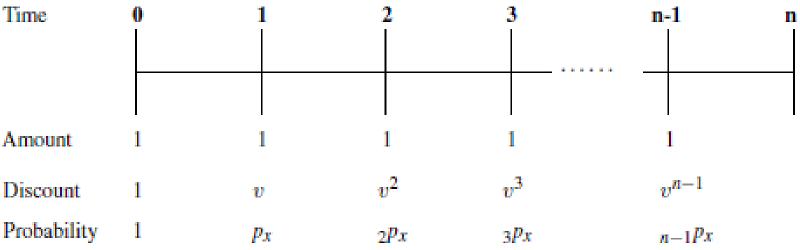


Figure 5.2 Time-line diagram for term life annuity-due.

5.4 Annual life annuities

5.4.2 Term annuity-due (3)

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

$$Y = \ddot{a}_{\overline{min(K_x+1, n)}|}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} = \mathbb{E}\left[Y\right] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \times {}_{k|}q_x + {}_{n}p_x \times \ddot{a}_{\overline{n}|}}$$

- Consider a **whole life immediate annuity** underwritten to (x) at time 0. It pays 1 annually in arrear, as long as (x) is alive.
- Benefit cash flow:

$$\sum_{t=1}^{K_x} (1, t) = \sum_{t=1}^{\infty} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y^* = \sum_{t=1}^{\infty} v^t \, 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{a_x \overset{\text{not.}}{=} \mathbb{E}\left[Y^*\right] = \ddot{a}_x - 1} \tag{5.9}$$

5.4 Annual life annuities

5.4.3 Immediate life annuities

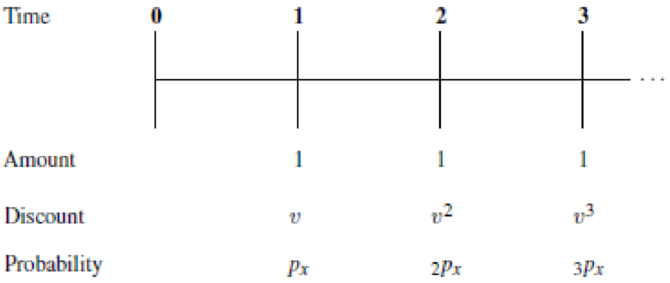


Figure 5.3 Time-line diagram for whole life immediate annuity.

5.4 Annual life annuities

5.4.3 Immediate life annuities

- Consider a **n-year term immediate annuity** underwritten to (x) at time 0. It pays 1 at times $1, 2, \ldots, n$, provided (x) is alive.
- Benefit cash flow:

$$\sum_{t=1}^{\min(K_x,n)} (1, t) = \sum_{t=1}^{n} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y^* = \sum_{t=1}^{n} v^t \ 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{a_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y^*\right] = \textstyle\sum_{t=1}^{n} v^t \ {}_t p_x} \qquad (5.11)$$

- Relation:

$$\boxed{a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \ {}_n p_x} \qquad (5.12)$$

5.4 Annual life annuities

5.4.4 Immediate life annuities

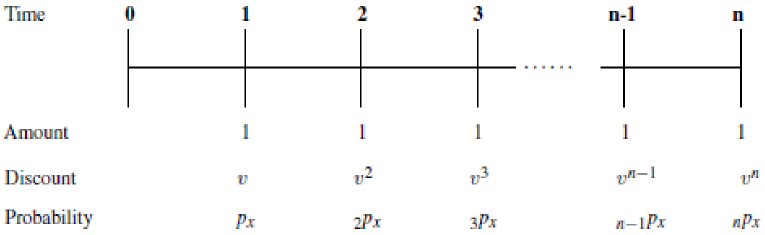


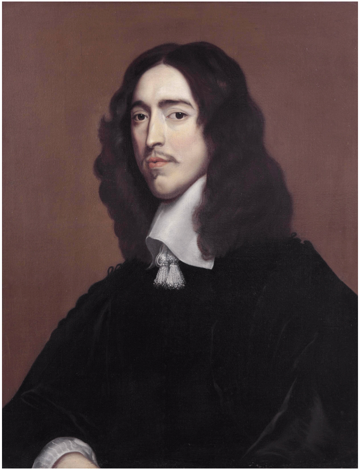
Figure 5.4 Time-line diagram for term life immediate annuity.

5.4 Annual life annuities

Some history

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \ {}_k p_x$$

Johan de Witt (1671)



Edmond Halley (1693).

5.4 Annual life annuities

Some history

- In second half of 17th century, states and cities often raised money for public purposes by the sale of lifelong annuities to their residents. This led to huge *unfunded liabilities*.
- **Johan de Witt** (1625 - 1672):
  - First to use a (hypothetical) life table and interest rates to determine the value of a life annuity:
    - *Waardye van Lijfrenten naar Proportie van Los Renten* (1671).
  - At that time, he was prime minister of the State of Holland.
  - In 1672, he and his brother were cruelly lynched by Orangists.

5.4 Annual life annuities

Some history

- **Edmond Halley** (1656 - 1742):
  - Was in 1690 asked by the Royal Society In London to estimate the value of the liabilities related to lifelong annuities.
  - In 1693, he published an article in 'Phylosophical Transactions of the Royal Society', in which
    - he displayed one of the first reliable lifetables based on demographic data,
    - and used these mortality rates and interest rates to determine the value of a life annuity.
  - Assisted and motivated Isaac Newton to publish his famous book 'Principles'.

5.5 Annuities payable continuously

Whole life continuous annuity (1)

- Consider an annuity underwritten to (x) at time 0. It pays continuously at a rate of 1 per year as long as (x) is alive.
- Benefit cash flow:

$$\int_0^{T_x} (dt, t)$$

- Present value:

$$Y = \int_0^{T_x} v^t \, dt = \bar{a}_{\overline{T_x}|} = \frac{1 - v^{T_x}}{\delta} \tag{5.13}$$

- Actuarial value:

$$\boxed{\overline{a}_x \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1 - \mathbb{E}\left[v^{T_x}\right]}{\delta} = \frac{1 - \overline{A}_x}{\delta}} \tag{5.14}$$

5.5 Annuities payable continuously

Whole life continuous annuity (2)

- Benefit cash flow:

$$\int_0^{T_x} (dt, t) = \int_0^{\infty} \left(1_{\{T_x > t\}}\ dt,\ t\right)$$

- Present value:

$$Y = \int_0^{\infty} e^{-\delta t}\ 1_{\{T_x > t\}}\ dt$$

- Actuarial value:

$$\boxed{\overline{a}_x = \mathbb{E}\left[Y\right] = \int_0^{\infty} e^{-\delta t}\ {}_t p_x\ dt} \qquad (5.15)$$

5.5 Annuities payable continuously

Whole life continuous annuity (2)

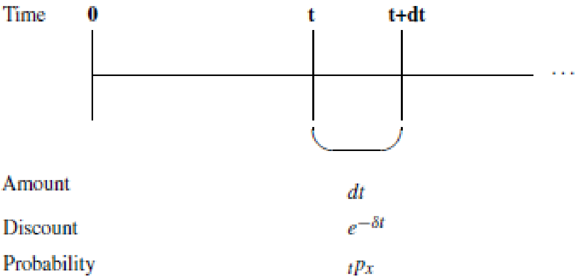


Figure 5.5 Time-line diagram for continuous whole life annuity.

5.5 Annuities payable continuously

Whole life continuous annuity (3)

- Benefit cash flow:

$$\int_0^{T_x} (dt, t)$$

- Present value:

$$Y = \int_0^{T_x} e^{-\delta t} \, dt = \bar{a}_{\overline{T_x}|}$$

- Actuarial value:

$$\boxed{\bar{a}_x = \mathbb{E}[Y] = \int_0^\infty \bar{a}_{\overline{t}|} \; {}_tp_x \; \mu_{x+t} \; dt}$$

5.5 Annuities payable continuously

Term continuous annuity (1)

- Consider an annuity underwritten to (x) at time 0. It pays continuously at a rate of 1 per year, for a period of n years and provided (x) is alive.
- Benefit cash flow:

$$\int_0^{\min(T_x,\ n)} (dt, t)$$

- Present value:

$$Y = \int_0^{\min(T_x,\ n)} e^{-\delta t}\ dt = \bar{a}_{\overline{min(T_x, n)}|} = \frac{1 - v^{\min(T_x,\ n)}}{\delta}$$

- Actuarial value:

$$\boxed{\bar{a}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{\min(T_x,\ n)}\right]}{\delta} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}} \qquad (5.16)$$

5.5 Annuities payable continuously

Term continuous annuity (2)

- Benefit cash flow:

$$\int_0^{\min(T_x,\ n)} (dt, t) = \int_0^n \left(1_{\{T_x>t\}}\ dt,\ \ t\right)$$

- Present value:

$$Y = \int_0^n e^{-\delta t}\ 1_{\{T_x>t\}}\ dt$$

- Actuarial value:

$$\boxed{\bar{a}_{x:\overline{n}|} = \mathbb{E}\left[Y\right] = \int_0^n e^{-\delta t}\ {}_t p_x\ dt} \tag{5.17}$$

5.5 Annuities payable continuously

Term continuous annuity (3)

- Benefit cash flow:

$$\int_0^{\min(T_x,\ n)} (dt, t)$$

- Present value:

$$Y = \int_0^{\min(T_x,\ n)} e^{-\delta t}\ dt = \bar{a}_{\overline{min(T_x,n)}|}$$

- Actuarial value:

$$\boxed{\bar{a}_{x:\overline{n}|} = \mathbb{E}\left[Y\right] = \int_0^n \bar{a}_{\overline{t}|}\ {}_t p_x\ \mu_{x+t}\ dt + \bar{a}_{\overline{n}|} \times\ {}_n p_x}$$

5.6 Annuities payable 1/m-thly

5.6.1 Introduction

- Recall: Future lifetime of (x) in years, rounded down to the lower 1/m-th of the year:

$$K_x^{(m)} = \frac{1}{m} \lfloor m T_x \rfloor$$

- Recall: Annuity-due with $1/m$-thly payments:

$$\ddot{a}^{(m)}_{\overline{(j+1)/m|}} = \frac{1}{m} \sum_{k=0}^{j} v^{k/m} = \frac{1 - v^{\frac{j+1}{m}}}{d^{(m)}}$$

5.6 Annuities payable 1/m-thly

5.6.2 Whole life annuities payable 1/m-thly (1)

- Consider an annuity underwritten to (x) at time 0. It pays an amount of 1 per year, payable in advance m times per year, throughout the lifetime of (x).
- Benefit cash flow:

$$\sum_{k=0}^{m\,K_x^{(m)}} \left(\frac{1}{m}, \frac{k}{m}\right)$$

- Present value:

$$Y = \frac{1}{m} \sum_{k=0}^{m\,K_x^{(m)}} v^{k/m} = \ddot{a}_{\overline{K_x^{(m)}+1/m}|}^{(m)} = \frac{1 - v^{K_x^{(m)}+\frac{1}{m}}}{d^{(m)}}$$

- Actuarial value:

$$\boxed{\ddot{a}_x^{(m)} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{K_x^{(m)}+\frac{1}{m}}\right]}{d^{(m)}} = \frac{1-A_x^{(m)}}{d^{(m)}}} \qquad (5.18)$$

5.6 Annuities payable 1/m-thly

5.6.2 Whole life annuities payable 1/m-thly (2)

- Actuarial value:

$$\boxed{\ddot{a}_x^{(m)} = \frac{1}{m} \sum_{t=0}^{\infty} v^{t/m} \; {}_{\frac{t}{m}}p_x} \qquad (5.19)$$

- Annuity-immediate vs. annuity-due:

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} \qquad (5.20)$$

5.6 Annuities payable 1/m-thly

5.6.2 Whole life annuities payable 1/m-thly (2)

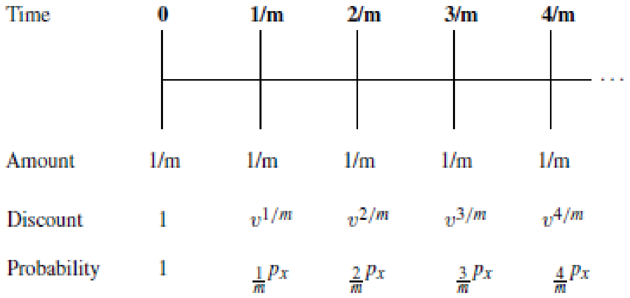


Figure 5.6 Time-line diagram for whole life $1/m$thly annuity-due.

5.6 Annuities payable 1/m-thly

5.6.2 Whole life annuities payable 1/m-thly: example

- In 1965, at age 90, **Jeanne Calment** sold her apartment by a *life annuity sale*.
- The buyer, André-François Raffray, was going to receive the apartment at the death of the seller:

$$\left(\text{apartment}, \ K_{90}^{(12)} + \frac{1}{12}\right)$$

- In return, Jeanne was going to receive a whole life annuity of 2500 fr. per month:

$$\sum_{k=0}^{12\,K_{90}^{(12)}} \left(2500, \frac{k}{12}\right)$$

- Jeann Calment died in 1997, aged 122.

5.6 Annuities payable 1/m-thly

Term annuities payable 1/m-thly (1)

- Consider an annuity underwritten to (x) at time 0, paying 1 per year, payable in advance m times per year, throughout the lifetime of (x), limited to a maximum of n years.
- Benefit cash flow:

$$\sum_{k=0}^{\min\left(m K_x^{(m)},\ mn-1\right)} \left(\frac{1}{m}, \frac{k}{m}\right)$$

- Present value:

$$Y = \ddot{a}^{(m)}_{\overline{min(K_x^{(m)}+1/m,n)}|} = \frac{1 - v^{\min\left(K_x^{(m)}+1/m,\ n\right)}}{d^{(m)}}$$

- Actuarial value:

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{\min\left(K_x^{(m)}+1/m,\ n\right)}\right]}{d^{(m)}} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}} \qquad (5.21)$$

5.6 Annuities payable 1/m-thly

Term annuities payable 1/m-thly (2)

- Actuarial value:

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\sum_{r=0}^{mn-1} v^{r/m} \; {}_{\frac{r}{m}}p_x} \tag{5.22}$$

- Annuity-immediate vs. annuity-due:

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m}\left(1 - v^n \; {}_np_x\right) \tag{5.23}$$

5.6 Annuities payable 1/m-thly

Term annuities payable 1/m-thly (2)

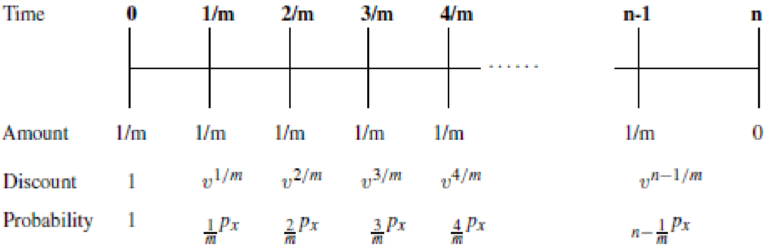


Figure 5.7 Time-line diagram for term life $1/m$thly annuity-due.

5.7 Comparison of annuities by payment frequency

Table 5.1. *Values of* a_x , $a_x^{(4)}$, $\bar{a}_x$, $\ddot{a}_x^{(4)}$ *and* $\ddot{a}_x$.

x	a_x	$a_x^{(4)}$	$\bar{a}_x$	$\ddot{a}_x^{(4)}$	$\ddot{a}_x$
20	18.966	19.338	19.462	19.588	19.966
40	17.458	17.829	17.954	18.079	18.458
60	13.904	14.275	14.400	14.525	14.904
80	7.548	7.917	8.042	8.167	8.548

- Technical basis:
  Standard Ultimate Survival Model and interest of 5%.
- Ordering:
  $$a_x < a_x^{(4)} < \bar{a}_x < \ddot{a}_x^{(4)} < \ddot{a}_x$$
- Reasons for this ordering:
  - Time value of money.
  - Payments only due upon survival.

5.7 Comparison of annuities by payment frequency

Example 5.2

- Calculate values of

$$a_{x:\overline{10}|},\ a^{(4)}_{x:\overline{10}|},\ \ddot{a}_{x:\overline{10}|},\ \ddot{a}^{(4)}_{x:\overline{10}|} \text{ and } \bar{a}^{(4)}_{x:\overline{10}|}$$

  for $x = 20$, 40, 60 and 80.

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: 5%.

5.7 Comparison of annuities by payment frequency

Example 5.2

Solution:

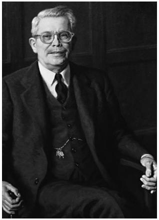
Table 5.2. *Values of* $a_{x:\overline{10}|}$, $a^{(4)}_{x:\overline{10}|}$, $\bar{a}_{x:\overline{10}|}$, $\ddot{a}^{(4)}_{x:\overline{10}|}$ *and* $\ddot{a}_{x:\overline{10}|}$.

x	$a_{x:\overline{10}|}$	$a^{(4)}_{x:\overline{10}|}$	$\bar{a}_{x:\overline{10}|}$	$\ddot{a}^{(4)}_{x:\overline{10}|}$	$\ddot{a}_{x:\overline{10}|}$
20	7.711	7.855	7.904	7.952	8.099
40	7.696	7.841	7.889	7.938	8.086
60	7.534	7.691	7.743	7.796	7.956
80	6.128	6.373	6.456	6.539	6.789

Life annuities

Samuel Huebner (1882 - 1964)

- *'Annuitants are long livers. Freedom from financial worry and fear, and contentment with a double income, are conducive to longevity. ... I am inclined to believe that annuities serve in old age, much the same economic purpose that periodic medical examinations do during the working years of life.'*
- *'Why exist on \$600, assuming 3% interest on \$20 000, and then live in fear, when \$1 600 may be obtained annually at age 65, through an annuity for all of life and minus all the fear?*



5.8 Deferred annuities

- Consider an annuity underwritten to (x) at time 0, with lifelong anual payments of 1 in advance, commencing at age $x + u$ (u is a non-negative integer).
- Benefit cash flow:

$$\sum_{k=u}^{K_x} (1, k)$$

- Actuarial value:

$$\boxed{{}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|}} \tag{5.25}$$

- Relation via actuarial discounting:

$$ {}_{u|}\ddot{a}_x = {}_uE_x \; \ddot{a}_{x+u} \tag{5.26}$$

5.8 Deferred annuities

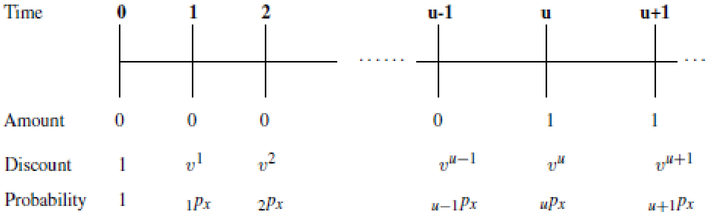


Figure 5.8 Time-line diagram for deferred annual annuity-due.

5.8 Deferred annuities

Some more relations

- Deferred term immediate annuity:

$$ {}_{u|}a_{x:\overline{n|}} = {}_{u}E_x \times a_{x+u:\overline{n|}} $$

- Deferred annuity-due payable $1/m$-thly:

$$ {}_{u|}\ddot{a}_x^{(m)} = {}_{u}E_x \times \ddot{a}_{x+u}^{(m)} \tag{5.27} $$

- Term annuity-due:

$$ \ddot{a}_{x:\overline{n|}} = \ddot{a}_x - {}_{n}E_x \times \ddot{a}_{x+n} \tag{5.28} $$

- Term annuity-due payable $1/m$-thly:

$$ \ddot{a}_{x:\overline{n|}}^{(m)} = \ddot{a}_x^{(m)} - {}_{n}E_x \times \ddot{a}_{x+n}^{(m)} \tag{5.29} $$

- Term-annuity with continuous payments:

$$ \overline{a}_{x:\overline{n|}} = \sum_{u=0}^{n-1} {}_{u|}\overline{a}_{x:\overline{1|}} \tag{5.31} $$

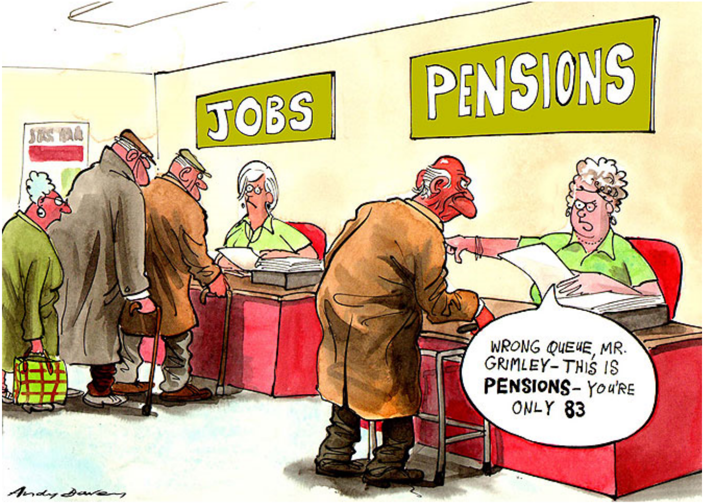
5.8 Deferred annuities

Example 5.3

- Consider the following notations:
  - Y_1 = PV r.v. of a u-year deferred whole life annuity-due,
  - Y_2 = PV r.v. of a u-year term annuity-due,
  - Y_3 = PV r.v. of a whole life annuity-due.
- Show that
$$Y_3 = Y_1 + Y_2$$
- Assume annual payments.

5.8 Deferred annuities

Increasing retirement age



5.8 Deferred annuities

Increasing retirement age

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: 3%.
- Pension reform: pension age is increased from 65 to 67.
- AV at age 65 of pension of 1 per year:
  - when pension starts at age 65:

$$\ddot{a}_{65} = 16.440$$

  - when pension starts at age 67:

$$_{2|}\ddot{a}_{65} = 14.474$$

- Relative decrease of pension liability for (65):

$$_{2|}\ddot{a}_{65} = 88\% \times \ddot{a}_{65}$$

  - In addition, (65) has to pay social security contributions between ages 65 and 67.

5.9 Guaranteed annuities

- Consider an annuity-due of 1 per year annually to (x), which is guaranteed for a period of n years.
- Benefit cash flow:

$$\sum_{k=0}^{n-1}(1,k)+\sum_{k=n}^{K_x}(1,k)$$

- Present value:

$$Y=\ddot{a}_{\overline{n}|}+\sum_{k=n}^{K_x}v^k$$

- Actuarial value:

$$\boxed{\ddot{a}_{\overline{\overline{x:\overline{n}|}}}\stackrel{\text{not.}}{=}\mathbb{E}\left[Y\right]=\ddot{a}_{\overline{n}|}+{}_nE_x\times\ddot{a}_{x+n}} \qquad (5.32)$$

- Guaranteed annuity with monthly payments:

$$\ddot{a}^{(12)}_{\overline{\overline{x:\overline{n}|}}}\stackrel{\text{not.}}{=}\ddot{a}^{(12)}_{\overline{n}|}+{}_nE_x\times\ddot{a}^{(12)}_{x+n}$$

5.9 Guaranteed annuities

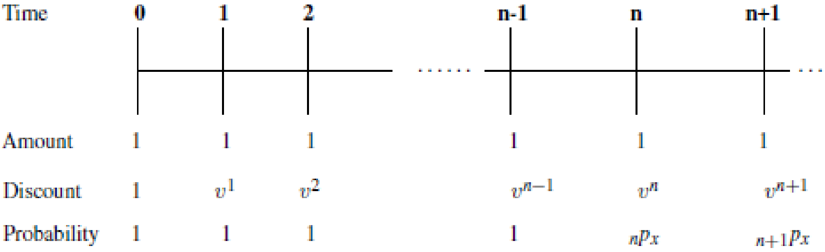


Figure 5.9 Time-line diagram for guaranteed annual annuity-due.

5.9 Guaranteed annuities

Example 5.4

- A pension plan member is entitled to a pension with EPV given by
$$12\ 000 \times \ddot{a}_{65}^{(12)}$$
- Alternatively, he can opt for a guaranteed annuity with EPV given by
$$12\ B \times \ddot{a}_{\overline{65:\overline{10|}}}^{(12)}$$
- Determine B, such that both EPV's are equal.
- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: 5%.

5.10 Increasing annuities

5.10.1 Arithmetically increasing annuities

- Consider an increasing annuity-due with a payment of $t+1$ at times $t = 0, 1, 2, \dots$ provided (x) is alive at time t.
- Benefit cash flow:

$$\sum_{t=0}^{\infty} \left((t+1) \times 1_{\{T_x > t\}},\ t \right)$$

- Actuarial value:

$$\boxed{(I\ddot{a})_x \overset{\text{not.}}{=} \sum_{t=0}^{\infty} v^t\ (t+1)\ {}_tp_x} \qquad (5.33)$$

5.10 Increasing annuities

5.10.1 Arithmetically increasing annuities

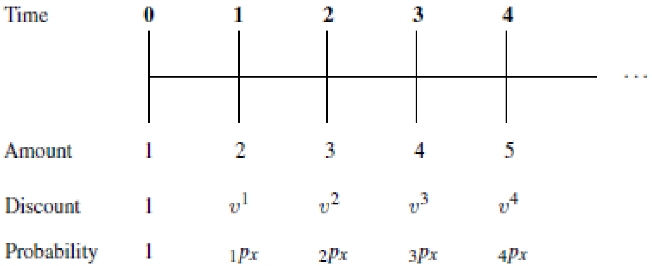


Figure 5.10 Time-line diagram for arithmetically increasing annual annuity-due.

5.10 Increasing annuities

5.10.1 Arithmetically increasing annuities

- Consider an increasing annuity-due with a payment of $t+1$ at times $t = 0, 1, 2, \ldots, n-1$, provided (x) is alive at time t.
- Benefit cash flow:

$$\sum_{t=0}^{n-1} \left((t+1) \times 1_{\{T_x > t\}},\ t \right)$$

- Actuarial value:

$$\boxed{(I\ddot{a})_{x:\overline{n}|} \overset{\text{not.}}{=} \sum_{t=0}^{n-1} v^t \ (t+1)\ {}_tp_x} \tag{5.34}$$

5.10 Increasing annuities

5.10.1 Arithmetically increasing annuities

- Consider a continuous annuity, with a total payment equal to t in the t-th year, $t = 1, 2, \ldots, n$, equally spread over the year, provided (x) is alive.
- Benefit cash flow:

$$\int_0^n \left(\lfloor t+1 \rfloor \times 1_{\{T_x > t\}} \, dt, \; t \right)$$

- Actuarial value:

$$\boxed{(I\bar{a})_{x:\overline{n}|} \overset{\text{not.}}{=} \sum_{m=0}^{n-1} (m+1) \; {}_{m|}\bar{a}_{x:\overline{1}|}}$$

5.10 Increasing annuities

5.10.1 Arithmetically increasing annuities

- Consider a continuous annuity, with a payment of $t\ dt$ in the interval $(t, t + dt)$, $0 < t < n$, provided (x) is alive.
- Benefit cash flow:

$$\int_0^n \left(t \times 1_{\{T_x > t\}}\ dt,\ t\right)$$

- Actuarial value:

$$\boxed{\left(\bar{I}\bar{a}\right)_{x:\overline{n}|} \overset{\text{not.}}{=} \int_0^n t\ e^{-\delta t}\ {}_tp_x\ dt} \qquad (5.35)$$

5.10 Increasing annuities

5.10.1 Arithmetically increasing annuities

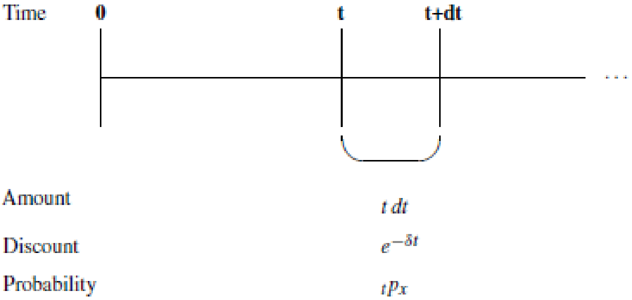


Figure 5.11 Time-line diagram for increasing continuous whole life annuity.

5.10 Increasing annuities

5.10.2 Geometrically increasing annuities

Example 5.5

- Consider an annuity-due with annual payments where the amount of the annuity is $(1+j)^t$ at times $t = 0, 1, 2, \dots, n-1$, provided (x) is alive at that time.
- Benefit cash flow:

$$\sum_{t=0}^{n-1} \left((1+j)^t \times 1_{\{T_x > t\}},\ t \right)$$

- Show that the EPV of this cash flow is given by:

$$\boxed{\sum_{t=0}^{n-1} (1+j)^t \times v^t \times {}_t p_x = \ddot{a}_{x:\overline{n|}\ i^*}}$$

with

$$i^* = \frac{i-j}{1+j}$$

5.10 Increasing annuities

5.10.2 Geometrically increasing annuities

Example 5.5

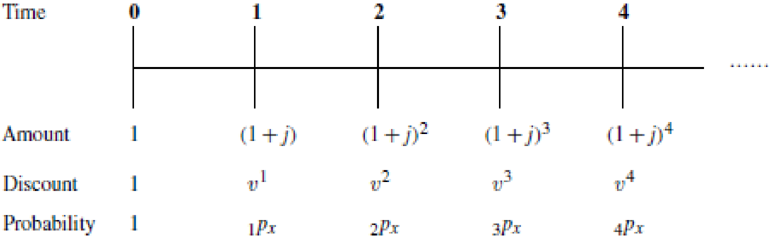


Figure 5.12 Time-line diagram for geometrically increasing annual annuity-due.

5.11 Evaluating annuity functions

5.11.1 Recursions

- Let ω be the first integer age such that $l_\omega = 0$. Then,

$$q_{\omega-1} = 1$$

- **Yearly annuity-due**:
  - Initial value: $\ddot{a}_{\omega-1} = 1$.
  - Backward recursion: for $x = \omega - 2,\ \omega - 3, \ldots,$

$$\boxed{\ddot{a}_x = 1 + v\ p_x\ \ddot{a}_{x+1}} \tag{5.36}$$

- $1/m$-**thly annuity-due**:
  - Initial value: $\ddot{a}^{(m)}_{\omega-1/m} = \frac{1}{m}$.
  - Backward recursion: for $x = \omega - \frac{2}{m},\ \omega - \frac{3}{m}, \ldots,$

$$\boxed{\ddot{a}^{(m)}_x = \frac{1}{m} + v^{1/m}\ {}_{1/m}p_x\ \ddot{a}^{(m)}_{x+1/m}} \tag{5.37}$$

5.11 Evaluating annuity functions

5.11.2 Applying the UDD assumption

- How to evaluate the EPV of $1/m$-thly and continuous annuities, given only the EPVs of yearly annuities?
- Recall:

$$A_x^{(m)} \overset{\text{UDD}}{=} \frac{i}{i^{(m)}} A_x \qquad \text{and} \qquad \overline{A}_x \overset{\text{UDD}}{=} \frac{i}{\delta} A_x$$

and

$$\ddot{a}_x = \frac{1 - A_x}{d}, \qquad \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}, \qquad \bar{a}_x = \frac{1 - \overline{A}_x}{\delta}$$

- Lifelong annuity-due with $1/m$-thly payments:

$$\boxed{\ddot{a}_x^{(m)} \overset{\text{UDD}}{=} \alpha(m)\ \ddot{a}_x - \beta(m)}$$

with

$$\alpha(m) = \frac{i\ d}{i^{(m)}\ d^{(m)}} \qquad \text{and} \qquad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}\ d^{(m)}} \quad (5.38)$$

5.11 Evaluating annuity functions

5.11.2 Applying the UDD assumption

- Limiting values:

$$\delta = \lim_{m \to \infty} i^{(m)} = \lim_{m \to \infty} d^{(m)}$$

- Lifelong annuity with continuous payments:

$$\boxed{\bar{a}_x \overset{\text{UDD}}{=} \frac{i\ d}{\delta^2}\ \ddot{a}_x - \frac{i-\delta}{\delta^2}}$$

- Term annuity-due with $1/m$-thly payments:

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} \overset{\text{UDD}}{=} \alpha\,(m)\ \ddot{a}_{x:\overline{n}|} - \beta\,(m)\,(1 - {}_nE_x)} \tag{5.39}$$

5.11 Evaluating annuity functions

5.11.2 Applying the UDD assumption

- Approximations for $\alpha(m)$ and $\beta(m)$:

$$\alpha(m) \approx 1 \qquad \text{and} \qquad \beta(m) \approx \frac{m-1}{2m}$$

- Approximation for $1/m$-thly annuity:

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n|}} \approx \ddot{a}_{x:\overline{n|}} - \tfrac{m-1}{2m}\left(1 - {}_{n}E_x\right)} \tag{5.40}$$

- Approximation for continuous annuity:

$$\boxed{\bar{a}^{(m)}_{x:\overline{n|}} \approx \ddot{a}_{x:\overline{n|}} - \tfrac{1}{2}\left(1 - {}_{n}E_x\right)}$$

5.11 Evaluating annuity functions

5.11.3 Woolhouse's formula (read in book)

5.12 Numerical illustrations

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: $i = 0.1$

Table 5.3. *Values of* $\ddot{a}^{(12)}_{x:\overline{10}|}$ *for* $i = 0.1$.

x	Exact	UDD	W2	W3	W3*
20	6.4655	6.4655	6.4704	6.4655	6.4655
30	6.4630	6.4630	6.4679	6.4630	6.4630
40	6.4550	6.4550	6.4599	6.4550	6.4550
50	6.4295	6.4294	6.4344	6.4295	6.4295
60	6.3485	6.3482	6.3535	6.3485	6.3485
70	6.0991	6.0982	6.1044	6.0990	6.0990
80	5.4003	5.3989	5.4073	5.4003	5.4003
90	3.8975	3.8997	3.9117	3.8975	3.8975
100	2.0497	2.0699	2.0842	2.0497	2.0496

- Exact: $\ddot{a}^{(12)}_{x:\overline{10}|}$
- UDD: $\ddot{a}^{(12)}_{x:\overline{10}|} \overset{\text{UDD}}{=} \alpha(12)\,\ddot{a}_{x:\overline{10}|} - \beta(12)\left(1 - {}_{10}E_x\right)$
- W2: $\ddot{a}^{(12)}_{x:\overline{10}|} \approx \ddot{a}_{x:\overline{10}|} - \frac{11}{24}\left(1 - {}_{10}E_x\right)$

5.12 Numerical illustrations

- Technical basis:
  - Mortality: Standard Ultimate Survival Model.
  - Interest: $i = 0.05$

Table 5.4. *Values of* $\ddot{a}^{(2)}_{x:\overline{25}|}$ *for* $i = 0.05$.

x	Exact	UDD	W2	W3	W3*
20	14.5770	14.5770	14.5792	14.5770	14.5770
30	14.5506	14.5505	14.5527	14.5506	14.5506
40	14.4663	14.4662	14.4684	14.4663	14.4663
50	14.2028	14.2024	14.2048	14.2028	14.2028
60	13.4275	13.4265	13.4295	13.4275	13.4275
70	11.5117	11.5104	11.5144	11.5117	11.5117
80	8.2889	8.2889	8.2938	8.2889	8.2889
90	4.9242	4.9281	4.9335	4.9242	4.9242
100	2.4425	2.4599	2.4656	2.4424	2.4424

- Exact: $\ddot{a}^{(2)}_{x:\overline{25}|}$
- UDD: $\ddot{a}^{(2)}_{x:\overline{25}|} \overset{\text{UDD}}{=} \alpha(2)\,\ddot{a}_{x:\overline{25}|} - \beta(2)\left(1 - {}_{25}E_x\right)$
- W2: $\ddot{a}^{(2)}_{x:\overline{25}|} \approx \ddot{a}_{x:\overline{25}|} - \frac{1}{4}\left(1 - {}_{25}E_x\right)$

5.13 Functions for select lives

- Throughout this chapter we assumed an ultimate survival model.
- Results can easily be adapted to the case of a select survival model.
- **Continuous life annuities and endowment assurances**:
  - Life annuity:

$$\bar{a}_{[x]+k:\overline{n}|} = \int_0^n e^{-\delta t} \; {}_t p_{[x]+k} \; dt$$

  - Endowment insurance:

$$\bar{A}_{[x]+k:\overline{n}|} = \int_0^n e^{-\delta t} \, {}_t p_{[x]+k} \; \mu_{[x]+k+t} \; + \; {}_n E_{[x]+k}$$

  - Relation:

$$\bar{a}_{[x]+k:\overline{n}|} = \frac{1 - \bar{A}_{[x]+k:\overline{n}|}}{\delta}$$

5.13 Functions for select lives

- **Yearly and $1/m-$thly annuities**:

$$\ddot{a}_{[x]+k} = \sum_{t=0}^{\infty} v^t \ {}_t p_{[x]+k} \quad \text{and} \quad \ddot{a}^{(m)}_{[x]+k} = \frac{1}{m} \sum_{t=0}^{\infty} v^{t/m} \ {}_{\frac{t}{m}} p_{[x]+k}$$

  - Approximations:

$$\ddot{a}^{(m)}_{[x]+k} \overset{\text{UDD}}{=} \alpha(m)\, \ddot{a}_{[x]+k} - \beta(m) \approx \ddot{a}_{[x]+k} - \frac{m-1}{2m}$$

- **Example 5.6:**
  - Technical basis:
    - Mortality: Standard Select Survival Model.
    - Interest: $i = 0.05$.
  - Assumption: $q_{131} = 1$
  - Question: Produce a table showing values of $\ddot{a}_{[x]}$, $\ddot{a}_{[x]+1}$ and $\ddot{a}_{x+2}$ for $x = 20, 21, \ldots, 80$.

5.14 Notes and further reading

(read in book)