Life Insurance Mathematics

Premium calculation[1]

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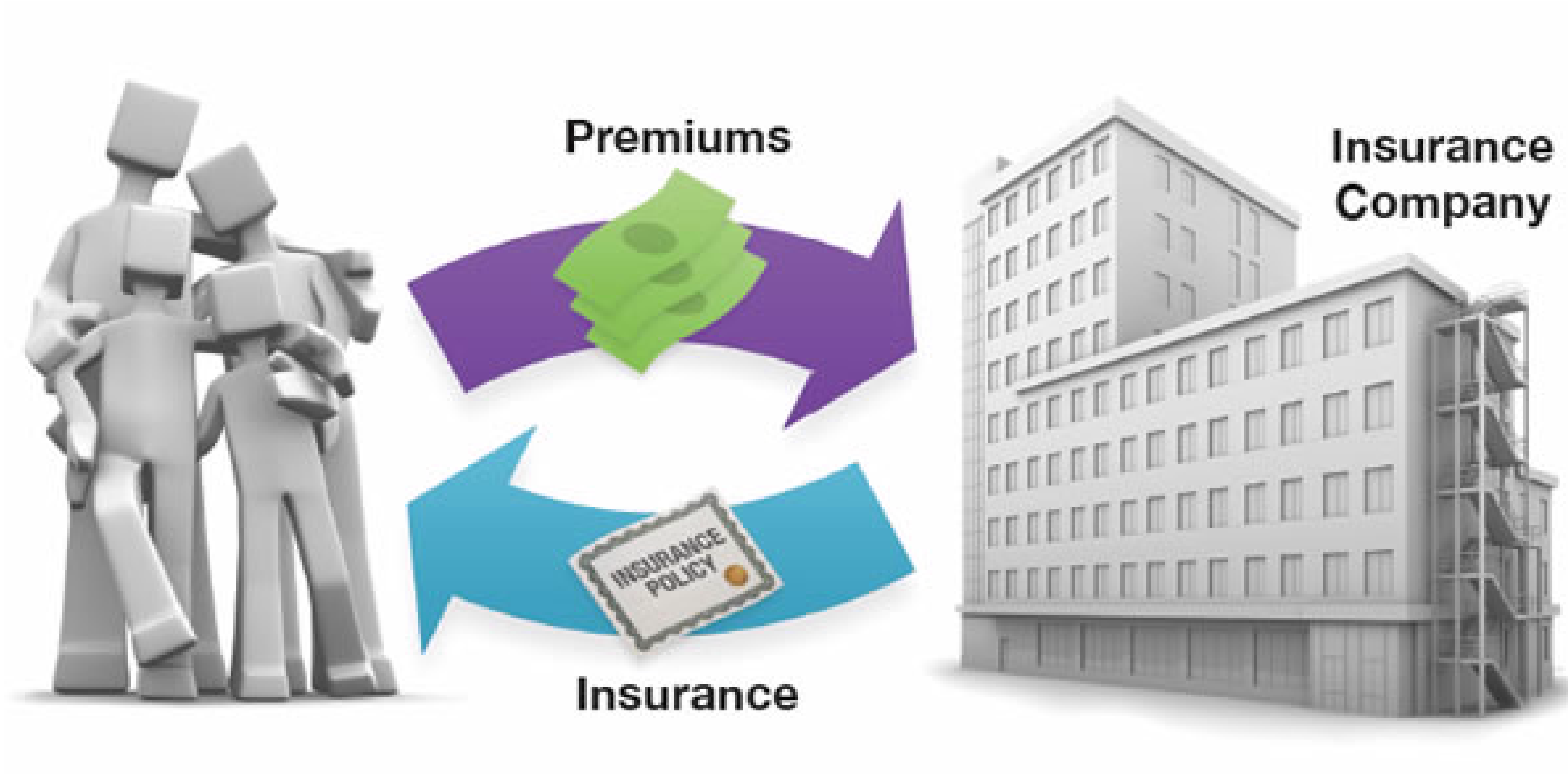
[1] Based on Chapter 6 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).

6.1 Summary

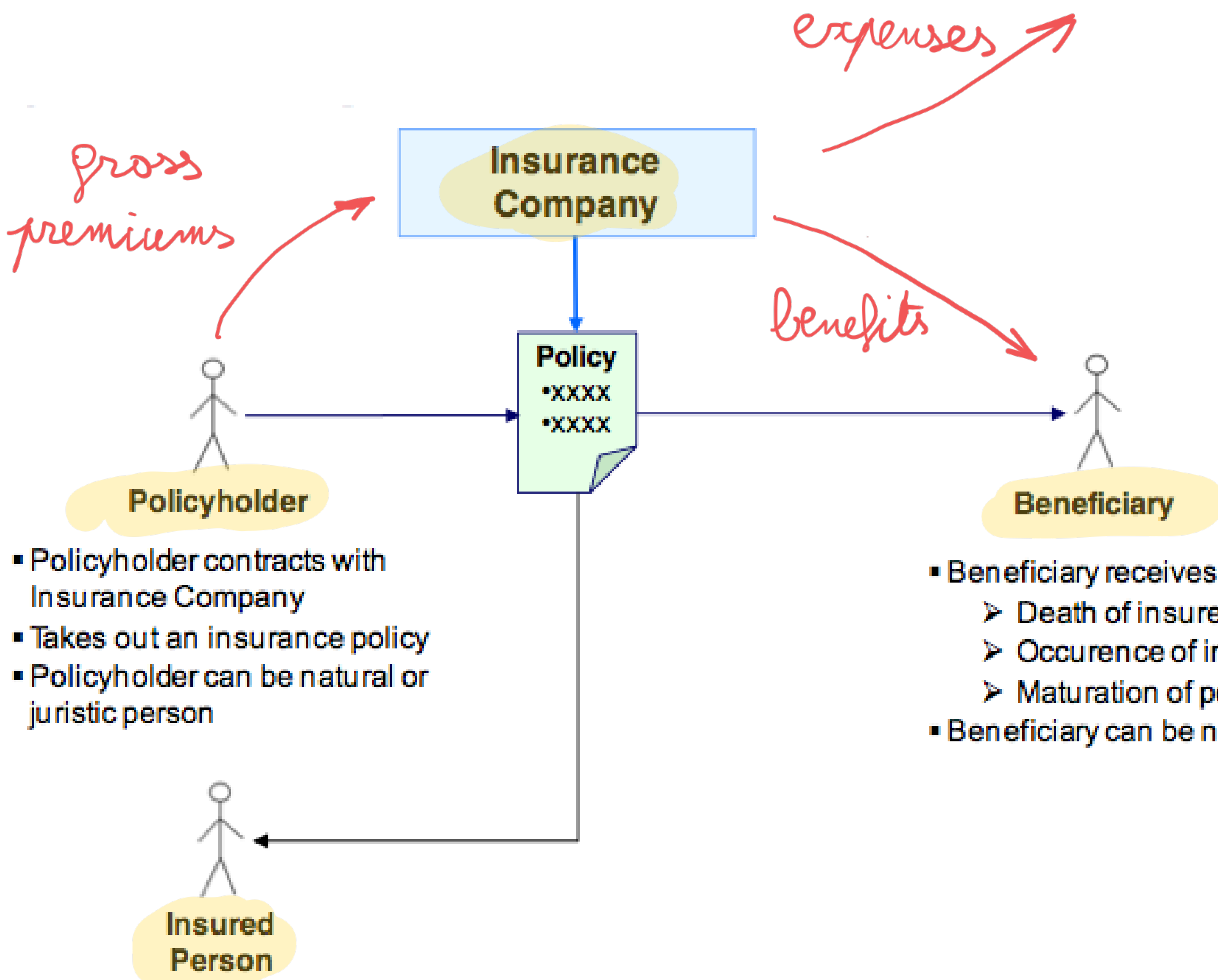
- Net and gross premiums.
- The loss at issue random variable.
- Premium calculation principles:
  - Equivalence premium principle.
  - Portfolio percentile premium principle.
- Profit.
- Premiums for non-standard (increased) risks.

6.2 Preliminaries

- A life insurance contract is a **complicated swap**:



6.2 Preliminaries



Policyholder

- Policyholder contracts with Insurance Company
- Takes out an insurance policy
- Policyholder can be natural or juristic person

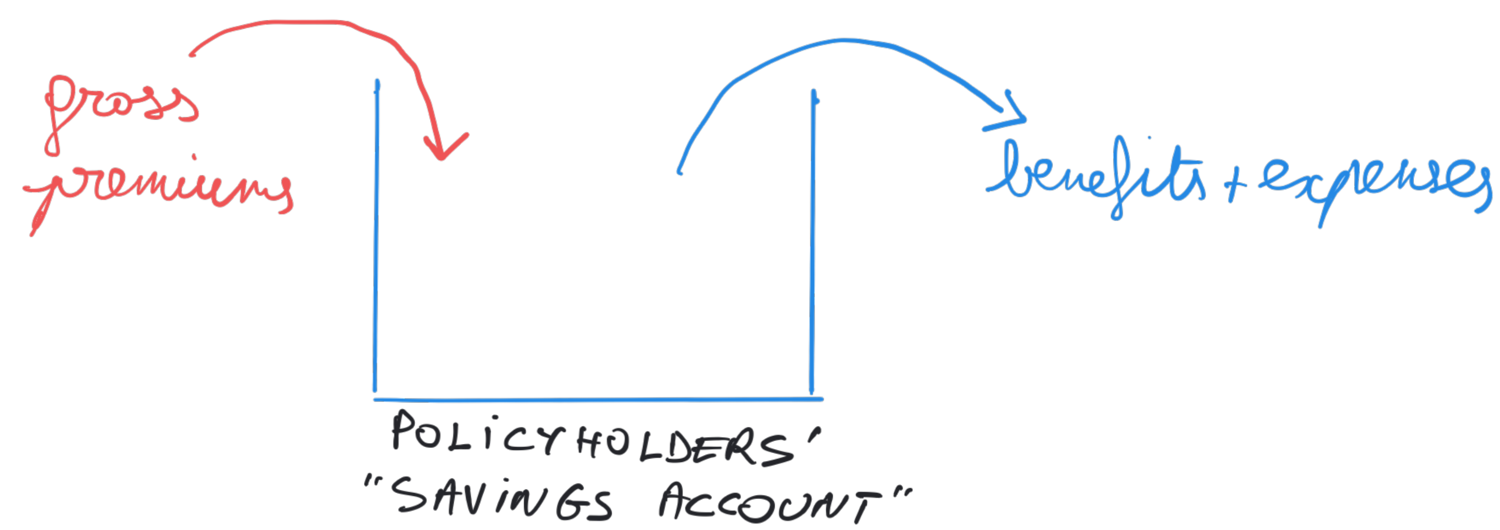
Beneficiary

- Beneficiary receives the payout on:
  - Death of insured person
  - Occurence of insured event
  - Maturation of policy (Deferred Annuity)
- Beneficiary can be natural or juristic person

Insured Person

- The life on which the policy is taken out on
- Must be a natural person
- Policyholder must (should) have an interest in the life of the insured person

6.2 Preliminaries



- Premiums:
  - *Net premiums*: cover benefits.
  - *Gross premiums*: cover benefits and expenses. = net premiums + loadings for expenses
  - *Single premium* vs. *regular premiums*.
- The premium paying term:
  - May start before ...
  - Should not start later than ...
  - May end before ...
  - Should not end later than ...

  ... the term in which benefits are payable.
- Premiums cease upon death of the insured.

6.2 Preliminaries

6.2.1 Assumptions

- Technical basis in this chapter (used in the examples):
  - *Standard Select Survival Model*:
    - The Standard Ultimate Survival Model:

$$\mu_x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$$

    - Two - year select period:

$$\mu_{[x]+s} = 0.9^{2-s} \mu_{x+s}, \qquad 0 \leq s \leq 2$$

  - *Interest:* 5%.
- Tables: See Appendix D on pages 735 - 739.
  - Life table functions: Table D.1.
  - Annuity and insurance functions: Tables D.2, D.3 and D.4.

6.3 The loss at issue r.v.

- A **life insurance contract** gives rise to the following cash flow streams:
  - benefits
  - expenses
  - net premiums
  - premium loadings for expenses
- $\text{PV}_0 \overset{\text{not.}}{=}$ Present Value at contract initiation ($=$ time 0).
- **Net loss at issue**: $L_0 = PV$ (if no confusion possible)

$$L_0^n \overset{\text{def.}}{=} \text{PV}_0 \, [\text{benefit outgo}] - \text{PV}_0 \, [\text{net premium income}]$$

- **Gross loss at issue**:

$$L_0^g \overset{\text{def.}}{=} \text{PV}_0 \, [\text{benefit outgo}] + \text{PV}_0 \, [\text{expenses}] - \text{PV}_0 \, [\text{gross premium income}]$$

- The r.v.'s L_0^n and L_0^g are functions of T_x.

6.3 The loss at issue r.v.

- **Convention**:
  - Let $\underline{C}$ denote a cash flow stream.
  - The Expected Present Value of $\underline{C}$ is denoted by C:

$$\boxed{C = \text{EPV}\left[\underline{C}\right]}$$

- **Examples**:
  - Whole life insurance:

$$\underline{A}_x \overset{\text{not.}}{=} (1,\ K_x + 1) \Rightarrow A_x = \text{EPV}\left[\underline{A}_x\right]$$

  - Annuity:

$$\underline{\ddot{a}}_x \overset{\text{not.}}{=} \sum_{k=0}^{K_x} (1, k) \Rightarrow \ddot{a}_x = \text{EPV}\left[\underline{\ddot{a}}_x\right]$$

  - We introduce similar notations for the cash flow streams of all life insurances and annuities introduced in previous chapters.

6.3 The loss at issue r.v.

Example 6.1 - whole life insurance

- Consider the net life insurance contract with

$$\boxed{\text{benefits} = S\,\bar{A}_{[60]}}$$

and

$$\boxed{\text{net premiums} = P\,\ddot{a}_{[60]:\overline{20}|}}$$

- Write down an expression for L_0^n in terms of $T_{[60]}$.

6.3 The loss at issue r.v.

Example 6.2 - solution

- **Benefits**:

$$\boxed{\text{benefits} = S\ \overline{A}_{[60]} = S \times \left(1,\ T_{[60]}\right)}$$

  - Present Value:

$$\text{PV}\left[\text{benefits}\right] = S\ v^{T_{[60]}}$$

- **Premiums**:

$$\boxed{\text{net premiums} = P\ \ddot{a}_{[60]:\overline{20}|} = P \times \sum_{k=0}^{\min\left(K_{[60]},19\right)} (1,\ k)}$$

  - Present Value:

$$\text{PV}\left[\text{net premiums}\right] = P \times \sum_{k=0}^{\min\left(K_{[60]},19\right)} v^k = P \times \ddot{a}_{\overline{min(K_{[60]}+1,20)}|}$$

6.3 The loss at issue r.v.

Example 6.2 - solution

- **Loss at issue**:

$$L_0^n = \text{PV}\left[\text{benefit outgo}\right] - \text{PV}\left[\text{net premium income}\right]$$

or

$$\boxed{L_0^n = S\ v^{T_{[60]}} - P\ \ddot{a}_{\overline{min(K_{[60]}+1,20)}|}}$$

6.4 The equivalence principle premium

6.4.1 Net premiums

- **Equivalence premium principle**:

$$\boxed{\mathbb{E}\left[L_0^n\right] = 0}$$

  - Equivalently,

$$\boxed{\text{EPV}[\text{benefit outgo}] = \text{EPV}[\text{net premium income}]} \quad ((6.1))$$

- The equivalence principle is the common premium principle in traditional life insurance.
- In this course, it is the *default* premium principle.

6.4 The equivalence principle premium

A whole life insurance contract

- Consider the net life insurance contract with

$$\boxed{\text{benefits} = S\ \bar{A}_{x}}$$

and

$$\boxed{\text{net premiums} = P\ \ddot{a}_{x}}$$

- Determine the annual net premium P.

6.4 The equivalence principle premium

A whole life insurance contract (cont'd)

- Loss:

$$L_0^n = \text{PV}\left[S\ \overline{A}_x\right] - \text{PV}\left[P\ \ddot{a}_x\right]$$

- Equivalence premium principle:

$$\mathbb{E}\left[L_0^n\right] = 0 \Rightarrow P = S\ \frac{\overline{A}_x}{\ddot{a}_x}$$

$S \cdot \overline{A}_x - P \cdot \ddot{a}_x = 0$

- P is an increasing function of x:
  - $\overline{A}_x$ increases in x
  - $\ddot{a}_x$ decreases in x

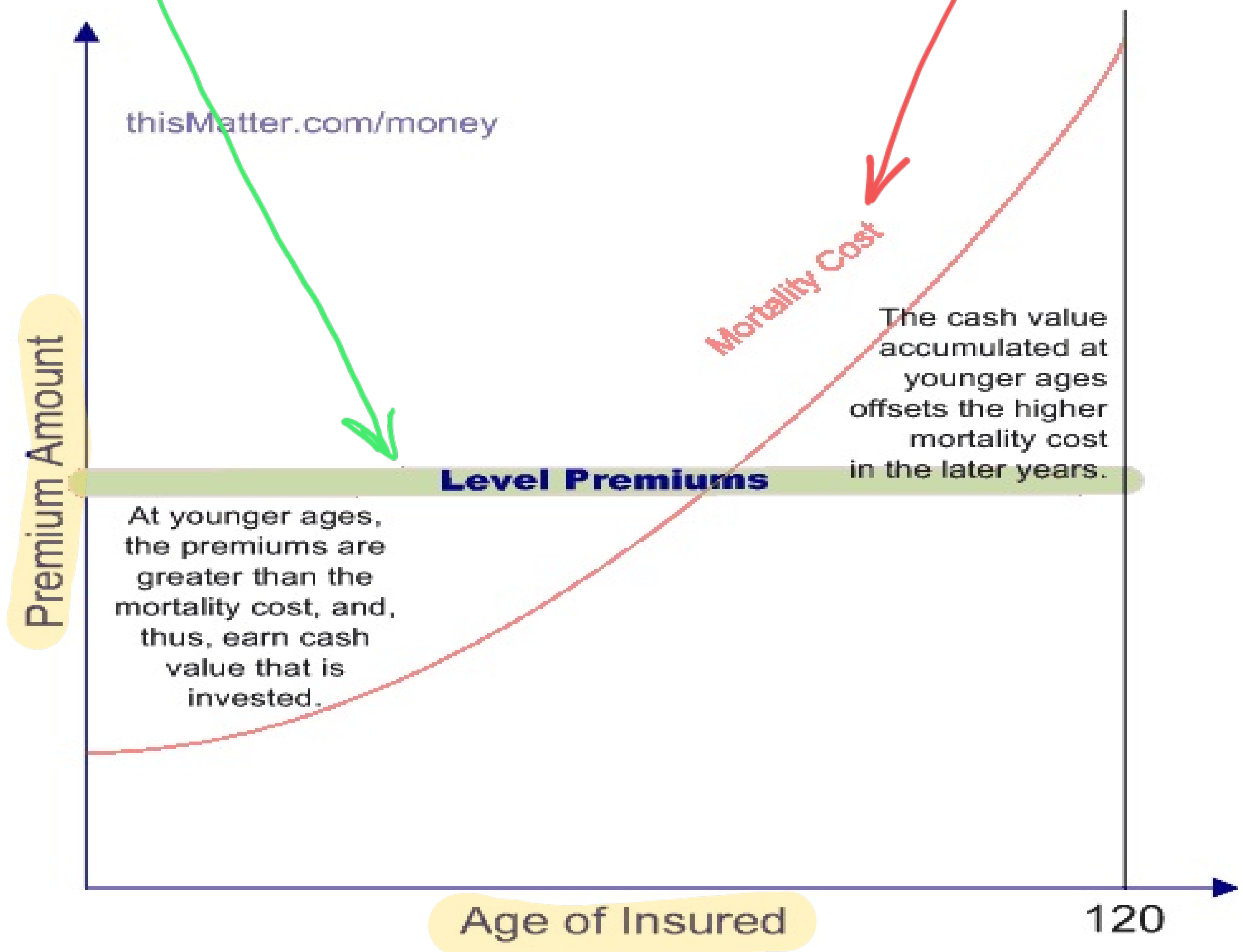
"For someone your age, the yearly premium on a $5,000 policy is $8,000."

6.4 The equivalence principle premium

A whole life insurance contract (cont'd)

- Level premium vs. cost of one-year cover:

$$P = S\,\frac{\overline{A}_x}{\ddot{a}_x} \quad \text{vs.} \quad \text{cost of cover in year } (k, k+1] = S \times \overline{A}^{\,1}_{x+k:\overline{1}|}$$



6.4 The equivalence principle premium

A whole life insurance contract (cont'd)

- Benefits: $50\ 000\ A_{25}$ Premiums: $P \times \ddot{a}_{25}$
- Technical basis:
  - Standard Select Survival Model.
  - Interest: 2.5%.
- Net yearly premium:

$$P = 50\ 000 \frac{A_{25}}{\ddot{a}_{25}} \approx 365$$

- Compare:

6.4 The equivalence principle premium

Example 6.2 - Endowment insurance

- Consider the net life insurance contract with

$$\boxed{\text{benefits} = S\, A_{[x]:\overline{n}|}}$$

and

$$\boxed{\text{net premiums} = P\, \ddot{a}_{[x]:\overline{n}|}}$$

- Determine expressions for:
  - the net loss at issue L_0^n,
  - the mean of L_0^n,
  - the annual net premium for the contract.

6.4 The equivalence principle premium

Example 6.2 - solution

- Net loss at issue:

$$L_0^n = \text{PV}\left[S\ \underline{A}_{[x]:\overline{n}|}\right] - \text{PV}\left[P\ \underline{\ddot{a}}_{[x]:\overline{n}|}\right]$$

$$= S \times v^{\min(K_{[x]}+1,n)} - P \times \ddot{a}_{\overline{\min(K_{[x]}+1;n)}|}$$

- Mean of L_0^n:

$$\mathbb{E}\left[L_0^n\right] = S\ A_{[x]:\overline{n}|} - P\ \ddot{a}_{[x]:\overline{n}|}$$

- Annual net premium:

$$\mathbb{E}\left[L_0^n\right] = 0 \Rightarrow \boxed{P = S\ \frac{A_{[x]:\overline{n}|}}{\ddot{a}_{[x]:\overline{n}|}}} \tag{6.2}$$

$$d \times \ddot{a}_{[x]:\overline{n}|} + A_{[x]:\overline{n}|} = 1$$

- Other expression for P:

$$P = S\ \left(\frac{1}{\ddot{a}_{[x]:\overline{n}|}} - d\right)$$

6.4 The equivalence principle premium

Example 6.3 - Deferred annuity

- Consider the net life insurance contract with

$$\text{benefits} = X \; {}_{n|}\ddot{a}^{(12)}_{[x]}$$

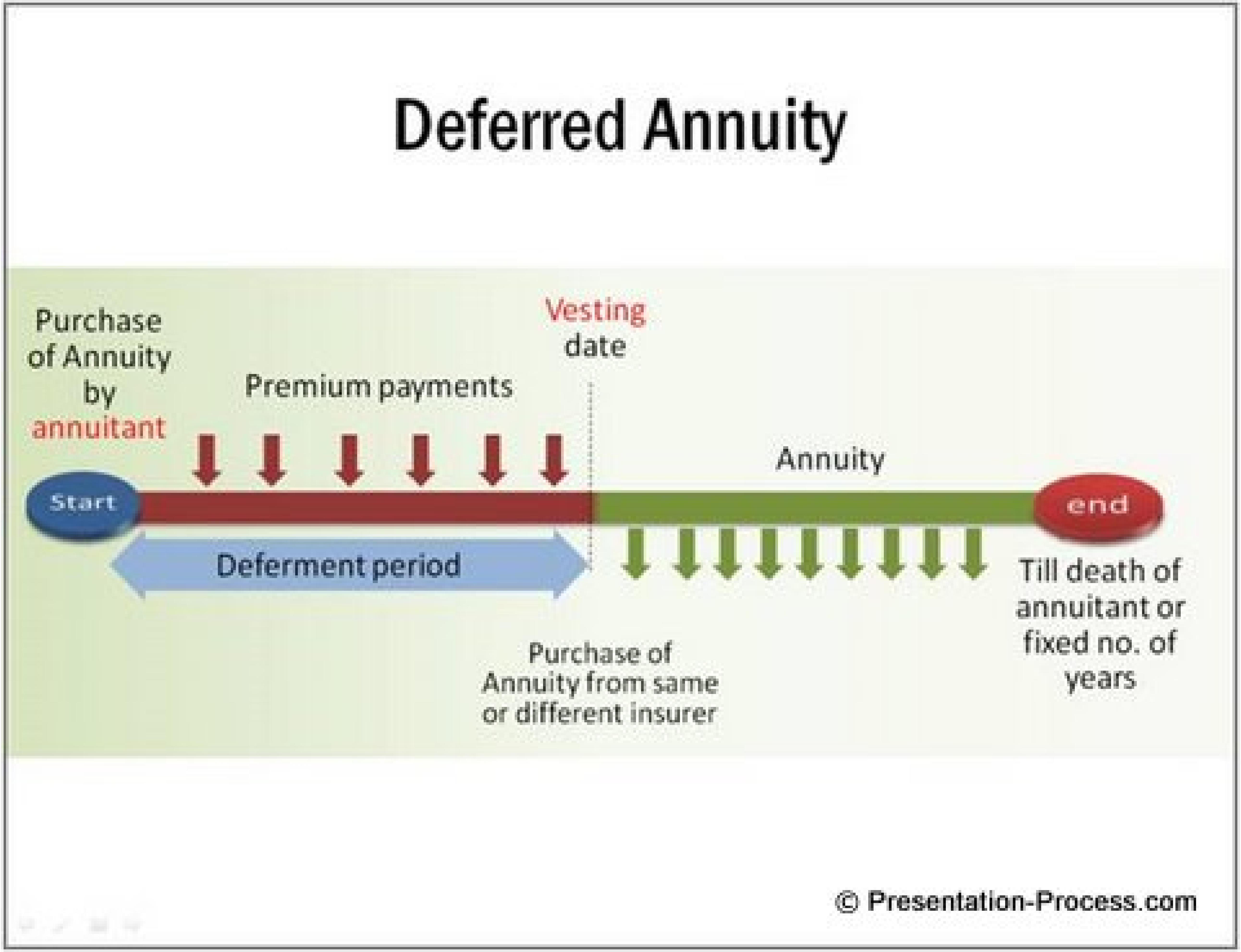
and

$$\text{net premiums} = 12 \; P \; \ddot{a}^{(12)}_{[x]:\overline{n}|}$$

- Questions:
  - Write down L_0^n in terms of lifetime r.v.'s for (x).
  - Derive an expression for P.

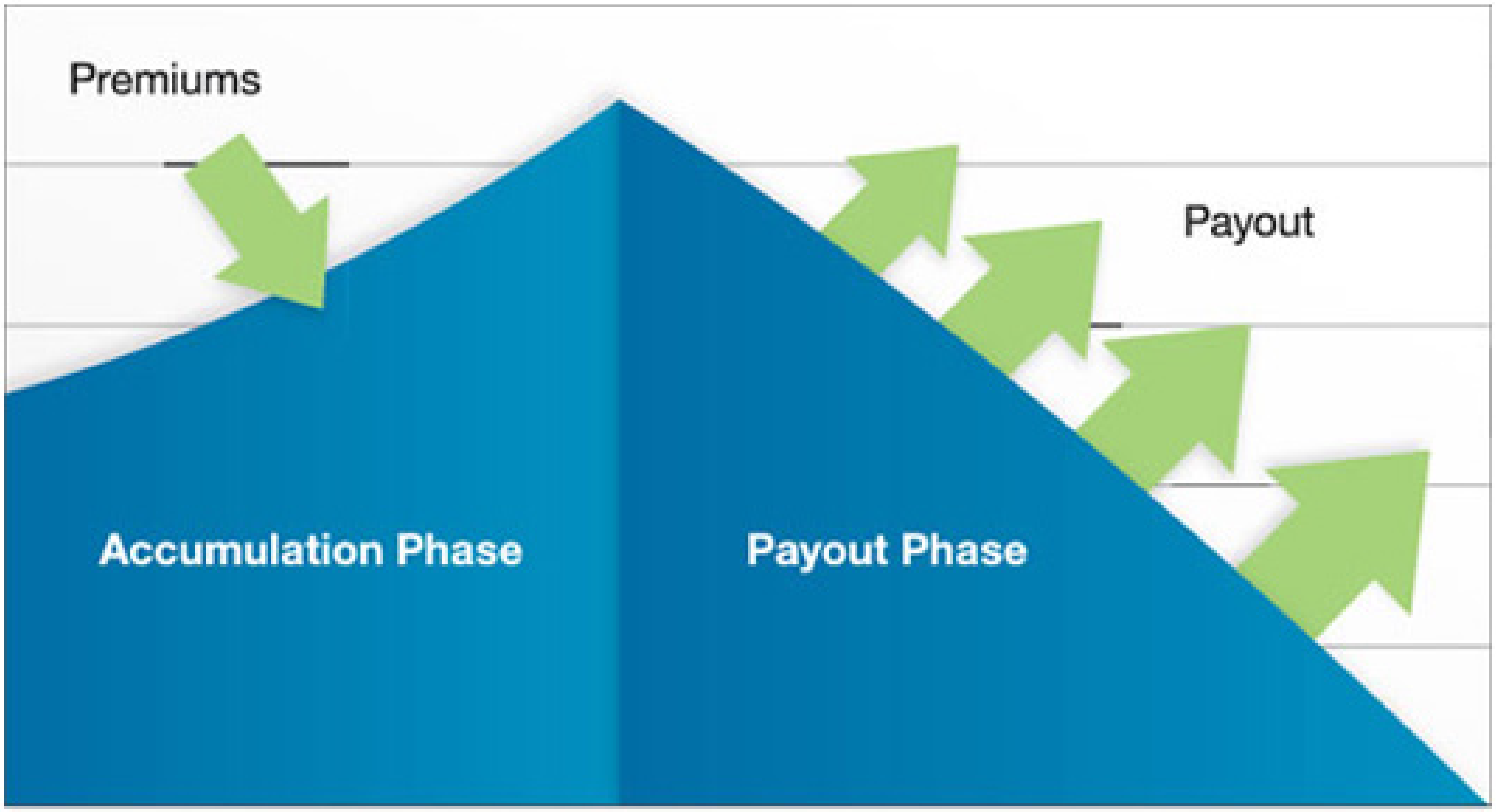
6.4 The equivalence principle premium

Example 6.3



6.4 The equivalence principle premium

Example 6.3 - Deferred annuity



6.4 The equivalence principle premium

Example 6.3 - solution

- Net future loss:

$$L_0^n = \mathrm{PV}\left[X \; {}_{n|}\ddot{a}_{[x]}^{(12)}\right] - \mathrm{PV}\left[12P \; \ddot{a}_{[x]:\overline{n}|}^{(12)}\right]$$

- Monthly premium:

$$E[L_0^n] = X \cdot {}_{n|}\ddot{a}_{[x]}^{(12)} - 12P \; \ddot{a}_{[x]:\overline{n}|}^{(12)} = 0$$

$$P = \frac{X}{12} \; \frac{{}_nE_{[x]} \; \ddot{a}_{[x]+n}^{(12)}}{\ddot{a}_{[x]:\overline{n}|}^{(12)}}$$

6.4 The equivalence principle premium

Example 6.3'

- Consider now the net life insurance contract with

$$\text{benefits} = S\,\bar{A}^{\,1}_{[x]:\overline{n}|} + X\;{}_{n|}\ddot{a}^{(12)}_{[x]}$$

and

$$\text{net premiums} = 12P\;\ddot{a}^{(12)}_{[x]:\overline{n}|}$$

- Questions:
  - Write down L_0^n in terms of lifetime r.v.'s for (x).
  - Derive an expression for P.

6.4 The equivalence principle premium

Example 6.3' - solution

- Net future loss:

$$L_0^n = \text{PV}\left[S\ \bar{A}^{\,1}_{[x]:\overline{n}|}\right] + \text{PV}\left[X\ {}_{n|}\underline{\ddot{a}}^{(12)}_{[x]}\right] - \text{PV}\left[12P\ \underline{\ddot{a}}^{(12)}_{[x]:\overline{n}|}\right]$$

- Monthly premium:

$E[L_0^n] = 0 \implies$

$$P = \frac{S\ \bar{A}^{\,1}_{[x]:\overline{n}|} + X\ {}_nE_{[x]}\ \ddot{a}^{(12)}_{[x]+n}}{12\ \ddot{a}^{(12)}_{[x]:\overline{n}|}}$$

6.4 The equivalence principle premium

Example 6.4

- Consider the net life insurance contract with

$$\text{benefits} = 100\ 000\ A_{[45]:\overline{20}|}$$

and

$$\text{net premiums} = P\ \ddot{a}^{(m)}_{[45]:\overline{20}|}$$

- Determine P in case m equals 1, 4 and 12, respectively.
- Technical basis:
  - Standard Select Survival Model.
  - Interest: 5%.

6.4 The equivalence principle premium

Example 6.4 - solution

- Net premiums:

$$\text{EPV}\left[\text{net premiums}\right] = P\ \ddot{a}^{(m)}_{[45]:\overline{20}|}$$

- Benefits:

$$\text{EPV}\left[\text{benefits}\right] = 100\ 000\ A_{[45]:\overline{20}|}$$

- Equivalence premium principle:

$E\left[L_0^n\right] = 0 \Rightarrow$

$$P = 100\ 000\ \frac{A_{[45]:\overline{20}|}}{\ddot{a}^{(m)}_{[45]:\overline{20}|}}$$

6.4 The equivalence principle premium

A pure endowment with premium refund

6.4 The equivalence principle premium

A pure endowment with premium refund

- Net premiums:

$$\text{EPV}\left[\text{net premiums}\right] = P\ \ddot{a}_{x:\overline{n}|}$$

- Benefits:

$$\text{EPV}\left[\text{benefits}\right] = S\ {}_nE_x + P\ \ (IA)^{1}_{x:\overline{n}|}$$

- Equivalence principle:

$$E[L_0^n] = 0 \Rightarrow \boxed{P = S\ \frac{{}_nE_x}{\ddot{a}_{x:\overline{n}|} - (IA)^{1}_{x:\overline{n}|}}}$$

$$= S\ \frac{{}_nE_x}{\ddot{a}_{x:\overline{n}|}} + S \times {}_nE_x \times \left(\frac{1}{\ddot{a}_{x:\overline{n}|} - (IA)^{1}_{x:\overline{n}|}} - \frac{1}{\ddot{a}_{x:\overline{n}|}}\right)$$

6.4 The equivalence principle premium

6.4.2 Gross premiums

(1) initial expenses
(= acquisition expenses)
(2) renewal expenses
(3) terminal expenses

- Gross life insurance contract:
  - Outgoing cash flows: **benefits** + **expenses**
  - Incoming cash flows: **gross premiums**
- **Gross loss at issue**:

$$L_0^g \overset{\text{def.}}{=} \text{PV}_0\,[\text{benefits}] + \text{PV}_0\,[\text{expenses}] - \text{PV}_0\,[\text{gross premiums}]$$

- **Equivalence premium principle**:

$$\mathbb{E}\left[L_0^g\right] = 0$$

  - Equivalently,

$$\text{EPV}_0\,[\text{gross premiums}] = \text{EPV}_0\,[\text{benefits}] + \text{EPV}_0\,[\text{expenses}] \tag{6.3}$$

6.4 The equivalence premium principle

Example 6.5 - Endowment insurance

- Consider the life insurance contract with

benefits $= 100\,000\ \bar{A}_{[30]:\overline{20}|}$, gross premiums $= P\ \ddot{a}_{[30]:\overline{25}|}$

- Questions:
  - Write down the gross loss at issue r.v.
  - Calculate the gross premium.
- Technical basis:
  - Standard Select Survival Model.
  - Assume UDD between integer ages.
  - Interest: 5%.
  - Expenses: ↳ initial expenses ↳ renewal expenses

$$(2\,000 + 0.475\ P, 0) + 0.025P\ddot{a}_{[30]:\overline{20}|}$$

6.4 The equivalence premium principle

Example 6.5 - solution

- Gross Future Loss:

$$L_0^g = 100000\ \mathrm{PV}\left[\bar{A}_{[30]:\overline{20}|}\right] + 2000 + 0.475P - 0.975\ \mathrm{PV}\left[P\ \ddot{a}_{[30]:\overline{20}|}\right]$$

$= +0.025 \times PV[P \times \ddot{a}_{[30]:\overline{20}|}] - PV[P \times \ddot{a}_{[30]:\overline{20}|}]$

- Equivalence principle: $E[L_0^g] = 0$

$$\Leftrightarrow \quad 100\ 000\ \bar{A}_{[30]:\overline{20}|} + 2\ 000 + 0.475\ P - 0.975\ P\ \ddot{a}_{[30]:\overline{20}|} = 0$$

- Gross premium:

$\bar{A}_{[30]:\overline{20}|} \overset{UDD}{=} \frac{i}{\delta} A^{1}_{[30]:\overline{20}|} + {}_{20}E_{30}$ (4.28)

$$P = \frac{100\ 000\ \bar{A}_{[30]:\overline{20}|} + 2\ 000}{0.975\ \ddot{a}_{[30]:\overline{20}|} - 0.475} = 3\ 260.60$$

6.4 The equivalence premium principle

Remark on Example 6.5 - New business strain

- Premium at policy issue:

$$P = 3\ 260.60$$

- Expenses at policy issue:

$$2\ 000 + 0.5\ P = 3\ 630.30 > P$$

- **New business strain**:
  - What? First year premium is insufficient to cover first year expenses. Regulation usually does not allow to book this shortfall as an asset.
  - Consequence:
    - The insurer needs funds (which he 'borrows' from shareholders) in order to be able to sell new policies.
    - This 'loan' is gradually paid off by the policyholder via expense loadings in his future premiums.
  - Triggers:
    - a high initial agent commission,
    - an aggressive portfolio growth strategy.

6.4 The equivalence premium principle

Example 6.6 - Term insurance

- Consider the life insurance contract with

benefits $= 50\,000\ \bar{A}^{\,1}_{[55]:\overline{10}|}$, gross premiums $= 12P\ \ddot{a}^{(12)}_{[55]:\overline{10}|}$

- Question:
  Calculate the monthly gross premium P.
- Technical basis:
  - Standard Select Survival Model, with UDD for fractional ages.
  - Interest: 5%.
  - Expenses:

$$(500, 0) + 0.10 \times 12P\ddot{a}^{(12)}_{[55]:\overline{1}|} + 0.01 \times 12P\ {}_{1|}\ddot{a}^{(12)}_{[55]:\overline{9}|}$$

6.4 The equivalence premium principle

Example 6.6 - solution

- Equivalence principle:

$$\text{EPV}\,[\text{benefits}] + \text{EPV}\,[\text{expenses}] = \text{EPV}\,[\text{gross premiums}]$$

with

$$\text{EPV}\,[\text{benefits}] = 50\,000\ \bar{A}^{\,1}_{[55]:\overline{10}|}$$

$$\text{EPV}\,[\text{expenses}] = 500 + 0.09 \times 12P \times \ddot{a}^{(12)}_{[55]:\overline{1}|} + 0.01 \times 12P \times \ddot{a}^{(12)}_{[55]:\overline{10}|}$$

$$\text{EPV}\,[\text{gross premiums}] = 12P\ \ddot{a}^{(12)}_{[55]:\overline{10}|}$$

UDD: $= \frac{i}{\delta} A^{\,1}_{[55]:\overline{10}|}$

UDD: $= \alpha(12) - \beta(12) \times (1 - {}_{1}E_{[55]})$

UDD: $= \alpha(12) \times \ddot{a}_{[55]:\overline{10}|} - \beta(12) \times (1 - {}_{12}E_{[55]})$

- Monthly premium:

$$\boxed{P = 18.99}$$

6.4 The equivalence premium principle

Example 6.7 - Variable life insurance

- Consider the life insurance contract with
  - Benefits:

$$\left(100\ 000 \times 1.025^{K_{[40]}},\ K_{[40]}+1\right)$$

  - Gross premiums:

$$P\ \ddot{a}_{[40]}$$

- Calculate the **annual gross premium** P.
- Technical basis:
  - Standard Select Survival Model.
  - $i = 5\%$.
  - Expenses:

$$(200, 0) + 0.05 \times P\left(\ddot{a}_{[40]} - 1\right)$$

6.4 The equivalence premium principle

Example 6.7 - solution

- Premiums:

$$\text{EPV}\left[\text{gross premiums}\right] = P\,\ddot{a}_{[40]} = 18.4596\ P$$

- Expenses:

$$\text{EPV}\left[\text{expenses}\right] = 200 + 0.05 \times P\left(\ddot{a}_{[40]} - 1\right) = 200 + 0.87298\ P$$

- Benefits: $\left(i^* = \frac{1.05}{1.025} - 1\right)$

$$\text{EPV}\left[\text{benefits}\right] = 100\ 000 \sum_{k=0}^{\infty} 1.025^k\ v^{k+1}\ {}_{k|}q_{[40]}$$

Example 4.8

$$\frac{1}{1+i^*} = \frac{1.025}{1.05}$$

$$= \frac{100\ 000}{1.025} A_{[40]i^*} = 32\ 816.71$$

- Annual premium:

$$\boxed{P = 1\ 877.38}$$

6.4 The equivalence premium principle

Example 6.8

- Consider the life insurance contract with

benefits $= 80\,000 \; {}_{15|}\ddot{a}^{(12)}_{[50]}$	gross premiums $= (P, 0)$

- <u>Question</u>:
  Calculate the gross single premium P.
- <u>Technical basis</u>:
  - Standard Select Survival Model, with UDD for fractional ages.
  - Interest: 5%.
  - Expenses:

$$(1\,000, 0) + \sum_{t=1}^{\infty} \left(20\ (1.01)^{t-1}\ 1_{\{T_{[50]}>t\}};\ t\right)$$

6.4 The equivalence premium principle

Example 6.8 - solution

- Benefits:

$$\text{EPV}\left[\text{benefits}\right] = 80\ 000\ {}_{15|}\ddot{a}^{(12)}_{[50]} = 483\ 303.2$$

- Expenses:

$$\begin{aligned}\text{EPV}\left[\text{expenses}\right] &= 1\ 000 + \frac{20}{1.01}\sum_{t=1}^{\infty}\left(\frac{1.01}{1.05}\right)^{t}\ {}_{t}p_{[50]} \\ &= 1\ 000 + \frac{20}{1.01}\left(\ddot{a}_{[50]j} - 1\right) \quad \text{with } j = 0.0396 \\ &= 1\ 365.4\end{aligned}$$

Example 5.5

$$\frac{1}{1+j} = \frac{1.01}{1.05}$$

- Single premium:

$$P = \text{EPV}\left[\text{benefits}\right] + \text{EPV}\left[\text{expenses}\right] = 484\ 669$$

6.5 Profit

(read in book)

6.6 The portfolio percentile premium principle

(read in book)

6.7 Extra risks

(read in book)

"Is this gonna raise my premium?"

6.8 Notes and further reading

(Read in book)

Systematic interest rate risk in life insurance

- It is impossible to provide a credible estimate of today's value of one Euro in 50 years' time.
- Traditional life insurance products guarantee minimum interest rates for periods of 50 years of more.
- Given the fact that there are no liquid financial instruments that make the valuation of such promises and the hedging of such risks possible, instead of extrapolating interest rates for such long periods, it would be more reasonable going forward to review the design of life insurance products.
- Interest rate guarantees could be defined for a period of say 10 years and would be reset at the end of such a period, taking into account the interest rate environment at that time.
- If life insurers continue to sell long dated guarantees, they should properly price the basis risk they run. The product would then probably be so expensive, that nobody would buy it.[2]

[2] Schnieper, R. (2018). Defining principles of a robust insurance solvency regime. *European Actuarial Journal*, 8, 169-196.