Life Insurance Mathematics

Policy values[1]

Jan Dhaene

[1] Based on Chapter 7 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).

7.1 Summary

- **Policy values**
  - for policies with annual (and $1/m$-thly) cash flows:
    - Definition.
    - Recursive calculation.
  - for policies with continuous cash flows:
    - Definition.
    - Thiele's differential equation.
- **Analysis of surplus** of a portfolio over an accounting period.
- **Asset share** of a policy at a given time.
- **Policy alterations**.

Introduction

- **Premium basis**
  = technical basis used for determining the premiums.

- **Convention**:
  Unless explicitly stated otherwise, the *premium basis* in all examples of this chapter uses the following survival model and interest:

  - **Standard Select Survival Model** (SSSM):

    - *Ultimate part*:

      $$\mu_x = 0.00022 + (2.7 \times 10^{-6}) \times (1.124)^x$$

    - *Select part*:

      $$\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s} \quad \text{for} \quad 0 \leq s \leq 2$$

  - **Interest**: 5%.

7.2 Policies with annual cash flows

7.2.1 The future loss r.v.

- Consider a **life insurance contract** underwritten on (x) at time 0.
- $\text{PV}_0 \overset{\text{not.}}{=}$ Present Value at contract initiation ($=$ time 0).
- **Net loss at issue**:

$$L_0^n = \text{PV}_0\,[\text{future benefits}] - \text{PV}_0\,[\text{future net premiums}]$$

$\hookrightarrow L_0^n \equiv L_0^n(T_{[x]})$

- **Gross loss at issue**:

$$L_0^g = \text{PV}_0\,[\text{future benefits}] + \text{PV}_0\,[\text{future expenses}] - \text{PV}_0\,[\text{future gross premiums}]$$

$\hookrightarrow L_0^g \equiv L_0^g(T_{[x]})$

- L_0^n and L_0^g are functions of $T_{[x]}$.

7.2 Policies with annual cash flows

7.2.1 The future loss r.v.

- Suppose that we are at time t and that the life insurance contract underwritten on (x) is still in force.
- $PV_t \overset{\text{not.}}{=}$ Present Value at time t.
- **Net future loss at time t:**

$$L_t^n = PV_t\,[\text{future benefits}] - PV_t\,[\text{future net premiums}]$$

↳ $L_t^n \equiv L_t^n(T_{[x]+t})$

- **Gross future loss at time t:**

$$L_t^g = PV_t\,[\text{future benefits}] + PV_t\,[\text{future expenses}] - PV_t\,[\text{future gross premiums}]$$

↳ $L_t^g \equiv L_t^g(T_{[x]+t})$

- L_t^n and L_t^g are functions of $T_{[x]+t}$.

7.2 Policies with annual cash flows

7.2.1 The future loss r.v.

Example 7.1 - An endowment contract

- Consider the **contract** with
  - Benefits:

    $$500\,000\ A_{[50]:\overline{20|}}$$

  - Net premiums:

    $$P\ \ddot{a}_{[50]:\overline{20|}}$$

- Determine the net premium P.
- Calculate $\mathbb{E}[L_t^n]$ for $t = 10$ and 11, just before the premium due at time t is paid.
- Assumption: The basis used for all calculations is the premium basis.

$E[L_t^n] = E[L_t^n(T_{[x]+t})]$

${}_t q_{[50]+10} = P[T_{[50]} < 10+t \mid T_{[50]} > 10]$

$= {}_{10|}q_{[50]} / {}_{10}p_{[50]}$

Example 7.1 - solution

$L_0^n = PV_0[500\ 000\ A_{[50]:\overline{20}|}] - PV_0[P \times \ddot{a}_{[50]:\overline{20}|}]$

- Annual net premium P:

$$\mathbb{E}[L_0^n] = 500\ 000\ A_{[50]:\overline{20}|} - P\ \ddot{a}_{[50]:\overline{20}|} = 0 \quad (7.1)$$

equivalence principle

$$\Rightarrow P = 15\ 114.33$$

- Expectation of net future loss at time 10:

$$\mathbb{E}[L_{10}^n] = 500\ 000\ A_{60:\overline{10}|} - P\ \ddot{a}_{60:\overline{10}|} = 190\ 339 \quad (7.2)$$

- closer to benefit
- less premiums

- Expectation of net future loss at time 11:

$$\mathbb{E}[L_{11}^n] = 500\ 000\ A_{61:\overline{9}|} - P\ \ddot{a}_{61:\overline{9}|} = 214\ 757$$

$L_{10}^n = PV_{10}[500\ 000 \times A_{60:\overline{10}|}] - PV_{10}[P \times \ddot{a}_{60:\overline{10}|}]$

7.2 Policies with annual cash flows

7.2.1 The future loss r.v.

More on Example 7.1

- **Notation**: (insured is assumed to be alive at t)

$$\text{EPV}_t\,[\text{future cash flow}] = \mathbb{E}_{T_{[x]+t}}[\text{PV}_t\,[\text{future cash flow}]]$$

- **Any year $t = 1, 2, \ldots, 19$**:

$$\begin{aligned} \text{EPV}_{t-1}[\text{Premiums in } [t-1, t)] &= P = 15\ 114.33 \\ \text{EPV}_{t-1}[\text{Benefits in } (t-1, t]] &= 500\ 000 \times v \times q_{[50]+t-1} < P \end{aligned}$$

- **Last year**:

$$\begin{aligned} \text{EPV}_{19}[\text{Premiums in } [19, 20)] &= P \\ \text{EPV}_{19}[\text{Benefits in } (19, 20]] &= 500\ 000 \times v = 476\ 190 > P \end{aligned}$$

More on Example 7.1

- **Excess in year** $t = 1, 2, \ldots, 20$: (Definition)

$$\boxed{\text{EPV}_{t-1}\left[\text{Premiums in } [t-1, t) - \text{Benefits in } (t-1, t]\right]}$$

- Excess in year $t = 1, 2, \ldots, 19$:

$$P - 500\ 000 \times v \times q_{[50]+t-1} > 0$$

- Excess in year 20:

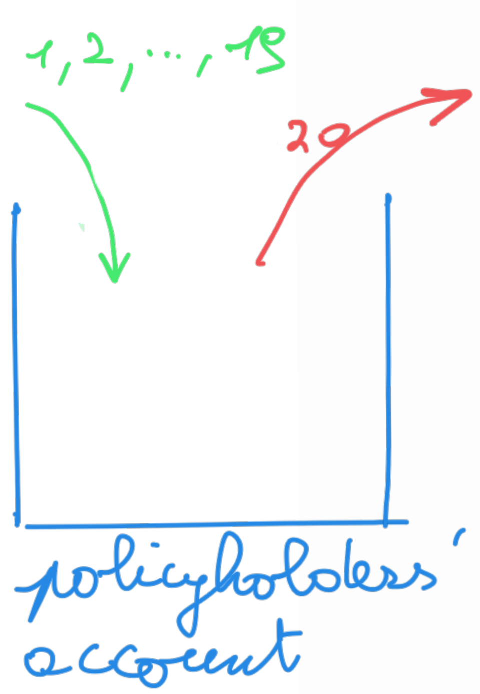
$$P - 500\ 000\ v < 0$$

7.2 Policies with annual cash flows

7.2.1 The future loss r.v.

More on Example 7.1

- The excesses in years $1, 2, \ldots, 19$ are used to build up insurer's assets, needed to fulfill his liabilities at time 20:



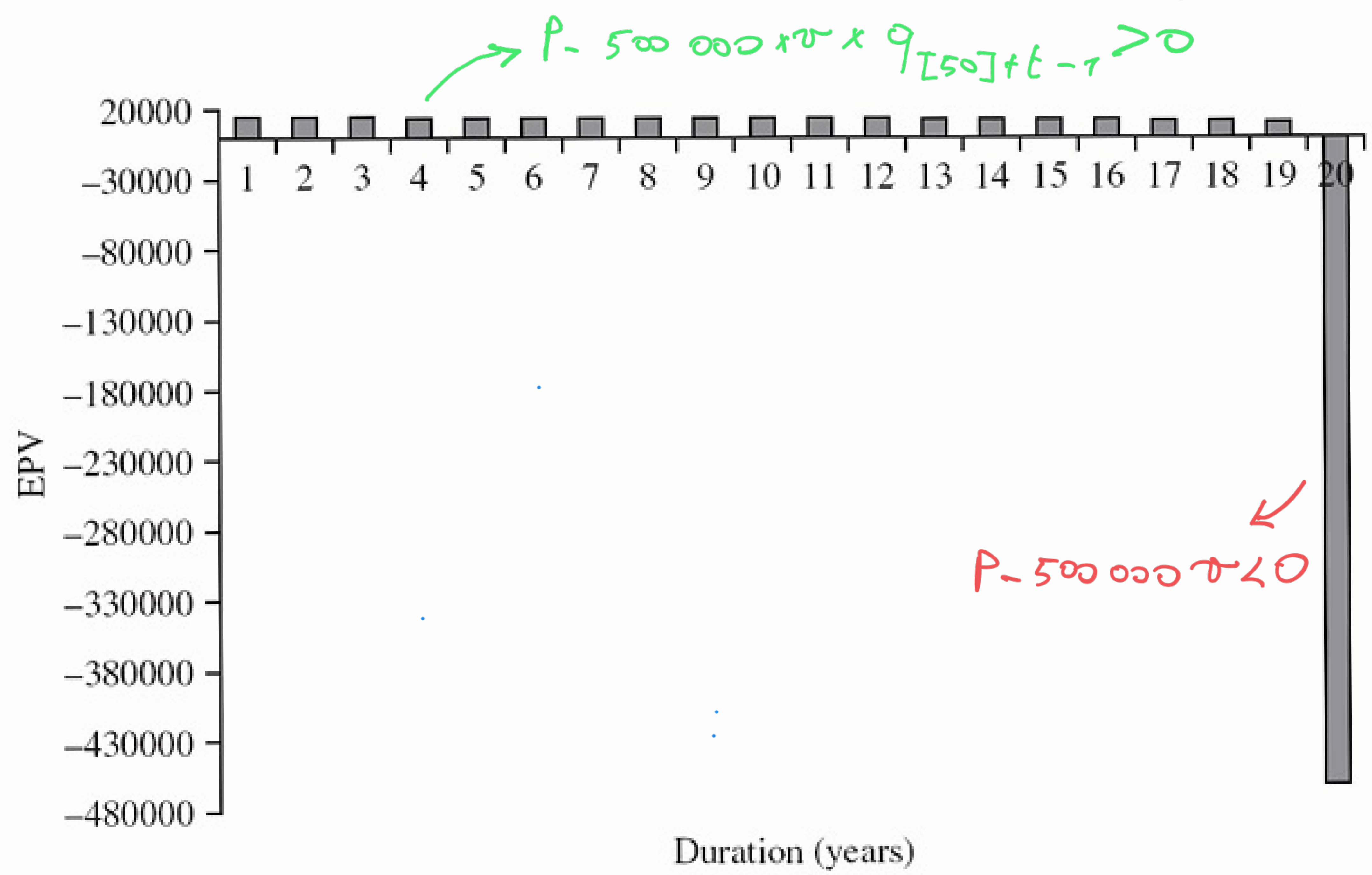


Figure 7.1 EPV of premiums minus claims for each year of a 20-year endowment insurance, sum insured $500 000, issued to (50).

A term insurance contract

- Contract:

  $$\text{Benefits} = 500\,000\; A^{\,1}_{[50]:\overline{20}|} \qquad \text{Premiums} = P\; \ddot{a}_{[50]:\overline{20}|}$$

- Compared with the endowment contract of Example 7.1, the term insurance contract:
  - has lower premiums,
  - has same benefits in first 19 years,
  - has lower yearly excesses in first 19 years.
  - has a higher (less negative) excess in 20th year.

A term insurance contract (cont'd)

- The positive excesses in the early years are used to build up the insurer's assets.
- These assets are needed in the later years when the yearly premium is not sufficient to cover the yearly benefit.

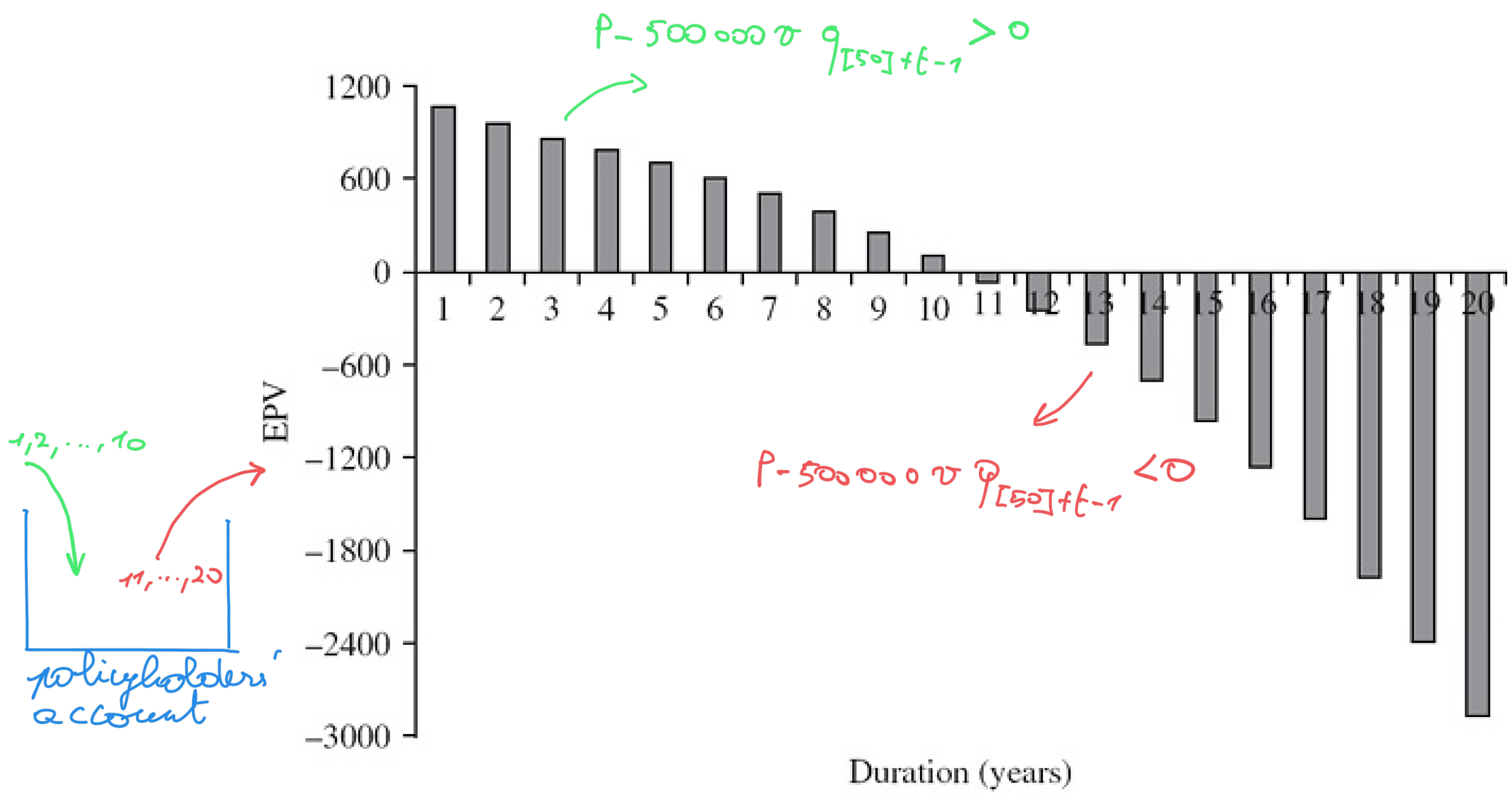


Figure 7.2 EPV of premiums minus claims for each year of a 20-year term insurance, sum insured $500 000, issued to (50).

Back to Example 7.1

- Suppose that at time 0 the insurer issues the following policy to each of N independent lives of age 50:

$$\boxed{\text{Benefits} = 500\,000\ \underline{A}_{[50]:\overline{20}|}} \qquad \boxed{\text{Premiums} = P\underline{\ddot{a}}_{[50]:\overline{20}|}}$$

- Suppose that we have arrived at at time 10.
- Suppose that the experience observed for these N policies over the first 10 years is precisely equal to the assumed premium basis at policy issue:
  - interest on investments = 5%,
  - observed mortality = SSSM.

7.2 Policies with annual cash flows

7.2.1 The future loss r.v. (back to Example 7.1)

- Accumulated value (at time 10) of past premium income:

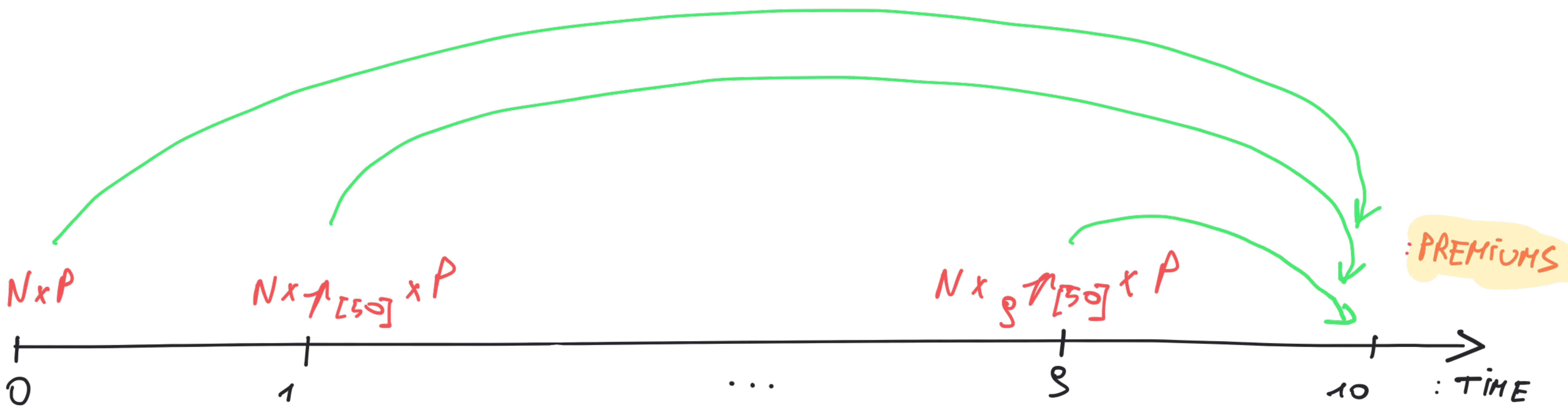
$$N \times P \times \left(1.05^{10} + p_{[50]} \times 1.05^{9} + \ldots + {}_{9}p_{[50]} \times 1.05\right)$$
$$= \quad 1.05^{10} \times N \times P \times \ddot{a}_{[50]:\overline{10}|}$$

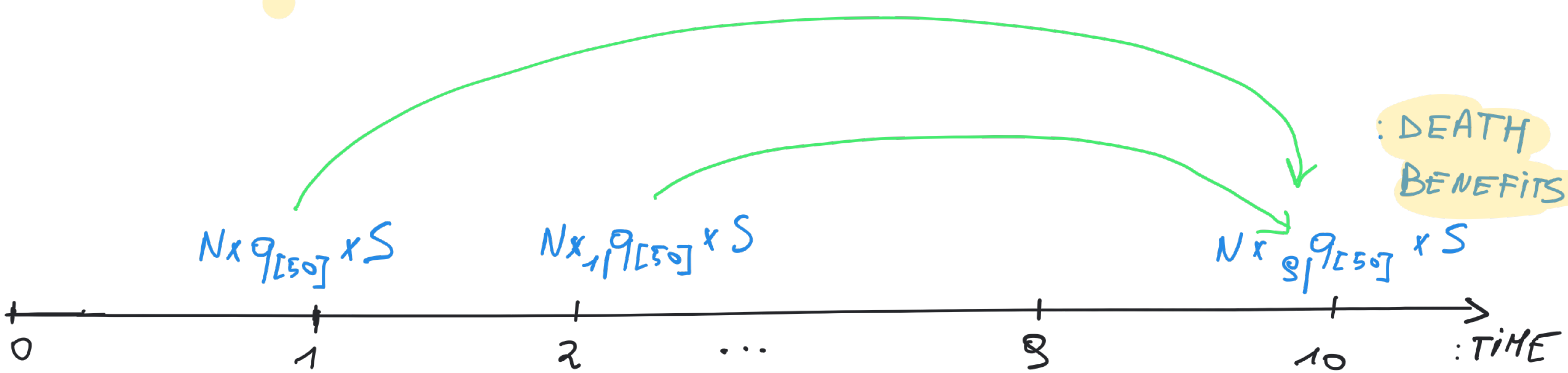
- Accumulated value (at time 10) of past benefit outgo:

$$N \times 500\ 000 \times \left(q_{[50]} \times 1.05^{9} + \ldots + {}_{9|}q_{[50]} \times 1.05^{0}\right)$$
$$= \quad 1.05^{10} \times N \times 500\ 000 \times A^{\ \ 1}_{[50]:\overline{10}|}$$

- Portfolio assets at time 10:

$$1.05^{10} \times N \times \left(P\ \ddot{a}_{[50]:\overline{10}|} - 500\ 000\ A^{\ \ 1}_{[50]:\overline{10}|}\right)$$
$$= \quad N \times 186\ 634$$


N x P
N x 1p[50] x P
N x 9p[50] x P
0
1
...
9
10
: TIME
: PREMIUMS


N x q[50] x S
N x 1|q[50] x S
N x 9|q[50] x S
0
1
2
...
9
10
: TIME
: DEATH BENEFITS

- **Asset share at time** 10:

definition

$$\text{AS}_{10} \overset{\downarrow}{=} \frac{\text{portfolio assets at time 10}}{\text{survivors at time 10}} = \frac{N \times 186\,634}{N \times {}_{10}p_{[50]}} = 190\,339$$

- The asset share at time at time 10 can be expressed as:

$$\text{AS}_{10} = \left(P\, \ddot{a}_{[50]:\overline{10}|} - 500\,000\, A^{\;\;1}_{[50]:\overline{10}|}\right) \times \left({}_{10}E_{[50]}\right)^{-1}$$

$$AS_{10} = \left(P\, \ddot{a}_{[50]:\overline{10}|} - 500\,000\, A^{\;\;1}_{[50]:\overline{10}|}\right) \times \frac{1.05^{10} \times N}{{}_{10}p_{[50]} \times N} =$$

7.2 Policies with annual cash flows

7.2.1 The future loss r.v.(back to Example 7.1)

- **Asset share at time** 10: (retrospective view)

$$\mathrm{AS}_{10} = \left(P\,\ddot{a}_{[50]:\overline{10}|} - 500\,000\,A^{\,1}_{[50]:\overline{10}|}\right) \times \left({}_{10}E_{[50]}\right)^{-1} = 190\,339$$

- **Expected future loss at time** 10: (prospective view)

$$\mathbb{E}[L^n_{10}] = 500\,000\,A_{60:\overline{10}|} - P\,\ddot{a}_{60:\overline{10}|} = 190\,339$$

- This **important equality** between 'available assets' ($\to AS_{10}$) and 'required assets' ($E[L^n_{10}]$) per policy in force at time 10 holds under following conditions:
  - premium determined according to the equivalence principle,
  - experience followed assumptions of premium basis,
  - $\mathbb{E}[L^n_{10}]$ calculated with premium basis.

- **Definition 7.2** The **gross premium policy value** for a policy still in force at time t is defined by

$$\boxed{{}_tV^g = \mathbb{E}\left[L_t^g\right]}$$

- **Equivalence relation**:

$$\boxed{{}_tV^g = \text{EPV}_t\left[\text{future (benefits + expenses - gross premiums)}\right]}$$

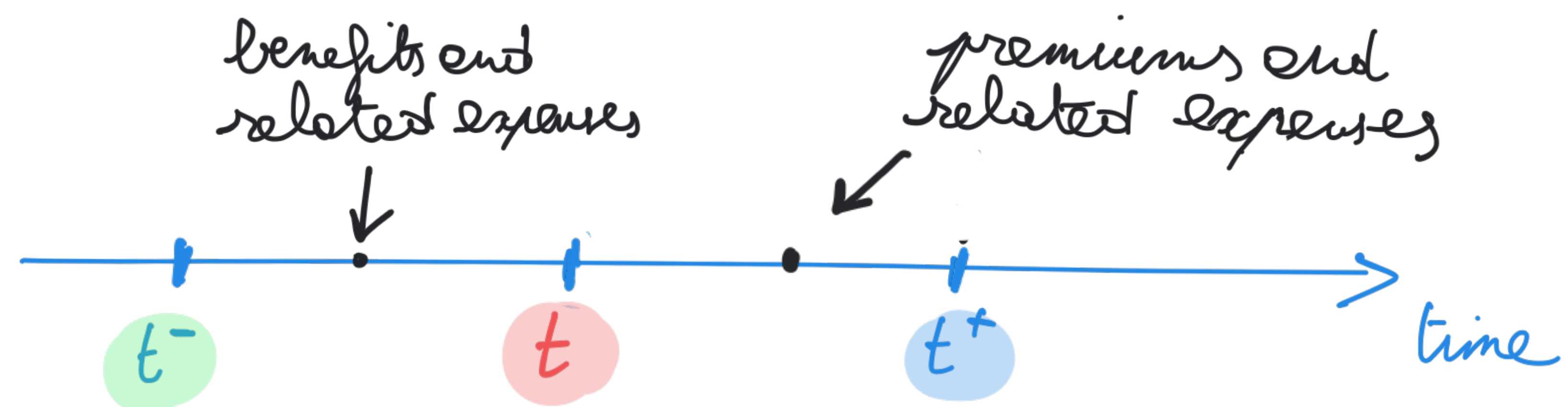
- **Interpretation**: ${}_tV^g$ is the actuarial value at time t of the insurer's future liability related to that policy, *given that the insured is still alive* at t.

  ${}_tV^g$ = prospective (mathematical) reserve

- **Calculation of ${}_tV^g$**:
  - Gross premiums, determined at policy issue with the premium basis, are given as input.
  - ${}_tV^g$ is then determined with the **policy value basis**.

7.2 Policies with annual cash flows

7.2.2 Policy values for policies with annual cash flows



Conventions:

- Time t, is the time *just after benefits* (and related expenses) and *just before premiums* (and related expenses) due at that time.
- Time t^- is the time *just before benefits* (and related expenses) and *just before premiums* (and related expenses) due at time t.
- Time t^+ is the time *just after benefits* (and related expenses) and *just after premiums* (and related expenses) due at time t.

7.2 Policies with annual cash flows

7.2.2 Policy values for policies with annual cash flows

- Suppose the policy is still in force at time t.
- The gross policy value ${}_tV^g$ is calculated:
  - **just after** the **survival benefits** (and related expenses) due at time t have been paid,
  - **just before** the **premiums** (and related expenses) due at t have been paid .
- Convention: If a policy has a finite term of n years, then

$$\boxed{{}_nV^g = {}_{n^-}V^g} = \lim_{t \to n^-} {}_tV^g$$

- $n-$year term insurance:

$${}_nV^g = 0$$

- $n-$year endowment insurance:

$${}_nV^g = \text{survival benefit at time } n$$

7.2 Policies with annual cash flows

7.2.2 Policy values for policies with annual cash flows

- **Definition 7.1** The **net premium policy value** for a policy still in force at time t is defined by

$$\boxed{{}_tV^n = \mathbb{E}\left[L_t^n\right]} = \text{ EPV}_t\left[\text{future (benefits - net premiums)}\right]$$

- **First method for calculating ${}_tV^n$:**
  - Net premiums, determined with the *premium basis*, are given as input.
  - ${}_tV^n$ is then determined with the **policy value basis**.
- **Second method for calculating ${}_tV^n$:**
  - Net premiums, determined with the *policy value basis*, are given as input.
  - ${}_tV^n$ is then determined with the **policy value basis**.
- **Remark**: Second method is a remainder of a time before modern computers. It is still widely used in the USA and is the standard approach in D,H,W (2020).

Example 7.2

- Contract:

$$\boxed{\text{benefits} = 100\,000\ A_{[50]}} \text{ and } \boxed{\text{gross premiums} = 1370\ \ddot{a}_{[50]}}$$

- Calculate ${}_5V^g$.
- Policy value basis at time 5:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$\boxed{0.125 \times 1300\ \ddot{a}_{55}}$$

Example 7.2 - solution

- Gross future loss r.v. at time 5:

$$
\begin{aligned}
{}_5L^g &= \mathrm{PV}_5\left[100\,000\ \underline{A}_{55} + 0.125P\ \underline{\ddot{a}}_{55} - P\ \underline{\ddot{a}}_{55}\right] \\
&= \mathrm{PV}_5\left[100\,000\ \underline{A}_{55} - 0.875\ P\ \underline{\ddot{a}}_{55}\right]
\end{aligned}
$$

- Gross policy value at time 5 :

$$
\begin{aligned}
{}_5V^g &= \mathbb{E}\left[L_5^g\right] = 100\,000\ A_{55} - 0.875\ P\ \ddot{a}_{55} \\
&= 4\,272.68
\end{aligned}
$$

Example 7.3

- Contract:

  benefits $= 100\,000\ A_{[60]:\overline{20|}}$ and gross premiums $= P\ \ddot{a}_{[60]:\overline{10|}}$

  with $P = 5\,200$.
- Calculate the gross policy values ${}_tV$ at times $t = 0$, 5 and 10.
- Policy value basis at time t: see below.

Example 7.3 - solution

- Policy value basis at time 0:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$\boxed{200\ \underline{A}_{[60]:\overline{20}|} + (0.05\ P;\ 0) + 0.05\ P\ \underline{\ddot{a}}_{[60]:\overline{10}|}}$$

- Gross policy value at time 0:

$$\begin{aligned} {}_0V^g &= 100\,200\ A_{[60]:\overline{20}|} + 0.05 \times P - 0.95 \times P\ \ddot{a}_{[60]:\overline{10}|} \\ &= 2\,023 > 0 \end{aligned}$$

policy value basis
more conservative
than premium basis

Example 7.3 - solution

- Policy value basis at time 5:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$\boxed{200\ A_{65:\overline{15}|} + 0.05\ P\ \ddot{a}_{65:\overline{5}|}}$$

- Gross policy value at time 5:

$$\begin{aligned} {}_5V^g &= 100\,200\ A_{65:\overline{15}|} - 0.95 \times P\ \ddot{a}_{65:\overline{5}|} \\ &= 29\,068 \end{aligned}$$

Example 7.3 - solution

- Policy value basis at time 10:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$\boxed{200\ \underline{A}_{70:\overline{10}|}}$$

- Gross policy value at time 10:

$$_{10}V^{g} = 100\,200\ A_{70:\overline{10}|} = 63\,073$$

Example 7.4

- Consider the **contract** with

$$\text{benefits} = 10\,000 \; {}_{10|}\ddot{a}_{[50]} + P \; (IA)^{\;1}_{[50]:\overline{10}|}$$

and

$$\text{gross premiums} = P \; \ddot{a}_{[50]:\overline{10}|} \quad \text{with} \quad P = 11\,900$$

- Calculate the gross policy values at times 0, 5, 15^- and 15^+.
- Policy value basis at time t: see below.

Example 7.4 - solution

- Policy value basis at time 0:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$(0.05P;\ 0) + 0.05 \times P\ \underline{\ddot{a}}_{[50]:\overline{10}|} + 100\ \underline{A}^{\ 1}_{[50]:\overline{10}|} + 25\ {}_{10|}\underline{\ddot{a}}_{[50]}$$

- Gross policy value at time 0:

$$\begin{aligned} {}_0V^g &= P\ (IA)^{\ 1}_{[50]:\overline{10}|} + 100\ A^{\ 1}_{[50]:\overline{10}|} + 10\,025\ {}_{10|}\ddot{a}_{[50]} \\ &\quad + 0.05 \times P - 0.95 \times P\ \ddot{a}_{[50]:\overline{10}|} \\ &= 485 \end{aligned}$$

Example 7.4 - solution

- Policy value basis at time 5:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$0.05 \times P \, \underline{\ddot{a}}_{55:\overline{5}|} + 100 \, \underline{A}^{1}_{55:\overline{5}|} + 25 \, {}_{5|}\underline{\ddot{a}}_{55}$$

- Gross policy value at time 5 :

$$\begin{aligned} {}_5V^g &= P \left((IA)^{1}_{55:\overline{5}|} + 5 \, A^{1}_{55:\overline{5}|} \right) + 100 \, A^{1}_{55:\overline{5}|} + 10\,025 \, {}_{5|}\ddot{a}_{55} \\ &\quad - 0.95 \times P \, \ddot{a}_{55:\overline{5}|} \\ &= 65\,470 \end{aligned}$$

Example 7.4 - solution

- Policy value basis at time 15:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$\boxed{25\ \ddot{a}_{65}}$$

- Gross policy value at time 15^-: just before annuity payment + related expense

$$_{15^-}V^g = 10\,025\ \ddot{a}_{65} = 135\,837$$

- Gross policy value at times 15 and 15^+: just after annuity payment and related expense

$$_{15}V = {}_{15^+}V^g = 10\,025\ a_{65} = 125\,812$$

Endowment insurance contract

- Benefits and net premiums:

$$\text{benefits} = 500\,000\ A_{[50]:\overline{20}|} \qquad \text{net premiums} = P\ \ddot{a}_{[50]:\overline{20}|}$$

- **Net policy values** $(t = 0, 1, \ldots, 19)$:

$${}_tV^n = 500\,000\ A_{[50]+t:\overline{20-t}|} - P\ \ddot{a}_{[50]+t:\overline{20-t}|}$$

  - Time $t = 0, 1, \ldots, 19$:

$${}_tV^n = {}_{t^-}V^n = {}_{t^+}V^n - P$$

  - Time 20:

$${}_{20}V^n = {}_{20^-}V^n = 500\,000$$

Endowment insurance contract (cont'd)

- Benefits and net premiums:

$$\text{benefits} = 500\,000\ A_{[50]:\overline{20}|}$$

$$\text{net premiums} = P\ \ddot{a}_{[50]:\overline{20}|}$$

- Policy values:

  ${}_tV^n$ $(t = 0, 1, \ldots, 19)$, calculated with premium basis:

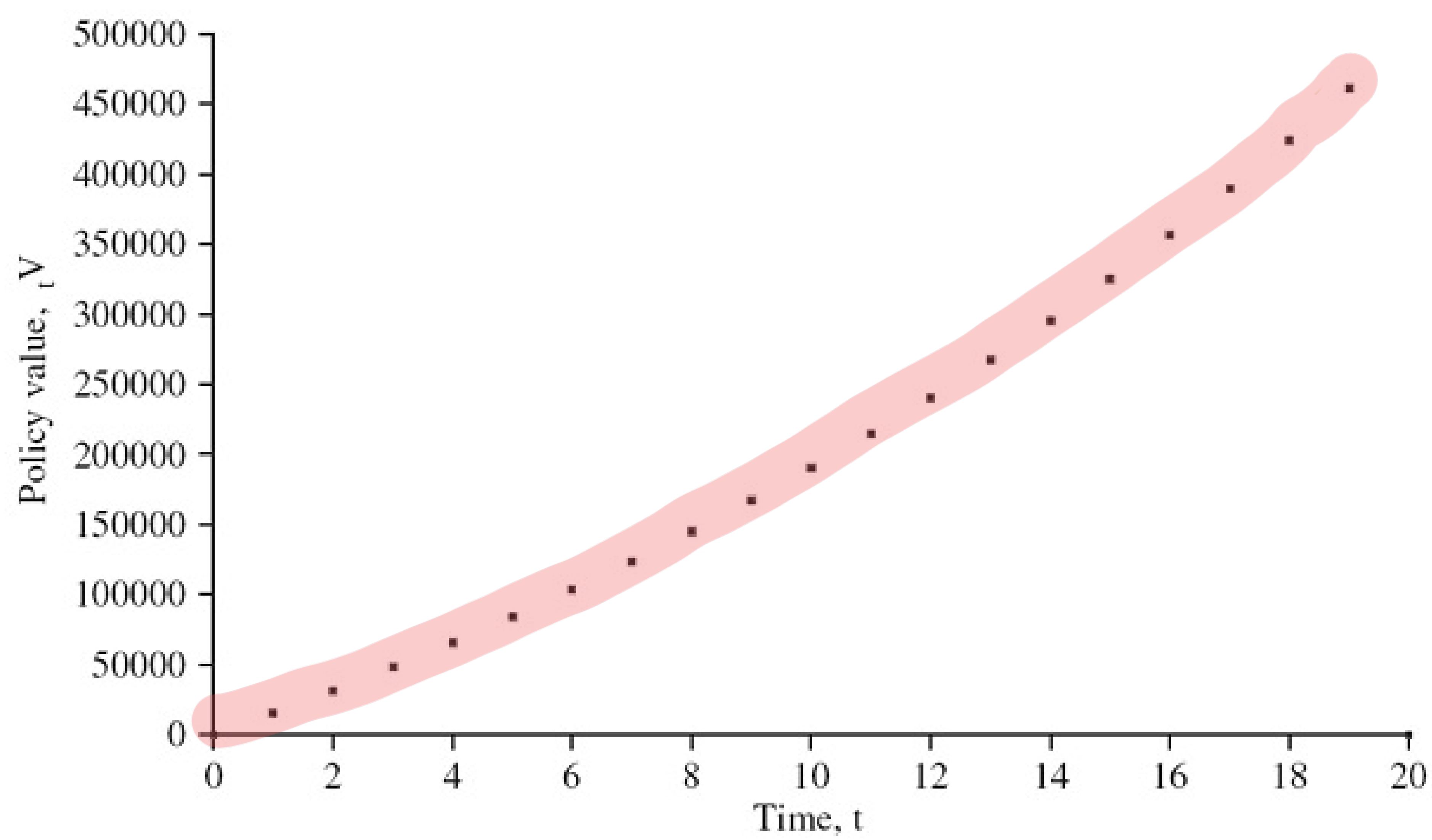
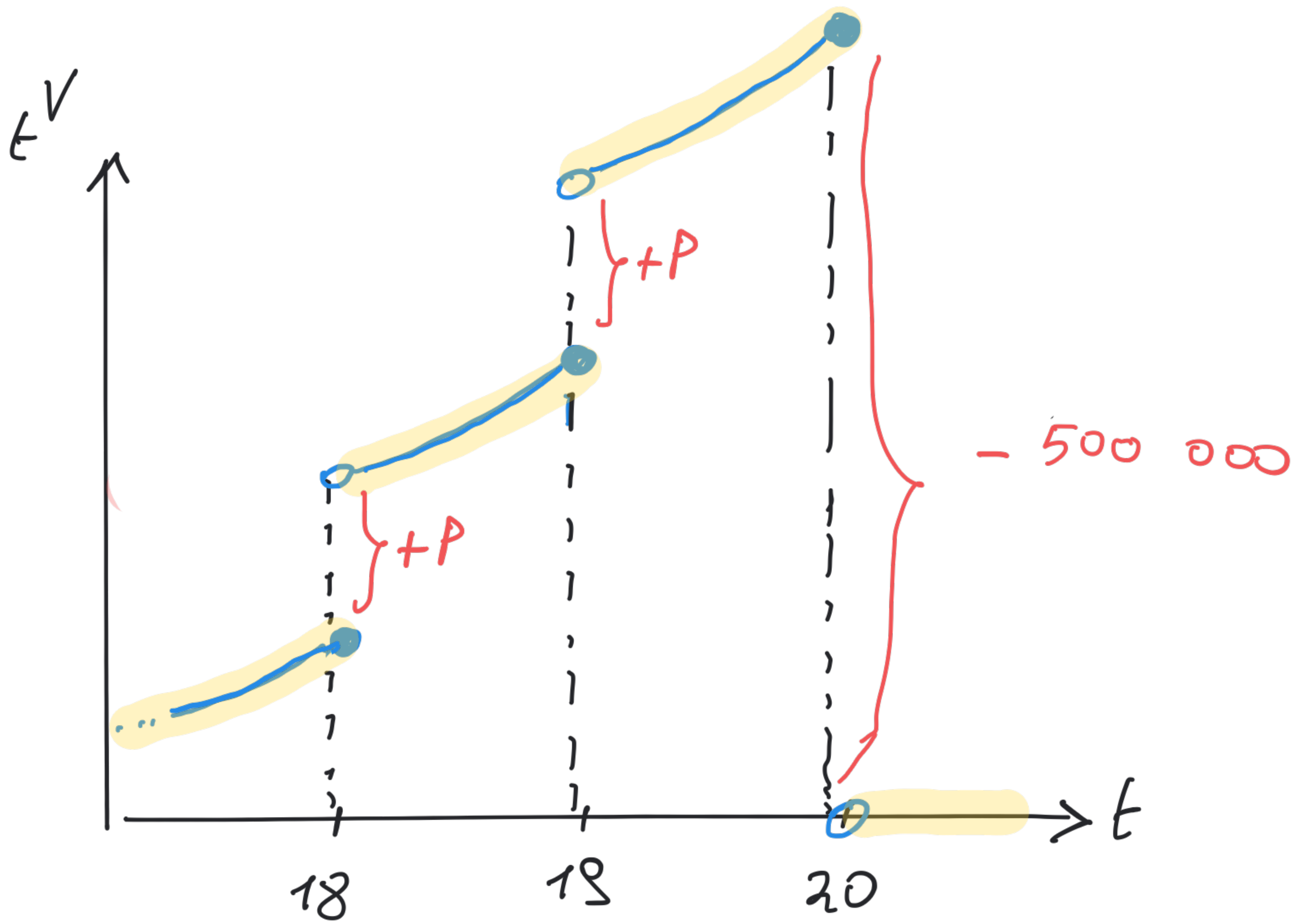


Figure 7.3 Policy values for each year of a 20-year endowment insurance, sum insured $500 000, issued to (50).

Endowment insurance contract: (cont'd)



$$_tV = 500\,000 \times A_{[50]+t:\overline{20-t|}} - P \times \ddot{a}_{[50]+t:\overline{20-t|}}$$

Endowment insurance contract (cont'd)

Term insurance contract

- Benefits and net premiums:

$$\text{benefits} = 500\,000\ A^{\ \ 1}_{[50]:\overline{20}|} \qquad \text{net premiums} = P\ \ddot{a}_{[50]:\overline{20}|}$$

- **Net policy values** $(t = 0, 1, \ldots, 19)$:

$$_tV^n = 500\,000\ A^{\ \ \ \ 1}_{[50]+t:\overline{20-t}|} - P\ \ddot{a}_{[50]+t:\overline{20-t}|}$$

  - Time $t = 0, 1, \ldots, 19$:

$$_tV^n = \ _{t^-}V^n = \ _{t^+}V^n - P$$

  - Time 20:

$$_{20}V^n = \ _{20^-}V^n = 0$$

7.2 Policies with annual cash flows

7.2.2 Policy values for policies with annual cash flows

Term insurance contract (cont'd)

- Benefits and net premiums:

$$\text{benefits} = 500\,000 \; A^{\;1}_{[50]:\overline{20}|}$$

$$\text{net premiums} = P \; \ddot{a}_{[50]:\overline{20}|}$$

- Policy values:
  ${}_tV^n$ $(t = 0, 1, \ldots, 19)$, calculated with premium basis:

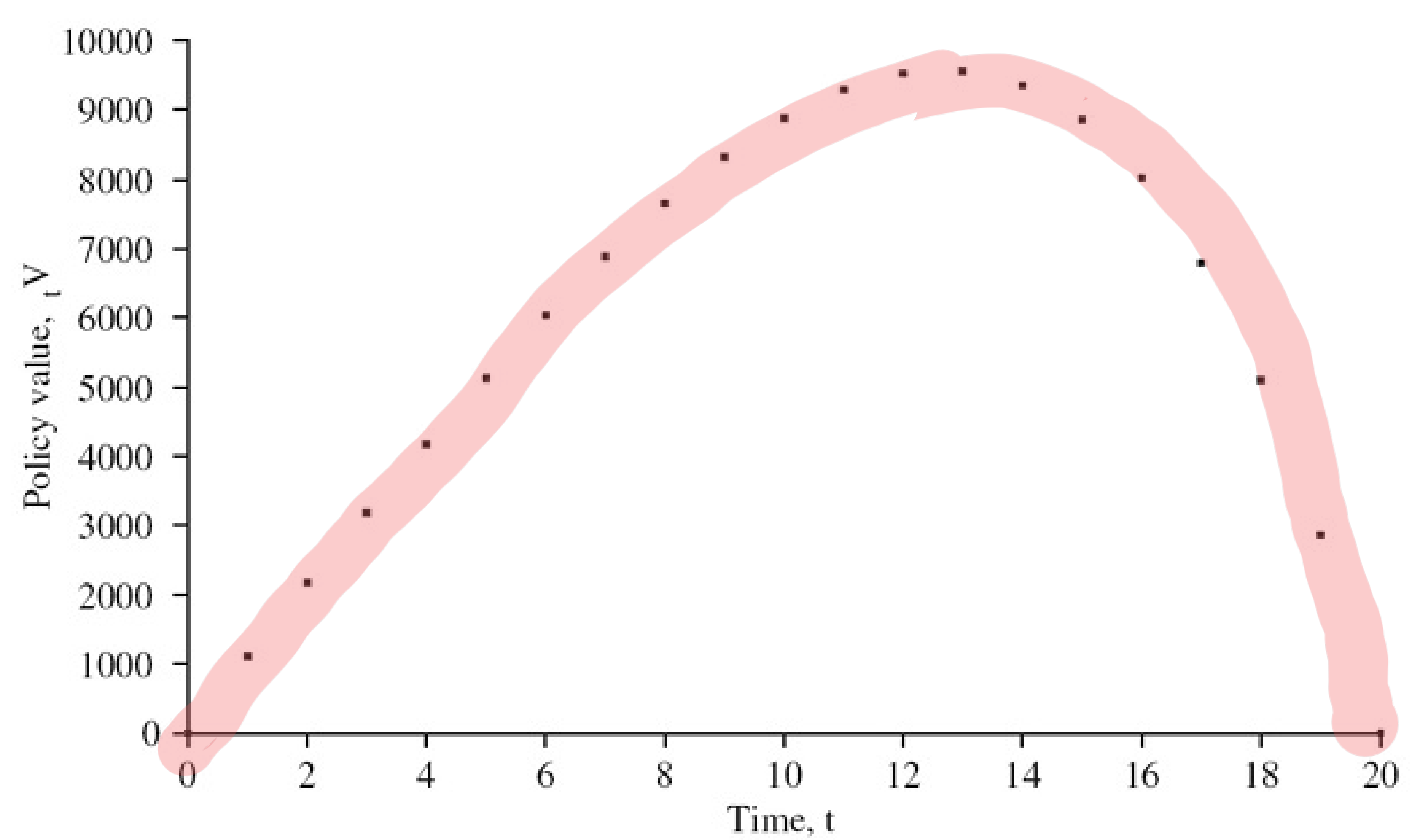
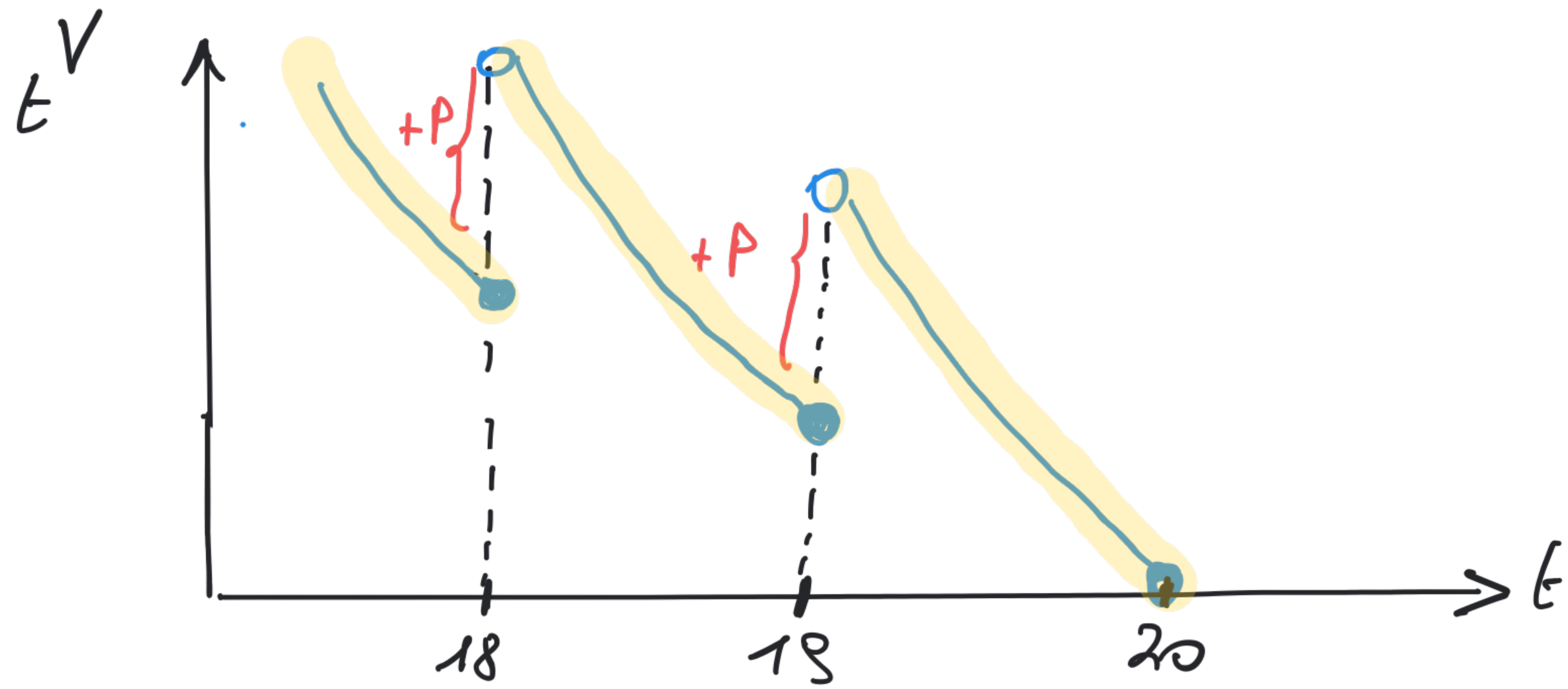


Figure 7.4 Policy values for each year of a 20-year term insurance, sum insured \$500 000, issued to (50).

Term insurance contract (cont'd)



$${}_tV = 500\,000 \times A^{\;1}_{[50]+t:\overline{20-t|}} - P \times \ddot{a}_{[50]+t:\overline{20-t|}}$$

Example 7.5

- Consider the **contract** with
  - Benefits:

$$S\, A_{[50]:\overline{20}|}$$

  - Net premiums:

$$P\, \ddot{a}_{[50]:\overline{20}|}$$

- Question: Determine a backward recursion for the policy values ${}_tV$, for $t = 0, 1, \ldots, 20$.
- Assumption: Policy values determined with fixed policy value basis (which may be different from premium basis).

Example 7.5 - solution

- **Policy value $_{20}V$**:

$$\boxed{_{20}V = S}$$

- **Policy value $_{19}V$**:
  - Definition:

$$_{19}V = S\,A_{[50]+19:\overline{1}|} - P$$

$$= S \times v - P$$

  - Recursion:

$$\boxed{\left(_{19}V + P\right)(1+i) = S}$$

  - Interpretation of recursion: Having arrived at time 19, the following equality must hold for each policy in force:

available assets at time 20^- = **required assets** at time 20^-

Example 7.5 - solution

- **Policy value $_tV$** at $t = 18, 17, \ldots, 0$:
  - Definition:

$$ {}_tV = S\, A_{[50]+t:\overline{20-t|}} - P\, \ddot{a}_{[50]+t:\overline{20-t|}} $$

  - Recursion: (Proof: see book page 232)

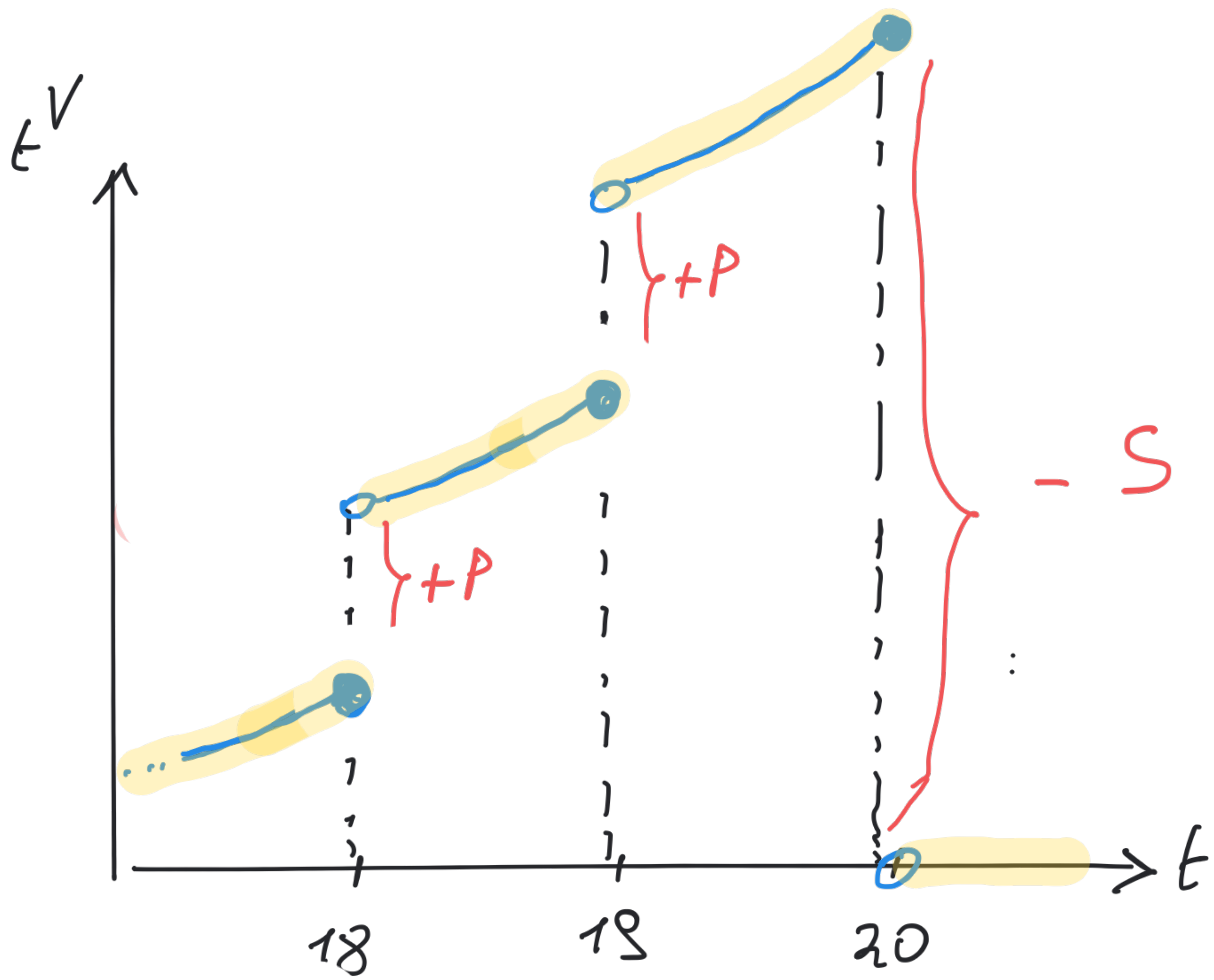
$$ ({}_tV + P)\,(1+i) = S \times q_{[50]+t} + {}_{t+1}V \times p_{[50]+t} \qquad (7.4) $$

  - Interpretation: Having arrived at time t, the following equality must hold for each policy in force:

available assets at time $t + 1^-$ = **required assets** at time $t + 1^-$

expected

Example 7.5 (continued)



From (7.4) : $${}_{19}V = \left({}_{18}V + P - S \times A^{\,1}_{68:\overline{1}|}\right) \times \left({}_{1}E_{68}\right)^{-1}$$

↳ - increasing effect due to premium
- decreasing effect due to death benefit
- increasing effect due to actuarial accumulation

Example 7.6

- Consider the **contract** with

$$\text{benefits} = 10\,000\ {}_{10|}\ddot{a}_{[50]} + P\ (IA)^{\ 1}_{[50]:\overline{10|}}$$

and

$$\text{gross premiums} = P\ \ddot{a}_{[50]:\overline{10|}} \quad \text{with} \quad P = 11\,900$$

- Question: Determine a backward recursion for the gross policy values ${}_tV$, for $t = 0, 1, \ldots, 9$.
- Policy value basis:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$(0.05P;\ 0) + 0.05 \times P\ \ddot{a}_{[50]:\overline{10|}} + 100\ A^{\ 1}_{[50]:\overline{10|}} + 25\ {}_{10|}\ddot{a}_{[50]}$$

Example 7.6 - solution

- **Policy value ${}_{10}V$:**

$$\boxed{{}_{10}V = 10\ 025\ a_{60}}$$

- **Policy value ${}_{9}V$:**
  - Definition:

$$ {}_{9}V = (10P + 100)\ A^{\ 1}_{59:\overline{1}|} + 10\ 025\ {}_{1|}\ddot{a}_{59} - 0.95\ P$$

  - Recursion:

$$\boxed{({}_{9}V + 0.95P)\,(1+i) = (10P + 100)\ q_{59} + (10\ 025 + {}_{10}V)\ p_{59}}$$

  - Interpretation: Having arrived at time 9, the following equality must hold for each policy in force:

$$\boxed{\textbf{available assets} \text{ at time } 10^- = \textbf{required assets} \text{ at time } 10^-}$$

Proof of recursion:

$${}_{9}V = (10P + 100) \times A^{1}_{59:\overline{1|}} + 10\,025\ {}_{1|}\ddot{a}_{59} - 0.95P$$

$$= (10P + 100) \times q_{59} \times v + 10\,025 \times p_{59} \times v \times \ddot{a}_{60} - 0.95P$$

$$\Longrightarrow ({}_{9}V + 0.95P)(1+i) = (10P + 100) \times q_{59} + 10\,025 \times (1 + a_{60}) \times p_{59}$$

$$= (10P + 100) \times q_{59} + (10\,025 + {}_{10}V) \times p_{59}$$

$\blacksquare$

Example 7.6 - solution (cont'd)

- **Policy value $_tV$** at $t = 8, \ldots, 1$:
  - Definition:

$$
\begin{aligned}
{}_tV \;=\;& P\,(\underline{IA})^{\;1}_{[50]+t:\overline{10-t}|} + (t \times P + 100)\;A^{\;1}_{[50+t]:\overline{10-t}|} \\
&+ 10\,025\;{}_{10-t|}\ddot{a}_{[50]+t} - 0.95\;P\;\underline{\ddot{a}}_{[50+t]:\overline{10-t}|}
\end{aligned}
$$

  - Recursion: (Proof: See book p. 233)

$$
({}_tV + 0.95P)\,(1+i) = ((t+1)\,P + 100)\;q_{[50]+t} + {}_{t+1}V\;p_{[50]+t} \quad (7.5)
$$

↳ REMARK: recursion (7.5) for $t = 8$ is NOT CORRECT in book.

- **Policy value $_0V$**:
  - Definition:

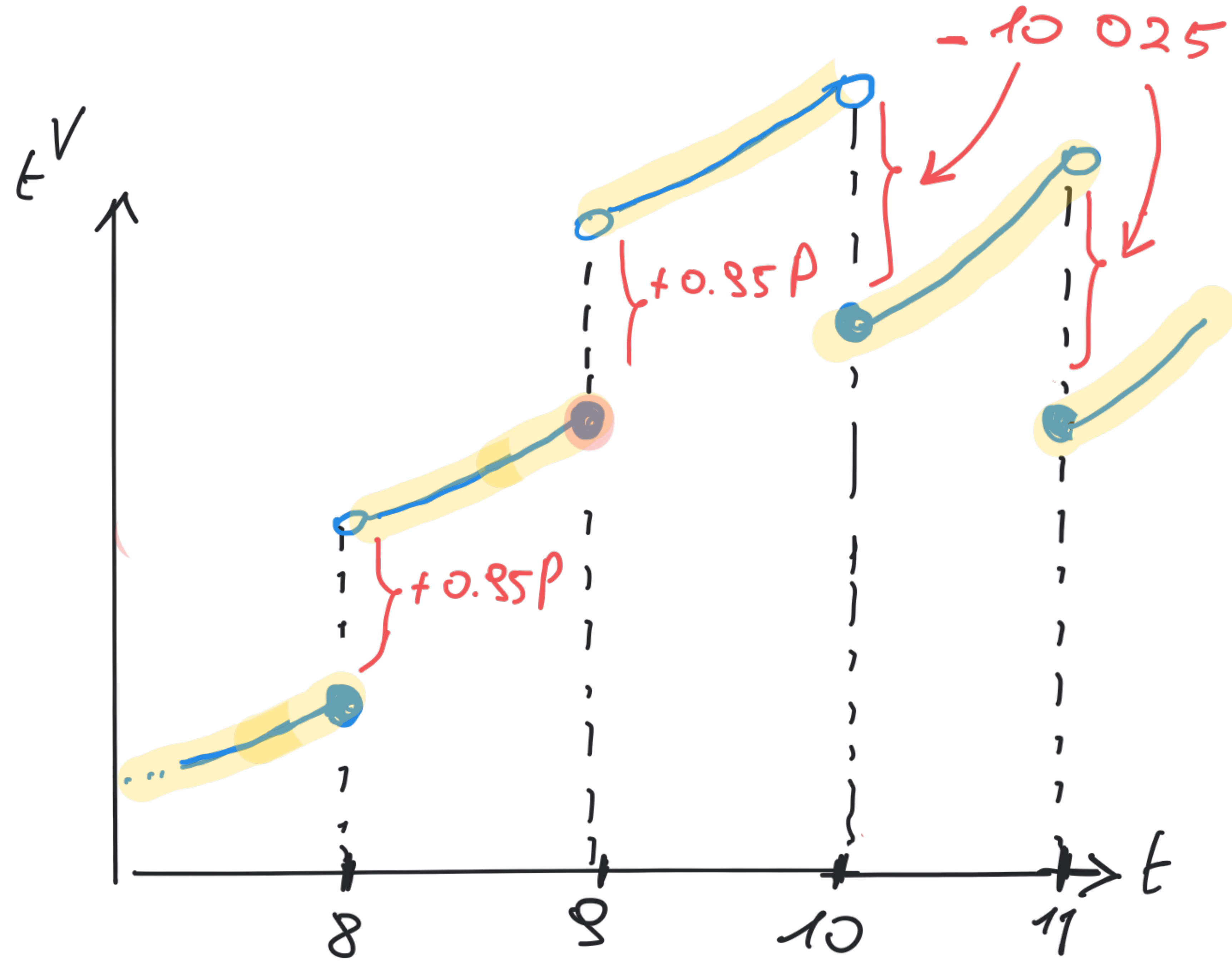
$$
{}_0V = P\;(\underline{IA})^{\;1}_{[50]:\overline{10}|} + 10\,000\;{}_{10|}\ddot{a}_{[50]+10} - P\;\underline{\ddot{a}}_{[50]:\overline{10}|} + \ldots
$$

↳ see slide 28

  - Recursion:

$$
({}_0V + 0.9P)\,(1+i) = (P + 100) \times q_{[50]} + {}_1V \times p_{[50]}
$$

Example 7.6 (continued)



From (7.5): $_{9}V = \left({}_{8}V + 0.95P - ({}_{8}P + 100) \times A^{1}_{58:\overline{7}|} \right) \times \left({}_{1}E_{58} \right)^{-1}$

We also have: $_{10}V = \left({}_{9}V + 0.95P - ({}_{10}P + 100) \times A^{1}_{59:\overline{7}|} \right) \times \left({}_{1}E_{59} \right)^{-1}$ $- 10\,025$

↳ from recursion for $_9V$

7.2 Policies with annual cash flows

7.2.3 Recursive formulae for policy values

A general contract

- Death benefits S_j + related expenses E_j:

$$\sum_{j=0}^{\infty} (S_{j+1} + E_{j+1}) \ {}_{j|}A^{\ 1}_{[x]:\overline{1|}}$$

- Gross premiums P_j - related expenses e_j:

$$\sum_{j=0}^{\infty} (P_j - e_j) \ {}_{j}E_{[x]}$$

- Suppose that the policy is still in force after t years.
- Determine the gross policy value ${}_tV$ as a function of ${}_{t+1}V$.
- Policy value basis for determining ${}_tV$ and ${}_{t+1}V$:
  - Probabilities $q_{[x]+j}$ for $j \geq t$.
  - Interest rates i_j in year $(j, j+1)$ for $j \geq t$.
  - Expences e_j for $j \geq t$ and E_j for $j > t$.

A general contract (cont'd)

- **Policy value ${}_tV$**: $(t = 0, 1, 2, \ldots)$

$$ {}_tV = \sum_{j=0}^{\infty} \left(S_{t+j+1} + E_{t+j+1}\right) \ {}_{j|}A^{\ \ 1}_{[x]+t:\overline{1}|} - \sum_{j=0}^{\infty} \left(P_{t+j} - e_{t+j}\right) \ {}_jE_{[x]+t} $$

- **Recursion**: (Proof: See book p. 235)

$$ \left({}_tV + P_t - e_t\right)\left(1 + i_t\right) = \left(S_{t+1} + E_{t+1}\right) \times q_{[x]+t} + {}_{t+1}V \times p_{[x]+t} \tag{7.7} $$

- **Interpretation**: Having arrived at time t, the following equality must hold for each policy in force:

available assets at time $t + 1^-$ = **required** assets at time $t + 1^-$

A general contract (cont'd)

- **Sum at Risk** at time $t+1$ (Death Strain at Risk, Net Amount at Risk):
  - Definition: (for $t = 0, 1, 2, \ldots$)

    what is needed for dyers
    − what is needed for survivors

$$\mathbf{D}_{t+1} = S_{t+1} + E_{t+1} - {}_{t+1}V$$

  - Recursion for policy values: (follows from (7.7), taking into account that $p = 1 - q$)

$$\left({}_tV + P_t - e_t\right)(1 + i_t) = {}_{t+1}V + \mathbf{D}_{t+1} \times q_{[x]+t} \qquad (7.8)$$

- Special case: a **pure savings contract**:

$$\Rightarrow {}_{t+1}V = \left({}_tV + P_t - e_t\right) \times (1 + i_t)$$

$$\mathbf{D}_{t+1} = 0, \qquad t = 0, 1, \ldots$$

A general contract (cont'd)

- **Savings premiums**:
  - Definition: (for $t = 0, 1, 2, \ldots$)

$$P_t^{\text{saving}} = {}_{t+1}V \times \frac{1}{1+i_t} - {}_tV$$

- **Risk premiums**:
  - Definition: (for $t = 0, 1, 2, \ldots$)

$$P_t^{\text{risk}} = q_{[x]+t} \times v \times \mathbf{D}_{t+1}$$

- **Decomposing the premiums**:

$$P_t - e_t = P_t^{\text{saving}} + P_t^{\text{risk}}$$

: follows from (7.8)

- Special case: **a pure savings contract.**

↳ $D_{t+1} = 0 \Rightarrow P_t^{\text{saving}} = P_t - e_t$

A more general contract (not in book)

- Death and survival benefits + related expenses:

$$\sum_{j=0}^{\infty}(S_{j+1}+E_{j+1})\ {}_{j|}A^{\ 1}_{[x]:\overline{1|}}+\sum_{j=0}^{\infty}(L_{j+1}+F_{j+1})\ {}_{j+1}E_{[x]}$$

- Gross premiums - related expenses:

$$\sum_{j=0}^{\infty}(P_j-e_j)\ {}_{j}E_{[x]}$$

- Suppose that the policy is still in force after t years.
- Determine the gross policy value ${}_tV$ in terms of ${}_{t+1}V$.
- Policy value basis for determining ${}_tV$ and ${}_{t+1}V$:
  - Probabilities $q_{[x]+j}$ for $j \geq t$.
  - Interest rates i_j in year $(j, j+1)$ for $j \geq t$.
  - Expences e_j for $j \geq t$ and E_j for $j > t$.

A more general contract (cont'd)

- **Policy value ${}_tV$**: (for $t = 0, 1, 2, \ldots$)

$$
\begin{aligned}
{}_tV &= \sum_{j=0}^{\infty} \left(S_{t+j+1} + E_{t+j+1}\right) \; {}_{j|}A^{\;\;1}_{[x]+t:\overline{1}|} \\
&\quad + \sum_{j=0}^{\infty} \left(L_{t+1+j} + F_{t+1+j}\right) \; {}_{j+1}E_{[x]+t} \\
&\quad - \sum_{j=0}^{\infty} \left(P_{t+j} - e_{t+j}\right) \; {}_{j}E_{[x]+t}
\end{aligned}
$$

- **Recursion**:

$$
\begin{aligned}
\left({}_tV + P_t - e_t\right)\left(1 + i_t\right) &= \left(S_{t+1} + E_{t+1}\right) \; q_{[x]+t} \\
&\quad + \left(L_{t+1} + F_{t+1} + {}_{t+1}V\right) \; p_{[x]+t}
\end{aligned}
$$

7.2 Policies with annual cash flows

7.2.3 Recursive formulae for policy values

A more general contract (cont'd)

- **Sum at Risk** (Death Strain Risk, Net Amount at Risk):
  - Definition: (for $t = 0, 1, 2, \ldots$)

$$\mathbf{D}_{t+1} = S_{t+1} + E_{t+1} - (L_{t+1} + F_{t+1} + {}_{t+1}V)$$

  - Recursion for policy values: (follows from recursion on previous slide)

$$({}_tV + P_t - e_t)(1 + i_t) = (L_{t+1} + F_{t+1} + {}_{t+1}V) + \mathbf{D}_{t+1}\, q_{[x]+t}$$

- Special cases:
  - **A pure savings contract**:

$$\mathbf{D}_{t+1} = 0, \qquad t = 0, 1, \ldots$$

⇒ policyholders' account grows as a savings account with interest i_t

  - **A pure survival contract**:

$$S_{t+1} + E_{t+1} = 0, \qquad t = 0, 1, \ldots \ .$$

⇒ $D_{t+1} \leq 0$ ⇒ policyholders' account grows faster than a savings account with interest i_t

7.2 Policies with annual cash flows

7.2.3 Recursive formulae for policy values

A more general contract (cont'd)

- **Savings premiums**:
  - Definition: (for $t = 0, 1, 2, \ldots$)

$$P_t^{\text{saving}} = (L_{t+1} + F_{t+1} + {}_{t+1}V) \times \frac{1}{1+i_t} - {}_tV$$

- **Risk premiums**:
  - Definition: (for $t = 0, 1, 2, \ldots$)

$$P_t^{\text{risk}} = \mathbf{D}_{t+1} \times \underline{A}^{\;\;1}_{[x]+t:\overline{1}|}$$

- **Decomposing the premiums**:

$$P_t - e_t = P_t^{\text{saving}} + P_t^{\text{risk}}$$

- Special cases:
  - **A pure savings contract**.
  - **A pure survival contract**.

↳ $P_t^{risk} = 0$ and $P_t^{saving} = P_t - e_t$

↳ $P_t^{risk} \leq 0$ and $P_t^{saving} \geq P_t - e_t$

7.2 Policies with annual cash flows

7.2.3 Recursive formulae for policy values

Example 7.7

- Benefits:

$$700\,000 \times {}_{20}E_{[50]} + \sum_{j=0}^{19} {}_jV \times {}_{j|}A^{\,1}_{[50]:\overline{1|}}$$

- Net premiums:

$$P\,\ddot{a}_{[50]:\overline{20|}} \quad \text{with} \quad P = 23\,500$$

- Question: Determine the policy value at time 15.
- Policy value basis: $i = 3.5\%$ and SSSM.
- Problem:

$${}_{15}V = 700\,000 \times {}_5E_{65} + \sum_{j=0}^{4} {}_{15+j}V \times {}_{j|}A^{\,1}_{65:\overline{1|}} - P \times \ddot{a}_{65:\overline{5|}}$$

7.2 Policies with annual cash flows

7.2.3 Recursive formulae for policy values

Example 7.7 - solution

- The policy value at time 20:

$$\boxed{{}_{20}V = 700\ 000}$$

- The policy value at time 19:

$$({}_{19}V + P) \times 1.035 = {}_{19}V \times q_{69} + 700\ 000 \times p_{69}$$

or

$$\boxed{{}_{19}V = 652\ 401}$$

- The policy value at time 18:

$$({}_{18}V + P) \times 1.035 = {}_{18}V \times q_{68} + {}_{19}V \times p_{68}$$

or

$$\boxed{{}_{18}V = 606\ 471}$$

- The policy value at times 17, 16, 15:

$$\boxed{{}_{15}V = \ldots = 478\ 063}$$

7.2 Policies with annual cash flows

7.2.4 Analysis of surplus

- Consider a portfolio of the 'more general contracts' underwritten to a group insureds of age x.
- **Recursion for policy values** (calculated with same basis):

$$\left({}_tV + P_t - e_t\right)\left(1 + i_t\right) = \left(S_{t+1} + E_{t+1}\right)\; q_{[x]+t}$$

$$+ \left(L_{t+1} + F_{t+1} + {}_{t+1}V\right)\; p_{[x]+t}$$

- **Interpretation**:
  - Suppose that at time t, the insurer sets aside ${}_tV$ for each 'surviving policy' and that experience in year $t+1$ follows the policy value basis.
  - Then at time $t+1^-$, the insurer will have the exact amount of funds needed to:
    - pay the *death benefits* (and related expenses) to all insureds who died in the past year,
    - the *survival benefits* (and related expenses) to all the survivors,
    - set up the *policy value* ${}_{t+1}V$ for all survivors.

7.2 Policies with annual cash flows

7.2.4 Analysis of surplus

- **In practice**, experienced *interest*, *mortality* and *expenses* in year $t+1$ will usually differ from the ones assumed in the policy value basis:

required assets at $t+1^-$

$$\left({}_tV + P_t - e_t^{\text{exp}}\right)\left(1 + i_t^{\text{exp}}\right) \neq \left(S_{t+1} + E_{t+1}^{\text{exp}}\right)\ q_{[x]+t}^{\text{exp}}$$

available assets at $t+1^-$

$$+ \left(L_{t+1} + F_{t+1}^{\text{exp}} + {}_{t+1}V\right)\ p_{[x]+t}^{\text{exp}}$$

- **Sources or profit or loss**:
  - Experienced expenses < assumed expensed: profit.
  - Experienced interest < assumed interest: loss.
  - Experienced mortality < assumed mortality: profit or loss.
- **Example 7.8**: **read in book**.

7.2 Policies with annual cash flows

7.2.5 Asset shares

- **Policy value**: ${}_tV$ = amount the insurer *needs to have* per policy in force at time t.
  - prospective view,
  - policy value basis.
- **Asset share**: AS_t = amount the insurer *does have* per policy in force at time t.
  - retrospective view,
  - experience basis.
- **Theorem**:
  - If the following conditions hold:
    - premiums calculated using the equivalence principle,
    - ${}_tV$ calculated using premium basis,
    - experience followed premium basis until time t,
  - then we have that

$$ {}_tV = AS_t $$

Corollary:

- Suppose benefits = $S \times A_{[50]:\overline{20|}}$

  and premiums = $P \times \ddot{a}_{[50]:\overline{20|}}$
- Suppose there are no expenses.
- Under the conditions of the theorem, one has:

$$ {}_{10}V = AS_{10} $$

PROOF OF COROLLARY:

see Example 7.1

- $_{10}V = S \times A_{60:\overline{10|}} - P \times \ddot{a}_{60:\overline{10|}}$
- $_{10}AS = \left(P \times \ddot{a}_{[50]:\overline{10|}} - S \times A^{1}_{[50]:\overline{10|}}\right) \times \left({}_{10}E_{[50]}\right)^{-1}$
- $S \times A_{[50]:\overline{20|}} = P \times \ddot{a}_{[50]:\overline{20|}}$: EQUIVALENCE PRINCIPLE

$$\Longleftrightarrow S \times A^{1}_{[50]:\overline{10|}} + S \times {}_{10}E_{[50]} \times A_{60:\overline{10|}}$$

$$= P \times \ddot{a}_{[50]:\overline{10|}} + P \times {}_{10}E_{[50]} \times \ddot{a}_{60:\overline{10|}}$$

■

7.2 Policies with annual cash flows

7.2.5 Asset shares

Example 7.9

- Consider the **contract** with

$$\text{benefits} = 10\,000 \ {}_{10|}\ddot{a}_{[50]} + P\,(IA)^{\,1}_{[50]:\overline{10}|}$$

and

$$\text{gross premiums} = P\,\ddot{a}_{[50]:\overline{10}|} \quad \text{with} \quad P = 11\,900$$

- Suppose:
  - at time 0, a total number of N such policies were sold,
  - we have arrived at time 5.

Example 7.9

- Insurer's **experience** over past 5 years for these policies:
  - Investment returns:

    $$0.048,\ 0.056,\ 0.052,\ 0.049 \text{ and } 0.047$$

  - Mortality rates:

    $$q_{[50]+t} = 0.0015 \text{ for } t = 0, 1, 2, 3, 4.$$

  - Expenses:
    - at policy issue: $0.15 \times P$,
    - at each premium payment (except time 0): $0.06 \times P$,
    - related to paying a death claim: 120.
- Question: Calculate AS_t at times $t = 0, 1, 2, 3, 4, 5$.

Example 7.9 - solution

- Asset share at time 0: $AS_0 = 0$.
- Asset share at time 1:
  - Total assets at time 0 is equal to 0.
  - Premium income at time 0: $0.85 \times P \times N = 10\ 115\ N$.
  - Total assets at time 1^- (before benefits and premiums): $10\ 115\ N \times 1.048 = 10\ 601\ N$.
  - Death claims at time 1: $(P + 120) \times (0.0015\ N) = 18\ N$.
  - Total assets at time 1 (after benefits, before premiums): $10\ 601\ N - 18\ N = 10\ 582\ N$.
  - Survivors at time 1: $N \times (1 - 0.0015) = 0.9985\ N$.
  - Asset share at time 1: $\frac{10\ 582\ N}{0.9985\ N} = 10\ 598 = AS_1$

Example 7.9 - solution (cont'd)

- Asset share in in any year:

Table 7.1. *Asset share calculation for Example 7.9.*

Year, t	Fund at start of year	Cash flow at start of year	Fund at end of year before death claims	Death claims and expenses	Fund at end of year	Survivors	AS_t
1	0	10 115 N	10 601 N	18 N	10 582 N	0.9985 N	10 598
2	10 582 N	11 169 N	22 970 N	36 N	22 934 N	0.9985^2 N	23 003
3	22 934 N	11 152 N	35 859 N	54 N	35 805 N	0.9985^3 N	35 967
4	35 805 N	11 136 N	49 241 N	71 N	49 170 N	0.9985^4 N	49 466
5	49 170 N	11 119 N	63 123 N	89 N	63 034 N	0.9985^5 N	63 509

7.3 Policy values for policies with 1/m-thly cash flows

Example 7.10

- Contract:

$$\text{Benefits} = 500\,000\ \underline{A}^{(12)}_{\overset{1}{[50]}:\overline{10}|}$$

$$\text{Gross premiums} = 4P\ \ddot{\underline{a}}^{(4)}_{[50]:\overline{5}|}$$

- Question:
  Determine (gross premium) policy values ${}_{2.75}V$, ${}_{3}V$ and ${}_{6.5}V$
- Policy value basis:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$0.1 \times 4P\ \ddot{\underline{a}}^{(4)}_{[50]:\overline{5}|}$$

7.3 Policy values for policies with 1/m-thly cash flows

Example 7.10 - solution

- Policy value at time 2.75:

$$ {}_{2.75}V = 500\,000 \times A^{(12)}_{\substack{1 \\ 52.75:\overline{7.25|}}} - 0.9 \times 4P \times \ddot{a}^{(4)}_{52.75:\overline{2.25|}} $$

- Policy value at time 3:

$$ {}_{3}V = 500\,000 \times A^{(12)}_{\substack{1 \\ 53:\overline{7|}}} - 0.9 \times 4P \times \ddot{a}^{(4)}_{53:\overline{2|}} $$

- Policy value at time 6.5:

$$ {}_{6.5}V = 500\,000 \times A^{(12)}_{\substack{1 \\ 56.5:\overline{3.5|}}} $$

Example 7.10 (continued)

- Contract:

$$\text{Benefits} = 500\,000\, A^{(12)}{}^{\;1}_{[50]:\overline{10}|}$$

$$\text{Gross premiums} = 4P\, \ddot{a}^{(4)}_{[50]:\overline{5}|}$$

↳ $P = 460$

- Question: Starting from (gross premium) policy value ${}_3V$, determine ${}_{\frac{35}{12}}V$, ${}_{\frac{34}{12}}V$ and ${}_{\frac{33}{12}}V$ recursively.
- Policy value basis:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$0.1 \times 4P\, \ddot{a}^{(4)}_{[50]:\overline{5}|}$$

Example 7.10 (continued) - solution

- Policy value at time $\frac{35}{12}$: (no premium date)

$$ {}_{\frac{35}{12}}V \times 1.05^{\frac{1}{12}} = 500\ 000 \times {}_{\frac{1}{12}}q_{50+\frac{35}{12}} + {}_{3}V \times {}_{\frac{1}{12}}p_{50+\frac{35}{12}} $$

↳ available assets at 3^- = required assets at 3^-

- Policy value at time $\frac{34}{12}$: (no premium date)

$$ {}_{\frac{34}{12}}V \times 1.05^{\frac{1}{12}} = 500\ 000 \times {}_{\frac{1}{12}}q_{50+\frac{34}{12}} + {}_{\frac{35}{12}}V \times {}_{\frac{1}{12}}p_{50+\frac{34}{12}} $$

↳ available assets at $35/12^-$ = required assets at $35/12^-$

- Policy value at time $\frac{33}{12}$: (premium date)

$$ \left({}_{\frac{33}{12}}V + 0.9 \times 460\right) 1.05^{\frac{1}{12}} = 500\ 000 \times {}_{\frac{1}{12}}q_{50+\frac{33}{12}} + {}_{\frac{34}{12}}V \times {}_{\frac{1}{12}}p_{50+\frac{33}{12}} \tag{7.9} $$

↳ available assets at $34/12^-$ = required assets at $34/12^-$

Example 7.11 (read in book).

Example 7.12

- Contract:

$$\text{Benefits} = 500\,000\; A^{(12)}_{\overset{1}{[50]}:\overline{10}|}$$

$$\text{Gross premiums} = 4P\; \ddot{a}^{(4)}_{[50]:\overline{5}|}$$

- Policy value basis:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

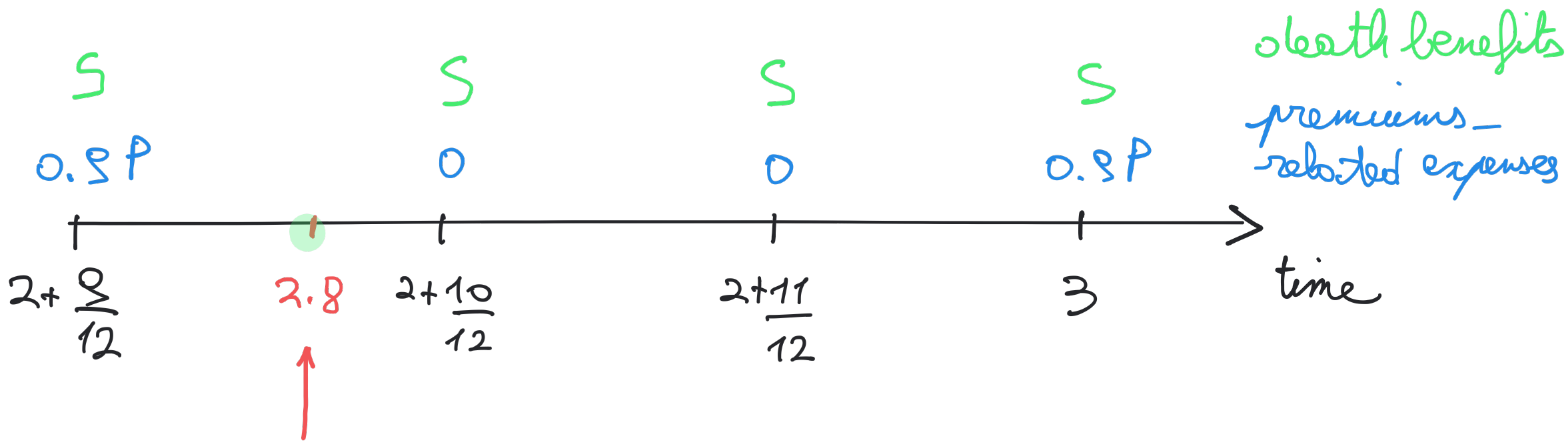
$$0.1 \times 4P\; \ddot{a}^{(4)}_{[50]:\overline{5}|}$$

- Question: Calculate the (gross premium) policy value ${}_{2.8}V$.

Example 7.12

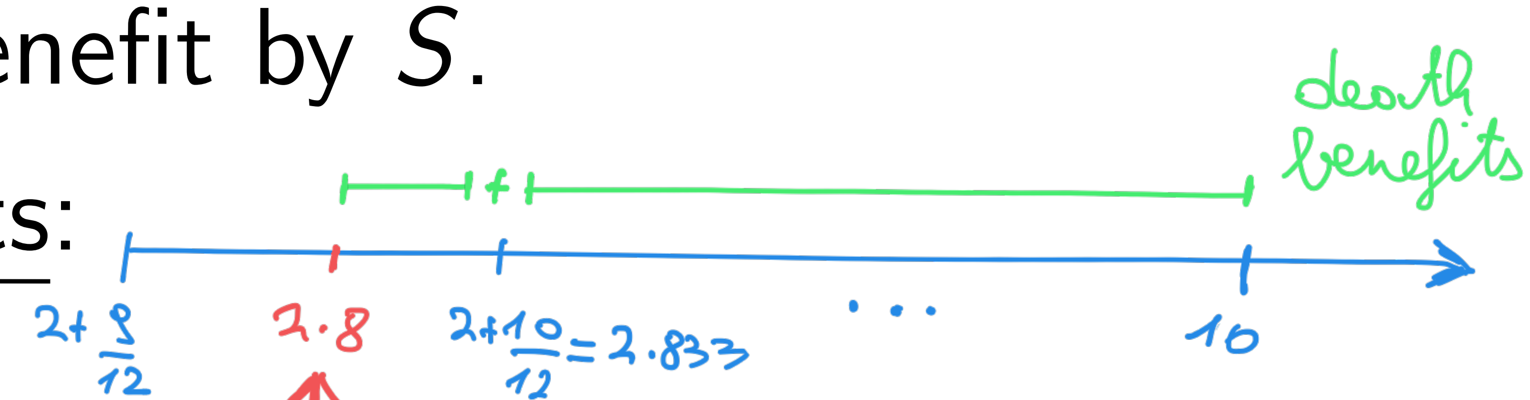
- Remark: Time 2.8 is not a cash flow payment date:

$$2 + \frac{9}{12} < 2.8 < 2 + \frac{10}{12}$$



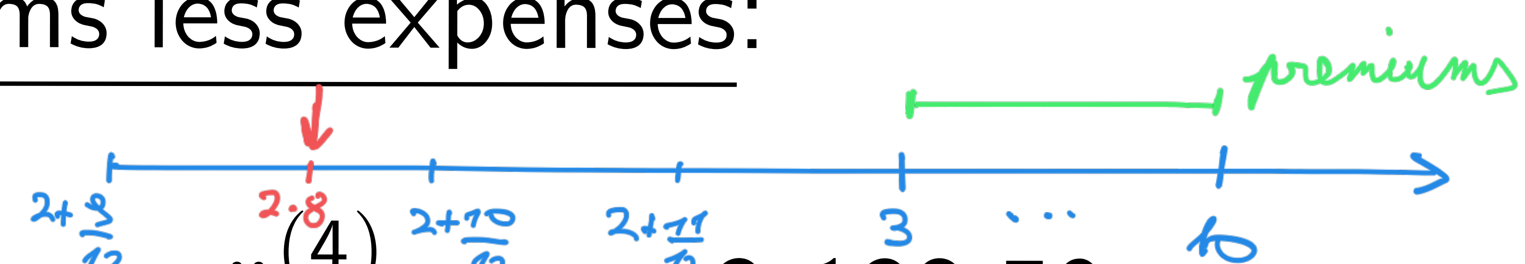
Example 7.12 - solution

- We denote the death benefit by S.
- EPV$_{2.8}$ of future benefits:



$$S \times v^{0.033} \times {}_{0.033}q_{52.8} + S \times {}_{0.033|}A^{(12)\,1}_{[52.8]:\overline{7.167}|} = 6\;614.75$$

- EPV$_{2.8}$ of future premiums less expenses:



$$0.9 \times 4P \times {}_{0.2|}\ddot{a}^{(4)}_{[52.8]:\overline{2}|} = 3\;138.59$$

- Policy value at time 2.8:

$${}_{2.8}V = 6\;614.75 - 3\;138.59 = 3\;476.16$$

Example 7.12 - solution

- In the following figure, the (gross premium) policy values ${}_tV$ at all durations t are shown for the policy in Example 7.12.
- After each premium payment, the function ${}_tV$ jumps upwards by an amount $0.9 \times P$.

$\hookrightarrow {}_{t^+}V = {}_tV + 0.9\,P$ $\quad (t = 0, \frac{1}{4}, \dots, \frac{19}{4})$

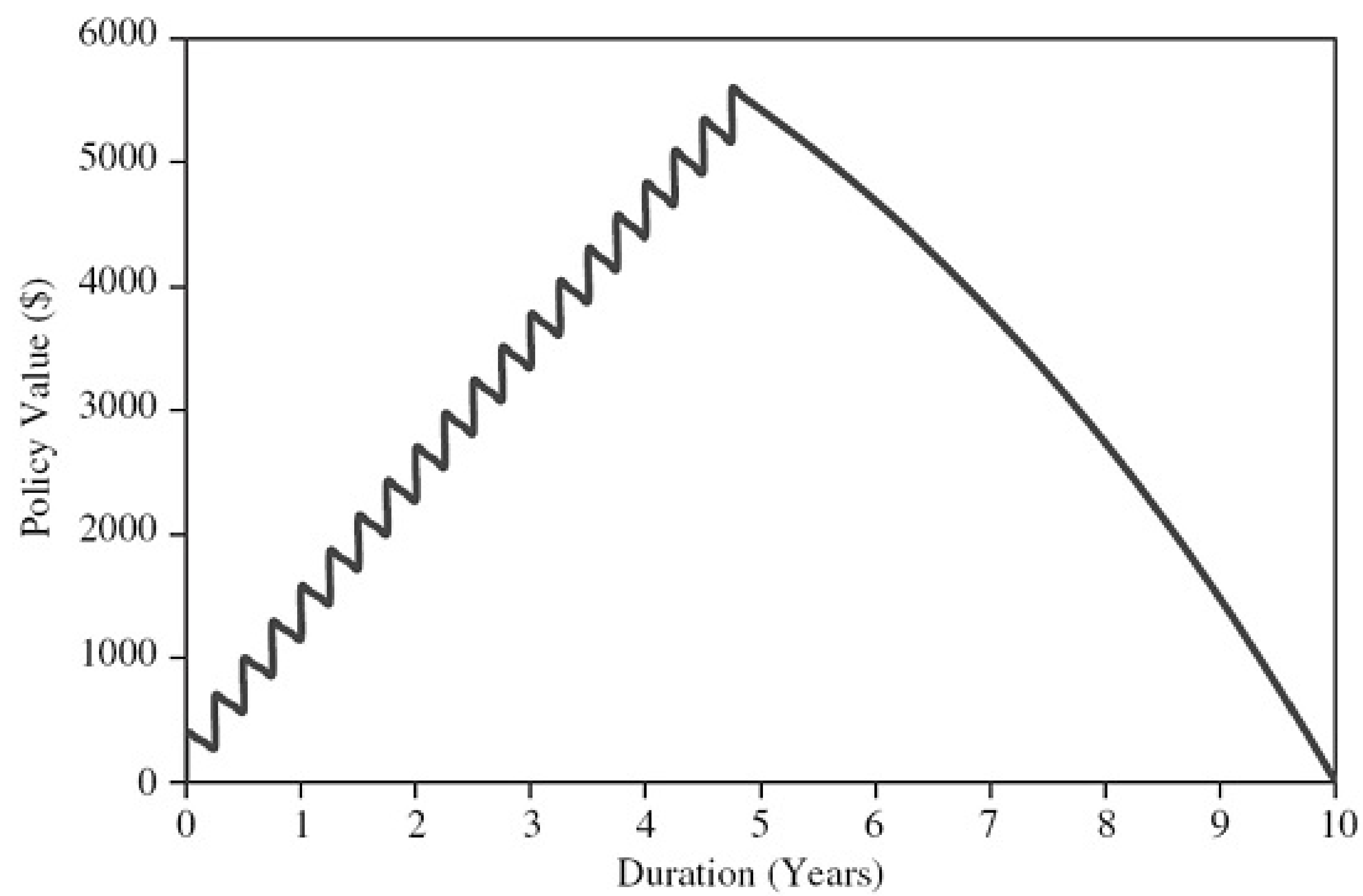


Figure 7.5 Policy values for the limited premium term insurance contract, Example 7.11.

7.3 Policy values for policies with 1/m-thly cash flows

7.3.2 Valuation between premium dates

- Contract:

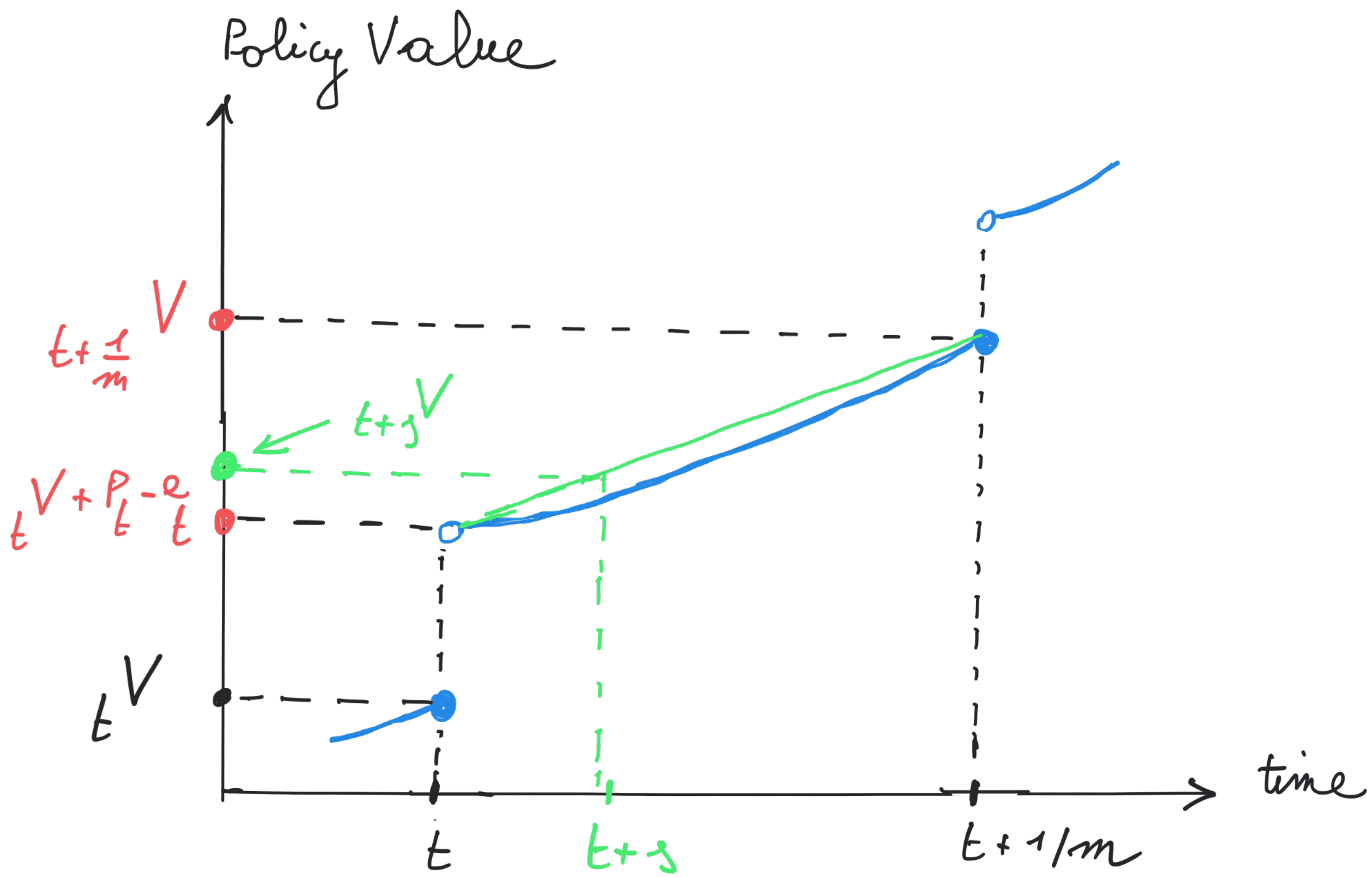
$$\text{Benefit} = \left(S_{K_{[x]}^{(m)} + \frac{1}{m}},\ K_{[x]}^{(m)} + \frac{1}{m} \right)$$

$$\text{Gross premiums} = \sum_{k=0}^{K_{[x]}^{(m)}} \left(P_{\frac{k}{m}},\ \frac{k}{m} \right)$$

$$\text{Expenses} = \sum_{k=0}^{K_{[x]}^{(m)}} \left(e_{\frac{k}{m}},\ \frac{k}{m} \right)$$

- Suppose that t is a premium date. $(t = 0, \frac{1}{m}, \frac{2}{m}, \cdots)$
- Using **linear interpolation**, the (gross premium) policy value ${}_{t+s}V$, with $0 < s \leq \frac{1}{m}$, is approximated by:

$$ {}_{t+s}V \approx (1 - m \times s) \times ({}_tV + P_t - e_t) + m \times s \times {}_{t+\frac{1}{m}}V $$



Linear interpolation: $_{t+s}V = a \times s + b$ $\quad (0 < s \leq \frac{1}{m})$

↳ at $s = 0$: $_tV + P_t - e_t = b$

↳ at $s = \frac{1}{m}$: $_{t+\frac{1}{m}}V = a \times \frac{1}{m} + b$

↳ Solving for a and b leads to the formula.

A general contract - discrete case

- Death benefit $S_{K_{[x]}+1}$ + related expense $E_{K_{[x]}+1}$:

$$\left(S_{K_{[x]}+1} + E_{K_{[x]}+1},\ K_{[x]}+1\right)$$

- Gross premiums P_t - related expenses e_t:

$$\sum_{t=0}^{K_{[x]}} \left(\left(P_t - e_t\right),\ t\right)$$

- Policy value basis: $(s \geq t)$
  - Probabilities $q_{[x]+s}$.
  - Interest rates i_s.
  - Expences e_s and E_s.

$D_{t+1} = S_{t+1} + E_{t+1} - {}_{t+1}V$

- **Recursion**: interest + premiums − expected death payment

(7.8) ⇒

$$ {}_{t+1}V - {}_{t}V = i_t \times {}_{t}V + (P_t - e_t) \times (1 + i_t) - D_{t+1} \times q_{[x]+t} \tag{7.10}$$

7.4 Policy values with continuous cash flows

7.4.1 Thiele's differential equation

Thorvald N. Thiele (1838–1910)

A general contract - continuous case

- Death benefit $S_{T_{[x]}}$ + related expense $E_{T_{[x]}}$:

$$\left(S_{T_{[x]}} + E_{T_{[x]}},\ T_{[x]} \right)$$

- Gross premiums P_t - related expenses e_t:

$$\int_0^{T_{[x]}} ((P_t - e_t)\ dt,\ t)$$

- Policy value basis: $(s \geq t)$
  - Forces-of-mortality $\mu_{[x]+s}$.
  - Interest intensities δ_s.
  - Expence functions e_s and E_s.

assumption: $S_s, E_s, P_s, e_s, \mu_{[x]+s}, \delta_s$ are all continuous functions of s

- **Thiele's differential equation**:

$D_t = S_t + E_t - {}_tV$

interest + premiums − expected death payment

$$\frac{d}{dt}\ {}_tV = \delta_t \times {}_tV + (P_t - e_t) - D_t \times \mu_{[x]+t} \tag{7.15}$$

7.4 Policy values with continuous cash flows

7.4.1 Numerical solution of Thiele's differential equation

(Read in book)

7.5 Policy alterations

A policyholder may request a **change in the terms of his policy**:

- At any time, the policyholder may **cancel his policy**:
  - The policy is said to *lapse* or to be *surrendered*.
  - For policies with a substantive investment objective (in addition to protection against death), at least part of the funds should be considered to be the policyholder's, under the stewardship of the insurer.
  - Therefore, it may be appropriate (or a legal obligation) for the insurer to 'attribute' a lump sum, called the *cash value* or the *surrender value*, to the policyholder if he wants to cancel his policy.

- The policyholder may **stop paying premiums**, but the policy is continued on a reduced basis:
  - The policy is said to be *paid-up*.
  - The reduced sum insured is called the paid-up sum insured.

7.5 Policy alterations

- A whole life insurance policy may be converted to a paid-up term insurance policy for the original sum insured.
- Some term insurance policies carry an option to convert to a whole life policy at certain times.
- Other possible policy alterations:
    - reducing or increasing premiums,
    - changing the amount of the benefits,
    - converting a whole life insurance to an endowment insurance,
    - converting a non-participating policy to a with-profit policy.
    - ...

7.5 Policy alterations

Surrender of a policy at time t :

- Let CV_t be the *cash surrender value* at time t.
- CV_t could be pre-specified (e.g. as a percentage of the sum insured),
- or CV_t could be based on the AS_t or ${}_tV$
- In case the policy is surrendered, setting CV_t equal to AS_t or ${}_tV$ may be over-generous:
  - Alteration of a contract involves **expenses**.
  - The insurer may have to realize assets, causing **liquidity risk**.
- In order to avoid **adverse selection** (selection against the insurer), CV_t may not (or only partially) be paid out, but (partially) used as a single premium for an altered contract for the policyholder. (Example: pure endowment)

7.5 Policy alterations

Policy alterations other than surrender at time t :

- The existing contract (with future gross premiums, benefits and expenses) is transformed into a new contract (with new future gross premiums, benefits and expenses).
- CV_t is considered as an extra premium at time t for the altered contract.
- **Equivalence relation** (at initiation of the altered contract):

$$\mathrm{CV}_t + \mathrm{EPV}_t \ [\text{future gross premiums of altered contract}]$$
$$=$$
$$\mathrm{EPV}_t \ [\text{future benefits and expenses of altered contract}]$$

7.5 Policy alterations

Example 7.14

- Consider the **contract** of Example 7.4 with

$$\text{benefits} = 10\,000 \ {}_{10|}\ddot{a}_{[50]} + P\ (IA)^{\ 1}_{[50]:\overline{10}|}$$

and

$$\text{gross premiums} = P\ \ddot{a}_{[50]:\overline{10}|} \qquad \text{with} \qquad P = 11\,900$$

- Basis used for policy values and policy alterations:
  - Interest: $i = 5\%$.
  - Survival model: SSSM.
  - Expenses:

$$(0.05P;\ 0) + 0.05 \times P\ \ddot{a}_{[50]:\overline{10}|} + 100\ A^{\ 1}_{[50]:\overline{10}|} + 25\ {}_{10|}\ddot{a}_{[50]}$$

- Suppose we have arrived at time 5.
- Insurer's experience over first 5 years: see Example 7.9.

7.5 Policy alterations

Example 7.14 (continued)

(a) **Surrendering the policy**:

- Suppose that the policy is surrendered at time 5.
- Calculate the cash surrender value, assuming the insurer uses

$$CV_5 = 0.9 \times AS_5 - 200$$

- Calculate the cash surrender value, assuming the insurer uses

$$\mathrm{CV}_5 = 0.9 \times \ {}_5\mathrm{V} - 200$$

- **Solution**:

$$\mathrm{CV}_5 = 0.9 \times \mathrm{AS}_5 - 200 = 0.9 \times 63\ 509 - 200 = 56\ 958$$

and

$$\mathrm{CV}_5 = 0.9 \times \ {}_5\mathrm{V} - 200 = 0.9 \times 65\ 470 - 200 = 58\ 723$$

7.5 Policy alterations

Example 7.14 (continued)

(b) Transforming the policy in a paid–up policy:

- Suppose that at time 5, the contract is altered as follows:

$$\text{altered future premium level} = 0$$

$$\text{altered future benefits} = X \times {}_{5|}\ddot{a}_{55} + 5P \times A^{\;\;1}_{[55]:\overline{5|}}$$

- Calculate the reduced annuity X in case $\mathrm{CV}_5 = 56\ 958$.
- **Solution**:
  - Equivalence relation:

$$CV_5 = (5P + 100) \times A^{\;\;1}_{[55]:\overline{5|}} + (X + 25) \times {}_{5|}\ddot{a}_{55}$$

  - We find that $X = 4\ 859$.

7.5 Policy alterations

Example 7.14 (continued)

(c) Transforming the deferred annuity in a pure endowment:

- Suppose that at time 5, the contract is altered as follows:

$$\text{altered gross premiums} = P \times \ddot{a}_{55:\overline{5}|} \quad \text{with} \quad P = 11\,900$$

$$\text{altered future benefits} = P \times \left((IA)^{\;1}_{55:\overline{5}|} + 5 \times A^{\;1}_{55:\overline{5}|} \right) + S \times {}_5E_{55}$$

- In case S is paid, an expense of 100 is charged.
- Determine S in case $\text{CV}_5 = 56\,958$.
- **Solution**:
  - Equivalence relation:

$$\text{CV}_5 + 0.95 \times P \times \ddot{a}_{55:\overline{5}|}$$

$$=$$

$$P \times (IA)^{\;1}_{55:\overline{5}|} + (5 \times P + 100) \times A^{\;1}_{55:\overline{5}|} + (S + 100) \times {}_5E_{55}$$

  - We find that $S = 138\,314$.

7.5 Policy alterations

Example 7.15 (read in book).

7.6 Retrospective policy values

(read in book)

7.7 Negative policy values

(read in book).

7.8 Deferred acquisition expenses and modified premium reserves

(read in book)

7.9 Other reserves

(read in book)

7.10 Notes and further reading

(read in book)