

# Life Insurance Mathematics

## Policy values<sup>1</sup>

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<sup>1</sup>Based on Chapter 7 in 'Actuarial Mathematics for Life Contingent Risks' by David C.M. Dickson, Mary R. Hardy and Howard R. Waters, Cambridge University Press, 2020 (third edition).

## 7.1 Summary

- **Policy values**

- for policies with annual (and  $1/m$ -thly) cash flows:

- Definition.
    - Recursive calculation.

- for policies with continuous cash flows:

- Definition.
    - Thiele's differential equation.

- **Analysis of surplus** of a portfolio over an accounting period.

- **Asset share** of a policy at a given time.

- **Policy alterations.**

# Introduction

- **Premium basis**

= technical basis used for determining the premiums.

- **Convention:**

Unless explicitly stated otherwise, the *premium basis* in all examples of this chapter uses the following survival model and interest:

- **Standard Select Survival Model** (SSSM):

- *Ultimate part:*

$$\mu_x = 0.00022 + (2.7 \times 10^{-6}) \times (1.124)^x$$

- *Select part:*

$$\mu_{[x]+s} = 0.9^{2-s} \mu_{x+s} \quad \text{for } 0 \leq s \leq 2$$

- **Interest:** 5%.

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

- Consider a **life insurance contract** underwritten on  $(x)$  at time 0.
- $PV_0 \stackrel{\text{not.}}{=} \text{Present Value at contract initiation (= time 0).}$
- **Net loss at issue:**

$$L_0^n = PV_0 [\text{future benefits}] - PV_0 [\text{future net premiums}]$$

- **Gross loss at issue:**

$$L_0^g = PV_0 [\text{future benefits}] + PV_0 [\text{future expenses}] \\ - PV_0 [\text{future gross premiums}]$$

- $L_0^n$  and  $L_0^g$  are functions of  $T_{[x]}$ .

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

- Suppose that we are at time  $t$  and that the life insurance contract underwritten on  $(x)$  is still in force.
- $PV_t^{\text{not.}}$  Present Value at time  $t$ .
- **Net future loss at time  $t$ :**

$$L_t^n = PV_t [\text{future benefits}] - PV_t [\text{future net premiums}]$$

- **Gross future loss at time  $t$ :**

$$L_t^g = PV_t [\text{future benefits}] + PV_t [\text{future expenses}] \\ - PV_t [\text{future gross premiums}]$$

- $L_t^n$  and  $L_t^g$  are functions of  $T_{[x]+t}$ .

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### Example 7.1 - An endowment contract

- Consider the contract with

- Benefits:

$$500\,000 \, A_{[50]:\overline{20}|}$$

- Net premiums:

$$P \, \ddot{a}_{[50]:\overline{20}|}$$

- Determine the net premium  $P$ .
- Calculate  $\mathbb{E}[L_t^n]$  for  $t = 10$  and  $11$ , just before the premium due at time  $t$  is paid.
- Assumption: The basis used for all calculations is the premium basis.

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### Example 7.1 - solution

- Annual net premium  $P$ :

$$\mathbb{E}[L_0^n] = 500\,000 A_{[50]:\overline{20}|} - P \ddot{a}_{[50]:\overline{20}|} = 0 \quad (7.1)$$

$$\Rightarrow P = 15\,114.33$$

- Expectation of net future loss at time 10:

$$\mathbb{E}[L_{10}^n] = 500\,000 A_{60:\overline{10}|} - P \ddot{a}_{60:\overline{10}|} = 190\,339 \quad (7.2)$$

- Expectation of net future loss at time 11:

$$\mathbb{E}[L_{11}^n] = 500\,000 A_{61:\overline{9}|} - P \ddot{a}_{61:\overline{9}|} = 214\,757$$

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### More on Example 7.1

- Notation:

$$\text{EPV}_t [\text{future cash flow}] = \mathbb{E}_{T_{[x]+t}} [\text{PV}_t [\text{future cash flow}]]$$

- Any year  $t = 1, 2, \dots, 19$ :

$$\text{EPV}_{t-1} [\text{Premiums in } [t-1, t)] = P = 15\,114.33$$

$$\text{EPV}_{t-1} [\text{Benefits in } (t-1, t)] = 500\,000 \times v \times q_{[50]+t-1} < P$$

- Last year:

$$\text{EPV}_{19} [\text{Premiums in } [19, 20)] = P$$

$$\text{EPV}_{19} [\text{Benefits in } (19, 20)] = 500\,000 \times v = 476\,190 > P$$



## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### More on Example 7.1

- Excess in year  $t = 1, 2, \dots, 20$ :

$$\text{EPV}_{t-1} [\text{Premiums in } [t-1, t) - \text{Benefits in } (t-1, t]]$$

- Excess in year  $t = 1, 2, \dots, 19$ :

$$P - 500\,000 \times v \times q_{[50]+t-1} > 0$$

- Excess in year 20:

$$P - 500\,000 v < 0$$

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### More on Example 7.1

- The excesses in years 1, 2, ..., 19 are used to build up insurer's assets, needed to fulfill his liabilities at time 20:

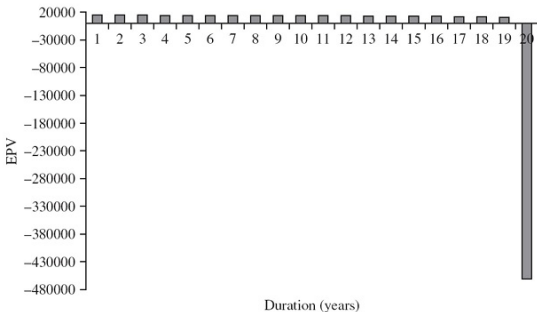


Figure 7.1 EPV of premiums minus claims for each year of a 20-year endowment insurance, sum insured \$500 000, issued to (50).

## 7.3 Policies with annual cash flows

### 7.3.1 The future loss r.v.

#### A term insurance contract

- Contract:

$$\text{Benefits} = 500\,000 \, A_{[50]:\overline{20}|}^1$$

$$\text{Premiums} = P \, \ddot{a}_{[50]:\overline{20}|}$$

- Compared with the endowment contract of Example 7.1, the term insurance contract:
  - has lower premiums,
  - has same benefits in first 19 years,
  - has lower yearly excesses in first 19 years.
  - has a higher (less negative) excess in 20th year.

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### A term insurance contract (cont'd)

- The positive excesses in the early years are used to build up the insurer's assets.
- These assets are needed in the later years when the yearly premium is not sufficient to cover the yearly benefit.

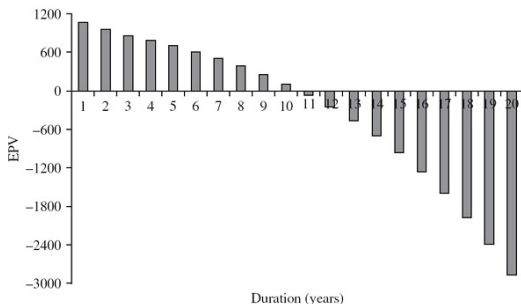


Figure 7.2 EPV of premiums minus claims for each year of a 20-year term insurance, sum insured \$500 000, issued to (50).

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.

#### Back to Example 7.1

- Suppose that at time 0 the insurer issues the following policy to each of  $N$  independent lives of age 50:

$$\text{Benefits} = 500\,000 \, \underline{A}_{[50]:\overline{20}|}$$

$$\text{Premiums} = P \, \underline{\ddot{a}}_{[50]:\overline{20}|}$$

- Suppose that we have arrived at at time 10.
- Suppose that the experience observed for these  $N$  policies over the first 10 years is precisely equal to the assumed premium basis at policy issue:
  - interest on investments = 5%,
  - observed mortality = SSSM.

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v. (back to Example 7.1)

- Accumulated value (at time 10) of past premium income:

$$\begin{aligned} & N \times P \times \left( 1.05^{10} + p_{[50]} \times 1.05^9 + \dots + {}_9p_{[50]} \times 1.05^0 \right) \\ &= 1.05^{10} \times N \times P \times \ddot{a}_{[50]:\overline{10}|} \end{aligned}$$

- Accumulated value (at time 10) of past benefit outgo:

$$\begin{aligned} & N \times 500\,000 \times \left( q_{[50]} \times 1.05^9 + \dots + {}_9q_{[50]} \times 1.05^0 \right) \\ &= 1.05^{10} \times N \times 500\,000 \times A_{[50]:\overline{10}|}^1 \end{aligned}$$

- Portfolio assets at time 10:

$$\begin{aligned} & 1.05^{10} \times N \times \left( P \ddot{a}_{[50]:\overline{10}|} - 500\,000 A_{[50]:\overline{10}|}^1 \right) \\ &= N \times 186\,634 \end{aligned}$$

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.(back to Example 7.1)

- **Asset share at time 10:**

$$AS_{10} = \frac{\text{portfolio assets at time 10}}{\text{survivors at time 10}} = \frac{N \times 186\,634}{N \times {}_{10}p_{[50]}} = 190\,339$$

- The asset share at time at time 10 can be expressed as:

$$AS_{10} = \left( P \ddot{a}_{[50]:\overline{10}|} - 500\,000 A_{[50]:\overline{10}|}^1 \right) \times \left( {}_{10}E_{[50]} \right)^{-1}$$

## 7.2 Policies with annual cash flows

### 7.2.1 The future loss r.v.(back to Example 7.1)

- **Asset share at time 10:**

$$AS_{10} = \left( P \ddot{a}_{[50]:\overline{10}|} - 500\,000 A_{[50]:\overline{10}|}^1 \right) \times \left( {}_{10}E_{[50]} \right)^{-1} = 190\,339$$

- **Expected future loss at time 10:**

$$\mathbb{E}[L_{10}^n] = 500\,000 A_{60:\overline{10}|} - P \ddot{a}_{60:\overline{10}|} = 190\,339$$

- This **important equality** between 'available assets' and 'required assets' per policy in force at time 10 holds under following conditions:
  - premium determined according to the equivalence principle,
  - experience followed assumptions of premium basis,
  - $\mathbb{E}[L_{10}^n]$  calculated with premium basis.



## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

- **Definition 7.2** The **gross premium policy value** for a policy still in force at time  $t$  is defined by

$${}_tV^g = \mathbb{E}[L_t^g]$$

- **Equivalence relation:**

$${}_tV^g = \text{EPV}_t[\text{future (benefits + expenses - gross premiums)}]$$

- **Interpretation:**  ${}_tV^g$  is the actuarial value at time  $t$  of the insurer's future liability related to that policy, *given that the insured is still alive at  $t$ .*
- **Calculation of  ${}_tV^g$ :**
  - Gross premiums, determined at policy issue with the premium basis, are given as input.
  - ${}_tV^g$  is then determined with the **policy value basis**.

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Conventions:

- Time  $t$ , is the time *just after benefits* (and related expenses) and *just before premiums* (and related expenses) due at that time.
- Time  $t^-$  is the time *just before benefits* (and related expenses) and *just before premiums* (and related expenses) due at time  $t$ .
- Time  $t^+$  is the time *just after benefits* (and related expenses) and *just after premiums* (and related expenses) due at time  $t$ .

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

- Suppose the policy is still in force at time  $t$ .
- The gross policy value  ${}_tV^g$  is calculated:
  - **just after** the **survival benefits** (and related expenses) due at time  $t$  have been paid,
  - **just before** the **premiums** (and related expenses) due at  $t$  have been paid .
- Convention: If a policy has a finite term of  $n$  years, then

$${}_nV^g = {}_{n-}V^g$$

- $n$ -year term insurance:

$${}_nV^g = 0$$

- $n$ -year endowment insurance:

$${}_nV^g = \text{survival benefit at time } n$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

- **Definition 7.1** The **net premium policy value** for a policy still in force at time  $t$  is defined by

$$\boxed{{}_tV^n = \mathbb{E}[L_t^n]} = \text{EPV}_t[\text{future (benefits - net premiums)}]$$

- **First method for calculating  ${}_tV^n$ :**

- Net premiums, determined with the *premium basis*, are given as input.
- ${}_tV^n$  is then determined with the **policy value basis**.

- **Second method for calculating  ${}_tV^n$ :**

- Net premiums, determined with the *policy value basis*, are given as input.
- ${}_tV^n$  is then determined with the **policy value basis**.

- **Remark:** Second method is a remainder of a time before modern computers. It is still widely used in the USA and is the standard approach in D,H,W (2020).

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.2

- Contract:

$$\boxed{\text{benefits} = 100\,000 \, \underline{A}_{[50]}} \quad \text{and} \quad \boxed{\text{gross premiums} = 1370 \, \ddot{a}_{[50]}}$$

- Calculate  ${}_5V^g$ .
- Policy value basis at time 5:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$\boxed{0.125 \times 1300 \, \ddot{a}_{55}}$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.2 - solution

- Gross future loss r.v. at time 5:

$$\begin{aligned} {}_5L^g &= PV_5 [100\,000 \, \underline{A}_{55} + 0.125P \, \underline{\ddot{a}}_{55} - P \, \underline{\ddot{a}}_{55}] \\ &= PV_5 [100\,000 \, \underline{A}_{55} - 0.875 P \, \underline{\ddot{a}}_{55}] \end{aligned}$$

- Gross policy value at time 5 :

$$\begin{aligned} {}_5V^g &= \mathbb{E} [L_5^g] = 100\,000 \, A_{55} - 0.875 P \, \ddot{a}_{55} \\ &= 4\,272.68 \end{aligned}$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.3

- Contract:

$$\boxed{\text{benefits} = 100\,000 \, A_{[60]:\overline{20}|}} \quad \text{and} \quad \boxed{\text{gross premiums} = P \, \ddot{a}_{[60]:\overline{10}|}}$$

with  $P = 5\,200$ .

- Calculate the gross policy values  ${}_tV$  at times  $t = 0, 5$  and  $10$ .
- Policy value basis at time  $t$ : see below.

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.3 - solution

- Policy value basis at time 0:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$200 A_{[60]:\overline{20}|} + (0.05 P; 0) + 0.05 P \ddot{a}_{[60]:\overline{10}|}$$

- Gross policy value at time 0:

$$\begin{aligned} {}_0V^g &= 100 - 200 A_{[60]:\overline{20}|} + 0.05 \times P - 0.95 \times P \ddot{a}_{[60]:\overline{10}|} \\ &= 2023 > 0 \end{aligned}$$



## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.3 - solution

- Policy value basis at time 5:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$200 \bar{A}_{65:\overline{15}|} + 0.05 P \ddot{a}_{65:\overline{5}|}$$

- Gross policy value at time 5:

$$\begin{aligned} {}_5V^g &= 100 - 200 \bar{A}_{65:\overline{15}|} - 0.95 \times P \ddot{a}_{65:\overline{5}|} \\ &= 29\,068 \end{aligned}$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.3 - solution

- Policy value basis at time 10:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$200 \, \underline{A}_{70:\overline{10}|}$$

- Gross policy value at time 10:

$${}_{10}V^g = 100 - 200 \, \underline{A}_{70:\overline{10}|} = 63\,073$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.4

- Consider the **contract** with

$$\text{benefits} = 10\,000 \, {}_{10|}\ddot{a}_{[50]} + P (\underline{IA})^1_{[50]:\overline{10}|}$$

and

$$\text{gross premiums} = P \, \ddot{a}_{[50]:\overline{10}|} \quad \text{with} \quad P = 11\,900$$

- Calculate the gross policy values at times 0, 5,  $15^-$  and  $15^+$ .
- Policy value basis at time  $t$ : see below.

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.4 - solution

- Policy value basis at time 0:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$(0.05P; 0) + 0.05 \times P \ddot{a}_{[50]:\overline{10}|} + 100 A_{[50]:\overline{10}|}^1 + 25 {}_{10|}\ddot{a}_{[50]}$$

- Gross policy value at time 0:

$$\begin{aligned} {}_0V^g &= P (IA)_{[50]:\overline{10}|}^1 + 100 A_{[50]:\overline{10}|}^1 + 10 \cdot 0.25 {}_{10|}\ddot{a}_{[50]} \\ &\quad + 0.05 \times P - 0.95 \times P \ddot{a}_{[50]:\overline{10}|} \\ &= 485 \end{aligned}$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.4 - solution

- Policy value basis at time 5:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$0.05 \times P \ddot{a}_{55:\overline{5}|} + 100 A_{55:\overline{5}|}^1 + 25 {}_5|\ddot{a}_{55}$$

- Gross policy value at time 5 :

$$\begin{aligned} {}_5V^g &= P \left( (IA)_{55:\overline{5}|}^1 + 5 A_{55:\overline{5}|}^1 \right) + 100 A_{55:\overline{5}|}^1 + 10\,025 {}_5|\ddot{a}_{55} \\ &\quad - 0.95 \times P \ddot{a}_{55:\overline{5}|} \\ &= 65\,470 \end{aligned}$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Example 7.4 - solution

- Policy value basis at time 15:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$\boxed{{}_{25}\ddot{a}_{65}}$$

- Gross policy value at time  $15^-$ :

$${}_{15^-}V^g = 10\,025 \ddot{a}_{65} = 135\,837$$

- Gross policy value at times 15 and  $15^+$ :

$${}_{15}V = {}_{15^+}V^g = 10\,025 a_{65} = 125\,812$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### **Endowment insurance contract**

- Benefits and net premiums:

$$\text{benefits} = 500\,000 \, A_{[50]:\overline{20}|}$$

$$\text{net premiums} = P \, \ddot{a}_{[50]:\overline{20}|}$$

- Net policy values ( $t = 0, 1, \dots, 19$ ):

$${}_tV^n = 500\,000 \, A_{[50]+t:\overline{20-t}|} - P \, \ddot{a}_{[50]+t:\overline{20-t}|}$$

- Time  $t = 0, 1, \dots, 19$ :

$${}_tV^n = {}_{t-}V^n = {}_{t+}V^n - P$$

- Time 20:

$${}_{20}V^n = {}_{20-}V^n = 500\,000$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Endowment insurance contract (cont'd)

- Benefits and net premiums:

$$\text{benefits} = 500\,000 \, A_{[50]:\overline{20}|}$$

$$\text{net premiums} = P \, \ddot{a}_{[50]:\overline{20}|}$$

- Policy values:

${}_tV^n$  ( $t = 0, 1, \dots, 19$ ), calculated with premium basis:

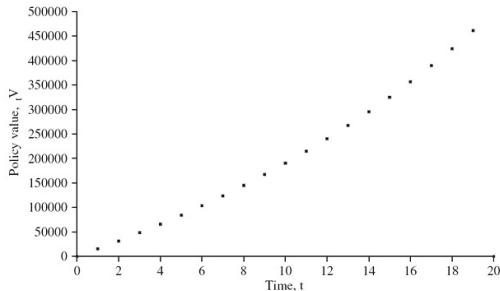


Figure 7.3 Policy values for each year of a 20-year endowment insurance, sum insured \$500 000, issued to (50).



## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### **Endowment insurance contract** (cont'd)



## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Term insurance contract

- Benefits and net premiums:

$$\text{benefits} = 500\,000 A_{[50]:\overline{20}|}^1$$

$$\text{net premiums} = P \ddot{a}_{[50]:\overline{20}|}$$

- Net policy values ( $t = 0, 1, \dots, 19$ ):

$${}_tV^n = 500\,000 A_{[50]+t:\overline{20-t}|}^1 - P \ddot{a}_{[50]+t:\overline{20-t}|}$$

- Time  $t = 0, 1, \dots, 19$ :

$${}_tV^n = {}_{t-}V^n = {}_{t+}V^n - P$$

- Time 20:

$${}_{20}V^n = {}_{20-}V^n = 0$$

## 7.2 Policies with annual cash flows

### 7.2.2 Policy values for policies with annual cash flows

#### Term insurance contract (cont'd)

- Benefits and net premiums:

$$\text{benefits} = 500\,000 A^1_{\overline{[50]:20}|}$$

$$\text{net premiums} = P \ddot{a}_{\overline{[50]:20}|}$$

- Policy values:

${}_tV^n$  ( $t = 0, 1, \dots, 19$ ), calculated with premium basis:

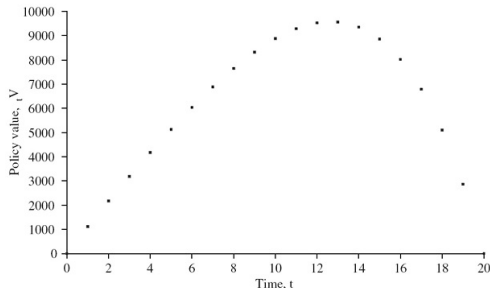


Figure 7.4 Policy values for each year of a 20-year term insurance, sum insured \$500,000, issued to (50).

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.5

- Consider the **contract** with

- Benefits:

$$S \, \underline{A}_{[50]:\overline{20}|}$$

- Net premiums:

$$P \, \underline{\ddot{a}}_{[50]:\overline{20}|}$$

- Question: Determine a backward recursion for the policy values  ${}_tV$ , for  $t = 0, 1, \dots, 20$ .
- Assumption: Policy values determined with fixed policy value basis (which may be different from premium basis).

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.5 - solution

- Policy value  ${}_{20}V$ :

$${}_{20}V = S$$

- Policy value  ${}_{19}V$ :

- Definition:

$${}_{19}V = S \underline{A}_{[50]+19:\overline{1}} - P$$

- Recursion:

$$({}_{19}V + P)(1 + i) = S$$

- Interpretation of recursion: Having arrived at time 19, the following equality must hold for each policy in force:

**available assets** at time  $20^-$  = **required assets** at time  $20^-$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.5 - solution

- Policy value  ${}_tV$  at  $t = 18, 17, \dots, 0$ :

- Definition:

$${}_tV = S \, \overline{A}_{[50]+t:\overline{20-t}|} - P \, \overline{\ddot{a}}_{[50]+t:\overline{20-t}|}$$

- Recursion:

$$\boxed{({}_tV + P)(1 + i) = S \times q_{[50]+t} + {}_{t+1}V \times p_{[50]+t}} \quad (7.4)$$

- Interpretation: Having arrived at time  $t$ , the following equality must hold for each policy in force:

$$\boxed{\text{available assets at time } t + 1^- = \text{required assets at time } t + 1^-}$$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.6

- Consider the **contract** with

$$\text{benefits} = 10\,000 \, {}_{10|}\ddot{a}_{[50]} + P (\underline{IA})^1_{[50]:\overline{10}}$$

and

$$\text{gross premiums} = P \, {}_{\ddot{a}_{[50]:\overline{10}}} \quad \text{with} \quad P = 11\,900$$

- Question: Determine a backward recursion for the gross policy values  ${}_tV$ , for  $t = 0, 1, \dots, 9$ .
- Policy value basis:
  - Interest:  $i = 5\%$ .
  - Survival model: SSSM.
  - Expenses:

$$(0.05P; 0) + 0.05 \times P \, {}_{\ddot{a}_{[50]:\overline{10}}} + 100 \, A^1_{[50]:\overline{10}} + 25 \, {}_{10|}\ddot{a}_{[50]}$$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.6 - solution

- Policy value  $_{10}V$ :

$$_{10}V = 10\,025\, a_{60}$$

- Policy value  $_9V$ :

- Definition:

$$_9V = (10P + 100) A_{59:\overline{1}|}^1 + 10\,025\, {}_1|\ddot{a}_{59} - 0.95\, P$$

- Recursion:

$$({}_9V + 0.95P)(1 + i) = (10P + 100) q_{59} + (10\,025 + {}_{10}V) p_{59}$$

- Interpretation: Having arrived at time 9, the following equality must hold for each policy in force:

$$\text{available assets at time } 10^- = \text{required assets at time } 10^-$$



## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.6 - solution (cont'd)

- Policy value  ${}_tV$  at  $t = 8, \dots, 1$ :

- Definition:

$${}_tV = P (\underline{IA})_{[50]+t:\overline{10-t}}^1 + (t \times P + 100) A_{[50+t]:\overline{10-t}}^1 \\ + 10\,025 {}_{10-t}|\ddot{a}_{[50]+t} - 0.95 P \ddot{a}_{[50+t]:\overline{10-t}}$$

- Recursion:

$$({}_tV + 0.95P)(1+i) = ((t+1)P + 100) q_{[50]+t} + {}_{t+1}V p_{[50]+t}$$

- Policy value  ${}_0V$ :

- Definition:

$${}_0V = P (\underline{IA})_{[50]:\overline{10}}^1 + 10\,000 {}_{10}|\ddot{a}_{[50]+10} - P \ddot{a}_{[50]:\overline{10}} + \dots$$

- Recursion:

$$({}_0V + 0.9P)(1+i) = (P + 100) \times q_{[50]} + {}_1V \times p_{[50]}$$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### A general contract

- Death benefits  $S_j$  + related expenses  $E_j$ :

$$\sum_{j=0}^{\infty} (S_{j+1} + E_{j+1}) {}_j|A_{[x]:1}^1$$

- Gross premiums  $P_j$  - related expenses  $e_j$ :

$$\sum_{j=0}^{\infty} (P_j - e_j) {}_jE_{[x]}$$

- Suppose that the policy is still in force after  $t$  years.
- Determine the gross policy value  ${}_tV$  as a function of  ${}_{t+1}V$ .
- Policy value basis for determining  ${}_tV$  and  ${}_{t+1}V$ :
  - Probabilities  $q_{[x]+j}$  for  $j \geq t$ .
  - Interest rates  $i_j$  in year  $(j, j+1)$  for  $j \geq t$ .
  - Expenses  $e_j$  for  $j \geq t$  and  $E_j$  for  $j > t$ .

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### A general contract (cont'd)

- Policy value  ${}_tV$ : ( $t = 0, 1, 2, \dots$ )

$${}_tV = \sum_{j=0}^{\infty} (S_{t+j+1} + E_{t+j+1}) {}_j|A_{[x]+t:\overline{1}}^1 - \sum_{j=0}^{\infty} (P_{t+j} - e_{t+j}) {}_jE_{[x]+t}$$

- Recursion:

$$({}_tV + P_t - e_t)(1 + i_t) = (S_{t+1} + E_{t+1}) \times q_{[x]+t} + {}_{t+1}V \times p_{[x]+t} \quad (7.7)$$

- Interpretation: Having arrived at time  $t$ , the following equality must hold for each policy in force:

**available assets** at time  $t + 1^- =$  **required** assets at time  $t + 1^-$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### A general contract (cont'd)

- **Sum at Risk** at time  $t + 1$  (Death Strain at Risk, Net Amount at Risk):
  - Definition: (for  $t = 0, 1, 2, \dots$ )

$$\mathbf{D}_{t+1} = S_{t+1} + E_{t+1} - {}_{t+1}V$$

- Recursion for policy values:

$$({}_tV + P_t - e_t)(1 + i_t) = {}_{t+1}V + \mathbf{D}_{t+1} \times q_{[x]+t} \quad (7.8)$$

- Special case: a pure savings contract:

$$\mathbf{D}_{t+1} = 0, \quad t = 0, 1, \dots$$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### **A general contract (cont'd)**

- **Savings premiums:**

- Definition: (for  $t = 0, 1, 2, \dots$ )

$$P_t^{\text{saving}} = {}_{t+1}V \times \frac{1}{1+i_t} - {}_tV$$

- **Risk premiums:**

- Definition: (for  $t = 0, 1, 2, \dots$ )

$$P_t^{\text{risk}} = q_{[x]+t} \times \frac{1}{1+i_t} \times D_{t+1}$$

- **Decomposing the premiums:**

$$P_t - e_t = P_t^{\text{saving}} + P_t^{\text{risk}}$$

- Special case: **a pure savings contract.**

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### **A more general contract** (not in book)

- Death and survival benefits + related expenses:

$$\sum_{j=0}^{\infty} (S_{j+1} + E_{j+1}) {}_j|A_{[x]:1}^1 + \sum_{j=0}^{\infty} (L_{j+1} + F_{j+1}) {}_j|E_{[x]}$$

- Gross premiums - related expenses:

$$\sum_{j=0}^{\infty} (P_j - e_j) {}_j|E_{[x]}$$

- Suppose that the policy is still in force after  $t$  years.
- Determine the gross policy value  ${}_tV$  in terms of  ${}_{t+1}V$ .
- Policy value basis for determining  ${}_tV$  and  ${}_{t+1}V$ :
  - Probabilities  $q_{[x]+j}$  for  $j \geq t$ .
  - Interest rates  $i_j$  in year  $(j, j+1)$  for  $j \geq t$ .
  - Expenses  $e_j$  for  $j \geq t$  and  $E_j$  for  $j > t$ .

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### A more general contract (cont'd)

- Policy value  ${}_tV$ : (for  $t = 0, 1, 2, \dots$ )

$$\begin{aligned} {}_tV &= \sum_{j=0}^{\infty} (S_{t+j+1} + E_{t+j+1}) {}_j|A_{[x]+t:\overline{1}}^1 \\ &\quad + \sum_{j=0}^{\infty} (L_{t+1+j} + F_{t+1+j}) {}_j|E_{[x]+t} \\ &\quad - \sum_{j=0}^{\infty} (P_{t+j} - e_{t+j}) {}_j|E_{[x]+t} \end{aligned}$$

- Recursion:

$$\begin{aligned} ({}_tV + P_t - e_t)(1 + i_t) &= (S_{t+1} + E_{t+1}) {}_0|q_{[x]+t} \\ &\quad + (L_{t+1} + F_{t+1} + {}_{t+1}V) {}_0|p_{[x]+t} \end{aligned}$$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### A more general contract (cont'd)

- **Sum at Risk** (Death Strain Risk, Net Amount at Risk):

- Definition: (for  $t = 0, 1, 2, \dots$ )

$$\mathbf{D}_{t+1} = S_{t+1} + E_{t+1} - (L_{t+1} + F_{t+1} + {}_{t+1}V)$$

- Recursion for policy values:

$$({}_tV + P_t - e_t)(1 + i_t) = (L_{t+1} + F_{t+1} + {}_{t+1}V) + \mathbf{D}_{t+1} q_{[x]+t}$$

- Special cases:

- A pure savings contract:

$$\mathbf{D}_{t+1} = 0, \quad t = 0, 1, \dots$$

- A pure survival contract:

$$S_{t+1} + E_{t+1} = 0, \quad t = 0, 1, \dots$$



## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### **A more general contract** (cont'd)

- **Savings premiums:**

- Definition: (for  $t = 0, 1, 2, \dots$ )

$$P_t^{\text{saving}} = (L_{t+1} + F_{t+1} + {}_{t+1}V) \times \frac{1}{1+i_t} - {}_tV$$

- **Risk premiums:**

- Definition: (for  $t = 0, 1, 2, \dots$ )

$$P_t^{\text{risk}} = D_{t+1} \times A_{[x]+t:\overline{1}}^1$$

- **Decomposing the premiums:**

$$P_t - e_t = P_t^{\text{saving}} + P_t^{\text{risk}}$$

- **Special cases:**

- **A pure savings contract.**
- **A pure survival contract.**

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### Example 7.7

- Benefits:

$$700\,000 \times {}_{20}E_{[50]} + \sum_{j=0}^{19} {}_jV \times {}_j|A^1_{[50]:\overline{1}|}$$

- Net premiums:

$$P \ddot{a}_{[50]:\overline{20}|} \quad \text{with} \quad P = 23\,500$$

- Question: Determine the policy value at time 15.
- Policy value basis:  $i = 3.5\%$  and SSSM.
- Problem:

$${}_{15}V = 700\,000 \times {}_5E_{65} + \sum_{j=0}^4 {}_{15+j}V \times {}_j|A^1_{65:\overline{1}|} - P \times \ddot{a}_{65:\overline{5}|}$$

## 7.2 Policies with annual cash flows

### 7.2.3 Recursive formulae for policy values

#### **Example 7.7 - solution**

- The policy value at time 20:

$${}_{20}V = 700\,000$$

- The policy value at time 19:

$$({}_{19}V + P) \times 1.035 = {}_{19}V \times q_{69} + 700\,000 \times p_{69}$$

or

$${}_{19}V = 652\,401$$

- The policy value at time 18:

$$({}_{18}V + P) \times 1.035 = {}_{18}V \times q_{68} + {}_{19}V \times p_{68}$$

or

$${}_{18}V = 606\,471$$

- The policy value at times 17, 16, 15:

$${}_{15}V = \dots = 478\,063$$

## 7.2 Policies with annual cash flows

### 7.2.4 Analysis of surplus

- Consider a portfolio of the 'more general contracts' underwritten to a group insureds of age  $x$ .
- **Recursion for policy values** (calculated with same basis):

$$\begin{aligned}({}_tV + P_t - e_t)(1 + i_t) &= (S_{t+1} + E_{t+1}) q_{[x]+t} \\ &\quad + (L_{t+1} + F_{t+1} + {}_{t+1}V) p_{[x]+t}\end{aligned}$$

- **Interpretation:**

- Suppose that at time  $t$ , the insurer sets aside  ${}_tV$  for each 'surviving policy' and that experience in year  $t + 1$  follows the policy value basis.
- Then at time  $t + 1^-$ , the insurer will have the exact amount of funds needed to:
  - pay the death benefits (and related expenses) to all insureds who died in the past year,
  - the survival benefits (and related expenses) to all the survivors,
  - set up the policy value  ${}_{t+1}V$  for all survivors.

## 7.2 Policies with annual cash flows

### 7.2.4 Analysis of surplus

- **In practice**, experienced *interest*, *mortality* and *expenses* in year  $t + 1$  will usually differ from the ones assumed in the policy value basis:

$$\begin{aligned}({}_tV + P_t - e_t^{\text{exp}})(1 + i_t^{\text{exp}}) &\neq (S_{t+1} + E_{t+1}^{\text{exp}}) q_{[x]+t}^{\text{exp}} \\ &\quad + (L_{t+1} + F_{t+1}^{\text{exp}} + {}_{t+1}V) p_{[x]+t}^{\text{exp}}\end{aligned}$$

- **Sources or profit or loss:**

- Experienced expenses < assumed expensed: profit.
- Experienced interest < assumed interest: loss.
- Experienced mortality < assumed mortality: profit or loss.

- **Example 7.8:** read in book.

## 7.2 Policies with annual cash flows

### 7.2.5 Asset shares

- **Policy value**:  ${}_tV$  = amount the insurer needs to have per policy in force at time  $t$ .
  - prospective view,
  - policy value basis.
- **Asset share**:  $AS_t$  = amount the insurer does have per policy in force at time  $t$ .
  - retrospective view,
  - experience basis.
- **Theorem**:
  - If the following conditions hold:
    - premiums calculated using the equivalence principle,
    - ${}_tV$  calculated using premium basis,
    - experience followed premium basis until time  $t$ ,
  - then we have that

$${}_tV = AS_t$$

## 7.2 Policies with annual cash flows

### 7.2.5 Asset shares

#### Example 7.9

- Consider the **contract** with

$$\text{benefits} = 10\,000 \, {}_{10|}\ddot{a}_{[50]} + P (\underline{IA})_{[50]:\overline{10}|}^1$$

and

$$\text{gross premiums} = P \, \ddot{a}_{[50]:\overline{10}|} \quad \text{with} \quad P = 11\,900$$

- Suppose:
  - at time 0, a total number of  $N$  such policies were sold,
  - we have arrived at time 5.

## 7.2 Policies with annual cash flows

### 7.2.5 Asset shares

#### Example 7.9

- Insurer's **experience** over past 5 years for these policies:

- Investment returns:

0.048, 0.056, 0.052, 0.049 and 0.047

- Mortality rates:

$$q_{[50]+t} = 0.0015 \text{ for } t = 0, 1, 2, 3, 4.$$

- Expenses:

- at policy issue:  $0.15 \times P$ ,
- at each premium payment (except time 0):  $0.06 \times P$ ,
- related to paying a death claim: 120.

- Question: Calculate  $AS_t$  at times  $t = 0, 1, 2, 3, 4, 5$ .



## 7.2 Policies with annual cash flows

### 7.2.5 Asset shares

#### Example 7.9 - solution

- Asset share at time 0:  $AS_0 = 0$ .
- Asset share at time 1:
  - Total assets at time 0 is equal to 0.
  - Premium income at time 0:  $0.85 \times P \times N = 10\,115\,N$ .
  - Total assets at time  $1^-$  (before benefits and premiums):  
 $10\,115\,N \times 1.048 = 10\,601\,N$ .
  - Death claims at time 1:  $(P + 120) \times (0.0015\,N) = 18\,N$ .
  - Total assets at time 1 (after benefits, before premiums):  
 $10\,601\,N - 18\,N = 10\,582\,N$ .
  - Survivors at time 1:  $N \times (1 - 0.0015) = 0.9985\,N$ .
  - Asset share at time 1:  $\frac{10\,582\,N}{0.9985\,N} = 10\,598$ .

## 7.2 Policies with annual cash flows

### 7.2.5 Asset shares

#### Example 7.9 - solution (cont'd)

- Asset share in any year:

Table 7.1. *Asset share calculation for Example 7.9.*

Year, $t$	Fund at start of year	Cash flow at start of year	Fund at end of year before death claims	Death claims and expenses	Fund at end of year	Survivors	$AS_t$
1	0	10 115 $N$	10 601 $N$	18 $N$	10 582 $N$	0.9985 $N$	10 598
2	10 582 $N$	11 169 $N$	22 970 $N$	36 $N$	22 934 $N$	0.9985 <sup>2</sup> $N$	23 003
3	22 934 $N$	11 152 $N$	35 859 $N$	54 $N$	35 805 $N$	0.9985 <sup>3</sup> $N$	35 967
4	35 805 $N$	11 136 $N$	49 241 $N$	71 $N$	49 170 $N$	0.9985 <sup>4</sup> $N$	49 466
5	49 170 $N$	11 119 $N$	63 123 $N$	89 $N$	63 034 $N$	0.9985 <sup>5</sup> $N$	63 509

## 7.3 Policy values for policies with 1/m-thly cash flows

### Example 7.10

- Contract:

$$\text{Benefits} = 500\,000 \, \underline{A}_{[50]:\overline{10}|}^{(12)}$$

$$\text{Gross premiums} = 4P \, \ddot{a}_{[50]:\overline{5}|}^{(4)}$$

with  $P = 460$ .

- Question:

Determine (gross premium) policy values  ${}_{2.75}V$ ,  ${}_3V$  and  ${}_{6.5}V$

- Policy value basis:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$0.1 \times 4P \, \ddot{a}_{[50]:\overline{5}|}^{(4)}$$

## 7.3 Policy values for policies with 1/m-thly cash flows

### Example 7.10 - solution

- Policy value at time 2.75:

$${}_{2.75}V = 500\,000 \times A_{52.75:\overline{7.25}|}^{(12)} - 0.9 \times 4P \times \ddot{a}_{52.75:\overline{2.25}|}^{(4)}$$

- Policy value at time 3:

$${}_3V = 500\,000 \times A_{53:\overline{7}|}^{(12)} - 0.9 \times 4P \times \ddot{a}_{53:\overline{2}|}^{(4)}$$

- Policy value at time 6.5:

$${}_{6.5}V = 500\,000 \times A_{56.5:\overline{3.5}|}^{(12)}$$

## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.1 Recursions with 1/m-thly cash flows

#### Example 7.10 (continued)

- Contract:

$$\text{Benefits} = 500\,000 \, {}_1A_{[50]:10}^{(12)}$$

$$\text{Gross premiums} = 4P \, {}_{[50]:5}^{(4)}$$

- Question: Starting from (gross premium) policy value  ${}_3V$ , determine  ${}_{\frac{35}{12}}V$ ,  ${}_{\frac{34}{12}}V$  and  ${}_{\frac{33}{12}}V$  recursively.
- Policy value basis:
  - Interest:  $i = 5\%$ .
  - Survival model: SSSM.
  - Expenses:

$$0.1 \times 4P \, {}_{[50]:5}^{(4)}$$

## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.1 Recursions for 1/m-thly cash flows

#### Example 7.10 (continued) - solution

- Policy value at time  $\frac{35}{12}$ :

$$\frac{35}{12} V \times 1.05^{\frac{1}{12}} = 500\,000 \times \frac{1}{12} q_{50+\frac{35}{12}} + {}_3V \times \frac{1}{12} p_{50+\frac{35}{12}}$$

- Policy value at time  $\frac{34}{12}$ :

$$\frac{34}{12} V \times 1.05^{\frac{1}{12}} = 500\,000 \times \frac{1}{12} q_{50+\frac{34}{12}} + \frac{35}{12} V \times \frac{1}{12} p_{50+\frac{34}{12}}$$

- Policy value at time  $\frac{33}{12}$ :

$$\left( \frac{33}{12} V + 0.9 \times 460 \right) 1.05^{\frac{1}{12}} = 500\,000 \times \frac{1}{12} q_{50+\frac{33}{12}} + \frac{34}{12} V \times \frac{1}{12} p_{50+\frac{33}{12}} \quad (7.9)$$

## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.2 Valuation between premium dates

**Example 7.11** (read in book).

**Example 7.12**

- Contract:

$$\text{Benefits} = 500\,000 \, \underline{A}_{[50]:\overline{10}|}^{(12)}$$

$$\text{Gross premiums} = 4P \, \ddot{a}_{[50]:\overline{5}|}^{(4)}$$

- Policy value basis:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$0.1 \times 4P \, \ddot{a}_{[50]:\overline{5}|}^{(4)}$$

- Question: Calculate the (gross premium) policy value  ${}_{2.8}V$ .

## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.2 Valuation between premium dates

#### Example 7.12

- Remark: Time 2.8 is not a cash flow payment date:

$$2 + \frac{9}{12} < 2.8 < 2 + \frac{10}{12}$$



## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.2 Valuation between premium dates

#### Example 7.12 - solution

- We denote the death benefit by  $S$ .
- EPV<sub>2.8</sub> of future benefits:

$$S \times v^{0.033} \times {}_{0.033}q_{52.8} + S \times {}_{0.033}|A_{52.8:\overline{7.167}|}^{(12)} = 6\,614.75$$

- EPV<sub>2.8</sub> of future premiums less expenses:

$$0.9 \times 4P \times {}_{0.2}|\ddot{a}_{52.8:\overline{2}|}^{(4)} = 3\,138.59$$

- Policy value at time 2.8:

$${}_{2.8}V = 6\,614.75 - 3\,138.59 = 3\,476.16$$

## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.2 Valuation between premium dates

#### Example 7.12 - solution

- In the following figure, the (gross premium) policy values  ${}_tV$  at all durations  $t$  are shown for the policy in Example 7.12.
- After each premium payment, the function  ${}_tV$  jumps upwards by an amount  $0.9 \times P$ .

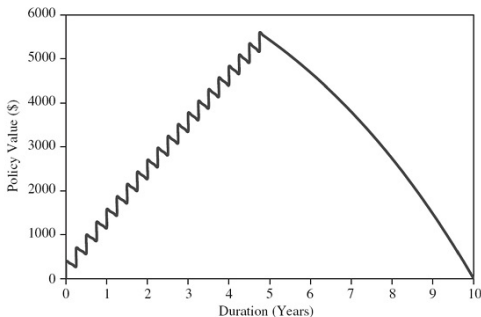


Figure 7.5 Policy values for the limited premium term insurance contract, Example 7.11

## 7.3 Policy values for policies with 1/m-thly cash flows

### 7.3.2 Valuation between premium dates

- Contract:

$$\text{Benefit} = \left( S_{K_{[x]}^{(m)} + \frac{1}{m}}, K_{[x]}^{(m)} + \frac{1}{m} \right)$$

$$\text{Gross premiums} = \sum_{k=0}^{K_{[x]}^{(m)}} \left( P_{\frac{k}{m}}, \frac{k}{m} \right)$$

$$\text{Expenses} = \sum_{k=0}^{K_{[x]}^{(m)}} \left( e_{\frac{k}{m}}, \frac{k}{m} \right)$$

- Suppose that  $t$  is a premium date.
- Using **linear interpolation**, the (gross premium) policy value  $_{t+s}V$ , with  $0 < s \leq \frac{1}{m}$ , is approximated by:

$$_{t+s}V \approx (1 - m \times s) \times ({}_tV + P_t - e_t) + m \times s \times {}_{t+\frac{1}{m}}V$$

## 7.4 Policy values with continuous cash flows

### 7.4.1 Thiele's differential equation

#### A general contract - discrete case

- Death benefit  $S_{K_{[x]}+1}$  + related expense  $E_{K_{[x]}+1}$ :

$$\left( S_{K_{[x]}+1} + E_{K_{[x]}+1}, K_{[x]} + 1 \right)$$

- Gross premiums  $P_t$  - related expenses  $e_t$ :

$$\sum_{t=0}^{K_{[x]}} ((P_t - e_t), t)$$

- Policy value basis:  $(s \geq t)$

- Probabilities  $q_{[x]+s}$ .
- Interest rates  $i_s$ .
- Expenses  $e_s$  and  $E_s$ .

- **Recursion:**

$$\boxed{_{t+1}V - {}_tV = i_t \times {}_tV + (P_t - e_t) \times (1 + i_t) - D_{t+1} \times q_{[x]+t}} \quad (7.10)$$

## 7.4 Policy values with continuous cash flows

### 7.4.1 Thiele's differential equation

#### **A general contract - continuous case**

- Death benefit  $S_{T_{[x]}}$  + related expense  $E_{T_{[x]}}$ :

$$\left( S_{T_{[x]}} + E_{T_{[x]}}, T_{[x]} \right)$$

- Gross premiums  $P_t$  - related expenses  $e_t$ :

$$\int_0^{T_{[x]}} ((P_t - e_t) dt, t)$$

- Policy value basis:  $(s \geq t)$ 
  - Forces-of-mortality  $\mu_{[x]+s}$ .
  - Interest intensities  $\delta_s$ .
  - Expense functions  $e_s$  and  $E_s$ .
- Thiele's differential equation:

$$\boxed{\frac{d}{dt} {}_tV = \delta_t \times {}_tV + (P_t - e_t) - D_t \times \mu_{[x]+t}} \quad (7.15)$$

## 7.4 Policy values with continuous cash flows

### 7.4.1 Numerical solution of Thiele's differential equation

(Read in book)

## 7.5 Policy alterations

A policyholder may request a **change in the terms of his policy**:

- At any time, the policyholder may **cancel his policy**:
  - The policy is said to lapse or to be surrendered.
  - For policies with a substantive investment objective (in addition to protection against death), at least part of the funds should be considered to be the policyholder's, under the stewardship of the insurer.
  - Therefore, it may be appropriate (or a legal obligation) for the insurer to 'attribute' a lump sum, called the *cash value* or the *surrender value*, to the policyholder if he wants to cancel his policy.
- The policyholder may **stop paying premiums**, but the policy is continued on a reduced basis:
  - The policy is said to be paid-up.
  - The reduced sum insured is called the paid-up sum insured.

## 7.5 Policy alterations

- A whole life insurance policy may be converted to a paid-up term insurance policy for the original sum insured.
- Some term insurance policies carry an option to convert to a whole life policy at certain times.
- Other possible policy alterations:
  - reducing or increasing premiums,
  - changing the amount of the benefits,
  - converting a whole life insurance to an endowment insurance,
  - converting a non-participating policy to a with-profit policy.
  - ...



## 7.5 Policy alterations

### Surrender of a policy at time $t$ :

- Let  $CV_t$  be the cash surrender value at time  $t$ .
- $CV_t$  could be pre-specified (e.g. as a percentage of the sum insured),
- or  $CV_t$  could be based on the  $AS_t$  or  ${}_tV$
- In case the policy is surrendered, setting  $CV_t$  equal to  $AS_t$  or  ${}_tV$  may be over-generous:
  - Alteration of a contract involves **expenses**.
  - The insurer may have to realize assets, causing **liquidity risk**.
- In order to avoid **adverse selection** (selection against the insurer),  $CV_t$  may not (or only partially) be paid out, but (partially) used as a single premium for an altered contract for the policyholder.

## 7.5 Policy alterations

### Policy alterations other than surrender at time $t$ :

- The existing contract (with future gross premiums, benefits and expenses) is transformed into a new contract (with new future gross premiums, benefits and expenses).
- $CV_t$  is considered as an extra premium at time  $t$  for the altered contract.
- **Equivalence relation** (at initiation of the altered contract):

$$\begin{aligned} & CV_t + EPV_t \text{ [future gross premiums of altered contract]} \\ = & \\ & EPV_t \text{ [future benefits and expenses of altered contract]} \end{aligned}$$

## 7.5 Policy alterations

### Example 7.14

- Consider the **contract** of Example 7.4 with

$$\text{benefits} = 10\,000 \, {}_{10|}\ddot{a}_{[50]} + P \, (IA)_{[50]:\overline{10}|}^1$$

and

$$\text{gross premiums} = P \, \ddot{a}_{[50]:\overline{10}|} \quad \text{with} \quad P = 11\,900$$

- Basis used for policy values and policy alterations:

- Interest:  $i = 5\%$ .
- Survival model: SSSM.
- Expenses:

$$(0.05P; 0) + 0.05 \times P \, \ddot{a}_{[50]:\overline{10}|} + 100 \, A_{[50]:\overline{10}|}^1 + 25 \, {}_{10|}\ddot{a}_{[50]}$$

- Suppose we have arrived at time 5.
- Insurer's experience over first 5 years: see Example 7.9.

## 7.5 Policy alterations

### Example 7.14 (continued)

#### (a) Surrendering the policy:

- Suppose that the policy is surrendered at time 5.
- Calculate the cash surrender value, assuming the insurer uses

$$CV_5 = 0.9 \times AS_5 - 200$$

- Calculate the cash surrender value, assuming the insurer uses

$$CV_5 = 0.9 \times {}_5V - 200$$

- **Solution:**

$$CV_5 = 0.9 \times AS_5 - 200 = 0.9 \times 63\,509 - 200 = 56\,958$$

and

$$CV_5 = 0.9 \times {}_5V - 200 = 0.9 \times 65\,470 - 200 = 58\,723$$

## 7.5 Policy alterations

### Example 7.14 (continued)

#### (b) Transforming the policy in a paid-up policy:

- Suppose that at time 5, the contract is altered as follows:

$$\boxed{\text{altered future premium level} = 0}$$

$$\boxed{\text{altered future benefits} = X \times {}_5|\ddot{a}_{55} + 5P \times A_{[55]:\overline{5}}^1}$$

- Calculate the reduced annuity  $X$  in case  $CV_5 = 56\,958$ .

- **Solution:**

- Equivalence relation:

$$CV_5 = (5P + 100) \times A_{[55]:\overline{5}}^1 + (X + 25) \times {}_5|\ddot{a}_{55}$$

- We find that  $X = 4\,859$ .

## 7.5 Policy alterations

### Example 7.14 (continued)

#### (c) Transforming the deferred annuity in a pure endowment:

- Suppose that at time 5, the contract is altered as follows:

$$\boxed{\text{altered gross premiums} = P \times \ddot{a}_{55:\overline{5}|}} \quad \text{with} \quad P = 11\,900$$

$$\boxed{\text{altered future benefits} = P \times \left( (\underline{IA})_{55:\overline{5}|}^1 + 5 \times \underline{A}_{55:\overline{5}|}^1 \right) + S \times {}_5E_{55}}$$

- In case  $S$  is paid, an expense of 100 is charged.
- Determine  $S$  in case  $CV_5 = 56\,958$ .
- **Solution:**

- Equivalence relation:

$$\begin{aligned} & CV_5 + 0.95 \times P \times \ddot{a}_{55:\overline{5}|} \\ &= \\ & P \times (\underline{IA})_{55:\overline{5}|}^1 + (5 \times P + 100) \times \underline{A}_{55:\overline{5}|}^1 + (S + 100) \times {}_5E_{55} \end{aligned}$$

- We find that  $S = 138\,314$ .

## 7.5 Policy alterations

**Example 7.15** (read in book).

## 7.6 Retrospective policy values

(read in book)



## 7.7 Negative policy values

(read in book).

## 7.8 Deferred acquisition expenses and modified premium reserves

(read in book)

## 7.9 Other reserves

(read in book)

## 7.10 Notes and further reading

(read in book)