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Fair dynamic valuation of insurance liabilities via convex hedging

Ze Chen [a,b,c,*], Bingzheng Chen [d], Jan Dhaene [e], Tianyu Yang [f]

[a] School of Finance, Renmin University of China, Beijing 100872, China
[b] China Insurance Institute, Renmin University of China, Beijing 100872, China
[c] Institute for Global Private Equity, Tsinghua University, Beijing 100084, China
[d] School of Economics and Management, Tsinghua University, Beijing 100084, China
[e] Actuarial Research Group, AFI, Faculty of Economics and Business, KU Leuven, Leuven, Belgium
[f] School of Finance, Shanghai University of Finance and Economics, Shanghai 200433, China

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ABSTRACT

A general class of fair dynamic valuations, which are model-consistent (mark-to-model), market-consistent (mark-to-market) and time-consistent, was introduced by Barigou et al. (2019) in a multi-period setting. In this paper, we generalize the convex hedging approach proposed in Dhaene et al. (2017) to a multi-period framework and investigate the realization of fair dynamic valuations via a convex hedge-based (CHB) approach. We show that the classes of fair dynamic valuations and CHB dynamic valuations are equivalent. Moreover, we show how to implement the CHB dynamic valuations based on two specific classes of convex hedging techniques, i.e. the quadratic and exponential convex hedging.

1. Introduction

Recent solvency regulations for the insurance industry, such as the Swiss Solvency Test and Solvency II, have required insurance companies to apply a fair valuation of liabilities. To consider and be consistent with the information provided by financial markets, any replicable (hedgeable) part of a claim must be valuated at the price of its replicating (hedging) portfolio. The remaining part is then valuated by an appropriate risk margin (e.g., based on cost-of-capital arguments). As the hedgeable part of a claim is usually not uniquely determined, different feasible hedging or valuation approaches are possible.

Barigou et al. (2019) proposed the *fair dynamic valuation* approach in a multi-period setting, which is model-consistent (mark-to-model for claims independent of financial market evolutions), market-consistent (mark-to-market for hedgeable parts of claims) and time-consistent. This approach is implemented through a backward iteration scheme of hedge-based valuations, and thus it largely relies on the adopted hedging technique.

In this study, we investigate the fair dynamic valuation of insurance liabilities using the convex hedging approach in a multi-period setting. Our study makes three major contributions to the body of research on this topic. First, we extend the framework of fair dynamic valuation by linking the concept of conventional actuarial and financial valuation to the model- and market-consistency. This integration makes the fair dynamic valuation framework become full-fledged.

Second, we build a theory of convex dynamic valuation by extending the single-period convex hedging technique proposed by Dhaene et al. (2017) and the fair dynamic valuation framework of Barigou et al. (2019). We propose convex hedge-based (CHB) dynamic valuation based on convex hedging. The convex hedging technique determines the hedging strategy such that the claim and value of hedging portfolio are ‘close to each other’ within the goal of minimizing the $\mathbb{P}$-expectation of the given convex function $u(x)$. We prove that the class of CHB dynamic valuation is equivalent to the class of fair dynamic valuation and can be characterized in terms of a CHB dynamic hedger.

Last, we illustrate that the proposed CHB dynamic valuation approach is a practical tool for obtaining fair dynamic valuation of liabilities. The major advantage of the convex hedging technique lies in that it transforms the determination of an appropriate hedging technique into the selection of a proper suitable convex function. The choice of the convex function $u(x)$ determines how deviations between the liability and the hedging portfolio outcome, x, are punished. One particular convex function is the quadratic function $u(x) = x^2$, in which case the hedging is the well-known mean–variance (MV) hedging. In this study, we illustrate some practical classes of convex functions, including MV and exponential hedging. Furthermore, we apply several CHB

* Corresponding author at: School of Finance, Renmin University of China, Beijing 100872, China.
E-mail addresses: zechen@ruc.edu.cn (Z. Chen), chenbzh@sem.tsinghua.edu.cn (B. Chen), Jan.Dhaene@kuleuven.be (J. Dhaene), ytysem2012@163.com (T. Yang).

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dynamic valuations to valuate variable annuities, an interesting example of a hybrid liability with both financial and actuarial risk, as an illustration. The numerical results show that our CHB dynamic valuation is a practical technique.

This study is related to the extensive literature on market-consistent, actuarial, and time-consistent valuations. Market-consistency requires that the value of any purely hedgeable part of a financial payoff should be equal to the amount necessary to hedge it, see e.g. Malamud et al. (2008), Tsanakas et al. (2013), Wüthrich et al. (2013), Pelsser and Stadje (2014), Delong et al. (2019a,b) and Dhaene et al. (2017). An actuarial valuation is typically based on the real-world measure $\mathbb{P}$, and it involves a subjective actuarial judgment on the choice of the model.[1] Moreover, time-consistency binds valuations at different time points in a consistent way along a time-horizon. Time-consistent valuations have been largely studied and we refer to Acciaio and Penner (2011) for an overview.

The remainder of this paper is structured as follows. In Section 2, we define the general framework of fair dynamic valuation. In Section 3, we introduce the equivalence between the classes of fair dynamic valuations and the CHB dynamic valuations. In Section 4, we present some practical examples of convex dynamic hedging: mean–variance and exponential hedging. Section 5 concludes the paper.

2. General framework of fair dynamic valuation

In this section, we revisit the general framework of fair dynamic valuation introduced in Barigou et al. (2019). Though the related concepts are well developed and investigated, this section contributes by enriching the fair dynamic valuation framework. After introducing the combined financial–actuarial setting in Section 2.1 and basic concepts in Section 2.2, we supplement the concept of actuarial and financial t-valuation, and further integrate them into the fair dynamic valuation framework in Section 2.3. Finally, the fair dynamic valuations and hedgers are revisited in Section 2.4.

2.1. Combined financial–actuarial setting

Following Barigou et al. (2019), we consider a setting consisting of financial and actuarial risks, modeled by the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathbb{P}$ is the physical probability measure. We consider a discrete time setting with the set of time points given by $\eta = \{0, 1, \dots, T\}$, with the current time being 0 and the maturity of liability being T. The finite and discrete time filtration is $\mathbb{G} = \{\mathcal{G}_t\}_{t\in\eta}$, where σ-algebra $\mathcal{G}_t$, $t \in \eta$, represents the general information available up to and including time t.[2]

We assume that there are $n+1$ non-dividend assets traded in a liquid, transparent and arbitrage-free financial market.[3] We describe the price processes of the traded assets by the $(n+1)$-dimensional stochastic process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\}_{t\in\eta}$. The vector $\boldsymbol{Y}(t)$, $t \in \eta$, represents the time$-t$ prices of all tradable assets, that is, $\boldsymbol{Y}(t) = \left(Y^{(0)}(t), Y^{(1)}(t), \dots, Y^{(n)}(t)\right)$. The price process $\boldsymbol{Y}$ is adapted to the filtration $\mathbb{G}$, which means that $\boldsymbol{Y}(t)$ is $\mathcal{G}_t$-measurable, for any $t = 0, 1, \dots, T$.[4] In particular, the asset 0 is a zero-coupon bond paying an amount of 1 at maturity T. Its price at time t, denoted by $B(t, T)$, is given by

$$Y^{(0)}(t) = B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right], \qquad \text{for any } t = 0, 1, \dots, T-1.$$

We will call the insurance liabilities due at time t as t-claims, which are $\mathcal{G}_t$-measurable r.v.s.[5] Furthermore, the set of all t-claims defined on $(\Omega, \mathbb{G}, \mathcal{G})$ is denoted by $\mathcal{C}_t$. In this paper, we consider pricing T-claims, i.e. insurance liabilities due at time T. Hereafter, a T-claim is generally denoted by $S(T)$, or simply S if no confusion would arise.

2.2. Basic concepts

First, we introduce the concept of trading strategy. A *time$-t$ trading strategy* (also called a *time$-t$ dynamic portfolio*), $t \in \{0, \dots, T-1\}$, is an $(n+1)$-dimensional *predictable* process $\boldsymbol{\theta}_t = \{\boldsymbol{\theta}_t(\tau)\}_{\tau\in\{t+1,\dots,T\}}$ with respect to the filtration $\mathbb{G}$. Its predictability requirement means that

$$\boldsymbol{\theta}_t(\tau) \text{ is } \mathcal{G}_{\tau-1} - \text{measurable}, \qquad \text{for any } \tau = t+1, \dots, T.$$

We introduce the notations $\boldsymbol{\theta}_t(\tau) = \left(\theta_t^{(0)}(\tau), \theta_t^{(1)}(\tau) \dots, \theta_t^{(n)}(\tau)\right)$ for the components of $\boldsymbol{\theta}_t(\tau)$, the quantity $\theta_t^{(i)}(\tau)$ is the number of units invested in asset i in time period τ, specifically in the time interval $(\tau-1, \tau]$.[6]

A time$-\tau$ trading strategy $\boldsymbol{\theta}_t(\tau)$ is only set up at time τ till $\tau+1$, and then the portfolio is rebalanced to implement $\boldsymbol{\theta}_t(\tau+1)$. The $\mathcal{G}_{\tau-1}$-measurability requirement means that the portfolio composition $\boldsymbol{\theta}_t(\tau)$ for the time period τ follows from the general information available up to and including time $\tau-1$. A time$-t$ trading strategy $\boldsymbol{\theta}_t$ is said to be *self-financing* if

$$\boldsymbol{\theta}_t(\tau)\cdot\boldsymbol{Y}(\tau) = \boldsymbol{\theta}_t(\tau+1)\cdot\boldsymbol{Y}(\tau), \qquad \text{for any } \tau = t+1, \dots, T-1. \quad (1)$$

That is, no capital is injected or withdrawn at any rebalancing moment $\tau = t+1, \dots, T-1$. The set of self-financing time$-t$ trading strategies is Θ_t. Taking into account (1), the time$-T$ value of any self-financing time$-t$ strategy $\boldsymbol{\theta}_t \in \Theta_t$ can be expressed as

$$\boldsymbol{\theta}_t(T)\cdot\boldsymbol{Y}(T) = \boldsymbol{\theta}_t(t+1)\cdot\boldsymbol{Y}(t) + \sum_{\tau=t+1}^{T}\boldsymbol{\theta}_t(\tau)\cdot\Delta\boldsymbol{Y}(\tau), \quad (2)$$

with $\Delta\boldsymbol{Y}(\tau) = \boldsymbol{Y}(\tau) - \boldsymbol{Y}(\tau-1)$. In this formula, $\boldsymbol{\theta}_t(\tau)\cdot\Delta\boldsymbol{Y}(\tau)$ is the change in the market value of the investment portfolio in the time period τ, i.e. between time $\tau-1$ (just after rebalancing) and time τ (just before rebalancing).

A simple example of a self-financing time$-t$ trading strategy is the static trading strategy $\boldsymbol{\beta}_t$ consisting of buying at time t one unit of the zero-coupon bond $B(t,T)$, which pays 1 at T, and holding it until maturity T. Another special self-financing time$-t$ trading strategy $\boldsymbol{0}_t$ corresponds to a null investment, i.e. $\boldsymbol{0}_t(\tau) = (0, 0, \dots, 0)$ for all $\tau = t+1, \dots, T$.

1 See e.g. Kaas et al. (2008) for non-life insurance and Norberg (2014) for life insurance.

2 All the random variables (r.v.s) and stochastic processes are defined on this filtered probability space and the equality between r.v.s is understood in the $\mathbb{P}$—almost sure sense. Furthermore, we assume that the second moments of all r.v.s exist under $\mathbb{P}$.

3 For a detailed mathematical introduction, see Dhaene et al. (2017) and Barigou and Dhaene (2019), Barigou et al. (2019).

4 The filtration $\mathbb{G}$ may simply coincide with the filtration generated by the price process $\boldsymbol{Y}$. However, we consider a more general setting, where $\mathbb{G}$ is not only related to the price history of traded assets, but may also contain additional information such as that related to non-tradable claims or the survival index of a particular population.

5 Barigou et al. (2019) provided a discussion and examples on the measurability of insurance liabilities with incoming information over time.

6 The investment stays constant during the time interval $(\tau-1, \tau]$ until its next rebalancing at time u. We refer to Barigou et al. (2019) for more detailed introduction of strategy rebalancing setting. Here, the $\mathcal{G}_{\tau-1}$-measurability requirement means that the portfolio composition $\boldsymbol{\theta}_t(\tau)$ for the time period τ follows from the general information available up to and including time $\tau-1$.

Table 1
t-valuations and t-hedgers, $t = 0, 1, 2, \ldots, T-1$.

	t-valuation ρ_t	t-hedger $\boldsymbol{\theta}_t$
Definition	Mapping $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$ is a t-valuation if it is normalized and translation invariant.	Mapping $\boldsymbol{\theta}_t : \mathcal{C}_T \to \Theta_t$ is a t-hedger if it is normalized and translation invariant.
Normalization	$\rho_t[0] = 0$.	$\boldsymbol{\theta}_{t,0} = \mathbf{0}_t$.
Translation invariance	$\rho_t[S+a] = \rho_t[S] + B(t,T)a$, for any $S \in \mathcal{C}_T$ and $a \in \mathcal{C}_t$ payable at T.	$\boldsymbol{\theta}_{t,S+a} = \boldsymbol{\theta}_{t,S} + a\boldsymbol{\beta}_t$, for any $S \in \mathcal{C}_T$ and $a \in \mathcal{C}_t$ payable at T.

In addition, we revisit two important building blocks of fair dynamic valuation, t-valuation and t-hedger. A t-valuation ρ_t (t-hedger $\boldsymbol{\theta}_t$) assigns to each T-claim a $\mathcal{G}_t$-measurable random variable $\rho_t[S]$ (a self-financing time$-t$ trading strategy $\boldsymbol{\theta}_{t,S} \in \Theta_t$) that represents the value (hedging strategy) of the T-claim S at time t, given the available information at time t. The value $\rho_t[S]$ is a t-claim, which is a deterministic value (random variable) at (before) time t, and $\boldsymbol{\theta}_{t,S}$ is called a t-hedge for S with a value $\boldsymbol{\theta}_{t,S}(t) \cdot \boldsymbol{Y}(t)$ at time t.

Table 1 summarizes the definitions of t-valuation and t-hedger.

Now, we revisit the notions of dynamic valuation and dynamic hedger introduced in Barigou et al. (2019).

Definition 1 (*Dynamic Valuation*)**.** A dynamic valuation is a sequence $(\rho_t)_{t=0}^{T-1}$ where for each $t = 0, 1, \ldots, T-1$, ρ_t is a t-valuation.

Definition 2 (*Dynamic Hedger*)**.** A dynamic hedger is a sequence $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ where for each $t = 0, 1, \ldots, T-1$, $\boldsymbol{\theta}_t$ is a t-hedger.

2.3. Fair t-valuations

In this section, we enrich the fair dynamic valuation approach by integrating the long-standing actuarial and financial valuation principle into the framework of fair t-valuation. The approaches used to valuate contingent claims under in the insurance and finance contexts are different. The conventional way of setting insurance premium consists of expected loss built on the Law of Large Numbers and some necessary loadings, see for instance Gerber (1979), Bowers (1986), and Bühlmann et al. (1996). In this sense, the conventional insurance premium is under the physical measure $\mathbb{P}$. However, the core of valuation in the finance context is no-arbitrage. This widely acknowledged principle of financial valuation implies that claims should be valuated under a risk-neutral equivalent martingale measure (EMM) $\mathbb{Q}$. In the following, we denote the expectation conditional on $\mathcal{G}_t$ by $\mathbb{E}_t^{\mathbb{P}}$ and $\mathbb{E}_t^{\mathbb{Q}}$, respectively.

First, we define the class of actuarial t-valuation, which generalizes insurance premium principles in the traditional insurance context.

Definition 3 (*Actuarial t-valuation*)**.** An actuarial t-valuation $\mathcal{A}_t[S]$ is a t-valuation $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$, such that

$$\mathcal{A}_t[S] = B(t,T) \cdot \left(\mathbb{E}_t^{\mathbb{P}}[S] + RM_t[S]\right), \qquad \text{for any } S \in \mathcal{C}_T, \tag{3}$$

where the mapping $RM_t : \mathcal{C}_T \to \mathcal{C}_t$ is $\mathbb{P}$-law invariant and $\mathbb{P}$-independent of time$-t$ and future asset prices $\mathbf{Y}_t = \{\boldsymbol{Y}(\tau)\}_{\tau \in \{t,\ldots,T\}}$.

The mark-to-model condition (3) requires that the mechanism of actuarial t-valuation should be independent of the information from the financial market since time t under the $\mathbb{P}$ measure. It is a generalization of various insurance methods in practice, e.g. the variance principle and the standard deviation principle.[7] One particular example of actuarial t-valuation is the standard deviation principle,

$$\mathcal{A}_t[S] = B(t,T) \cdot \left(\mathbb{E}_t^{\mathbb{P}}[S] + \alpha\sigma_t^{\mathbb{P}}[S]\right),$$

with $\sigma_t^{\mathbb{P}}[S] := \sqrt{Var^{\mathbb{P}}[S \mid \mathcal{G}_t]}$ and $\alpha > 0$.

Second, let us step from the actuarial valuation method to the financial valuation method, and introduce the financial t-valuation. Its financial valuation condition (4) shows that claims should be valuated under a risk-neutral EMM $\mathbb{Q}$.

Definition 4 (*Financial t-valuation*)**.** A financial t-valuation $\mathcal{F}_t[S]$ is a t-valuation $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$, such that

$$\mathcal{F}_t[S] = B(t,T) \cdot \mathbb{E}_t^{\mathbb{Q}}[S], \qquad \text{for any } S \in \mathcal{C}_T, \tag{4}$$

where $\mathbb{Q}$ is an EMM.

At time t, based on the extent to which insurance claims can be hedged by tradable assets, Barigou et al. (2019) define two special types of T-claims: t-orthogonal T-claims and t-hedgeable T-claims (see Table 2). Hereafter, we denote the set of all t-orthogonal T-claims by $\mathcal{O}_T^t$, and the set of all time-t hedgeable T-claims by $\mathcal{H}_T^t$. It is intuitive that the suitable t-valuations applied to the class of t-orthogonal T-claim $S^{\perp}$ and t-hedgeable T-claim S^h should be actuarial t-valuation and financial t-valuation, respectively. T-claims are often neither t-orthogonal nor t-hedgeable, but are correlated with the market price of tradable assets. This most common type of T-claim, t-hybrid T-claim, is partially hedgeable by tradable assets.

Some recent regulations, such as the Swiss Solvency Test and Solvency II, have realized the importance of the financial risk embedded in hybrid insurance claims and adopted the so-called market-consistent valuation. Dhaene et al. (2017) and Barigou et al. (2019) proposed fair t-valuation, which merges both model-consistency and market-consistency (see Fig. 1). Model-consistency is a property of t-valuation concerning valuating orthogonal claims.[8] Model-consistent t-valuation 'identifies' the orthogonal claims, and applies actuarial t-valuation, which is completely 'independent' of the financial market. In addition, market-consistency 'identifies' the hedgeable parts of any claims, stating that the valuation of any hedgeable parts should be based on the market price.[9] Market-consistent t-valuation is 'independent' of actuarial models, but depends on the information of financial market. Table 3 summarizes the mathematical definitions of model-consistent, market-consistent and fair t-valuations. Therefore, we can see that the fair t-valuation approach meets all the requirements in Table 2.

7 See e.g. Bowers (1986), Kaas et al. (2008) and Norberg (2014).

8 To avoid concept misunderstandings, we remark that the model-consistent condition in our paper is introduced as 'actuarial condition' in Barigou et al. (2019). Thus, the actuarial t-valuation by (3) in our paper is a subclass of that in Barigou et al. (2019).

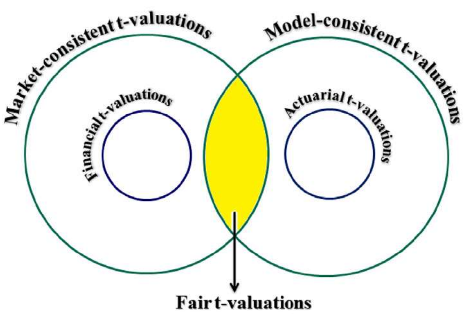
9 Some identical or similar conditions can be found in the literature (Kupper et al., 2008; Malamud et al., 2008; Artzner and Eisele, 2010; Pelsser and Stadje, 2014).

Table 2
T-claims: types and proper t-valuations, $t = 0, 1, 2, \ldots, T-1$.

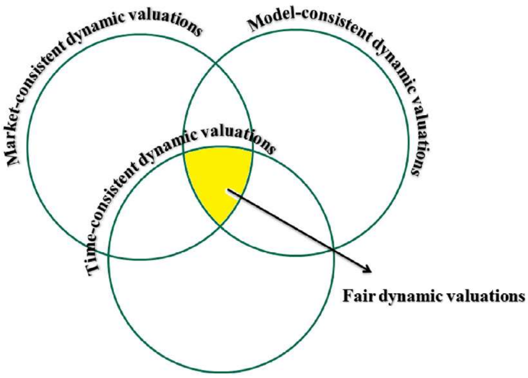
	Definition	Proper t-valuation
t-orthogonal T-claim $S^{\perp}$	A T-claim which is $\mathbb{P}$-independent of the stochastic process $\mathbf{Y}_{t+1} = \{\mathbf{Y}(\tau)\}_{\tau\in\{t+1,\ldots,T\}}$. Notation: $S^{\perp} \perp \mathbf{Y}_{t+1}$.	Actuarial t-valuation.
t-hedgeable T-claim S^h	A T-claim which can be replicated by a time$-t$ self-financing strategy $\boldsymbol{\theta}_t \in \Theta_t : S^h = \boldsymbol{\theta}_t(T) \cdot \mathbf{Y}(T)$.	Financial t- valuation.
t-hybrid T-claim	A T-claim which is neither t-hedgeable nor t-orthogonal.	Fair t-valuation.

Table 3
Model-consistent, market-consistent and fair t-valuations and t-hedgers, $t = 0, 1, 2, \ldots, T-1$.

	t-valuation ρ_t	t-hedger $\boldsymbol{\theta}_t$
Model-consistency	ρ_t is a *model-consistent t-valuation* if there exists an actuarial t-valuation $\mathcal{A}_t$ such that $\rho_t\left[S^{\perp}\right] = \mathcal{A}_t\left[S^{\perp}\right]$, for any $S^{\perp} \in \mathcal{O}_T^t$.	$\boldsymbol{\theta}_t$ is a *model-consistent t-hedger* if there exists a model-consistent t-valuation ρ_t such that $\boldsymbol{\theta}_{t,S^{\perp}} = \frac{\rho_t[S^{\perp}]}{B(t,T)}\boldsymbol{\beta}_t$, for any $S^{\perp} \in \mathcal{O}_T^t$.
Market-consistency	ρ_t is a *market-consistent t-valuation* if $\rho_t\left[S + S^h\right] = \rho_t[S] + \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} S^h\right]$, for any $S \in \mathcal{C}_T$ and $S^h \in \mathcal{H}_T^t$.	$\boldsymbol{\theta}_t$ is a *market-consistent t-hedger* if $\boldsymbol{\theta}_{t,S+S^h} = \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h}$, for any $S \in \mathcal{C}_T$ and $S^h \in \mathcal{H}_T^t$.
Fairness	ρ_t is a *fair t-valuation* if it is both model- and market-consistent.	$\boldsymbol{\theta}_t$ is a *fair t-hedger* if it is both model- and market-consistent.



(a) Fair t−valuation.



(b) Fair dynamic valuation.

Fig. 1. Classes of t-valuation and dynamic valuation.

Though the approach of fair t- and dynamic valuation is developed, we contribute a missing piece to the framework: the link between conventional actuarial (financial) t-valuation and model-consistent (market-consistent) t-valuation. As shown in Fig. 1, the classes of actuarial and financial t-valuations are exclusive to each other. The two important subclasses of t-valuations, model-consistent and market-consistent t-valuations, extend the classes of actuarial and financial t-valuations into broader ones, respectively. In this sense, actuarial and financial t-valuations are particular types of model-consistent and market-consistent t-valuations. We revisit the classes of model-consistent, market-consistent and fair t-hedgers in Table 3.

2.4. Fair dynamic valuations

In this section, we revisit the concept and conclusion of fair dynamic valuation in Barigou et al. (2019), which incorporates time-consistency. Time-consistency is a concept that couples different static t- valuations, which means that the same time$-t$ value is assigned to a T-claim regardless of whether it is calculated in one step or two steps backward in time. The definition of time-consistent valuation in Table 4 is often named the 'recursiveness' or 'tower property' definition.[10] The definition of time-consistent dynamic hedger is introduced similarly on the basis of time-consistent dynamic valuation.

First, we introduce an equation that appears in the definition of time-consistent valuation and is often used in the remainder of the paper. For a t-valuation for T-claims S, consider a trading strategy that invests $\rho_t\,[S]$ at time t in the zero-coupon bond $B(t, T)$, for $t = 0, 1, \ldots, T-1$. Obviously, the initial investment at time t of this trading strategy is $\rho_t\,[S]$, and its time$-T$ value $\widetilde{\rho_t}$ satisfies that

$$\widetilde{\rho_t}\,[S] = \frac{\rho_t\,[S]}{B(t,T)}. \tag{5}$$

[10] See e.g. Cheridito and Kupper (2011), Acciaio and Penner (2011) and Föllmer and Schied (2011) for the discrete time case, and see Frittelli and Gianin (2004), Delbaen et al. (2010), Pelsser and Stadje (2014) and Feinstein and Rudloff (2015) for the continuous case. In addition, there are some weaker notions of time-consistency in the literature, see e.g. Roorda et al. (2005) and Kriele and Wolf (2014).

Table 4
Model-consistent, market-consistent, time-consistent and fair dynamic valuations and hedgers.

	Dynamic valuation $(\rho_t)_{t=0}^{T-1}$	Dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$
Model-consistency	$(\rho_t)_{t=0}^{T-1}$ is a *model-consistent dynamic valuation* if any ρ_t is a model-consistent t-valuation.	$(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is a *model-consistent dynamic hedger* if any $\boldsymbol{\theta}_t$ is a model-consistent t-hedger.
Market-consistency	$(\rho_t)_{t=0}^{T-1}$ is a *market-consistent dynamic valuation* if any ρ_t is a market-consistent t-valuation.	$(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is a *market-consistent dynamic hedger* if any $\boldsymbol{\theta}_t$ is a market-consistent t-hedger.
Time-consistency	$(\rho_t)_{t=0}^{T-1}$ is a *time-consistent dynamic valuation* if $\rho_0, \rho_1, \dots, \rho_{T-1}$ are connected in the following way: $\rho_t [S] = \rho_t [\widetilde{\rho}_{t+1} [S]]$, for any $S \in \mathcal{C}_T$ and $t = 0, 1, \dots, T-2$.	$(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is a *time-consistent dynamic hedger* if $\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{T-1}$ are connected in the following way: $\boldsymbol{\theta}_{t,S} = \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}$, for any $S \in \mathcal{C}_T$ and $t = 0, 1, \dots, T-2$.
Fairness	$(\rho_t)_{t=0}^{T-1}$ is a *fair dynamic valuation* if it is model-, market- and time-consistent.	$(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is a *fair dynamic hedger* if it is model-, market- and time-consistent.

The time$-T$ value of the t-valuation $\rho_t [S]$ works to compare t-valuations at different times.

Fig. 1 shows that fair dynamic valuation (hedger) merges the properties of model-consistent, market-consistent and time-consistent valuations (hedgers). Model-consistent and market-consistent dynamic valuations (hedgers) are natural generalizations of model-consistent and market-consistent t-valuations (hedgers). Similarly, model-consistent and market-consistent dynamic hedgers are also natural generalizations of those of t-hedgers.

Merging the notions of model-consistent, market-consistent and time-consistent dynamic properties leads to the concept of fair dynamic valuation (hedger). Table 4 summarizes some of the important properties of fair dynamic valuations and hedgers.

3. Fair dynamic valuation via convex hedging

Barigou et al. (2019) proved that a dynamic valuation $(\rho_t)_{t=0}^{T-1}$ is fair if and only if there exists a fair dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that

$$\rho_t [S] = \boldsymbol{\mu}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T.$$

In this section, we propose a general convex hedge-based (CHB) dynamic valuation approach. We prove that the class of CHB valuations is equivalent to the class of fair dynamic valuations.

3.1. Convex t-hedger and valuation

To begin with, we extend the convex hedger of Dhaene et al. (2017) under a single-period framework to our multi-period setting.

Definition 5 (*Convex t-hedger*)**.** Consider a strictly convex non-negative function u with $u(0) = 0$. The t-hedger $\boldsymbol{\theta}_t^u$ determined via

$$\boldsymbol{\theta}_{t,S}^u = \arg\min_{\boldsymbol{\mu}_t \in \Theta_t} \mathbb{E}_t^{\mathbb{P}} \left[u \left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S \right) \right], \qquad \text{for any } S \in \mathcal{C}_T, \tag{6}$$

is called a convex t-hedger (with convex function u).

As we assume that the time$-T$ value of any time$-t$ trading strategy is square-integrable, a solution to the optimization problem (6) exists, see for instance Černý and Kallsen (2009). The convex t-hedger attaches the hedge $\boldsymbol{\theta}_{t,S}^u$ to any claim S, such that the time$-T$ value of the claim and hedging portfolio are 'close to each other' in the sense that the $\mathbb{P}$-expectation of the u-value of their difference is minimized. The choice of the convex function u determines how severe deviations are punished.

In the following theorem, we show that any convex t-hedger is a fair t-hedger.

Theorem 1. *Convex t-hedger $\boldsymbol{\theta}_t^u$ is a fair t-hedger with the underlying model-consistent t-valuation $\rho_t^u \left[S^\perp\right]$ given by*

$$\rho_t^u \left[S^\perp\right] = B(t,T) \cdot [\mathbb{E}_t^{\mathbb{P}}(S^\perp) + \arg\min_{s\in\mathbb{R}} \mathbb{E}_t^{\mathbb{P}} \left[u \left(s - \mathbb{E}^{\mathbb{P}}(S^\perp) - S^\perp \right) \right]], \tag{7}$$

for any $S^\perp \in \mathcal{O}_T$.

Proof. Consider the t-hedger $\boldsymbol{\theta}_t^u$ defined in (6). We have to prove that $\boldsymbol{\theta}_t^u$ satisfies the market- and model-consistent conditions in the definition of a fair t-hedge.
(a) For any t-hedgeable claim $S^h \in \mathcal{H}_T^t$, which can be replicated by a time$-t$ self-financing strategy $\boldsymbol{\theta}_t \in \Theta_t$ such that $S^h = \boldsymbol{\theta}_{t,S^h} \cdot \boldsymbol{Y}(T)$, we have that

$$\begin{aligned}\boldsymbol{\theta}_{t,S+S^h}^u &= \arg\min_{\boldsymbol{\mu}_t \in \Theta_t} \mathbb{E}_t^{\mathbb{P}} \left[u \left(\left(\boldsymbol{\mu}_t(T) - \boldsymbol{\theta}_{t,S^h}(T) \right) \cdot \boldsymbol{Y}(T) - S \right) \right] \\ &= \boldsymbol{\theta}_{t,S^h} + \arg\min_{\boldsymbol{\mu}_t' \in \Theta_t} \mathbb{E}_t^{\mathbb{P}} \left[u \left(\boldsymbol{\mu}_t'(T) \cdot \boldsymbol{Y}(T) - S \right) \right] \\ &= \boldsymbol{\theta}_{t,S^h} + \boldsymbol{\theta}_{t,S}^u,\end{aligned}$$

which means that the market-consistency condition is satisfied.
(b) Consider any t-orthogonal T-claim $S^\perp \in \mathcal{O}_T^t$. Notice that

$$\mathbb{E}_t^{\mathbb{P}}(S^\perp) + \arg\min_{s\in\mathbb{R}} \mathbb{E}_t^{\mathbb{P}} \left[u \left(s - \mathbb{E}^{\mathbb{P}}(S^\perp) - S^\perp \right) \right] = \arg\min_{s\in\mathbb{R}} \mathbb{E}_t^{\mathbb{P}} \left[u \left(s - S^\perp \right) \right].$$

Taking into account the independence of $S^\perp$ and $\boldsymbol{Y}$ as well as Jensen's inequality, we find that for any trading strategy $\boldsymbol{\mu} \in \Theta$, a convex function $u(x)$ satisfies

$$\mathbb{E}_t^{\mathbb{P}} \left[u \left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S^\perp \right) \mid S^\perp \right] \geq u \left(\boldsymbol{\mu}_t(T) \cdot \mathbb{E}_t^{\mathbb{P}} \left[\boldsymbol{Y}(T) \right] - S^\perp \right).$$

Taking expectations on both sides leads to

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}} \left[u \left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S^\perp \right) \right] &\geq \mathbb{E}_t^{\mathbb{P}} \left[u \left(\boldsymbol{\mu}_t(T) \cdot \mathbb{E}_t^{\mathbb{P}} \left[\boldsymbol{Y}(T) \right] - S^\perp \right) \right] \\ &\geq \mathbb{E}_t^{\mathbb{P}} \left[u \left(\widetilde{\rho}_t \left[S^\perp\right] - S^\perp \right) \right],\end{aligned}$$

which holds for any $\boldsymbol{\mu}_t \in \Theta_t$. Notice that $\widetilde{\rho}_t \left[S^\perp\right]$ can be rewritten as

$$\widetilde{\rho}_t [S] = \left(\rho_t \left[S^\perp\right], 0, \dots, 0 \right) \cdot \boldsymbol{Y}(T),$$

with the relation between $\rho_t \left[S^\perp\right]$ and $\widetilde{\rho}_t [S]$ indicated in (5). As $\left(\rho_t \left[S^\perp\right], 0, \dots, 0 \right)$ is an element of Θ_t, we find that

$$\boldsymbol{\theta}_{t,S^\perp}^u = \frac{\rho_t \left[S^\perp\right]}{B(t,T)} \boldsymbol{\beta}_t. \tag{8}$$

It is easy to verify that ρ_t^u is a model-consistent valuation satisfying

$$\rho_t^u\left[S^{\perp}\right]=B(t,T)\cdot[\mathbb{E}_t^{\mathbb{P}}(S^{\perp})+\arg\min_{s\in\mathbb{R}}\mathbb{E}^{\mathbb{P}}\left[u\left(s-\mathbb{E}_t^{\mathbb{P}}(S^{\perp})-S^{\perp}\right)\right]],$$
$$\text{for any } S^{\perp}\in\mathcal{O}_T.$$

Thus, we can conclude that the model-consistency condition is also satisfied. ■

Definition 6 (*Convex Hedge-Based t-hedger*)**.** A t-hedger $\boldsymbol{\theta}_t:\mathcal{C}_T\to\Theta_t$ defined by

$$\boldsymbol{\theta}_{t,S}^{CHB}=\boldsymbol{\theta}_{t,S}^{u}+\pi_t[S-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)]\,\boldsymbol{\beta}_t,$$
$$\text{for any } S\in\mathcal{C}_T \text{ and } t=0,1,\dots,T-2, \tag{9}$$

with underlying convex t-hedger $\boldsymbol{\theta}_t^u$ and model-consistent t-valuation π_t, is called a convex hedge-based t-hedger (CHB t-hedger).

A CHB t-hedger $\boldsymbol{\theta}_t^{CHB}$ is determined by its underlying convex t-hedger $\boldsymbol{\theta}_t^u$ first, augmented by a model-consistent t-hedger $\pi_t\cdot\boldsymbol{\beta}_t$ which invests in the zero-coupon bonds. Due to the fact that the convex t-hedger $\boldsymbol{\theta}_t^u$ is fair, we find that any CHB t-hedger $\boldsymbol{\theta}_t^{CHB}$ is a fair t-hedger.

Corollary 1. *Any CHB t-hedger is a fair t-hedger.*

Proof. Consider the CHB t-hedger $\boldsymbol{\theta}_t^{CHB}$ given in (9). In order to show that $\boldsymbol{\theta}_t^{CHB}$ is fair, we have to verify whether it is both market-consistent and model-consistent.

(i) Let $S\in\mathcal{C}_T$ and $S^h\in\mathcal{H}_T^t$ with $\boldsymbol{\theta}_t\in\Theta_t$ such that $S^h=\boldsymbol{\theta}_{t,S^h}(T)\cdot\boldsymbol{Y}(T)$. We have that

$$\boldsymbol{\theta}_{t,S+S^h}^{u}=\boldsymbol{\theta}_{t,S}^{u}+\boldsymbol{\theta}_{t,S^h},$$

taking into account this additivity relation, we find that

$$\begin{aligned}\boldsymbol{\theta}_{t,S+S^h}^{CHB}&=\boldsymbol{\theta}_{t,S+S^h}^{u}+\pi_t[S+S^h-\boldsymbol{\theta}_{t,S+S^h}^{u}(T)\cdot\boldsymbol{Y}(T)]\,\boldsymbol{\beta}_t\\&=\boldsymbol{\theta}_{t,S}^{u}+\boldsymbol{\theta}_{t,S^h}+\pi_t[S+S^h-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)-\boldsymbol{\theta}_{t,S^h}(T)\cdot\boldsymbol{Y}(T)]\boldsymbol{\beta}_t\\&=\boldsymbol{\theta}_{t,S}^{u}+\pi_t[S-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)]\,\boldsymbol{\beta}_t+\boldsymbol{\theta}_{t,S^h}\\&=\boldsymbol{\theta}_{t,S}^{CHB}+\boldsymbol{\theta}_{t,S^h}.\end{aligned}$$

Hence, $\boldsymbol{\theta}_t^{CHB}$ is market-consistent.

(ii) Let $S^{\perp}\in\mathcal{O}_T$. From (8), we know that $\boldsymbol{\theta}_{t,S^{\perp}}^{u}=\frac{\rho_t[S^{\perp}]}{B(t,T)}\boldsymbol{\beta}_t$. Taking into account the translation-invariance of π_t leads to

$$\begin{aligned}\boldsymbol{\theta}_{t,S^{\perp}}^{CHB}&=\boldsymbol{\theta}_{t,S^{\perp}}^{u}+\pi_t[S^{\perp}-\boldsymbol{\theta}_{t,S^{\perp}}^{u}(T)\cdot\boldsymbol{Y}(T)]\,\boldsymbol{\beta}_t\\&=\frac{\rho_t\left[S^{\perp}\right]}{B(t,T)}\boldsymbol{\beta}_t+\pi_t[S^{\perp}-\frac{\rho_t\left[S^{\perp}\right]}{B(t,T)}]\boldsymbol{\beta}_t\\&=\pi_t[S^{\perp}]\,\boldsymbol{\beta}_t.\end{aligned}$$

Given that π_t is a model-consistent t-valuation, we find that $\boldsymbol{\theta}_t^{CHB}$ is model-consistent. Therefore, any CHB t-hedger $\boldsymbol{\theta}_t^{CHB}$ is both market-consistent and model-consistent, and hence, fair. ■

Next, we define convex hedge-based t-valuations.

Definition 7 (*Convex Hedge-Based t-valuation*)**.** The t-valuation $\rho_t:\mathcal{C}_T\to\mathcal{C}_t$, $t=0,1,\dots,T-1$, defined by

$$\rho_t\,[S]=\boldsymbol{\theta}_{t,S}^{u}(t+1)\cdot\boldsymbol{Y}(t)+\pi_t[S-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)], \tag{10}$$

with underlying convex t-hedger $\boldsymbol{\theta}_t^u$ and model-consistent t-valuation π_t, is called a convex hedge-based t-valuation (CHB t-valuation).

In the following theorem we show that the classes of fair t-valuations and CHB t-valuations are equivalent.

Theorem 2. *A mapping $\rho_t:\mathcal{C}_T\to\mathcal{C}_t$, $t=0,1,\dots,T-1$, is a CHB t-valuation if and only if it is a fair t-valuation.*

Proof. (a) Consider the CHB t-valuation ρ_t defined in (10). The CHB t-valuation ρ_t can be represented as the time t value of the following CHB t-hedger $\boldsymbol{\theta}_t^{CHB}$,

$$\begin{aligned}\rho_t\,[S]&=(\boldsymbol{\theta}_{t,S}^{u}(t+1)+\pi_t[S-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)]\boldsymbol{\beta}_t)\cdot\boldsymbol{Y}(t)\\&=\boldsymbol{\theta}_{t,S}^{CHB}(t+1)\cdot\boldsymbol{Y}(t).\end{aligned}$$

In order to show that ρ_t is fair, we have to verify whether ρ_t is both market-consistent and model-consistent.

(i) Let $S\in\mathcal{C}_T$ and $S^h\in\mathcal{H}_T^t$ with $\boldsymbol{\theta}_t\in\Theta_t$ such that $S^h=\boldsymbol{\theta}_{t,S^h}(T)\cdot\boldsymbol{Y}(T)$. By Corollary 1, we find that

$$\begin{aligned}\rho_t\left[S+S^h\right]&=\boldsymbol{\theta}_{t,S+S^h}^{CHB}(t+1)\cdot\boldsymbol{Y}(t)\\&=(\boldsymbol{\theta}_{t,S}^{CHB}(t+1)+\boldsymbol{\theta}_{t,S^h}(t+1))\cdot\boldsymbol{Y}(t)\\&=\rho_t\,[S]+\boldsymbol{\theta}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t).\end{aligned}$$

Hence, ρ_t is market-consistent.

(ii) Let $S^{\perp}\in\mathcal{O}_T$. By Corollary 1, we know that

$$\begin{aligned}\rho_t\left[S^{\perp}\right]&=\boldsymbol{\theta}_{t,S^{\perp}}^{CHB}(t+1)\cdot\boldsymbol{Y}(t)\\&=(\pi_t[S^{\perp}]\,\boldsymbol{\beta}_t)\cdot\boldsymbol{Y}(t)\\&=\pi_t[S^{\perp}].\end{aligned}$$

Given that π_t is a model-consistent t-valuation, we find that ρ_t is model-consistent.

(b) Consider a fair t-valuation ρ_t. Let $\boldsymbol{\theta}_{t,S}^{u}\cdot\boldsymbol{Y}(T)$ be the time$-T$ value of a t-convex hedge of the T-claim S, e.g. determined via the underlying quadratic function $u(x)=x^2$. By the market-consistency property, we immediately find that

$$\begin{aligned}\rho_t\,[S]&=\rho_t\left[\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)+\left(S-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)\right)\right]\\&=\boldsymbol{\theta}_{t,S}^{u}(t+1)\cdot\boldsymbol{Y}(t)+\rho_t\left[S-\boldsymbol{\theta}_{t,S}^{u}(T)\cdot\boldsymbol{Y}(T)\right].\end{aligned}$$

Given that ρ_t is fair, it is also model-consistent. Hence, we can conclude that the fair t-valuation ρ_t is a CHB t-valuation.

Thus, for any convex t-hedger $\boldsymbol{\theta}_t^u$, the CHB t-valuation is fair. ■

3.2. Convex dynamic hedger and valuation

In the previous section, we introduced the convex t-hedgers and valuations. In this section, we interpret the time-consistency property under the framework of convex hedge-based dynamic hedgers and valuations.

Definition 8 (*Convex Hedge-Based Dynamic Hedger*)**.** The dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ where for each $t=0,1,\dots,T-1$, $\boldsymbol{\theta}_t$ is a CHB t-hedger and connected in the following way:

$$\boldsymbol{\theta}_{t,S}=\boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]},\qquad\text{for any } S\in\mathcal{C}_T \text{ and } t=0,1,\dots,T-2, \tag{11}$$

with $(t+1)$-valuation $\rho_{t+1}\,[S]=\boldsymbol{\theta}_{t+1,S}\cdot\boldsymbol{Y}(t+1)$, is called a convex hedge-based dynamic hedger (CHB dynamic hedger).

After having introduced the concept of CHB dynamic hedger, we now define convex hedge-based dynamic valuation (CHB dynamic valuation).

Definition 9 (*Convex Hedge-Based Dynamic Valuation*)**.** The dynamic valuation $(\rho_t)_{t=0}^{T-1}$ where all ρ_t are CHB t-valuations and connected in the following way:

$$\rho_t\,[S]=\rho_t\,[\widetilde{\rho}_{t+1}\,[S]]\,,\qquad\text{for any } S\in\mathcal{C}_T \text{ and } t=0,1,\dots,T-2, \tag{12}$$

is called a convex hedge-based dynamic valuation (CHB dynamic valuation).

In the following theorem, we prove that a fair dynamic valuation can be characterized in terms of a CHB dynamic hedger.

Theorem 3. *A dynamic valuation $(\rho_t)_{t=0}^{T-1}$ is a fair dynamic valuation if and only if there exists a CHB dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that*

$$\rho_t[S] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, \ldots, T-1. \tag{13}$$

Proof. (a) Suppose that $(\rho_t)_{t=0}^{T-1}$ is a fair dynamic valuation. First, by Theorem 2, we know that fair t-valuation ρ_{T-1} is a CHB t-valuation. That is, there exist a convex t-hedger $\boldsymbol{\theta}_{T-1}^u$ and a model-consistent t-valuation π_{T-1} such that

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}^u(T)\cdot\boldsymbol{Y}(T-1)+\pi_{T-1}[S-\boldsymbol{\theta}_{T-1,S}^u(T)\cdot\boldsymbol{Y}(T)], \qquad \text{for any } S \in \mathcal{C}_T.$$

Second, we construct a dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ which is model-consistent, market-consistent and time-consistent based on convex t-hedger $\boldsymbol{\theta}_t^u$. Let us set

$$\boldsymbol{\mu}_{T-1,S} = \boldsymbol{\theta}_{T-1,S}^u + \pi_{T-1}\left[S - \boldsymbol{\theta}_{T-1,S}^u(T)\cdot\boldsymbol{Y}(T)\right]\boldsymbol{\beta}_{T-1}.$$

Obviously, $\boldsymbol{\mu}_{T-1}$ is a fair $(T-1)$-hedger and

$$\rho_{T-1}[S] = \boldsymbol{\mu}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1), \qquad \text{for any } S \in \mathcal{C}_T.$$

Then, by definition the $(T-2)$-valuation $\rho_{T-2}[S]$ is

$$\begin{aligned}\rho_{T-2}[\widetilde{\rho}_{T-1}[S]] &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]}^u(T-1)\cdot\boldsymbol{Y}(T-2)\\ &\quad + \pi_{T-2}[S-\boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]}^u(T)\cdot\boldsymbol{Y}(T)],\\ &= (\boldsymbol{\theta}_{T-2,S}^u(T-1)\\ &\quad + \pi_{T-2}\left[S-\boldsymbol{\theta}_{T-2,S}^u(T)\cdot\boldsymbol{Y}(T)\right]\boldsymbol{\beta}_{T-2})\cdot\boldsymbol{Y}(T-2)\\ &= \boldsymbol{\mu}_{T-2,S}(T-1)\cdot\boldsymbol{Y}(T-2),\end{aligned}$$

where $\boldsymbol{\mu}_{T-2}$ is a CHB $(T-2)$-hedger. Hence, $\rho_{T-2}[S]$ is equivalent to the $(T-2)$- value of hedger $\boldsymbol{\mu}_{T-2}$. Starting from a CHB t-hedger $\boldsymbol{\mu}_{T-1}$, we construct the time-consistent adaptation

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\mu}_{t,\widetilde{\rho}_{t+1}[S]}, \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, \ldots, T-2.$$

Iteratively, we can show this CHB dynamic hedger $(\boldsymbol{\mu}_{t,S})_{t=0}^{T-1}$ satisfies $\rho_t[S] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)$.
(b) Consider the CHB dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ defined in (11). From Theorem 2, we know that for any $t = 0, 1, \ldots, T-1$, the CHB t-valuation $\rho_t = \boldsymbol{\theta}_{t,S}\cdot\boldsymbol{Y}(t)$ is both model-consistent and market-consistent, and hence, fair. Moreover, from the fact that $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ is time-consistent, we have

$$\begin{aligned}\rho_t[S] &= \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)\\ &= \boldsymbol{\mu}_{t,\widetilde{\rho}_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t)\\ &= \rho_t[\widetilde{\rho}_{t+1}[S]].\end{aligned}$$

Thus, $(\rho_t)_{t=0}^{T-1}$ is a fair dynamic valuation, which ends the proof. ■

Theorem 2 shows that the class of fair t-valuations is equivalent to the class of CHB t-valuations. In the following theorem, we extend this equivalence to dynamic valuations and show that any fair dynamic valuation can be expressed as a CHB dynamic valuation.

Theorem 4. *A dynamic valuation $(\rho_t)_{t=0}^{T-1}$ is a CHB dynamic valuation if and only if it is a fair dynamic valuation.*

Proof. (a) Consider the CHB dynamic valuation $(\rho_t)_{t=0}^{T-1}$ defined in Eq. (12). From Theorem 2, we know that any CHB t-valuation ρ_t is both market-consistent and model-consistent. Moreover, from the definition of CHB dynamic valuation in Eq. (12), $(\rho_t)_{t=0}^{T-1}$ satisfies the time-consistent condition. Thus, $(\rho_t)_{t=0}^{T-1}$ is a fair dynamic valuation.
(b) Consider a fair dynamic valuation $(\rho_t)_{t=0}^{T-1}$. For $t = 0, 1, \ldots, T-1$, any t-valuation ρ_t is a fair t-valuation, and hence, a CHB t-valuation by Theorem 2. Given that $(\rho_t)_{t=0}^{T-1}$ is fair, it is also time-consistent. That is, all t-valuations involved are connected as:

$$\rho_t[S] = \rho_t[\widetilde{\rho}_{t+1}[S]], \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, \ldots, T-2.$$

Thus, we can conclude that $(\rho_t)_{t=0}^{T-1}$ is a CHB dynamic valuation. ■

In Theorem 4 we have proven the equivalence between the classes of CHB dynamic valuations and fair dynamic valuations. This result is a generalization of Theorem 3 in Barigou and Dhaene (2019) as the mean–variance hedging is a special case of convex hedging. The equivalences provided by Theorems 3 and 4 lead to the conclusion that a CHB dynamic valuation can be characterized in terms of a CHB dynamic hedger. That is, any dynamic valuation $(\rho_t)_{t=0}^{T-1}$ is a CHB dynamic valuation if and only if there exists a CHB dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that

$$\rho_t[S] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, \ldots, T-1.$$

To sum up, under the convex hedging approach, determining the time$-t$ fair value of a T-claim S departs from splitting this claim into the value of its convex hedge and remaining claim:

$$S = \boldsymbol{\theta}_{t,S}^u(T)\cdot\boldsymbol{Y}(T) + \left(S - \boldsymbol{\theta}_{t,S}^u(T)\cdot\boldsymbol{Y}(T)\right).$$

The trading strategy $\boldsymbol{\theta}_{t,S}^u$ hedges the claim S under a certain convex function optimization goal. The fair t-valuation of S is then the sum of the financial market price of the hedge $\boldsymbol{\theta}_{t,S}^u$ and the model-consistent value of the remaining claim $S-\boldsymbol{\theta}_{t,S}^u(T)\cdot\boldsymbol{Y}(T)$. In the time horizon, the general procedure to determine the fair dynamic valuation of a T-claim S via the convex hedging approach is based on the following backward iterations scheme,[11]

1. $(T-1)$-valuation $\rho_{T-1}[S]$ is determined by combining a convex hedge portfolio and remaining non-hedged risk priced via a model-consistent $(T-1)$-valuation π_{T-1}.
2. At any time t, the t-valuation $\rho_t[S]$ is determined iteratively by $\rho_t[S] = \rho_t[\widetilde{\rho}_{t+1}[S]]$, for $t = 0, 1, \ldots, T-2$, which requires the convex hedge and model-consistent risk margin.

4. Convex dynamic hedging: Some practical examples

In this section, we illustrate some practical convex dynamic hedging techniques. As introduced in Section 3, the convex dynamic valuation approach largely relies on the choice of specific convex function. In Section 4.1, we introduce two applicable classes of convex functions and corresponding convex hedgers: mean–variance and exponential convex hedgers. In Section 4.2, we investigate some properties of loss averse exponential t-hedger, an applicable hedging technique first proposed in this work. In Section 4.3, we use these two classes of convex hedgers to conduct CHB dynamic valuations.

[11] The backward iteration scheme for obtaining fair dynamic valuations was also introduced in Barigou et al. (2019). However, in this paper we specifically adopt the convex hedging technique.

4.1. Mean–variance and exponential hedging

In this section, we introduce two specific classes of convex hedging: mean–variance (MV) and exponential hedging. The underlying convex function of MV hedging is the quadratic function, while that of exponential hedging become exponential functions. Both types of functions 'punish' the closer hedge deviations relatively less than the farther ones, in order to obtain the best hedging.

4.1.1. (Loss averse) MV t-hedger

MV hedging is a technique of approximating, with minimal mean squared error, a given payoff by the final value of a trading strategy. MV hedging is widely used because of its simplicity and nice properties, see e.g. Thomson (2005) and Dahl and Møller (2006). The minimization function of the MV hedging is the quadratic function, without differentiating the loss and gain deviations. The definition of MV t-hedger is as follows:

Definition 10 (*Mean–Variance t-hedger*)**.** The convex t-hedger determined via

$$\boldsymbol{\theta}^{MV}_{t,S} = \arg\min_{\boldsymbol{\mu}_t \in \Theta_t} \mathbb{E}^{\mathbb{P}}_t \left[\left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S \right)^2 \right],$$

for any $S \in \mathcal{C}_T$ and $t = 0, 1, \ldots, T-1$,

is called the mean–variance (MV) t-hedger.

We define the deviation between the outcomes of the hedging portfolio and insurance claim at time T,

$$x_S = \boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S.$$

Thus, x_S is a random variable to be observed at time T. The $x_S < 0$ cases represent losses of insurers, and the opposite $x_S > 0$ cases indicate gains. Notice that MV t-hedger indifferently punishes the gains and losses.

Loss aversion is an important concept in decision theory and prospect theory, referring to that for decision makers a loss of a certain amount leads to losing more satisfaction than the satisfaction from a gain of the equivalent amount (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). Chen et al. (2020) propose the following definition of loss averse mean–variance (LAMV) hedging.

Definition 11 (*Loss Averse Mean–Variance t-hedger*)**.** The convex t-hedger determined via

$$\boldsymbol{\theta}^{LAMV}_{t,S} = \arg\min_{\boldsymbol{\mu}_t \in \Theta_t} \mathbb{E}^{\mathbb{P}}_t \left[u \left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S \right) \right],$$

for any $S \in \mathcal{C}_T$ and $t = 0, 1, \ldots, T-1$, with

$$u(x_S) = \begin{cases} x_S^2 & x \geqslant 0 \\ \lambda \cdot x_S^2 & x < 0 \end{cases}, \lambda > 1, \tag{14}$$

is called a loss averse mean–variance (LAMV) t-hedger.

The LAMV t-hedger is more sensitive to losses than to gains. It punishes losses more than gains. The LAMV's loss aversion coefficient λ indicates the degree of aversion toward negative deviations. Chen et al. (2020) investigate the properties of LAMV hedging and its application in fair dynamic valuation.

4.1.2. (Loss averse) exponential t-hedger

Exponential functions also fall into the category of convex functions. In this subsection, we define the exponential convex hedger with an underlying exponential function.

Definition 12 (*Exponential t-hedger*)**.** The convex t-hedger determined via

$$\boldsymbol{\theta}^{E}_{t,S} = \arg\min_{\boldsymbol{\mu}_t \in \Theta_t} \mathbb{E}^{\mathbb{P}}_t \left[u \left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S \right) \right],$$

for any $S \in \mathcal{C}_T$ and $t = 0, 1, \ldots, T-1$, with

$$u(x_S) = \exp(\alpha |x_S|) - 1, \quad \text{for any } x_S,$$

is called an exponential t-hedger.

As $|x_S|$ represents the absolute value of a deviation and the convex function is exponentially increasing, thus a higher α indicates that larger deviations are relatively more severely punished. Hereafter, we call this effect of α the *tails aversion* coefficient.

Note that the exponential t-hedger is different from the exponential hedging technique employed in studies on the exponential utility indifference valuation and hedging strategies, see for instance Musiela and Zariphopoulou (2004) and Mania et al. (2005). The major difference lies in that positive and negative deviations, x_S and $-x_S$ for $x_S > 0$, are punished equivalently by the exponential t-hedger though these two approaches punish all deviations. However, the exponential utility indifference approach punishes one side relatively less than the other as it favors gains.

Now, we compare the MV t-hedger with the exponential t-hedger. Both t-hedgers are symmetric in the sense that positive and negative deviations, x_S and $-x_S$, are punished equivalently if the absolute values of deviation are equal. However, they differ in their attitudes toward small and large deviations. Consider $c > 0$ such that

$$\exp(\alpha |c|) - 1 = c^2,$$

then we know that

$$\exp(\alpha |x_S|) - 1 \leq x_S^2, \text{ for } |x_S| \leq c,$$

$$\exp(\alpha |x_S|) - 1 > x_S^2, \text{ for } |x_S| > c.$$

This comparison indicates that the exponential t-hedger punishes large deviations $|x_S| > c$ more severely than MV t-hedger. While, the exponential t-hedger punishes the small deviations $|x_S| \leq c$ less severely than MV t-hedger. This is because the growth of exponential functions is much larger that of quadratic ones. For instance, consider the following deviations: $2x_S > x_S > 0$, we have

$$\frac{\exp(2\alpha x_S) - 1}{\exp(\alpha x_S) - 1} \approx \exp[\alpha x_S] \text{ and } \frac{(2x_S)^2}{x_S^2} = 4. \tag{15}$$

Eq. (15) implies that the growth rate of the exponential t-hedger's punishment could be much higher than that of MV t-hedger's when the scale of deviation x_S is large.

Therefore, hereafter we adopt α, the tails aversion coefficient, to measure exponential t-hedger's aversion toward large deviations. Here, we call large deviations, both positive and negative ones, the tails.

Definition 13 (*Loss Averse Exponential t-hedger*)**.** The convex t-hedger determined via

$$\boldsymbol{\theta}^{LAE}_{t,S} = \arg\min_{\boldsymbol{\mu}_t \in \Theta_t} \mathbb{E}^{\mathbb{P}}_t \left[u \left(\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) - S \right) \right],$$

for any $S \in \mathcal{C}_T$ and $t = 0, 1, \ldots, T-1$, with

$$u(x_S) = \begin{cases} \exp(\alpha |x_S|) - 1 & x_S \geqslant 0 \\ \exp(\gamma |x_S|) - 1 & x_S < 0 \end{cases}, \gamma \geq \alpha > 0, \tag{16}$$

is called a loss-averse exponential (LAE) t-hedger.

4.2. Some properties of LAE t-hedger

Chen et al. (2020) proposed the $\mathbb{P}$-*symmetric* property for t-hedgers. $\mathbb{P}$-*symmetric* t-hedger hedges 'symmetrically' toward any liability $S \in \mathcal{C}_T$ (payout cashflows) and a corresponding asset $-S$ (income cashflows).

Definition 14. A t-hedger $\boldsymbol{\theta}_t$ is $\mathbb{P}$-*symmetric* if

$$\boldsymbol{\theta}_{t,S} = -\boldsymbol{\theta}_{t,-S}, \quad \text{for any claim } S \in \mathcal{C}_T.$$

Chen et al. (2020) also showed that LAMV t-hedger is $\mathbb{P}$-*symmetric* if and only if $\lambda = 1$. Since $\gamma \geq \alpha > 0$, we define the LAE's loss aversion as $\lambda_E = \frac{\gamma}{\alpha}$. λ_E represents the degree that loss ($x_S < 0$) deviations are relatively more severely punished than gains ($x_S \geqslant 0$). The following proposition proves that the LAE t-hedger $\boldsymbol{\theta}_t^{LAE}$ is $\mathbb{P}$-*symmetric* if and only if $\lambda_E = 1$.

Proposition 1. *The LAE t-hedger $\boldsymbol{\theta}_t^{LAE}$ is* $\mathbb{P}$-symmetric *if and only if $\lambda_E = 1$.*

Proof. For any $S \in \mathcal{C}_T$, the first order conditions for LAE t-hedger to minimize $\mathbb{E}_t^{\mathbb{P}}[u(S - \boldsymbol{\mu} \cdot \boldsymbol{Y}(T))]$ are

$$\mathbb{E}_t^{\mathbb{P}}\{\exp[\alpha(\boldsymbol{\mu}(T)\cdot \boldsymbol{Y}(T) - S)] \cdot \alpha I_{\{\boldsymbol{\mu}(T)\cdot\boldsymbol{Y}(T)\geq S\}} - \exp[\gamma(S - \boldsymbol{\mu}(T)\cdot \boldsymbol{Y}(T))] \cdot \gamma I_{\{\boldsymbol{\mu}(T)\cdot\boldsymbol{Y}(T)< S\}}\} \cdot Y^{(i)}(T) = 0,$$

for $i = 0, 1, \ldots, n$. As the asset 0 is risk-free with $Y^{(0)}(T) = 1$, we have

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot \alpha I_{\{x_S\geq 0\}} - \exp(\gamma|x_S|)\cdot \gamma I_{\{x_S<0\}}] = 0. \tag{17}$$

(1) On the one hand, when $\lambda_E = 1$, namely $\alpha = \gamma$, Eq. (17) becomes

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot \alpha I_{\{x_S\geq 0\}} - \exp(\alpha|x_S|)\cdot \alpha I_{\{x_S<0\}}] = 0. \tag{18}$$

where $x_S = \boldsymbol{\theta}_{t,S}^{LAE} \cdot \boldsymbol{Y}(T) - S$. Denote $x_{-S} = \boldsymbol{\theta}_{t,-S}^{LAE} \cdot \boldsymbol{Y}(T) - (-S)$, then for $-S$ Eq. (17) becomes

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_{-S}|)\cdot I_{\{x_{-S}\geq 0\}} - \exp(\alpha|x_{-S}|)\cdot I_{\{x_{-S}<0\}}] = 0. \tag{19}$$

From Eq. (18), we know that $\boldsymbol{\theta}_{t,-S}^{LAE} = -\boldsymbol{\theta}_{t,S}^{LAE}$ is a feasible solution of Eq. (19), as in this case $I_{\{x_S\geq 0\}} = I_{\{\boldsymbol{x}_{-S}<0\}}$ and $I_{\{x_S<0\}} = I_{\{x_{-S}\geq 0\}}$. Due to the convexity of $u(x)$, thus we have $\boldsymbol{\theta}_{t,-S}^{LAE} = -\boldsymbol{\theta}_{t,S}^{LAE}$, for any $S \in \mathcal{C}_T$.
(2) On the other hand, if $\boldsymbol{\theta}_{t,-S}^{LAE} = -\boldsymbol{\theta}_{t,S}^{LAE}$ for any $S \in \mathcal{C}_T$, Eq. (17) for $\boldsymbol{\theta}_{t,S}^{LAE}$ and $\boldsymbol{\theta}_{t,-S}^{LAE}$ are given by

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot \alpha I_{\{x_S\geq 0\}} - \exp(\gamma|x_S|)\cdot \gamma I_{\{x_S<0\}}] = 0, \tag{20}$$

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_{-S}|)\cdot \alpha I_{\{x_{-S}\geq 0\}} - \exp(\gamma|x_{-S}|)\cdot \gamma I_{\{x_{-S}<0\}}] = 0. \tag{21}$$

As $x_{-S} = -x_S$, summing Eqs. (20) and (21) leads to

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot \alpha - \exp(\gamma|x_S|)\cdot \gamma] = 0, \ \text{for any } S \in \mathcal{C}_T. \tag{22}$$

Thus, as $\gamma \geq \alpha > 0$, Eq. (22) clearly implies that $\alpha = \gamma$ and then $\lambda_E = 1$. ■

The following corollary shows that LAE t-hedger differentiates the gain and loss deviations.

Corollary 2. *For any $S \in \mathcal{C}_T$, the LAE t-hedger $\boldsymbol{\theta}_{t,S}^{LAE}$ satisfies*

$$\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\,|\,x_S \geq 0]\cdot \Pr\{x_S \geq 0\} \geq \mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\,|\,x_S < 0]\cdot \Pr\{x_S < 0\}. \tag{23}$$

Proof. For the LAE t-hedger with $\gamma \geq \alpha > 0$, we have

$$\begin{aligned}
0 &\leq \mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot \alpha I_{\{x_S\geq 0\}} - \exp(\alpha|x_S|)\cdot \alpha I_{\{x_S<0\}}] \\
&= \alpha \cdot \{\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot I_{\{x_S\geq 0\}}] - \mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot I_{\{x_S<0\}}]\} \\
&= \alpha \cdot \{\mathbb{E}_t^{\mathbb{P}}[\exp(\alpha|x_S|)\cdot \frac{I_{\{x_S\geq 0\}}}{\Pr\{x_S\geq 0\}}] \\
&\quad \cdot \Pr\{x_S \geq 0\} - \exp[(\alpha|x_S|)\cdot \frac{I_{\{x_S<0\}}}{\Pr\{x_S<0\}}]\cdot \Pr\{x_S<0\}\},
\end{aligned}$$

which proves equation (23). ■

It is clear that both sides of Eq. (23) are equal if $\lambda_E = 1$, and the left side is greater than the right side if $\lambda_E > 1$. Compared with $\lambda_E = 1$, a higher proportion of the $\boldsymbol{\theta}_{t,S}^{LAE}$ deviation punishment comes from gains ($x_S \geq 0$) than losses ($x_S < 0$) when $\lambda_E > 1$. Chen et al. (2020) discussed the effect of loss aversion on deviations when using the LAMV hedging technique.

4.3. Quadratic and exponential dynamic valuations

After having introduced the specific convex t-hedgers, we now define their corresponding CHB t-valuations: MV hedge-based (MVHB), LAMV hedge-based (LAMVHB), exponential hedge-based (EHB), LAE hedge-based (LAEHB) t- valuations, as well as dynamic valuations.

Definition 15 (*MVHB, LAMVHB, EHB, LAEHB t- and Dynamic Valuation*)**.**Consider a convex t-valuation $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$, $t = 0, 1, \ldots, T-1$, defined by

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}(t) + \pi_t[S - \boldsymbol{\theta}_{t,S}(T)\cdot \boldsymbol{Y}(T)], \qquad \text{for any } S \in \mathcal{C}_T,$$

where $\boldsymbol{\theta}_t$ is a convex t-hedger and π_t is a model-consistent t-valuation; and a dynamic valuation $(\rho_t)_{t=0}^{T-1}$ where all ρ_t are connected in the following way:

$$\rho_t[S] = \rho_t[\widetilde{\rho}_{t+1}[S]], \qquad \text{for any } S \in \mathcal{C}_T \text{ and } t = 0, 1, \ldots, T-2.$$

- ρ_t is an MVHB t-valuation if $\boldsymbol{\theta}_t$ is an MV t-hedger; and $(\rho_t)_{t=0}^{T-1}$ is an MVHB dynamic valuation if any ρ_t is an MVHB t-valuation.
- ρ_t is an LAMVHB t-valuation if $\boldsymbol{\theta}_t$ is an LAMV t-hedger; and $(\rho_t)_{t=0}^{T-1}$ is an LAMVHB dynamic valuation if any ρ_t is an LAMVHB t-valuation.
- ρ_t is an EHB t-valuation if $\boldsymbol{\theta}_t$ is an exponential t-hedger; and $(\rho_t)_{t=0}^{T-1}$ is an EHB dynamic valuation if any ρ_t is an EHB t-valuation.
- ρ_t is an LAEHB t-valuation if $\boldsymbol{\theta}_t$ is an LAE t-hedger; and $(\rho_t)_{t=0}^{T-1}$ is an LAEHB dynamic valuation if any ρ_t is an LAEHB t-valuation.

The above-defined t- and dynamic valuations require the choice of model-consistent t-valuation π_t. In this study, we consider the widely-used cost-of-capital approach, which is also adopted by the Solvency II regulation. The cost-of-capital risk margin is the following model-consistent t-valuation:

$$\pi_t[S] = e^{-r}\left[\mathbb{E}_t^{\mathbb{P}}[S] + i_{coc}\cdot VaR_p^{\mathbb{P}}[S]\right],$$

where $VaR_p^{\mathbb{P}}$ is the Value-at-Risk measure and $i_{coc} = 0.06$.
From Theorem 4 we know that the MVHB, LAMVHB, EHB and LAEHB dynamic valuations are all particular CHB dynamic valuations; hence, they are also fair dynamic valuations.

Corollary 3. *Any MVHB dynamic valuation $(\rho_t^{MVHB})_{t=0}^{T-1}$, LAMVHB dynamic valuation $(\rho_t^{LAMVHB})_{t=0}^{T-1}$, EHB dynamic valuation $(\rho_t^{EHB})_{t=0}^{T-1}$, and LAEHB dynamic valuation $(\rho_t^{LAEHB})_{t=0}^{T-1}$ is a fair dynamic valuation.*

4.4. Numerical illustration

In this section, we provide a simple numerical illustration which determines convex dynamic valuation of a portfolio of variable annuity contracts. The purpose of our numerical illustration is to show how the convex dynamic valuation approach can be implemented to valuate equity-linked liabilities in practice, rather than to select the most appropriate convex hedging or to analyze the implications for pricing variable annuities.

Our numerical example has some similarities with the one in Barigou et al. (2019) and Chen et al. (2020). Barigou et al. (2019) investigated a simple equity-linked life-insurance contract and Chen et al. (2020) illustrated a ratchet guaranteed benefit payoff. We benefit from these two studies by adopting their simulation setting and calculation technique.

It is important to remind of the distinction of our illustration. We implement and compare the EHB and LAEHB dynamic valuations that are first proposed in this work. Since the MVHB dynamic valuation in Barigou et al. (2019) and LAMVHB dynamic valuation in Chen et al. (2020) are particular types of convex dynamic valuation, we also include them in our simulation.

4.4.1. Application to a portfolio of variable annuity contracts

We consider pricing variable annuity contracts with GMAB and GMDB riders. The GMAB rider guarantees the minimum amount received by the annuitant after the accumulation period, protecting the annuity value from market fluctuations; the GMDB rider protects against the risk of early death during the accumulation phase. For simplicity, we assume that there are only a risk-free asset $Y^{(0)}(t)$ with a constant rate r and a risky asset $Y^{(1)}(t)$, $t = 0, 1, \dots, T$, in the financial market. Thus, we have $B(t,T) = e^{-r(T-t)}$. The specific simulation setting and calibration of the financial market and mortality process follow those of Barigou et al. (2019) and Chen et al. (2020). For more details, we refer to the Appendix.

Specifically, we consider a variable annuity payoff with the following payoff riders at time T used in Bacinello et al. (2011),[12]

1. GMAB rider: the insured who survives to maturity receives at T

   $$G^A = \max(Y^{(1)}(T), e^{rT});$$

2. GMDB rider: the insured who died at $t_i < T$ receives at T

   $$G^D = \max(Y^{(1)}(t_i), e^{rt_i}) \cdot e^{r(T-t_i)}.$$

If we denote the survival indicator of the insured by $\mathcal{I}(T)$, which equals 1 if the insured survives and 0 otherwise, thus the variable annuity payoff can be written as

$$\text{Payoff} = \mathcal{I}(T) \cdot G^A + (1 - \mathcal{I}(T)) \cdot G^D. \tag{24}$$

We consider pricing a portfolio of $l_x = 1000$ variable annuity contracts with GMAB and GMDB riders at time 0 with a maturity of $T = 10$ years.

4.4.2. Valuation results

In this section, we use the four classes of CHB dynamic valuations introduced above to determine the fair dynamic value of the time$-T$ variable annuity liability S and provide a numerical analysis. In our simulation, the CHB dynamic valuations of this liability are calculated on the basis of 10000 simulated scenarios. The calculation of the CHB t-hedgers and valuations is approximated using the Least Squares Monte Carlo (LSMC) approach.[13] The specific LSMC procedure and formula are given in the Appendix.

Fig. 2 presents the expected MVHB and LAMVHB dynamic valuations of the 10 000 simulated paths at different time points, and Fig. 3 shows that of EHB and LAEHB dynamic valuations. The overall relation between dynamic valuation and time t is jointly shaped by two trends: (1) it increases with t due to the upward trend of the risky asset; (2) it decreases with t as the risk margin value of remaining risk diminishes over time. In general, we observe a steady increase in these fair dynamic valuations, except a slightly decreasing trend in the LAMVHB dynamic valuation with $\lambda = 3$.

Effect of loss aversion. We first examine the effect of loss aversion embedded in the hedging technique of the LAMVHB and LAEHB dynamic valuations. Consistent with our expectations, the results show that a larger loss aversion coefficient λ or λ_E leads to higher hedging costs and valuation outcomes. This is because it costs more to construct a portfolio to avoid losses. Our result is in line with those of Chen et al. (2020) who proposed LAMVHB and investigated its properties. Moreover, Fig. 3 displays a similar conclusion that LAEHB dynamic value (with $\alpha = 0.10$ and $\lambda_E = 2$) is larger than EHB value (with $\alpha = 0.10$ and $\lambda_E = 1$).

Effect of tails aversion. Next, we study the effect of tails aversion on the EHB dynamic valuation. Fig. 3 compares the EHB dynamic valuations with $\alpha = 0.01$ and $\alpha = 0.10$. We find that the coefficient α increases the EHB dynamic value, suggesting that it is more costly to reach a relatively close hedging of large deviations. Compared with loss aversion, the cost of tails aversion is higher in our example. As a higher tails aversion reduces the large deviations and thus results in less remaining risk, the higher EHB valuation with $\alpha = 0.10$ further indicates that the tails aversion α leads to a higher hedging cost. Similar to the loss aversion of the hedging technique proposed in Chen et al. (2020), the tails aversion of the EHB valuation might be another feasible method to control the prudence of fair dynamic valuation.[14]

Our numerical results demonstrate that the CHB dynamic valuation approach is feasible and practical. We also contribute to the literature and illustrate one particular class of convex hedging techniques: (loss averse) exponential hedging.

5. Concluding remarks

It is challenging to determine the fair valuation of insurance liabilities in a multi-period framework, which is often a combination of hedgeable and unhedgeable risks. A fair dynamic valuation framework was proposed in Barigou et al. (2019) which merges model-consistent, market-consistent and time-consistent considerations. To implement the fair dynamic valuation, it is vital to determine the appropriate hedging technique.

In this study, we defined the concepts of actuarial and financial t-valuations, and then integrated them into the fair dynamic valuation framework. In addition, we investigated the fair dynamic valuation of insurance liabilities via the convex hedging approach in a multi-period dynamic investment setting. We proposed CHB dynamic valuations that extend the convex hedging and valuation of Dhaene et al. (2017) into a dynamic setting. We also showed that the class of fair dynamic valuations is equivalent to the class of CHB dynamic valuations.

12 The GMDB is normally paid upon the death of the insured, see Bacinello et al. (2011). To adjust this into our setting in which the liability is only payable at maturity time T, we assume the GMDB liability is invested in the risk-free asset from the death of policyholder until maturity.

13 This regression-based method was proposed by Carriere (1996) and Longstaff and Schwartz (2001) for the valuation of American-type options, and also employed by Barigou and Dhaene (2019) and Chen et al. (2020) to implement the fair dynamic hedging and valuation of insurance claims.

14 We refer to Chen et al. (2020) for a discussion on the prudence of fair dynamic valuation.

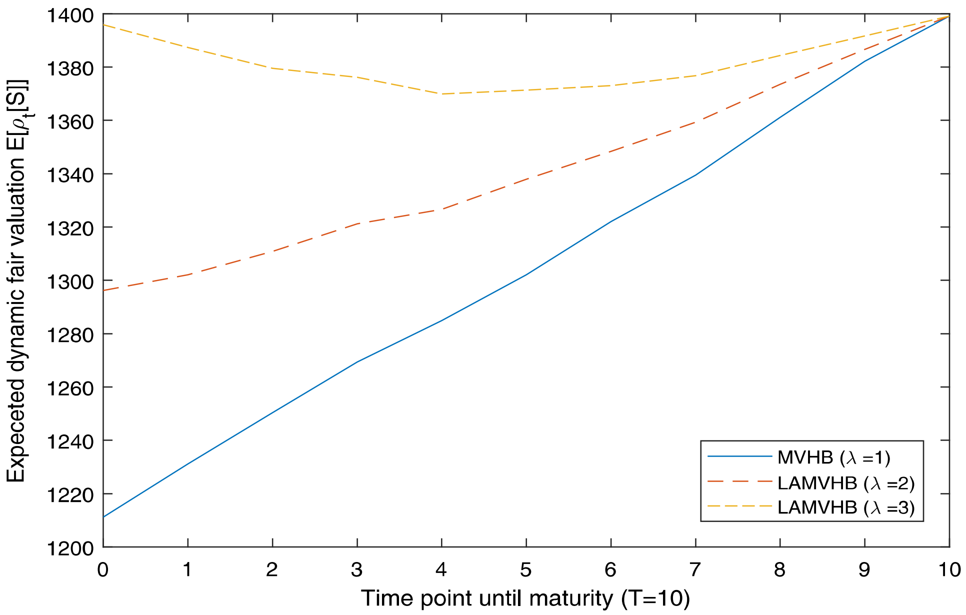


Fig. 2. Expected MVHB and LAMVHB dynamic valuations at different time points.

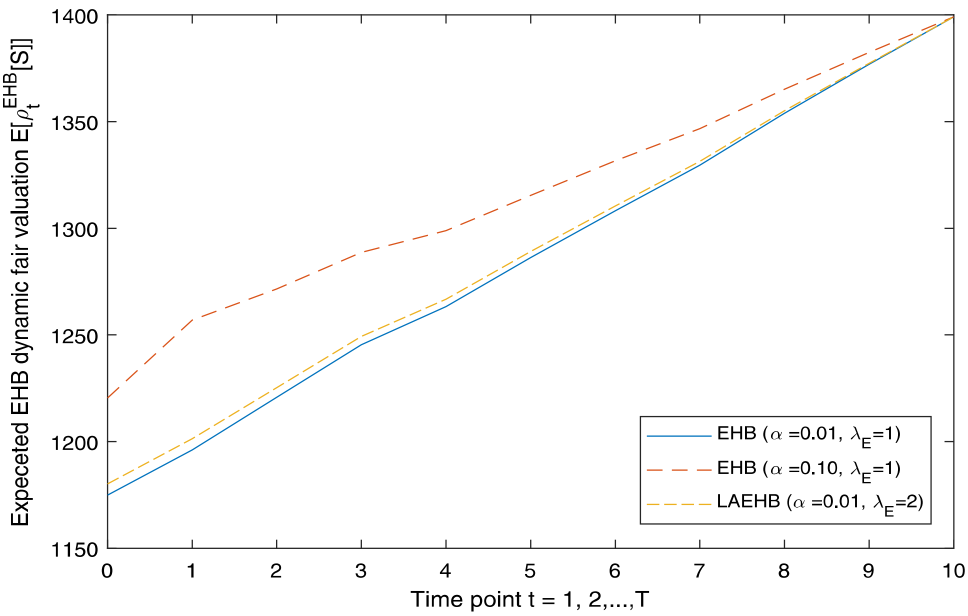


Fig. 3. Expected EHB and LAEHB dynamic valuations at different time points.

Moreover, the convex hedging approach allows the choice of appropriate convex functions to obtain a fair dynamic valuation. We illustrated how to implement CHB dynamic valuations with two particular classes of convex hedging technique: MV and exponential hedging. A simple numerical illustration of pricing variable annuity liabilities further showed that our CHB dynamic valuation provides a practical method for obtaining fair dynamic value of insurance liabilities.

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Appendix A. Simulation setting of financial market and mortality process

We briefly introduce the numerical simulation setting of the financial market and mortality process. In our simulation, we generate 10 000 scenarios of $Y^{(1)}(t)$ and $N(t)$ for $t = 1, \ldots, T$.

First, to simplify the illustration, we assume that the stock follows a geometric Brownian motion:

$$dY^{(1)}(t) = Y^{(1)}(t)\left(\mu dt + \sigma dZ_1(t)\right), \tag{25}$$

with the parameters $\mu, \sigma > 0$. The benefit payoff equals the maximum of the mean of the stock value from times 1 to T and a guaranteed amount K. Thus, the insurer faces liability S at time T:

$$S = N(T) \times \max\left(Y^{(1)}(T), K\right),$$

where $N(t), t = 0, 1, \ldots, T$ is a mortality process counting the number of survivals among an initial population of l_x insured of age x. Following Barigou et al. (2019) and Chen et al. (2020), the adopted parameters for the financial market are $r = 0.02$, $\mu = 0.07$, $\sigma = 0.3$.

Second, the mortality intensity is assumed to be stochastic and it follows the dynamics under the $\mathbb{P}$ measure given by

$$d\lambda_x(t) = c\lambda_x(t)dt + \xi dZ_2(t),$$

with $c, \xi > 0$. $Z_2(t)$ is a standard Brownian motion independent of $Z_1(t)$ in Eq. (25). The survival function is then defined by

$$S_x(t) := \mathbb{P}(T_x > t) = \exp\left(-\int_x^{x+t} \lambda_x(s)ds\right),$$

where T_x is the remaining lifetime of an individual aged x at time 0. Moreover, the deaths of individuals are assumed to be independent events, conditional on the knowledge of population mortality.[15]

Furthermore, we denote $N(t)$ as the number of survived insured at the end of year t, $D(t)$ as the number of deaths in year t. Then, the dynamics of the number of active contracts can be described as a nested binomial process as follows: $N(t+1) = N(t) - D(t+1)$ with $D(t+1)|N(t), q_{x+t} \sim Bin(N(t), q_{x+t})$. Here, q_{x+t} represents the one-year death probability:

$$q_{x+t} := \mathbb{P}(T_x \le t+1 | T_x > t) = 1 - \frac{S_x(t+1)}{S_x(t)}, \text{ for } t = 0, \ldots, T-1.$$

In the simulation, we adopt the parameter setting of Luciano et al. (2017) and set $\lambda_x(0) = 0.0087$, $c = 0.0750$, $\xi = 0.000597$, which correspond to 55-aged male in the UK.

Appendix B. LSMC simulation procedure

We introduce the simulation procedure of implementing of LSMC approach to obtain the CHB t-hedgers and valuations. The key idea of LSMC is to regress the conditional expectations on the cross-sectional information of the underlying risk drivers, as this can substantially reduce computation intensity in dynamic optimizations. For more detailed explanation, we refer to Barigou et al. (2019) and Chen et al. (2020) which have adopted the LSMC simulation procedure for fair dynamic valuation.

First, for any path i, $i = 1, 2, \ldots, 10\,000$, at any time $t = 0, 1, \ldots, T-1$, a number of 10 000 candidate scenarios of $N_c(t+1)$ and $Y_c^{(1)}(t+1)$ are generated on the basis of $N(t)$ and $Y^{(1)}(t)$. However, only one scenario is randomly chosen to be the simulated $(N(t+1), Y^{(1)}(t))$ in path i (unobservable at t). Second, at any time t of path i, the t-hedgers and valuations are based on the 10 000 candidate scenarios. At time t of each path, the conditional expectations are regressed over the risk drivers at time $t+1$ using a second-order least squares regression:

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}[S]\,|(N(t+1), Y^{(1)}(t+1))\right]\\&\quad\approx \beta_0 + \beta_1 N(t+1)Y^{(1)}(t+1) + \beta_2\left(N(t+1)Y^{(1)}(t+1)\right)^2,\end{aligned}$$

for all scenarios $(N_c(t+1), Y_c^{(1)}(t+1))$. After having β_0, β_1 and β_2, we can obtain the estimated $\rho_{t+1}^c[S]$ for all candidate scenarios. Here, the choice of the type and number of basis functions follows that of Barigou et al. (2019) and Chen et al. (2020). For a discussion of the basis functions and their implications on robustness and convergence, see Areal et al. (2008), Moreno and Navas (2003), and Stentoft (2012).

On this basis, we apply the CHB t-hedgers and valuations. The hedge is obtained by finding the optimal strategy minimizing the convex punishment function. For instance, the MVHB t-hedger is obtained with an MV optimization using these 10 000 candidate scenarios $(N_c(t+1), Y_c^{(1)}(t+1))$ and estimated $\rho_{t+1}^c[S]$. Finally, the expected dynamic valuations of this liability are the expected values of these 10 000 simulated scenarios.

[15] See Milevsky et al. (2006) for similar assumptions.

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