Systemic Risk: Conditional Distortion Risk Measures

Jan Dhaene
Faculty of Business and Economics
Katholieke Universiteit Leuven
Jan.Dhaene@kuleuven.be

Roger J. A. Laeven
Amsterdam School of Economics
University of Amsterdam, KU Leuven
and CentER
R.J.A.Laeven@uva.nl

Yiying Zhang*
Department of Mathematics
Southern University of Science and Technology
zhangyy3@sustech.edu.cn

December 5, 2021

Abstract

In this paper, we introduce the rich classes of conditional distortion (CoD) risk measures and distortion risk contribution (ΔCoD) measures as measures of systemic risk and analyze their properties and representations. The classes unify, and significantly extend, existing systemic risk measures such as the conditional Value-at-Risk, conditional Expected Shortfall, and risk contribution measures in terms of the VaR and ES. We provide sufficient conditions for two random vectors to be ordered by the proposed CoD-risk measures and ΔCoD-measures. These conditions are expressed using the conventional stochastic dominance, increasing convex/concave, dispersive, and excess wealth orders for the marginals and canonical positive/negative stochastic dependence notions.

Keywords: Distortion risk measures; Co-risk measures; Risk contribution measures; Systemic risk; Stochastic orders; Copulae.

JEL Classification: G22.

*Corresponding author.

1 Introduction

Risk measures are commonly used as capital requirements, i.e., as real-valued mappings from a class of financial positions determining the amount of risk capital to be held in reserve. The purpose of this risk capital is to make the risk given by the financial position taken by a financial institution, such as an insurance company or a bank, acceptable from a *microprudential* regulatory perspective. Prominent examples of risk measures are the Value-at-Risk (VaR) and the Expected Shortfall (ES).[1] Indeed, with the international adoption of financial regulatory frameworks, such as the Basel Capital Accords for banks and European Solvency Regulation for insurers, VaR has become the predominant measure of risk for financial institutions. Since its first adoption in the nineties of the previous century, an active area of research has analyzed the appealing and appalling properties of VaR—and, later, ES—, and has developed new alternative theories of risk measurement. These theories build on a rich literature in actuarial mathematics and decision theory and have nowadays reached a high level of mathematical and economic sophistication.

Over the past decade-and-a-half we have witnessed a myriad of transmissions of adverse economic events, at an international or even global scale. The interactions among risks in the form of stochastic interdependences play a central role in quantitative risk analysis; see Denuit et al. (2005), Embrechts et al. (2005), Genest et al. (2009), Kaas et al. (2009), Laeven (2009), Goovaerts et al. (2011), Asimit and Gerrard (2016) and the references therein in the context of risk aggregation under VaR and ES. Since the 2008/09 global financial crisis, risk measures have increasingly been employed not just to provide microprudential assessments of marginal risks or aggregate portfolio risks, but also to evaluate forms of systemic risk: from a *macroprudential* perspective we are interested in the systemic risk that a failure or loss of one entity spreads contagiously to other entities or even to the entire financial system. Indeed, the complex system of financial institutions in a competitive economy induces an undeniable presence of interconnectedness, in part caused by dynamic feedback relations within our highly interconnected financial-economic network.[2] This interconnectedness can cause a collapse in part of the system as a result of a contagious disruption due to the failure of a singular player. Thus, the potential threat of the failure of a singular player can have a reverberating effect on the security and stability of the system and the economy as a whole. In the literature, several papers have proposed different conditional risk (co-risk) measures and risk contribution measures to evaluate the systemic risks emerging from a group of financial institutions and the interactions among them; see Gourieroux and Monfort (2013), Girardi and Ergün (2013), Adrian and Brunnermeier (2016), Brownlees and Engle (2016), Acharya et al. (2017) and the references therein.

As a simple measure of systemic risk, Adrian and Brunnermeier (2016) analyze the

[1]Expected Shortfall is also referred to as Tail Value-at-Risk (TVaR) and Average Value-at-Risk (AVaR); see Definition 2.4 for the explicit definitions.

[2]See, e.g., Aït-Sahalia et al. (2015).

conditional VaR (CoVaR). It is defined as the VaR of one specific financial institution, conditional upon the occurrence of an event corresponding to a stress scenario to which another financial institution is exposed. The prefix "Co" is meant to refer to "conditional" (or "co-movement") and emphasizes the systemic nature of this measure of risk. In a sense, CoVaR provides a measure of a spillover effect. In related literature, Mainik and Schaanning (2014) introduce the conditional expected shortfall (CoES) and Acharya et al. (2017) propose the marginal expected shortfall (MES); see Definition 2.6. For a given choice of such co-risk measures, the associated risk contribution measure evaluates how a stress scenario for one component incrementally affects another component or the entire system. Examples of risk contribution measures including ΔCoVaR and ΔCoES can be found in Girardi and Ergün (2013), Mainik and Schaanning (2014), and Adrian and Brunnermeier (2016); see also Definition 2.7. Using data on U.S. financial institutions over the period 2005-2014, Kleinow et al. (2017) compare several commonly used metrics of systemic risk, including CoVaR and MES. They illustrate that the alternative measurement approaches may induce very different assessments of systemic risk. In particular, they show that different systemic risk measures may lead to contradicting evaluations of the riskiness of different types of financial institutions. Hence, the "dependence consistency" of systemic risk measures, their properties, and their representations require further analysis; see also the discussion in Mainik and Schaanning (2014).

In the context of comparisons of these co-risk measures and risk contribution measures, an interesting paper by Sordo et al. (2018) provides sufficient conditions to stochastically order two random vectors in terms of their CoVaR, CoES, ΔCoVaR, and ΔCoES, where the conditions are expressed using conventional stochastic orders for the marginals under some assumptions of positive dependence. Furthermore, Fang and Li (2018) investigate how the marginal distributions and the dependence structure affect the interactions among paired risks under the above co-risk measures and risk contribution measures. From a risk-theoretic perspective, the VaR and ES arise as special cases within the rich class of distortion risk measures (Yaari, 1987; Denuit et al., 2005, 2006; Dhaene et al., 2006; Goovaerts et al., 2010; Föllmer and Schied, 2011). Distortion risk measures satisfy several appealing (in fact, characterizing) properties including monotonicity, comonotonic additivity, and positive homogeneity. Besides, distortion risk measures are consistent with the usual stochastic order (i.e., first-order stochastic dominance), and, for concave distortion functions, with the increasing convex order (i.e., stop-loss order). Furthermore, concave distortion risk measures occur naturally as building blocks of law-invariant convex risk measures (see, e.g., Chapter 4 in Föllmer and Schied, 2011).

The aim of this paper is to introduce general and unified classes of conditional risk measures and risk contribution measures by means of distortion functions. This gives rise to *conditional distortion (CoD) risk measures* and *distortion risk contribution (ΔCoD) measures*. We analyze the properties and present representations of these new, rich classes of systemic risk measures, and provide a systematic treatment of their marginal and dependence consistency, hence reveal worst (best) case scenarios. In particular, we establish sufficient conditions for ordering two bivariate random vectors by the proposed systemic risk measures in terms of the canonical stochastic orders (e.g., first-order stochastic dominance, the increasing concave/convex order, the dispersive order, and the excess wealth order) between the marginals, dependence structure, distortion functions, and threshold quantiles. The interactions between paired risks are also investigated. Existing results in Mainik and Schaanning (2014), Sordo et al. (2018), and Fang and Li (2018) are generalized and extended.

In a somewhat related strand of the literature, Hoffmann et al. (2016) axiomatically introduce risk-consistent conditional systemic risk measures defined on multidimensional risks. This class consists of those conditional systemic risk measures that can be decomposed into a state-wise conditional aggregation and a univariate conditional risk measure. Their studies extend known results for unconditional risk measures on finite state spaces. Besides, Biagini et al. (2019) specify a general methodological framework for systemic risk measures via multidimensional acceptance sets and aggregation functions. Their approach yields systemic risk measures that can be given the interpretation of the minimal amount of cash that safeguards the aggregated system.

This paper is organized as follows. In Section 2, we recall some definitions and concepts. In Section 3, we introduce conditional distortion (CoD) risk measures and distortion risk contribution (ΔCoD) measures, and give some useful expressions employed in the sequel. Section 4 studies the comparison of two random vectors under CoD-risk measures, and provides sufficient conditions for their ordering in terms of the usual stochastic order, the increasing convex order, and the increasing concave order of marginals, under appropriate assumptions on the dependence structure, distortion functions, and threshold quantiles. In Section 5, we present sufficient conditions for comparison of the distortion risk contribution measures in terms of the dispersive order and the excess wealth order of marginals. Section 6 investigates the interactions between paired risks under our proposed CoD-risk measures and ΔCoD-measures. Section 7 and an Online Appendix provide some theoretical and numerical examples, respectively, to illustrate our main findings. Section 8 concludes the paper. All proofs are relegated to the Appendix.

2 Preliminaries

Let $\mathbb{R}_0^+ = [0, +\infty)$ and let $\mathbb{N}^+$ be the set of strictly positive natural numbers. Throughout this paper, the term "increasing" is used for "non-decreasing" and "decreasing" is used for "non-increasing". We use the expression '$X \sim F$' to denote that the random variable (r.v.) has distribution function (d.f.) F, and use $\underline{X}$ to denote the random vector $(X_1, \ldots, X_n)$. Expectations and density functions are assumed to exist when they appear.

2.1 Stochastic Ordering and Comparing Dependence

We denote by X and Y two r.v.'s with respective d.f.'s F and G, survival functions $\overline{F}$ and $\overline{G}$, and density functions f and g. Let $F^{-1}(p) = \inf\{x \in \mathbb{R} \mid F(x) \geq p\}$ and $G^{-1}(p) = \inf\{x \in \mathbb{R} \mid G(x) \geq p\}$ be the generalized inverses of the d.f.'s F and G, $p \in [0,1]$, respectively, where $\inf \emptyset = +\infty$ by convention.

Definition 2.1 (Univariate stochastic orders)**.** [3] *X is said to be smaller than Y in the*

(i) likelihood ratio order (denoted by $X \leq_{\rm lr} Y$) if $g(x)/f(x)$ is increasing in $x \in \mathbb{R}$;

(ii) hazard rate order (denoted by $X \leq_{\rm hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x \in \mathbb{R}$;

(iii) usual stochastic order (denoted by $X \leq_{\rm st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all $x \in \mathbb{R}$;

(iv) increasing convex order (denoted by $X \leq_{\rm icx} Y$) if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for any increasing and convex function $\phi : \mathbb{R} \to \mathbb{R}$;

(v) increasing concave order (denoted by $X \leq_{\rm icv} Y$) if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for any increasing and concave function $\phi : \mathbb{R} \to \mathbb{R}$;

(vi) dispersive order (denoted by $X \leq_{\rm disp} Y$) if $F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$, for all $0 < u \leq v < 1$;

(vii) excess wealth order (denoted by $X \leq_{\rm ew} Y$) if $\int_{F^{-1}(u)}^{\infty} \overline{F}(t){\rm d}t \leq \int_{G^{-1}(u)}^{\infty} \overline{G}(t){\rm d}t$, for all $u \in (0,1)$.

For comprehensive discussions on these partial orders, we refer the reader to the monographs by Denuit et al. (2005), Marshall and Olkin (2007), and Shaked and Shanthikumar (2007).

The following notions entail that, for a bivariate random vector (X, Y), larger values of one component are associated with larger values of the other, in some specific sense.

Definition 2.2 (Bivariate stochastic orders)**.** [4]

(i) The bivariate random vector (X, Y) is said to be totally positive of order 2 [reverse regular of order 2] (written as ${\rm TP}_2$ [${\rm RR}_2$]) if $[X|Y = y_1] \leq_{\rm lr} [\geq_{\rm lr}] [X|Y = y_2]$, for all $y_1 \leq y_2$, and $[Y|X = x_1] \leq_{\rm lr} [\geq_{\rm lr}] [Y|X = x_2]$, for all $x_1 \leq x_2$.

[3] As is well known, $X \leq_{\rm lr} Y \Longrightarrow X \leq_{\rm hr} Y \Longrightarrow X \leq_{\rm st} Y \Longrightarrow X \leq_{\rm icx\ [icv]} Y$. Furthermore, the dispersive order is stronger than (i.e., implies) the excess wealth order, and is a partial order used to compare the variabilities among two probability distributions.

[4] The following implications (with slight abuse of notation) are well known: (X, Y) is ${\rm TP}_2 \Longrightarrow X \uparrow_{\rm SI} Y$ $[Y \uparrow_{\rm SI} X] \Longrightarrow X \uparrow_{\rm RTI} Y$ $[Y \uparrow_{\rm RTI} X] \Longrightarrow (X, Y)$ is PQD, and (X, Y) is ${\rm RR}_2 \Longrightarrow X \uparrow_{\rm SD} Y$ $[Y \uparrow_{\rm SD} X] \Longrightarrow X \uparrow_{\rm RTD} Y$ $[Y \uparrow_{\rm RTD} X] \Longrightarrow (X, Y)$ is NQD. It is also clear that if (X, Y) is ${\rm TP}_2$ [${\rm RR}_2$] then it must be PDS [NDS]. Besides, (X, Y) is ${\rm TP}_2$ [${\rm RR}_2$] if and only if its copula C is ${\rm TP}_2$ [${\rm RR}_2$] (see Müller and Stoyan, 2002; Cai and Wei, 2012). For more detailed discussions, interested readers are referred to Barlow and Proschan (1975), Block et al. (1982), Joe (1997), and Denuit et al. (2005).

(ii) *X is said to be stochastically increasing [decreasing] in Y (written as $X \uparrow_{\text{SI [SD]}} Y$) if $[X|Y = y_1] \leq_{\text{st}} [\geq_{\text{st}}][X|Y = y_2]$, for all $y_1 \leq y_2$.*

(iii) *The bivariate random vector (X, Y) is said to be positively [negatively] dependent through stochastic ordering (PDS [NDS]) if $X \uparrow_{\text{SI [SD]}} Y$ and $Y \uparrow_{\text{SI [SD]}} X$.*

(iv) *X is said to be right tail increasing [decreasing] in Y (written as $X \uparrow_{\text{RTI [RTD]}} Y$) if $\mathbb{P}(X > x|Y > y)$ is increasing in $y \in \mathbb{R}$, for all $x \in \mathbb{R}$.*

(v) *The bivariate random vector (X, Y) is said to be positive [negative] quadrant dependent (PQD [NQD]) if, for all $(x, y) \in \mathbb{R}^2$, it holds that*

$$\mathbb{P}(X > x, Y > y) \geq [\leq]\mathbb{P}(X > x)\mathbb{P}(Y > y),$$

or equivalently,

$$\mathbb{P}(X \leq x, Y \leq y) \geq [\leq]\mathbb{P}(X \leq x)\mathbb{P}(Y \leq y).$$

Next, for a bivariate random vector (X, Y) with respective marginal d.f.'s F and G and joint d.f. H, it is well known that H admits the decomposition

$$H(x, y) = C(F(x), G(y)), \qquad x, y \in \mathbb{R},$$

where C, referred to as the copula of H, is a bivariate d.f. on $(0, 1)^2$ with uniform marginals (Sklar, 1959). If both F and G are continuous, then C is uniquely determined by $C(u, v) = H(F^{-1}(u), G^{-1}(v))$. We also denote by $\overline{C}$ the *joint tail function* for two uniform r.v.'s $U, V \sim U(0, 1)$ whose joint d.f. is the copula C, that is,

$$\overline{C}(u, v) = \mathbb{P}(U > u, V > v) = 1 - u - v + C(u, v), \quad (u, v) \in (0, 1)^2.$$

The joint tail function $\overline{C}$ should not be confused with the *survival copula* of U and V, which is defined as

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v), \quad (u, v) \in (0, 1)^2.$$

The survival copula $\hat{C}$ couples the joint survival function to its univariate margins (survival functions) in a manner completely analogous to how a copula links the joint d.f. to its margins. Clearly, $\overline{C}(u, v) = \hat{C}(1 - u, 1 - v)$.

We also recall the definition of concordance order (c.f. Definition 2.8.1 in Nelsen, 2007).

Definition 2.3. *Given two copulas C and C', C is said to be smaller than C' in concordance order (denoted by $C \prec C'$) if $C(u, v) \leq C'(u, v)$, for all $(u, v) \in (0, 1)^2$.*

Concordance order is also referred to as correlation order or positive quadrant dependence (PQD) order (see Dhaene and Goovaerts, 1996, 1997; Nelsen, 2007). It is a partial order as not every pair of copulas is concordance-comparable. Besides, the canonical scale-free dependence measures given by Kendall's tau and Spearman's rho are well known to be increasing with respect to the concordance order.

2.2 Distortion Risk Measures

We state the following definition:

Definition 2.4. *For a r.v. X with d.f. F, the* VaR *and* ES *at confidence level $\alpha \in (0,1)$ are defined as*

$$\mathrm{VaR}_\alpha[X] = F^{-1}(\alpha), \qquad \text{and} \qquad \mathrm{ES}_\alpha[X] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_p[X]\mathrm{d}p,$$

provided the integral exists.

VaR and ES occur as special cases of distortion risk measures (Yaari, 1987; Denuit et al., 2005, 2006; Dhaene et al., 2006; Goovaerts et al., 2010; Föllmer and Schied, 2011). In full generality, a *distortion function* $g : [0,1] \mapsto [0,1]$ is an increasing function such that $g(0) = 0$ and $g(1) = 1$. The set of all distortion functions is henceforth denoted by $\mathcal{G}$. A distortion risk measure is then defined as follows.

Definition 2.5. *For a distortion function $g \in \mathcal{G}$ and a r.v. X with d.f. F, the distortion risk measure D_g is defined as*

$$\mathrm{D}_g[X] = -\int_{-\infty}^0 [1 - g(\overline{F}(t))]\mathrm{d}t + \int_0^{+\infty} g(\overline{F}(t))\mathrm{d}t,$$

provided the integrals exist.

As is well known, a distortion risk measure is coherent, i.e., monotonic, translation invariant, positively homogeneous, *and* subadditive (see e.g., Föllmer and Schied, 2011; Laeven and Stadje, 2013) if and only if its distortion function is concave. The two prominent examples of distortion risk measures given by VaR_α and ES_α correspond to the distortion functions $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$ and $g(p) = \min\{1, \frac{p}{1-\alpha}\}$, for $\alpha \in (0,1)$, respectively. Obviously, the former distortion function is not continuous but only left-continuous, while the latter distortion function is continuous and concave but not differentiable everywhere. Besides, the incomplete beta function (Wang, 1995; Wirch and Hardy, 2001), the Wang distortion or rather Esscher-Girsanov transform (Goovaerts and Laeven, 2008), and the lookback distortion (Hürlimann, 2004) are commonly used special cases of distortion functions; see also Balbás et al. (2009). It is easily verified that for a r.v. X and any two distortion functions $g, g' \in \mathcal{G}$, $g \le g'$ implies that $\mathrm{D}_g[X] \le \mathrm{D}_{g'}[X]$.

Next, we introduce the notion of a dual distortion function. Consider a distortion function g and define $\overline{g} : [0,1] \mapsto [0,1]$ by $\overline{g}(p) = 1 - g(1-p)$, for $p \in [0,1]$. Obviously, $\overline{g}$ is also a distortion function, called the *dual distortion function* of g. It is well known that $\mathrm{D}_{\overline{g}}[X] = -\mathrm{D}_g[-X]$ (see Lemma 5 in Dhaene et al., 2012). Note that if g is left-continuous, then $\overline{g}$ is right-continuous (c.f. Theorems 4 and 6 in (Dhaene et al., 2012)). For a left-continuous distortion function g, the transformation of the tail function $\overline{F}$ of X given by $g(\overline{F}(x)) = g \circ \overline{F}(x)$ defines a new tail function associated to a r.v. X_g. It is the distorted counterpart of the r.v. X, induced by the distortion function g.

2.3 Co-Risk Measures and Risk Contribution Measures

In recent years, conditional risk (co-risk) measures are increasingly employed as measures of systemic risk. Prototypical examples of co-risk measures include the conditional Value-at-Risk (CoVaR), the conditional Expected Shortfall (CoES), and the marginal Expected Shortfall (MES). For a given co-risk measure, the corresponding risk contribution measure assesses the incremental effect of a stress scenario. Well-known examples of risk contribution measures include ΔCoVaR and ΔCoES. See e.g., Girardi and Ergün (2013), Mainik and Schaanning (2014), Adrian and Brunnermeier (2016), Acharya et al. (2017), and Karimalis and Nomikos (2018) for further details.

We state the following definitions:

Definition 2.6. *Let $\alpha, \beta \in (0,1)$. Then,*

$$\begin{aligned} \mathrm{CoVaR}_{\alpha,\beta}[Y|X] &= \mathrm{VaR}_\beta[Y|X > \mathrm{VaR}_\alpha[X]], \\ \mathrm{CoES}_{\alpha,\beta}[Y|X] &= \frac{1}{1-\beta}\int_\beta^1 \mathrm{CoVaR}_{\alpha,t}[Y|X]\mathrm{d}t, \\ \mathrm{MES}_\alpha[Y|X] &= \mathrm{CoES}_{\alpha,0}[Y|X] = \mathbb{E}[Y|X > \mathrm{VaR}_\alpha[X]]. \end{aligned}$$

One easily verifies that, with continuous marginals, the CoES can be represented through a conditional expectation of Y, in a manner similar to the familiar representation of ES:

$$\mathrm{CoES}_{\alpha,\beta}[Y|X] = \mathbb{E}[Y|X > \mathrm{VaR}_\alpha[X], Y > \mathrm{CoVaR}_{\alpha,\beta}[Y|X]].$$

To measure the risk contribution of X to Y, one may compare $\mathrm{CoVaR}_{\alpha,\beta}[Y|X]$, which is the VaR of Y conditional upon X being in a stress scenario, to $\mathrm{VaR}_\beta[Y]$, which evaluates Y unconditionally. Alternatively, one may replace the benchmark $\mathrm{VaR}_\beta[Y]$ by the conditional VaR of Y given that X exceeds its median (see Mainik and Schaanning, 2014; Adrian and Brunnermeier, 2016). The same applies to CoES, *mutatis mutandis*.

Definition 2.7. *Let $\alpha, \beta \in (0,1)$. Then,*

$$\begin{aligned} \Delta\mathrm{CoVaR}_{\alpha,\beta}[Y|X] &= \mathrm{CoVaR}_{\alpha,\beta}[Y|X] - \mathrm{VaR}_\beta[Y], \\ \Delta^{\mathrm{med}}\mathrm{CoVaR}_{\alpha,\beta}[Y|X] &= \mathrm{CoVaR}_{\alpha,\beta}[Y|X] - \mathrm{CoVaR}_{1/2,\beta}[Y|X], \\ \Delta\mathrm{CoES}_{\alpha,\beta}[Y|X] &= \mathrm{CoES}_{\alpha,\beta}[Y|X] - \mathrm{ES}_\beta[Y], \\ \Delta^{\mathrm{med}}\mathrm{CoES}_{\alpha,\beta}[Y|X] &= \mathrm{CoES}_{\alpha,\beta}[Y|X] - \mathrm{CoES}_{1/2,\beta}[Y|X]. \end{aligned}$$

3 Conditional Distortion Risk Measures and Distortion Risk Contribution Measures

Consider a bivariate random vector (X, Y) with marginal d.f.'s F, G and joint d.f. H. We define the conditional distortion (CoD) risk measure as follows; CoVaR, CoES, and MES occur as special cases.

Definition 3.1 (Conditional distortion risk measures). *For $g, h \in \mathcal{G}$,*

$$\begin{aligned} \mathrm{CoD}_{g,h}[Y|X] &= \mathrm{D}_h\left[Y|X > \mathrm{D}_g[X]\right] \\ &= -\int_{-\infty}^{0}[1 - h(\overline{F}_{Y|X>\mathrm{D}_g[X]}(y))]\mathrm{d}y + \int_{0}^{+\infty} h(\overline{F}_{Y|X>\mathrm{D}_g[X]}(y))\mathrm{d}y, \end{aligned}$$

where $\mathrm{D}_g[X]$ is as in Definition 2.5.

Remark 3.2. *(a) From Definition 3.1 it is apparent that CoD-risk measures evaluate the risk associated with $[Y|X > \mathrm{D}_g[X]]$, i.e., the risk Y subject to the conditioning event where X is in the stress scenario given by $X > \mathrm{D}_g[X]$. Furthermore, g and h are distortion functions imposed on the d.f.'s of X and $[Y|X > \mathrm{D}_g[X]]$, respectively.*

(b) One readily verifies that CoVaR and CoES (hence MES) occur as special cases in the class of CoD-risk measures. Indeed, if $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$ and

(i) $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, then $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{CoVaR}_{\alpha,\beta}[Y|X]$;

(ii) $h(p) = \min\{1, \frac{p}{1-\beta}\}$, then $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{CoES}_{\alpha,\beta}[Y|X]$;

(iii) $h(p) = p$, then $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{MES}_{\alpha}[Y|X]$.

Besides, two related types of conditional risk measures occur as follows: if $g(p) = \min\{1, \frac{p}{1-\alpha}\}$ and

(iv) $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, then $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{VaR}_{\beta}[Y|X > \mathrm{ES}_{\alpha}[X]]$;

(v) $h(p) = \min\{1, \frac{p}{1-\beta}\}$, then $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{ES}_{\beta}[Y|X > \mathrm{ES}_{\alpha}[X]]$, which is considered in Equation (10) of Boyle and Kim (2012).

(c) The class of CoD-risk measures is rich and is in general not restricted to satisfy (conditional) subadditivity. For example, if $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, it reduces to a conditional VaR, which is not subadditive in general. However, if h is concave then the CoD-risk measure inherits the subadditivity property of $\mathrm{D}_h[\cdot]$.

(d) Computing CoD-risk measures requires the conditional d.f. of Y conditioned upon $X > \mathrm{D}_g[X]$. Throughout we restrict attention to conditioning events of the form $X > \mathrm{D}_g[X]$ instead of $X = \mathrm{D}_g[X]$, as the former are probabilistically and statistically better behaved; see also the discussion in Girardi and Ergün (2013) and Mainik and Schaanning (2014).

To illustrate the generality of the class of CoD-risk measures, we present in Section 7 a collection of examples that goes well beyond the conditional risk measures considered in the literature.

For a given CoD-risk measure, we define two types of associated distortion risk contribution measures, $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X]$, as follows.

Definition 3.3 (Distortion risk contribution measures). *(i) Type-I: For $g, h \in \mathcal{G}$,*

$$\Delta\mathrm{CoD}_{g,h}[Y|X] = \mathrm{CoD}_{g,h}[Y|X] - \mathrm{D}_h[Y],$$

where $\mathrm{D}_h[Y]$ is as in Definition 2.5.

(ii) Type-II: For $g, \tilde{g}, h \in \mathcal{G}$,

$$\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] = \mathrm{CoD}_{g,h}[Y|X] - \mathrm{CoD}_{\tilde{g},h}[Y|X].$$

Remark 3.4. *(a) The class of distortion risk contribution measures in Definition 3.3(i) contains $\Delta\mathrm{CoVaR}$ and $\Delta\mathrm{CoES}$ as special cases. Indeed, if $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$ and*

(i) $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, then $\Delta\mathrm{CoD}_{g,h}[Y|X] = \Delta\mathrm{CoVaR}_{\alpha,\beta}[Y|X]$;

(ii) $h(p) = \min\{1, \frac{p}{1-\beta}\}$, then $\Delta\mathrm{CoD}_{g,h}[Y|X] = \Delta\mathrm{CoES}_{\alpha,\beta}[Y|X]$.

Furthermore, two related risk contribution measures (which seem to be new in the literature) arise as follows: if $g(p) = \min\{1, \frac{p}{1-\alpha}\}$ and

(iii) $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, then $\Delta\mathrm{CoD}_{g,h}[Y|X] = \mathrm{VaR}_\beta[Y|X > \mathrm{ES}_\alpha[X]] - \mathrm{VaR}_\beta[Y]$;

(iv) $h(p) = \min\{1, \frac{p}{1-\beta}\}$, then $\Delta\mathrm{CoD}_{g,h}[Y|X] = \mathrm{ES}_\beta[Y|X > \mathrm{ES}_\alpha[X]] - \mathrm{ES}_\beta[Y]$.

(b) Boyle and Kim (2012) consider the following risk contribution measure (see their Equation (8)): $\mathrm{ES}_\beta[Y|X = \mathrm{ES}_\alpha[X]] - \mathrm{ES}_\beta[Y]$, where the conditional event is $[X = \mathrm{ES}_\alpha[X]]$ instead of $[X > \mathrm{ES}_\alpha[X]]$. We restrict attention to conditioning events of the form $X > \mathrm{D}_g[X]$; cf. Remark 3.2(d).

(c) The class of distortion risk contribution measures in Definition 3.3(ii) contains $\Delta^{\mathrm{med}}\mathrm{CoVaR}$ and $\Delta^{\mathrm{med}}\mathrm{CoES}$ as special cases. More explicitly, if $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$, $\tilde{g}(p) = \mathbf{1}_{(1/2,1]}(p)$, and

(i) $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, then $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] = \Delta^{\mathrm{med}}\mathrm{CoVaR}_{\alpha,\beta}[Y|X]$;

(ii) $h(p) = \min\{1, \frac{p}{1-\beta}\}$, then $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] = \Delta^{\mathrm{med}}\mathrm{CoES}_{\alpha,\beta}[Y|X]$.

In the next theorems, we present some useful representations of our CoD-risk measures and (two types of) distortion risk contribution measures introduced in Definitions 3.1 and 3.3. These representations will play a key role in proving the subsequent comparison results.

Theorem 3.5. *Let $(U, V) \sim C$ where C is a copula of H. If F is continuous and strictly increasing, and h is left-continuous, then*

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1}(F^{-1}_{V|U>u_g}(p))\mathrm{d}\overline{h}(p), \tag{1}$$

where $u_g = F(\mathrm{D}_g[X])$, $\overline{h}(p) = 1 - h(1-p)$ for $p \in [0,1]$, and $F_{V|U>u}(v) = \frac{v - C(u,v)}{1-u}$ for $(u, v) \in (0,1)^2$.

Remark 3.6. *Consider the setup of Theorem 3.5. Define the generalized upper inverses* $G^{-1+}(p) = \sup\{x \in \mathbb{R} \mid G(x) \le p\}$ *and* $F^{-1+}_{V|U>u_g}(p) = \sup\{x \in \mathbb{R} \mid F_{V|U>u_g}(x) \le p\}$ *with* $\sup \emptyset = -\infty$ *by convention. Since the event* $\{F_{Y|X>\mathrm{D}_g[X]}(y) \le p\}$ *is equivalent to* $\{F_{V|U>u_g}(G(y)) \le p\}$, *we have* $F^{-1+}_{Y|X>\mathrm{D}_g[X]}(p) = G^{-1+}(F^{-1+}_{V|U>u_g}(p))$, *for* $p \in (0,1)$. *If now, under the setup of Theorem 3.5, h were right-continuous instead of left-continuous, then, by applying Theorem 4 in Dhaene et al. (2012), expression (1) can be modified as*

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1+}(F^{-1+}_{V|U>u_g}(p))\mathrm{d}\overline{h}(p).$$

Note that $F_{V|U>u_g}(v) = \frac{v-C(u_g,v)}{1-u_g}$. *If G is continuous and strictly increasing, and* $v - C(u,v)$ *is continuous and strictly increasing in* $v \in [0,1]$ *for any* $u \in (0,1)$ *(which implies that* $F_{V|U>u_g}$ *is continuous and strictly increasing), we have* $G^{-1+}(p) = G^{-1}(p)$ *and* $F^{-1+}_{V|U>u_g}(p) = F^{-1}_{V|U>u_g}(p)$, *which implies that the distortion function h in Theorem 3.5 can be either left-continuous or right-continuous (given that F is continuous and strictly increasing). Then, by applying Theorem 7 of Dhaene et al. (2012), h can be also assumed to be any general distortion function, i.e., a convex combination of left-continuous and right-continuous distortion functions.*

Corollary 3.7. *Under the setup of Theorem 3.5, if* $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$ *and*

- *(i)* $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$, *then* $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{CoVaR}_{\alpha,\beta}[Y|X] = G^{-1}(F^{-1}_{V|U>\alpha}(\beta))$;
- *(ii)* $h(p) = \min\{1, \frac{p}{1-\beta}\}$, *then* $\mathrm{CoD}_{g,h}[Y|X] = \mathrm{CoES}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 G^{-1}(F^{-1}_{V|U>\alpha}(p))\mathrm{d}p$; *cf. Girardi and Ergün (2013) and Mainik and Schaanning (2014).*

Based on Theorem 3.5, we obtain the following result, which gives representations for the proposed two types of distortion risk contribution measures.

Theorem 3.8. *Let* $(U,V) \sim C$ *where C is a copula of H. If F is continuous and strictly increasing, and h is left-continuous, then*

$$\Delta\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \left[G^{-1}(F^{-1}_{V|U>u_g}(p)) - G^{-1}(p)\right]\mathrm{d}\overline{h}(p), \tag{2}$$

$$\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \left[G^{-1}(F^{-1}_{V|U>u_g}(p)) - G^{-1}(F^{-1}_{V|U>u_{\tilde{g}}}(p))\right]\mathrm{d}\overline{h}(p), \tag{3}$$

where $u_g = F(\mathrm{D}_g[X])$, $u_{\tilde{g}} = F(\mathrm{D}_{\tilde{g}}[X])$, $\overline{h}(p) = 1 - h(1-p)$ *for* $p \in [0,1]$, *and* $F_{V|U>u}(v) = \frac{v-C(u,v)}{1-u}$ *for* $(u,v) \in (0,1)^2$.

4 Stochastic Orders and CoD-Risk Measures

Let (X, Y) and (X', Y') be two bivariate random vectors with marginals F, G and F', G' and copula C and C', respectively. Henceforth, we always assume that F, F', G and G' are continuous and strictly increasing and that the distortion functions for X, X', Y and Y' are left-continuous, to avoid cumbersome subtle technical discussions; cf. Remark 3.6.

4.1 The Risks Y and Y' Have Equal Marginal Distributions

This subsection provides sufficient conditions for the CoD-risk measures of the bivariate random vectors (X, Y) and (X', Y') to be ordered, when Y and Y' have common d.f.'s, i.e., $G = G'$. Our next result states that, in this setting, the CoD-risk measure preserves the ordering induced by "$\prec$" between the copulas and by the distortion functions applied to Y and Y', when assuming additionally that $F = F'$. In other words, the result states that more positive dependence together with a larger distortion function on the conditional risk event leads to a larger CoD-risk measure.

Theorem 4.1. *Suppose $F = F'$ and $G = G'$. Then, $C \prec C'$ and $h \leq h'$ imply that* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h'}[Y'|X']$.

The following result, which in contrast to Theorem 4.1 does not require $F = F'$, can easily be derived from (the proof of) Theorem 4.1 when g is the distortion function associated with VaR.

Corollary 4.2. *Suppose that $G = G'$ and $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$, for some $\alpha \in (0, 1)$. Then, $C \prec C'$ and $h \leq h'$ imply that* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h'}[Y'|X']$.

Remark 4.3. *(i) Due to the Fréchet-Hoeffding bounds, the worst (best) case dependence scenario in Theorem 4.1 and Corollary 4.2 occurs under comonotonicity (countermonotonicity). This stands in sharp contrast to worst VaR scenarios in risk aggregation; see e.g., Embrechts et al. (2005) and Laeven (2009).*

(ii) Theorem 4.1 and Corollary 4.2 do not hold as stated when X and X' adopt different distortion functions, i.e., when $\mathrm{CoD}_{g,h'}[Y'|X']$ is replaced by $\mathrm{CoD}_{g',h'}[Y'|X']$ with $g \neq g'$.

(iii) Under the additional assumption that $h(p) = h'(p) = \mathbf{1}_{(1-\beta,1]}(p)$, the result of Corollary 4.2 reduces to Theorem 3.4 of Mainik and Schaanning (2014).

To conclude this subsection, we investigate the effects of threshold quantiles of X and X' and the dependence structure among (X, Y) on the CoD-risk measures.

Theorem 4.4. *Suppose that $G = G'$. Let $u_g = F(\mathrm{D}_g[X])$ and $u_{g'} = F'(\mathrm{D}_{g'}[X'])$. Then* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g',h'}[Y'|X']$ *if $C \prec C'$, $h \leq h'$, and either one of the following two conditions holds:*

(i) $u_g \le u_{g'}$ *and* $Y \uparrow_{\mathrm{RTI}} X$ *or* $Y' \uparrow_{\mathrm{RTI}} X'$ *or both hold;*

(ii) $u_g \ge u_{g'}$ *and* $Y \uparrow_{\mathrm{RTD}} X$ *or* $Y' \uparrow_{\mathrm{RTD}} X'$ *or both hold.*

Theorem 4.4 states that, if X and Y are positively [negatively] dependent through $Y \uparrow_{\mathrm{RTI\ [RTD]}} X$, then a larger distortion function employed for risk Y, more concordance of the copula, together with a larger [smaller] threshold quantile adopted for risk X lead to a larger CoD-risk measure.

Remark 4.5. *(i) Suppose that* $h \le g$ *and* $Y \uparrow_{\mathrm{RTD}} X$*. Then, in light of Theorem 4.4(ii), we have* $\mathrm{CoD}_{g,h}[Y|X] \le \mathrm{CoD}_{h,g}[Y|X]$*. This means that if* Y *is negatively dependent of* X *through RTD, then a larger distortion function for* Y *and a smaller distortion function for* X *lead to a larger value of the CoD-risk measure.*

(ii) Let (X, Y) *and* (X', Y) *be two bivariate random vectors with the same copula* C*. Suppose that either (i)* $u_g \le u_{g'}$ *and* $Y \uparrow_{\mathrm{RTI}} X$ *or* $Y \uparrow_{\mathrm{RTI}} X'$ *or both hold, or (ii)* $u_g \ge u_{g'}$ *and* $Y \uparrow_{\mathrm{RTD}} X$ *or* $Y \uparrow_{\mathrm{RTD}} X'$ *or both hold. Then Theorem 4.4 implies that* $\mathrm{CoD}_{g,h}[Y|X] \le \mathrm{CoD}_{g',h}[Y|X']$ *for all* $h \in \mathcal{G}$*. This result states that* X' *is more relevant for* Y *than* X *if* Y *is positively [negatively] dependent of* X *(and/or* X'*) through RTI [RTD] and the threshold quantile of* X *is smaller [larger] than that of* X'*, which is consistent with the systemic relevance order proposed in Dhaene et al. (2020).*

4.2 The Risks Y and Y' Have Different Marginal Distributions

We now provide sufficient conditions for the CoD-risk measures of the bivariate random vectors (X, Y) and (X', Y') to be ordered when Y and Y' are allowed to have different d.f.'s.

Sordo and Ramos (2007) provides a useful characterization of the usual stochastic order and the increasing convex order as follows. In a similar manner, we give an equivalent characterization of the increasing concave order. All three serve as auxiliary results.

Lemma 4.6. *Let* X *and* Y *be two r.v.'s with continuous and strictly increasing d.f.'s* F *and* G*, respectively. Then,* $X \le_{\mathrm{st\ [icx,\ icv]}} Y$ *if and only if*

$$\int_0^1 F^{-1}(t)\mathrm{d}\phi(t) \le \int_0^1 G^{-1}(t)\mathrm{d}\phi(t),$$

for all increasing [increasing convex, increasing concave] $\phi : [0,1] \to [0,1]$*.*

Employing Lemma 4.6, we obtain the following comparison result.

Theorem 4.7. *Let* $F = F'$*,* $C \prec C'$*, and* $h \le h'$*.*

(i) Suppose that $X \uparrow_{\mathrm{SI}} Y$ *or* $X' \uparrow_{\mathrm{SI}} Y'$ *or both hold. Then,* $Y \leq_{\mathrm{st}\ [\mathrm{icx}]} Y'$ *implies that* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h'}[Y'|X']$ *for any* $g \in \mathcal{G}$ *and increasing [increasing concave]* $h, h' \in \mathcal{G}$.

(ii) Suppose that $X \uparrow_{\mathrm{SD}} Y$ *or* $X' \uparrow_{\mathrm{SD}} Y'$ *or both hold. Then,* $Y \leq_{\mathrm{icv}} Y'$ *implies that* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h'}[Y'|X']$ *for any* $g \in \mathcal{G}$ *and increasing convex* $h, h' \in \mathcal{G}$.

Remark 4.8. *Dhaene et al. (2020) defines the systemic contribution order in terms of the increasing convex order. More explicitly, consider the market of losses* $\underline{Z}$*, the corresponding microprudential regulation* $\underline{R}$*, and the aggregate residual loss level* $s \in \mathbb{R}_0^+$*. Individual loss* Z_j *is said to be "smaller in systemic contribution order" (denoted by* $Z_j \leq_{(\underline{R},s)-con} Z_k$*) than individual loss* Z_k *under microprudential regulation* $\underline{R}$ *and aggregate loss level* s*, if*

$$\left[(Z_j - R_j)_+ \mid \sum_{i=1}^{n} (Z_i - R_i)_+ > s\right] \leq_{\mathrm{icx}} \left[(Z_k - R_k)_+ \mid \sum_{i=1}^{n} (Z_i - R_i)_+ > s\right].$$

According to Theorem 4.7(i), if $X = X' = \sum_{i=1}^{n} (Z_i - R_i)_+$, $Y = (Z_j - R_j)_+$, $Y' = (Z_k - R_k)_+$, $C = C'$, $h = h'$*, and* $\sum_{i=1}^{n} (Z_i - R_i)_+ \uparrow_{\mathrm{SI}} (Z_j - R_j)_+$ *or* $\sum_{i=1}^{n} (Z_i - R_i)_+ \uparrow_{\mathrm{SI}} (Z_k - R_k)_+$ *or both hold, then* $(Z_j - R_j)_+ \leq_{\mathrm{icx}} (Z_k - R_k)_+$ *implies that*

$$\mathrm{CoD}_{g,h}\left[(Z_j - R_j)_+ \middle| \sum_{i=1}^{n} (Z_i - R_i)_+\right] \leq \mathrm{CoD}_{g,h}\left[(Z_k - R_k)_+ \middle| \sum_{i=1}^{n} (Z_i - R_i)_+\right],$$

for all concave h*, which is consistent with the definition* $Z_j \leq_{(\underline{R},s)-con} Z_k$ *when taking* $s = \mathrm{D}_g[\sum_{i=1}^{n} (Z_i - R_i)_+]$.

The following result, in contrast to Theorem 4.7 not requiring $F = F'$, can be derived from (the proof of) Theorem 4.7 when g corresponds to the distortion function of VaR.

Corollary 4.9. *Let* $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$, $C \prec C'$*, and* $h \leq h'$.

(i) Suppose that $X \uparrow_{\mathrm{SI}} Y$ *or* $X' \uparrow_{\mathrm{SI}} Y'$ *or both hold. Then,* $Y \leq_{\mathrm{st}\ [\mathrm{icx}]} Y'$ *implies that* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h'}[Y'|X']$ *for any increasing [increasing concave]* $h, h' \in \mathcal{G}$.

(ii) Suppose that $X \uparrow_{\mathrm{SD}} Y$ *or* $X' \uparrow_{\mathrm{SD}} Y'$ *or both hold. Then,* $Y \leq_{\mathrm{icv}} Y'$ *implies that* $\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h'}[Y'|X']$ *for any increasing convex* $h, h' \in \mathcal{G}$.

Remark 4.10. *In the special case that* $h(p) = h'(p) = \min\{1, \frac{p}{1-\beta}\}$*, the result of Corollary 4.9(i) reduces to Theorem 12 in Sordo et al. (2018).*

The next result generalizes Theorem 4.7 to the case of different d.f.'s of X and X', where we replace the condition '$F = F'$' by requiring that $F(\mathrm{D}_g[X]) \leq F'(\mathrm{D}_{g'}[X'])$.

Theorem 4.11. *Let* $C \prec C'$, $h \leq h'$*, and* $u_g \leq u_{g'}$*, where* $u_g = F(\mathrm{D}_g[X])$ *and* $u_{g'} = F'(\mathrm{D}_{g'}[X'])$.

(i) Suppose that (X, Y) *or* (X', Y') *or both are PDS. Then,* $Y \leq_{\text{st [icx]}} Y'$ *implies that* $\text{CoD}_{g,h}[Y|X] \leq \text{CoD}_{g',h'}[Y'|X']$ *for any increasing [increasing concave]* $h, h' \in \mathcal{G}$*.*

(ii) Suppose that (X, Y) *or* (X', Y') *or both are NDS. Then,* $Y \leq_{\text{icv}} Y'$ *implies that* $\text{CoD}_{g,h}[Y|X] \leq \text{CoD}_{g',h'}[Y'|X']$ *for any increasing convex* $h, h' \in \mathcal{G}$*.*

5 Stochastic Orders and Distortion Risk Contribution Measures

Like in Section 4, we consider two bivariate random vectors (X, Y) and (X', Y') with marginals F, G and F', G' and copula C and C', respectively. This section provides sufficient conditions for their distortion risk contribution measures to be ordered.

5.1 Dispersive Order and Distortion Risk Contribution Measures

The following lemma, adapted from Sordo et al. (2018), is helpful to establish our results that follow, which link the dispersive order among the marginals and distortion risk contribution measures.

Lemma 5.1. *Let* X *and* Y *be two r.v.'s with continuous and strictly increasing d.f.'s* F *and* G*, respectively. Let* h *be a convex distortion function and let* g *be another distortion function such that* $h(p) \geq g(p)$*, for all* $p \in [0, 1]$*. Denote by* X_h *[*Y_g*] the distorted r.v.'s induced from* X *[*Y*] by the distortion functions* h *[*g*]. If* $X \leq_{\text{disp}} Y$*, then*

(i) $F^{-1}_{X_h}(p) - F^{-1}(p) \geq F^{-1}_{Y_g}(p) - G^{-1}(p)$*, for* $p \in (0, 1)$*;*

(ii) $F^{-1}_{X_h}(p) - F^{-1}_{X_g}(p) \leq F^{-1}_{Y_h}(p) - F^{-1}_{Y_g}(p)$*, for* $p \in (0, 1)$*.*

5.1.1 Type-I Distortion Risk Contribution Measures: $\Delta\text{CoD}_{g,h}[Y|X]$

We first study sufficient conditions on the dependence structures of (X, Y) and (X', Y') and stochastic orderings between Y and Y' for comparing $\Delta\text{CoD}_{g,h}[Y|X]$ and $\Delta\text{CoD}_{g,h}[Y'|X']$.

Theorem 5.2. *Suppose that* $F = F'$ *and* $Y \leq_{\text{disp}} Y'$*.*

(i) If $C \prec C'$*, and* $X \uparrow_{\text{SI}} Y$ *or* $X' \uparrow_{\text{SI}} Y'$ *or both hold, then* $\Delta\text{CoD}_{g,h}[Y|X] \leq \Delta\text{CoD}_{g,h}[Y'|X']$ *for any* $g, h \in \mathcal{G}$*.*

(ii) If $C \succ C'$*, and* $X \uparrow_{\text{SD}} Y$ *or* $X' \uparrow_{\text{SD}} Y'$ *or both hold, then* $\Delta\text{CoD}_{g,h}[Y|X] \geq \Delta\text{CoD}_{g,h}[Y'|X']$ *for any* $g, h \in \mathcal{G}$*.*

The next theorem generalizes Theorem 5.2 to the case where X and X' may have different d.f.'s.

Theorem 5.3. *Let $u_g = F(\mathrm{D}_g[X])$ and $u_{g'} = F'(\mathrm{D}_{g'}[X'])$. Suppose that $Y \leq_{\mathrm{disp}} Y'$.*

(i) If C is PDS, $u_g \leq u_{g'}$, and $C \prec C'$, then $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g',h}[Y'|X']$ for any $h \in \mathcal{G}$.

(ii) If C is NDS, $u_g \geq u_{g'}$, and $C \succ C'$, then $\Delta\mathrm{CoD}_{g,h}[Y|X] \geq \Delta\mathrm{CoD}_{g',h}[Y'|X']$ for any $h \in \mathcal{G}$.

The following lemma is partially taken from Theorem 5 in Sordo et al. (2015) and the proof for the case of a convex distortion function can be established using similar arguments.

Lemma 5.4. *Let X be a r.v. and let g be a concave [convex] distortion function. Then $X \leq_{\mathrm{hr}} [\geq_{\mathrm{hr}}] X_g$, where X_g is the distorted r.v. induced from X by applying the distortion function g.*

Recall that a (nonnegative) r.v. X has an increasing [decreasing] failure rate (IFR [DFR]) if, and only if, its survival function $\overline{F}$ is log-concave [log-convex].

Theorem 5.5. *Assume that Y is DFR. Then $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g,h'}[Y|X]$ for any $g \in \mathcal{G}$ if either one of the following two conditions holds:*

(i) $X \uparrow_{\mathrm{SI}} Y$ and $h \leq h'$;

(ii) $X \uparrow_{\mathrm{SD}} Y$ and $h \geq h'$.

Remark 5.6. *If $X \uparrow_{\mathrm{SI}} Y$, Y is DFR, $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$, $h(p) = \min\{1, \frac{p}{1-\beta}\}$, and $h'(p) = \min\{1, \frac{p}{1-\beta'}\}$ such that $\beta \leq \beta'$, then Theorem 5.5 reduces to the result of Theorem 17 in Sordo et al. (2018).*

Upon combining Theorems 5.2 and 5.5, the following result can be obtained immediately. It generalizes the result of Corollary 19 in Sordo et al. (2018).

Corollary 5.7. *Suppose that $F = F'$, $X \uparrow_{\mathrm{SI}} Y$ or $X' \uparrow_{\mathrm{SI}} Y'$ or both hold, and Y or Y' or both are DFR. Then, $Y \leq_{\mathrm{disp}} Y'$, $C \prec C'$, and $h \leq h'$ imply that $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g,h'}[Y'|X']$ for any $g \in \mathcal{G}$.*

The following result generalizes the above result to the case of risks X and X' having different d.f.'s.

Theorem 5.8. *Let $u_g = F(\mathrm{D}_g[X])$ and $u_{g'} = F'(\mathrm{D}_{g'}[X'])$. Suppose that (X, Y) or (X', Y') or both are PDS, and Y or Y' or both are DFR. Then, $Y \leq_{\mathrm{disp}} Y'$, $C \prec C'$, $u_g \leq u_{g'}$, and $h \leq h'$ imply that $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g',h'}[Y'|X']$ for any $g \in \mathcal{G}$.*

5.1.2 Type-II Distortion Risk Contribution Measures: $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X]$

In this subsection, we turn our attention to the distortion risk contribution measure $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X]$.

Theorem 5.9. *Suppose that $C = C'$, $F = F'$ and $u_g \geq u_{\tilde{g}}$, where $u_g = F(\mathrm{D}_g[X])$ and $u_{\tilde{g}} = F(\mathrm{D}_{\tilde{g}}[X])$. Then, $Y \leq_{\mathrm{disp}} Y'$ and C is PDS imply that $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] \leq \Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y'|X']$ for any $h \in \mathcal{G}$.*

Remark 5.10. *Theorem 5.9 contains Theorem 20 of Sordo et al. (2018) as a special case when $\tilde{g}(p) = \mathbf{1}_{(1/2,1]}(p)$, $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$, and $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$ with $1/2 \leq \alpha \leq 1$. It is also worth noting that the condition that "C is TP_2" used there can be weakened by "C is PDS" as seen in Theorem 5.9. Besides, the condition $u_g \geq u_{\tilde{g}}$ is equivalent to $\mathrm{D}_g[X] \geq \mathrm{D}_{\tilde{g}}[X]$. Therefore, a sufficient condition for this inequality is to require $g \geq \tilde{g}$.*

The next result provides some other sufficient conditions in terms of the negative dependence structure of the copula.

Theorem 5.11. *Suppose that $C = C'$, $F = F'$, $u_g = F(\mathrm{D}_g[X])$, and $u_{\tilde{g}} = F(\mathrm{D}_{\tilde{g}}[X])$. Then, $Y \leq_{\mathrm{disp}} Y'$, C is NDS and $u_g \leq u_{\tilde{g}}$ imply that $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] \leq \Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y'|X']$ for any $h \in \mathcal{G}$.*

5.2 Excess Wealth Order and Distortion Risk Contribution Measures

For a r.v. X with d.f. F, Sordo (2008) establishes an equivalence characterization between the excess wealth order and the class of risk measures of the form

$$\mathrm{D}_{\phi_1,\phi_2}[X] = \int_0^1 F^{-1}(t)\mathrm{d}\phi_1(t) - \int_0^1 F^{-1}(t)\mathrm{d}\phi_2(t),$$

where ϕ_1 and ϕ_2 are two distortion functions.

Lemma 5.12. (Sordo, 2008) *Let X and Y be two r.v.'s with d.f.'s F and G, respectively. Then, $X \leq_{\mathrm{ew}} Y$ if and only if $\mathrm{D}_{\phi_1,\phi_2}[X] \leq \mathrm{D}_{\phi_1,\phi_2}[Y]$ for all $\mathrm{D}_{\phi_1,\phi_2}$ such that $\phi_2(t)$ and $\phi_1\phi_2^{-1}(t)$ are convex on $t \in [0,1]$.*

Theorem 5.13. *Suppose that $F = F'$, $X \uparrow_{\mathrm{SI}} Y$ or $X' \uparrow_{\mathrm{SI}} Y'$ or both hold, $h(t)$ is concave, and $\overline{h}(A(\overline{h}^{-1}(t)))$ is convex, where $A(t) = 1 - \frac{\overline{C}(u_g,t)}{1-u_g}$. Then, $Y \leq_{\mathrm{ew}} Y'$ and $C \prec C'$ imply that $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g,h}[Y'|X']$ for any $g \in \mathcal{G}$.*

It is interesting to study also sufficient conditions for the ordering of $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y'|X']$ by using the excess wealth order among the marginals. This is left as an open problem.

6 Interaction between Paired Risks

Recently, Fang and Li (2018) studied how the marginal d.f.'s and the dependence structure affect the interactions among paired risks under the CoVaR, CoES, ΔCoVaR, and ΔCoES measures. In this section, we shall establish some novel results for our CoD-risk measures and distortion risk contribution measures, which generalize the corresponding ones established in Fang and Li (2018).

Theorem 6.1. *Assume that $C(u,v)$ is symmetric, $u_g^X = F(\mathrm{D}_g[X])$, and $u_g^Y = G(\mathrm{D}_g[Y])$.*

(i) If $X \le_{\mathrm{st}} Y$, $u_g^X \ge u_g^Y$, and $Y \uparrow_{\mathrm{RTI}} X$, we have $\mathrm{CoD}_{g,h}[X|Y] \le \mathrm{CoD}_{g,h}[Y|X]$ for any $h \in \mathcal{G}$.

(ii) If $X \le_{\mathrm{st}} Y$, $u_g^X \le u_g^Y$, and $Y \uparrow_{\mathrm{RTD}} X$, we have $\mathrm{CoD}_{g,h}[X|Y] \le \mathrm{CoD}_{g,h}[Y|X]$ for any $h \in \mathcal{G}$.

(iii) If $X \le_{\mathrm{icx}} Y$, $u_g^X \ge u_g^Y$, and C is PDS, we have $\mathrm{CoD}_{g,h}[X|Y] \le \mathrm{CoD}_{g,h}[Y|X]$ for any concave $h \in \mathcal{G}$.

(iv) If $X \le_{\mathrm{icv}} Y$, $u_g^X \le u_g^Y$, and C is NDS, we have $\mathrm{CoD}_{g,h}[X|Y] \le \mathrm{CoD}_{g,h}[Y|X]$ for any convex $h \in \mathcal{G}$.

(v) If $X \le_{\mathrm{disp}} Y$, $u_g^X \ge u_g^Y$, and C is PDS, we have $\Delta\mathrm{CoD}_{g,h}[X|Y] \le \Delta\mathrm{CoD}_{g,h}[Y|X]$ for any $h \in \mathcal{G}$.

(vi) If $X \le_{\mathrm{disp}} Y$, $u_g^X \le u_g^Y$, and C is NDS, we have $\Delta\mathrm{CoD}_{g,h}[X|Y] \ge \Delta\mathrm{CoD}_{g,h}[Y|X]$ for any $h \in \mathcal{G}$.

The next result can be proved easily by using similar arguments as in the proof of Theorem 5.13, and thus we omit the proof for brevity.

Theorem 6.2. *Assume that $C(u,v)$ is symmetric and $u_g^X \ge u_g^Y$. If $X \le_{\mathrm{ew}} Y$, $C(u,v)$ is PDS, h is concave, and $\overline{h}(A_X(\overline{h}^{-1}(t)))$ is convex, where $A_X(t) = 1 - \frac{\overline{C}(u_g^X,t)}{1-u_g^X}$, then $\Delta\mathrm{CoD}_{g,h}[X|Y] \le \Delta\mathrm{CoD}_{g,h}[Y'|X']$ for any $g \in \mathcal{G}$.*

It would be of interest to obtain sufficient conditions for ordering co-risk measures and risk contribution measures when the copula is asymmetric. This research question is left as an open problem.

7 Examples

In this section, we present selected examples of the rich classes of conditional distortion risk measures and distortion risk contribution measures introduced in this paper, to illustrate their generality. In particular, we show that these classes naturally give rise to many interesting alternatives to CoVaR, CoES, and MES, simply by suitably choosing the respective distortion functions. For numerical examples that illustrate and validate our main comparison results, we refer to the Online Appendix provided as supplementary material.

Example 7.1. *(Dual-power function) Assume Y to be nonnegative and consider the distortion function $h(p) = 1 - (1-p)^k$, for $k \in \mathbb{N}^+$ and $p \in [0,1]$; see Eeckhoudt et al. (2020) and Eeckhoudt and Laeven (2020) and the references therein. Let $\widetilde{Y}_i^g$ be independent copies of the r.v. $[Y|X > \mathrm{D}_g[X]]$, for $i = 1, \ldots, k$. According to Definition 3.1, we then have*

$$\begin{aligned}\mathrm{CoD}_{g,h}[Y|X] &= \int_0^{+\infty} h(\overline{F}_{Y|X>\mathrm{D}_g[X]}(y))\mathrm{d}y = \int_0^{+\infty} [1 - F^k_{Y|X>\mathrm{D}_g[X]}(y)]\mathrm{d}y \\ &= \mathbb{E}[\max\{\widetilde{Y}_1^g, \ldots, \widetilde{Y}_k^g\}] = \mathbb{E}\left[\widetilde{Y}^g_{k:k}\right],\end{aligned}$$

where $\widetilde{Y}^g_{k:k}$ is the maximum order statistic of $\widetilde{Y}_1^g, \ldots, \widetilde{Y}_k^g$. This means that the CoD-risk measure can be represented as the expectation of the maximum order statistic computed from a set of i.i.d. r.v.'s with d.f. $F_{Y|X>\mathrm{D}_g[X]}$. Similarly, when $h(p) = p^k$ for $k \in \mathbb{N}^+$ and $p \in [0,1]$ (see Wang (1995)),

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^{+\infty} \overline{F}^k_{\widetilde{Y}^g}(y)\mathrm{d}y = \mathbb{E}\left[\widetilde{Y}^g_{1:k}\right],$$

where $\widetilde{Y}^g_{1:k}$ is the minimum order statistic of $\widetilde{Y}_1^g, \ldots, \widetilde{Y}_k^g$.

Example 7.2. *(Powers of dual-power functions) Consider the distortion function $h(p) = \left(1-(1-p)^k\right)^{\frac{1}{k}}$, for $k \in \mathbb{N}^+$ and $p \in [0,1]$; see Cherny and Madan (2009). Under the setting of Example 7.1, we have*

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^{+\infty} \left(1 - F^k_{Y|X>\mathrm{D}_g[X]}(y)\right)^{\frac{1}{k}} \mathrm{d}y = \int_0^{+\infty} \overline{F}^{\frac{1}{k}}_{\widetilde{Y}^g_{k:k}}(y)\mathrm{d}y,$$

where $\overline{F}_{\widetilde{Y}^g_{k:k}}$ is the survival function associated with the maximum order statistic $\widetilde{Y}^g_{k:k}$ arising from $\widetilde{Y}_1^g, \ldots, \widetilde{Y}_k^g$ defined in Example 7.1. Furthermore, $\overline{F}^{\frac{1}{k}}_{\widetilde{Y}^g_{k:k}}$ is the survival function associated with the r.v. $\overline{Y}^g$ which is such that the minimum over independent copies, $\min\{\overline{Y}_1^g, \ldots, \overline{Y}_k^g\}$, has the same distribution as $\max\{\widetilde{Y}_1^g, \ldots, \widetilde{Y}_k^g\}$. Similarly, when $h(p) = \left(1-(1-p)^{\frac{1}{k}}\right)^k$ for $k \in \mathbb{N}^+$ and $p \in [0,1]$,

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^{+\infty} \left(1 - F^{\frac{1}{k}}_{Y|X>\mathrm{D}_g[X]}(y)\right)^k \mathrm{d}y = \mathbb{E}\left[\min\left\{\hat{Y}_1^g, \ldots, \hat{Y}_k^g\right\}\right] = \mathbb{E}\left[\hat{Y}^g_{1:k}\right],$$

where $\hat{Y}^g_{1:k}$ is the minimum order statistic of i.i.d. r.v.'s $\hat{Y}_1^g, \ldots, \hat{Y}_k^g$ and the generic r.v. $\hat{Y}^g$ is such that $\max\{\hat{Y}_1^g, \ldots, \hat{Y}_k^g\}$ has the same distribution as $\widetilde{Y}^g$.

Example 7.3. *(Absolute deviation function) The absolute deviation principle in Denneberg (1990) corresponds to the following piecewise linear distortion function:*

$$h(p) = \begin{cases} (1+r)p, & 0 \le p < 0.5, \\ r + (1-r)p, & 0.5 \le p \le 1, \end{cases}$$

$0 < r < 1$. *Then, for a nonnegative r.v.* Y,

$$\begin{aligned} \mathrm{CoD}_{g,h}[Y|X] &= \int_{\overline{F}_{\widetilde{Y}^g}(y)\in[0,0.5)} (1+r)\overline{F}_{\widetilde{Y}^g}(y)\mathrm{d}y + \int_{\overline{F}_{\widetilde{Y}^g}(y)\in[0.5,1]} \left[r + (1-r)\overline{F}_{\widetilde{Y}^g}(y)\right] \mathrm{d}y \\ &= (1+r)\mathbb{E}[\widetilde{Y}^g] + r\overline{F}^{-1}_{\widetilde{Y}^g}(0.5) - 2r\int_{\overline{F}_{\widetilde{Y}^g}(y)\in[0.5,1]} \overline{F}_{\widetilde{Y}^g}(y)\mathrm{d}y. \end{aligned}$$

Example 7.4. *(Gini) Assume* Y *to be nonnegative and consider* $h(p) = (1+r)p - rp^2$, *for* $r \in [0,1]$ *and* $p \in [0,1]$*; see Denneberg (1990) for its connection to the Gini coefficient. One can verify that in this case,*

$$\begin{aligned} \mathrm{CoD}_{g,h}[Y|X] &= (1+r)\int_0^{+\infty} \overline{F}_{Y|X>\mathrm{D}_g[X]}(y)\mathrm{d}y - r\int_0^{+\infty} \overline{F}^2_{Y|X>\mathrm{D}_g[X]}(y)\mathrm{d}y \\ &= (1+r)\mathbb{E}[\widetilde{Y}^g] - r\mathbb{E}[\min\{\widetilde{Y}^g_1, \widetilde{Y}^g_2\}] \\ &= \mathbb{E}[\widetilde{Y}^g] + r\left(\mathbb{E}[\widetilde{Y}^g] - \mathbb{E}[\widetilde{Y}^g_{1:2}]\right) = \mathbb{E}[\widetilde{Y}^g] + r\left(\mathbb{E}[\widetilde{Y}^g_{2:2}] - \mathbb{E}[\widetilde{Y}^g]\right), \end{aligned}$$

where $\widetilde{Y}^g_{1:2}$ *and* $\widetilde{Y}^g_{2:2}$ *are the minimum and maximum of the independent copies* $\widetilde{Y}^g_1$ *and* $\widetilde{Y}^g_2$*, respectively.*

Other examples occur when the distortion function h is taken to be the well-known incomplete beta function (Wirch and Hardy (2001)), the lookback distortion (Hürlimann (2004)) given by

$$h(p) = p^r\left(1 - r\log(p)\right), \qquad r \in (0,1],$$

or the Esscher-Girsanov transform, which can be expressed as (Goovaerts and Laeven (2008) and Labuschagne and Offwood (2010))

$$h(p) = \Phi\left(\Phi^{-1}(p) + \mathfrak{h}\mathfrak{v}\right), \qquad \mathfrak{h} \in \mathbb{R}, \mathfrak{v} > 0,$$

with Φ the standard normal d.f.

The following example provides explicit expressions for the distortion risk contribution measures of Definition 3.3, under the setup of Example 7.1.

Example 7.5. *(i) Under the setup of Example 7.1, we have*

$$\Delta\mathrm{CoD}_{g,h}[Y|X] = \mathbb{E}[\widetilde{Y}^g_{k:k}] - \mathbb{E}\left[Y_{k:k}\right],$$

where $Y_{k:k}$ *is the maximum order statistic of* $Y_1, \ldots, Y_k$ *with* Y_i *being independent copies of* Y*, for* $i = 1, \ldots, k$.

(ii) *Similarly, under the setup of Example 7.1,*

$$\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] = \mathbb{E}[\widetilde{Y}^{g}_{k:k}] - \mathbb{E}[\widetilde{Y}^{\tilde{g}}_{k:k}],$$

where $\widetilde{Y}^{g}_{k:k}$ *and* $\widetilde{Y}^{\tilde{g}}_{k:k}$ *are the maximum order statistics of* $\widetilde{Y}^{g}_{1}, \ldots, \widetilde{Y}^{g}_{k}$ *and* $\widetilde{Y}^{\tilde{g}}_{1}, \ldots, \widetilde{Y}^{\tilde{g}}_{k}$*, respectively, and* $\widetilde{Y}^{\tilde{g}}_{i}$ *are independent copies of the r.v.* $[Y|X > \mathrm{D}_{\tilde{g}}[X]]$*, for* $i = 1, \ldots, k$*.*

Explicit expressions for the distortion risk contribution measures corresponding to Examples 7.2–7.4 occur *mutatis mutandis.*

In the following three examples, we provide closed-form expressions of the representation in Theorem 3.5 under various distributional assumptions on the pair (X, Y).

Example 7.6. *(Bivariate normal distribution) Assume that* $(X, Y) \sim N(\boldsymbol{\mu}, \Sigma)$*, where* $\boldsymbol{\mu} = (\mu_X, \mu_Y)^\top$*,* $\mu_X, \mu_Y \in \mathbb{R}$*,*

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix},$$

$\sigma_X, \sigma_Y > 0$*, and* $\rho \in (-1, 1)$*. Then, the copula of* (X, Y) *is given by*

$$C_\rho(u, v) \;=\; \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{2(1-\rho^2)}} \mathrm{d}s_1 \mathrm{d}s_2, \tag{4}$$

where $\Phi^{-1}(\cdot)$ *is the inverse of the standard normal d.f. It can be calculated that*

$$G^{-1}\left(F^{-1}_{V|U>u_g}(p)\right) = \mu_Y + \sigma_Y \Phi^{-1}\left(F^{-1}_{V|U>u_g}(p)\right),$$

where

$$F_{V|U>u_g}(v) = \frac{v - C_\rho(u_g, v)}{1 - u_g} \qquad \text{and} \qquad u_g = F(\mathrm{D}_g[X]). \tag{5}$$

Then, by applying Theorem 3.5, we obtain

$$\mathrm{CoD}_{g,h}[Y|X] = \mu_Y + \sigma_Y \int_0^1 \Phi^{-1}\left(F^{-1}_{V|U>u_g}(p)\right) \mathrm{d}\overline{h}(p). \tag{6}$$

In particular, we have the following two special cases. Let $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$*. If*

(i) $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$*, then (6) simplifies to*

$$\mathrm{CoVaR}_{\alpha,\beta}[Y|X] = \mu_Y + \sigma_Y \Phi^{-1}\left(F^{-1}_{V|U>\alpha}(\beta)\right);$$

cf. Mainik and Schaanning (2014).

(ii) $h(p) = \min\{1, \frac{p}{1-\beta}\}$*, then (6) simplifies to*

$$\mathrm{CoES}_{\alpha,\beta}[Y|X] = \mu_Y + \frac{\sigma_Y}{1-\beta}\int_\beta^1 \Phi^{-1}\left(F^{-1}_{V|U>\alpha}(p)\right)\mathrm{d}p.$$

Example 7.7. *(Bivariate log-normal distribution) Assume that* $(\log X, \log Y) \sim N(\boldsymbol{\mu}, \Sigma)$*, where* $\boldsymbol{\mu}$ *and* Σ *are as in Example 7.6. Then, the copula of* (X, Y) *is given in (4). Furthermore,*

$$G^{-1}\left(F^{-1}_{V|U>u_g}(p)\right) = e^{\mu_Y + \sigma_Y \Phi^{-1}\left(F^{-1}_{V|U>u_g}(p)\right)},$$

where $F_{V|U>u_g}(v)$ *and* u_g *are as in (5). From Theorem 3.5,*

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 e^{\mu_Y + \sigma_Y \Phi^{-1}\left(F^{-1}_{V|U>u_g}(p)\right)}\mathrm{d}\overline{h}(p). \tag{7}$$

In particular, with $g(p) = \mathbf{1}_{(1-\alpha,1]}(p)$*,*

(i) if $h(p) = \mathbf{1}_{(1-\beta,1]}(p)$*, (7) yields* $\mathrm{CoVaR}_{\alpha,\beta}[Y|X] = e^{\mu_Y + \sigma_Y \Phi^{-1}\left(F^{-1}_{V|U>\alpha}(\beta)\right)}$.

(ii) if $h(p) = \min\{1, \frac{p}{1-\beta}\}$*, (7) yields* $\mathrm{CoES}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 e^{\mu_Y + \sigma_Y \Phi^{-1}\left(F^{-1}_{V|U>\alpha}(p)\right)}\mathrm{d}p$.

Example 7.8. *(Bivariate Student t distribution) Assume that* $(X, Y) \sim t_\nu(\mathbf{0}, \Sigma)$*, where* $\Sigma = \begin{pmatrix}\rho & 1\\ 1 & \rho\end{pmatrix}$*,* $\rho \in (-1, 1)$*, and* $\nu > 0$ *is the common degrees of freedom of the marginals. Then, the copula of* (X, Y) *is given by*

$$C_{\rho,\nu}(u, v) = \int_{-\infty}^{t_\nu^{-1}(u)}\int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}}\left\{1 + \frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{\nu(1-\rho^2)}\right\}^{-\frac{\nu+2}{2}}\mathrm{d}s_1\mathrm{d}s_2,$$

where $t_\nu^{-1}(\cdot)$ *is the inverse of Student t d.f. with* ν *degrees of freedom. When* $\nu = 1$ *and* $\nu = 2$*, we can derive explicit formulas of CoD-risk measures, as follows:*

(i) If $\nu = 1$*, then*

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \tan\left(\pi\left(F^{-1}_{V|U>u_g}(p) - \frac{1}{2}\right)\right)\mathrm{d}\overline{h}(p),$$

where $F_{V|U>u_g}(v) = \frac{v - C_{\rho,\nu}(u_g, v)}{1-u_g}$.

(ii) If $\nu = 2$*, then*

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \frac{\sqrt{2}\left(F^{-1}_{V|U>u_g}(p) - \frac{1}{2}\right)}{\sqrt{F^{-1}_{V|U>u_g}(p)\left(1 - F^{-1}_{V|U>u_g}(p)\right)}}\mathrm{d}\overline{h}(p).$$

8 Conclusions

We have introduced the rich classes of conditional distortion (CoD) risk measures and distortion risk contribution (ΔCoD) measures, which include, and significantly extend, many of the existing measures proposed in the academic literature related to systemic risk. We have analyzed their properties and representations. We have given sufficient conditions for two random vectors to be ordered by the proposed measures, using the conventional stochastic order, the increasing convex [concave] order, the dispersive order, and the excess wealth order of marginals, under explicit assumptions of positive or negative dependence, distortion functions, and threshold quantiles. Several examples have been provided to illustrate our theoretical results.

This paper represents, of course, just the first step towards a systematic analysis of the class of conditional distortion risk measures, and opens up a novel area of investigation. Problems related to statistical inference, probabilistic analysis and evaluation, and refined worst case analysis are currently open.

This work is the second in a triplet on systemic risk by the same authors. Dhaene et al. (2020) introduces and investigates some new stochastic orders that can be applied in the context of systemic risk evaluation, whereas the present article introduces conditional distortion risk measures applicable in this context. In a future study, we will combine the results of both papers to attribute systemic risk to the different participants in a given market or economy with dependent risks.

Acknowledgements

We are very grateful to an Editor and two referees whose constructive comments and suggestions have significantly improved the paper. Jan Dhaene acknowledges the financial support of the Research Foundation Flanders (FWO) under grant GOC3817N. Roger Laeven acknowledges the financial support of the Netherlands Organization for Scientific Research under grant NWO VIDI. Yiying Zhang thanks the financial support and nice working environment of the Actuarial Research Group at KU Leuven and the Amsterdam Center of Excellence in Risk and Macro Finance at the University of Amsterdam during his visit. Yiying Zhang also acknowledges the financial support from the National Natural Science Foundation of China (No. 12101336) and the Natural Science Foundation of Tianjin (No. 20JCQNJC01740).

Supplementary Material

Supplement to "Systemic Risk: Conditional Distortion Risk Measures". In online supplementary material (Dhaene, Laeven and Zhang, 2021), we provide several numerical illustrations of our main theoretical results by specifying different copulas, distortion functions, and distribution functions of the risks.

Appendix

A Proofs

A.1 Proof of Theorem 3.5

Proof. Since F is continuous and strictly increasing, $(U,V)\sim C$, and the marginals of C are uniform, it follows that $\mathbb{P}(U>u_g)=\mathbb{P}(X>\mathrm{D}_g[X])$. Then, the d.f. of $[Y|X>\mathrm{D}_g[X]]$ can be written as

$$\begin{aligned}
F_{Y|X>\mathrm{D}_g[X]}(y) &= \mathbb{P}(Y\le y|X>\mathrm{D}_g[X])=\frac{\mathbb{P}(Y\le y,X>\mathrm{D}_g[X])}{\mathbb{P}(X>\mathrm{D}_g[X])}\\
&= \frac{\mathbb{P}(Y\le y)-\mathbb{P}(Y\le y,X\le \mathrm{D}_g[X])}{1-\mathbb{P}(X\le \mathrm{D}_g[X])}=\frac{G(y)-C(F(\mathrm{D}_g[X]),G(y))}{1-F(\mathrm{D}_g[X])}\\
&= F_{V|U>u_g}(G(y)),
\end{aligned}$$

which in turn implies that $F^{-1}_{Y|X>\mathrm{D}_g[X]}(p)=G^{-1}(F^{-1}_{V|U>u_g}(p))$ by using the argument that the event $\{F_{Y|X>\mathrm{D}_g[X]}(y)\ge p\}$ is equivalent to $\{F_{V|U>u_g}(G(y))\ge p\}$, for any $p\in(0,1)$. Hence, by applying Fubini's theorem and a change of variable (see Theorem 6 in Dhaene et al., 2012) one can verify that

$$\begin{aligned}
\mathrm{CoD}_{g,h}[Y|X] &= -\int_{-\infty}^{0}\left[1-h(1-F_{Y|X>\mathrm{D}_g[X]}(y))\right]\mathrm{d}y+\int_0^{+\infty}h(1-F_{Y|X>\mathrm{D}_g[X]}(y))\mathrm{d}y\\
&= \int_0^1 F^{-1}_{Y|X>\mathrm{D}_g[X]}(1-p)\mathrm{d}h(p)=\int_0^1 F^{-1}_{Y|X>\mathrm{D}_g[X]}(p)\mathrm{d}\overline{h}(p)\\
&= \int_0^1 G^{-1}(F^{-1}_{V|U>u_g}(p))\mathrm{d}\overline{h}(p).
\end{aligned}$$

Thus, the proof is established. ■

A.2 Proof of Theorem 4.1

Proof. Let $(U,V)\sim C$ and $(U',V')\sim C'$. From Theorem 3.5, we have

$$\mathrm{CoD}_{g,h}[Y|X]=\int_0^1 G^{-1}(F^{-1}_{V|U>u_g}(p))\mathrm{d}\overline{h}(p),\ \ \mathrm{CoD}_{g,h'}[Y'|X']=\int_0^1 G^{-1}(F^{-1}_{V'|U'>u_g}(p))\mathrm{d}\overline{h}'(p),$$

where $\overline{h}'(p)=1-h'(1-p)$ for $p\in[0,1]$.

We first show that $\mathrm{CoD}_{g,h}[Y|X]\le\mathrm{CoD}_{g,h}[Y'|X']$. Since $\overline{h}$ is increasing, this reduces to showing that $G^{-1}(F^{-1}_{V|U>u_g}(p))\le G^{-1}(F^{-1}_{V'|U'>u_g}(p))$, i.e., $F^{-1}_{V|U>u_g}(p)\le F^{-1}_{V'|U'>u_g}(p)$ for

$p \in (0,1)$. Thus, it suffices to show that $F_{V|U>u_g}(t) \geq F_{V'|U'>u_g}(t)$ for $t \in (0,1)$, that is,

$$\frac{t - C(u_g, t)}{1 - u_g} \geq \frac{t - C'(u_g, t)}{1 - u_g},$$

which is in fact guaranteed by the condition $C \prec C'$.

On the other hand, we can verify that $\overline{h}(0) = \overline{h}'(0) = 0$, $\overline{h}(1) = \overline{h}'(1) = 1$ and $\overline{h}'(p) \leq \overline{h}(p)$ because of $h(p) \leq h'(p)$ for $p \in [0,1]$. Then, by using integration by parts, one has

$$\begin{aligned} \mathrm{CoD}_{g,h}[Y'|X'] - \mathrm{CoD}_{g,h'}[Y'|X'] &= \int_0^1 G^{-1}(F^{-1}_{V'|U'>u_g}(p))\mathrm{d}(\overline{h}(p) - \overline{h}'(p)) \\ &= \int_0^1 (\overline{h}'(p) - \overline{h}(p))\mathrm{d}G^{-1}(F^{-1}_{V'|U'>u_g}(p)) \leq 0, \end{aligned}$$

which yields that $\mathrm{CoD}_{g,h}[Y'|X'] \leq \mathrm{CoD}_{g,h'}[Y'|X']$. Hence, the proof is established. ■

A.3 Proof of Theorem 4.4

Proof. We only give the proof of (i). The proof of (ii) can be established in a similar manner. We assume that $u_g \leq u_{g'}$ and $Y \uparrow_{\mathrm{RTI}} X$ (the other two cases follow similarly). Let $U = F(X)$ and $V = G(Y)$. In light of Theorem 3.5, we have

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1}(F^{-1}_{V|U>u_g}(p))\mathrm{d}\overline{h}(p).$$

By making use of a change of variable $p = F_{V|U>u_g}(t)$, we obtain

$$\begin{aligned} \mathrm{CoD}_{g,h}[Y|X] &= \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}\left(F^{-1}_{V|U>u_g}(t)\right) = \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}\left(\frac{t - C(u_g, t)}{1 - u_g}\right) \\ &= \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A_{u_g}(t)), \end{aligned}$$

where $A_{u_g}(t) = 1 - \frac{\overline{C}(u_g,t)}{1-u_g}$. Similarly, by letting $U' = F'(X')$ and $V' = G'(Y')$, we have

$$\mathrm{CoD}_{g',h'}[Y'|X'] = \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}'(A_{u_{g'}}(t)),$$

where $A_{u_{g'}}(t) = 1 - \frac{\overline{C}'(u_{g'},t)}{1-u_{g'}}$. Since $G = G'$, $\overline{h}'(A_{u_{g'}}(0)) = \overline{h}(A_{u_{g'}}(0)) = 0$, and $\overline{h}'(A_{u_{g'}}(1)) = \overline{h}(A_{u_{g'}}(1)) = 1$, we have

$$\mathrm{CoD}_{g',h'}[Y'|X'] - \mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1}(t)\mathrm{d}\left[\overline{h}'(A_{u_{g'}}(t)) - \overline{h}(A_{u_g}(t))\right]$$

$$= \int_0^1 \left[\overline{h}(A_{u_g}(t)) - \overline{h}'(A_{u_{g'}}(t))\right] \mathrm{d}G^{-1}(t). \qquad (8)$$

In order to show the nonnegativity of (8), it suffices to show that $\overline{h}(A_{u_g}(t)) \geq \overline{h}'(A_{u_{g'}}(t))$, for all $t \in [0,1]$. Since $h \leq h'$, one has $\overline{h}(A_{u_g}(t)) \geq \overline{h}'(A_{u_g}(t))$, for all $t \in [0,1]$. Thus, it is enough to show that $\overline{h}'(A_{u_g}(t)) \geq \overline{h}'(A_{u_{g'}}(t))$, for all $t \in [0,1]$, that is,

$$\frac{\overline{C}'(u_{g'}, t)}{1 - u_{g'}} \geq \frac{\overline{C}(u_g, t)}{1 - u_g}. \qquad (9)$$

Taking into account that $C \prec C'$, we have that

$$\frac{\overline{C}'(u_{g'}, t)}{1 - u_{g'}} \geq \frac{\overline{C}(u_{g'}, t)}{1 - u_{g'}}. \qquad (10)$$

Thus, by using (10), (9) can be established if we can show that

$$\frac{\overline{C}(u_{g'}, t)}{1 - u_{g'}} \geq \frac{\overline{C}(u_g, t)}{1 - u_g}. \qquad (11)$$

Since $u_g \leq u_{g'}$ and $Y \uparrow_{\mathrm{RTI}} X$, it holds that

$$\frac{\overline{C}(u_{g'}, t)}{1 - u_{g'}} = \mathbb{P}(V > t | U > u_{g'}) \geq \mathbb{P}(V > t | U > u_g) = \frac{\overline{C}(u_g, t)}{1 - u_g},$$

which proves (11) and thus the desired result is obtained. ■

A.4 Proof of Lemma 4.6

Proof. The proof for the usual stochastic order and the increasing convex order can be found in Sordo and Ramos (2007). We only prove the characterization of the increasing concave order.

Assume that $X \leq_{\mathrm{icv}} Y$. According to Theorem 4.A.1 in Shaked and Shanthikumar (2007), we know that $X \leq_{\mathrm{icv}} Y$ is equivalent to $-X \geq_{\mathrm{icx}} -Y$. Then, by using the equivalent characterization of the increasing convex order, it follows that

$$-X \geq_{\mathrm{icx}} -Y \Longleftrightarrow \int_0^1 F_{-X}^{-1}(t) \mathrm{d}\phi(t) \geq \int_0^1 F_{-Y}^{-1}(t) \mathrm{d}\phi(t),$$

for all increasing convex $\phi : [0,1] \to [0,1]$. Since F is continuous and strictly increasing, we have

$$\int_0^1 F_{-X}^{-1}(t) \mathrm{d}\phi(t) = -\int_0^1 F^{-1}(1-t) \mathrm{d}\phi(t),$$

and similarly for $-Y$. Thus, we have

$$X \leq_{\mathrm{icv}} Y \Longleftrightarrow -X \geq_{\mathrm{icx}} -Y \Longleftrightarrow \int_0^1 F^{-1}(1-t)\mathrm{d}\phi(t) \leq \int_0^1 G^{-1}(1-t)\mathrm{d}\phi(t),$$

that is

$$X \leq_{\mathrm{icv}} Y \Longleftrightarrow \int_0^1 F^{-1}(t)\mathrm{d}\psi(t) \leq \int_0^1 G^{-1}(t)\mathrm{d}\psi(t),$$

where $\psi(t) := 1-\phi(1-t)$ is increasing and concave on $[0,1]$. Hence, the proof is established. ■

A.5 Proof of Theorem 4.7

Proof. We only give the proof for the increasing convex ordering between Y and Y'. The proofs for the usual stochastic ordering and the increasing concave ordering can be obtained in a similar manner by using Lemma 4.6. Furthermore, we only consider the case of $X \uparrow_{\mathrm{SI}} Y$ since the proof can be carried out similarly for $X' \uparrow_{\mathrm{SI}} Y'$.

Let $U = F(X)$ and $V = G(Y)$. In light of Theorem 3.5, we have

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1}(F^{-1}_{V|U>u_g}(p))\mathrm{d}\overline{h}(p).$$

By a change of variable $p = F_{V|U>u_g}(t)$, we obtain

$$\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A(t)), \tag{12}$$

where $A(t) = 1 - \frac{\overline{C}(u_g,t)}{1-u_g}$. Note that $A(t)$ is increasing and convex in $t \in [0,1]$ since $\frac{\mathrm{d}A(t)}{\mathrm{d}t} \overset{\mathrm{sgn}}{=} -\frac{\partial \overline{C}(u_g,t)}{\partial t} = \mathbb{P}(U > u_g | V = t)$ is nonnegative and increasing in $t \in [0,1]$ because $X \uparrow_{\mathrm{SI}} Y$. On the other hand, it is easy to verify that $\overline{h}$ is also increasing convex due to the increasing concavity of h. Hence, we know $\overline{h}(A(t))$ is increasing and convex in $t \in [0,1]$. Similarly, by $F = F'$ we can obtain

$$\mathrm{CoD}_{g,h'}[Y|X] = \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}'(B(t)),$$

where $B(t) = 1 - \frac{\overline{C}'(u_g,t)}{1-u_g}$. The desired result boils down to showing that

$$\int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A(t)) \leq \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}'(B(t)). \tag{13}$$

On the one hand, by using Lemma 4.6, $Y \le_{\rm icx} Y'$ implies that

$$\int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A(t)) \le \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(A(t)). \tag{14}$$

On the other hand, $C \prec C'$ implies that $A(t) \ge B(t)$, and thus $\overline{h}(A(t)) \ge \overline{h}(B(t)) \ge \overline{h}'(B(t))$ since $h \le h'$. Because $\overline{h}(A(0)) = \overline{h}'(B(0)) = 0$ and $\overline{h}(A(1)) = \overline{h}'(B(1)) = 1$, we then have

$$\int_0^1 G'^{-1}(t)\mathrm{d}\left[\overline{h}(A(t)) - \overline{h}'(B(t))\right] = \int_0^1 \left[\overline{h}'(B(t)) - \overline{h}(A(t))\right] \mathrm{d}G'^{-1}(t) \le 0,$$

which means that

$$\int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(A(t)) \le \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}'(B(t)). \tag{15}$$

Upon combining (14) and (15), the desired result (13) is established. ■

A.6 Proof of Theorem 4.11

Proof. Note that if (X,Y) is PDS [NDS], then $X \uparrow_{\rm SI\ [SD]} Y$ and $Y \uparrow_{\rm SI\ [SD]} X$. The proof is then easily obtained by combining the proof methods in Theorems 4.4 and 4.7. ■

A.7 Proof of Lemma 5.1

Proof. The proof can be obtained by using similar arguments as in Lemma 14 in Sordo et al. (2018), and is thus omitted here for brevity. ■

A.8 Proof of Theorem 5.2

Proof. We only give the proof of (i) since the proof of (ii) is similar by applying Lemma 5.1(i). Suppose that $X \uparrow_{\rm SI} Y$. According to Theorem 3.8 and the proof of Theorem 3.5, we have

$$\Delta\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \left[F_{Y_h}^{-1}(p) - G^{-1}(p)\right] \mathrm{d}\overline{h}(p),$$
$$\Delta\mathrm{CoD}_{g,h}[Y'|X'] = \int_0^1 \left[F_{Y'_{h'}}^{-1}(p) - G'^{-1}(p)\right] \mathrm{d}\overline{h}(p),$$

where $Y_h = [Y|X > \mathrm{D}_g[X]]$ and $Y'_{h'} = [Y'|X' > \mathrm{D}_g[X]]$ are the distorted r.v.'s induced from Y and Y' by the concave distortion functions

$$h(p) = \frac{\overline{C}(F(\mathrm{D}_g[X]), 1-p)}{1 - F(\mathrm{D}_g[X])} \quad \text{and} \quad h'(p) = \frac{\overline{C}'(F(\mathrm{D}_g[X]), 1-p)}{1 - F(\mathrm{D}_g[X])}, \quad p \in [0,1]. \tag{16}$$

Since $C \prec C'$, it clearly holds that $h(p) \le h'(p)$ for all $p \in [0,1]$. From $Y \le_{\mathrm{disp}} Y'$ and Lemma 14 in Sordo et al. (2018), we have

$$F_{Y_h}^{-1}(p) - G^{-1}(p) \le F_{Y'_{h'}}^{-1}(p) - G'^{-1}(p), \quad \text{for } p \in (0,1),$$

which yields the desired result since $\overline{h}(p)$ is increasing in $p \in [0,1]$. ■

A.9 Proof of Theorem 5.5

Proof. We only give the proof for (i) since the proof can be established in a similar manner for (ii). According to Theorem 3.8 and the proof of Theorem 3.5, we have

$$\Delta\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \left[F_{Y_{\hat{h}}}^{-1}(p) - G^{-1}(p) \right] \mathrm{d}\overline{h}(p),$$

where $Y_{\hat{h}} = [Y|X > \mathrm{D}_g[X]]$ is a distorted r.v. induced from Y by the concave distortion function

$$\hat{h}(p) = \frac{\overline{C}(F(\mathrm{D}_g[X]), 1-p)}{1 - F(\mathrm{D}_g[X])}, \quad p \in [0,1].$$

Then, from Lemma 5.4, we have $Y \le_{\mathrm{hr}} Y_{\hat{h}}$. Since Y is DFR, it follows that $Y \le_{\mathrm{disp}} Y_{\hat{h}}$ upon invoking Theorem 3.B.20(a) of Shaked and Shanthikumar (2007). Therefore, it holds that

$$G^{-1}(p_2) - G^{-1}(p_1) \le F_{Y_{\hat{h}}}^{-1}(p_2) - F_{Y_{\hat{h}}}^{-1}(p_1), \quad \text{for } 0 < p_1 < p_2 < 1,$$

which implies that $F_{Y_{\hat{h}}}^{-1}(p) - G^{-1}(p)$ is increasing in $p \in (0,1)$. Since $\overline{h}(0) = \overline{h}'(0) = 0$ and $\overline{h}(1) = \overline{h}'(1) = 1$, we then have that

$$\begin{aligned}
\Delta\mathrm{CoD}_{g,h}[Y|X] - \Delta\mathrm{CoD}_{g,h'}[Y|X] &= \int_0^1 \left[F_{Y_{\hat{h}}}^{-1}(p) - G^{-1}(p) \right] \mathrm{d}[\overline{h}(p) - \overline{h}'(p)] \\
&= \int_0^1 [\overline{h}'(p) - \overline{h}(p)] \mathrm{d} \left[F_{Y_{\hat{h}}}^{-1}(p) - G^{-1}(p) \right] \le 0,
\end{aligned}$$

which yields the desired result. ■

A.10 Proof of Theorem 5.8

Proof. By using the proof method of Theorem 4.11, the result can be obtained from Theorem 5.3. ■

A.11 Proof of Theorem 5.9

Proof. In light of Theorem 3.8 and $F = F'$, one can observe that

$$\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 \left[F^{-1}_{Y_{\hat{h}_1}}(p) - F^{-1}_{Y_{\hat{h}_2}}(p)\right] \mathrm{d}\overline{h}(p),$$

$$\Delta^{\tilde{g}}\mathrm{CoD}_{g,h}[Y'|X'] = \int_0^1 \left[F^{-1}_{Y'_{\hat{h}_1}}(p) - F^{-1}_{Y'_{\hat{h}_2}}(p)\right] \mathrm{d}\overline{h}(p),$$

where $Y_{\hat{h}_1} = [Y|X > \mathrm{D}_g[X]]$ and $Y_{\hat{h}_2} = [Y|X > \mathrm{D}_{\tilde{g}}[X]]$ are the distorted r.v.'s induced from Y by the concave distortion function (this is due to the fact that C is PDS implies that $Y \uparrow_{\mathrm{SI}} X$)

$$\hat{h}_1(p) = \frac{\overline{C}(u_g, 1-p)}{1-u_g} \quad \text{and} \quad \hat{h}_2(p) = \frac{\overline{C}(u_{\tilde{g}}, 1-p)}{1-u_{\tilde{g}}}, \quad p \in [0,1], \tag{17}$$

and $Y'_{\hat{h}_1} = [Y'|X' > \mathrm{D}_g[X']]$ and $Y'_{\hat{h}_2} = [Y'|X' > \mathrm{D}_{\tilde{g}}[X']]$ are also the distorted r.v.'s induced from Y' by (17).

On the other hand, the condition that C is PDS implies that V is right tail increasing in U if $(U, V) \sim C$. Thus, we know

$$\frac{\overline{C}(u, 1-p)}{1-u} = \mathbb{P}(V > 1-p|U > u)$$

is increasing in $u \in [0,1)$ for $p \in [0,1]$. Therefore, it follows that $\hat{h}_1(p) \geq \hat{h}_2(p)$ for $p \in [0,1]$ because of $u_g \geq u_{\tilde{g}}$. Then, the desired result can be obtained from Lemma 14 in Sordo et al. (2018). ■

A.12 Proof of Theorem 5.11

Proof. Upon using Lemma 5.1(ii), the proof can be established in a similar manner to that of Theorem 5.9 and is thus omitted here for brevity. ■

A.13 Proof of Theorem 5.13

Proof. Assume that $X \uparrow_{\rm SI} Y$ (the case $X' \uparrow_{\rm SI} Y'$ can be dealt with analogously). Note that

$$\Delta\mathrm{CoD}_{g,h}[Y|X] = \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A(t)) - \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(t),$$

$$\Delta\mathrm{CoD}_{g,h}[Y'|X'] = \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(B(t)) - \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(t).$$

Since $h(t)$ is concave and $\overline{h}(A(\overline{h}^{-1}(t)))$ is convex, it follows from Lemma 5.12 that

$$\begin{aligned}\Delta\mathrm{CoD}_{g,h}[Y|X] &\le \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(A(t)) - \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(t)\\ &\le \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(B(t)) - \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(t) = \Delta\mathrm{CoD}_{g,h'}[Y'|X'],\end{aligned}$$

where the last inequality is due to the fact that $C \prec C'$ implies $\int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(A(t)) \le \int_0^1 G'^{-1}(t)\mathrm{d}\overline{h}(B(t))$. Hence, the proof is established. ■

A.14 Proof of Theorem 6.1

Proof. Proof of (i) and (ii): By using (12), the desired result is equivalent to showing that

$$\int_0^1 F^{-1}(t)\mathrm{d}\overline{h}(\tilde{A}(t)) \le \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A(t)),$$

where $A(t) = 1 - \frac{\overline{C}(u_g^X,t)}{1-u_g^X}$ and $\tilde{A}(t) = 1 - \frac{\overline{C}(t,u_g^Y)}{1-u_g^Y}$. Since $C(u,v)$ is symmetric, we have $\tilde{A}(t) = 1 - \frac{\overline{C}(u_g^Y,t)}{1-u_g^Y}$. By using $X \le_{\rm st} Y$, we have

$$\int_0^1 F^{-1}(t)\mathrm{d}\overline{h}(\tilde{A}(t)) \le \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(\tilde{A}(t)).$$

On the other hand, in light of $u_g^X \ge [\le] u_g^Y$ and $Y \uparrow_{\rm RTI\ [RTD]} X$, one can verify that $A(t) \le \tilde{A}(t)$. Thus, it holds that

$$\int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(\tilde{A}(t)) - \int_0^1 G^{-1}(t)\mathrm{d}\overline{h}(A(t)) = \int_0^1 [\overline{h}(A(t)) - \overline{h}(\tilde{A}(t))]\mathrm{d}G^{-1}(t) \le 0.$$

Hence, the proof is completed.

Proof of (iii) and (iv): In light of the proof of (i) and (ii) and the proof of Theorem 4.7, it is easy to see that both $\tilde{A}(t)$ and $A(t)$ are increasing and convex due to C being

PDS. Besides, $A(t) \leq \tilde{A}(t)$ due to $u_g^X \geq u_g^Y$ and C being PDS. Thus, the concavity of h implies that $\overline{h}(A(t))$ and $\overline{h}(\tilde{A}(t))$ are increasing and convex. Then, the proof of (iii) is completed by using Lemma 4.6 and the second part of the proof of (i) and (ii). Result (iv) can be proved in a similar manner and thus is omitted here.

Proof of (v) and (vi): The proof can be obtained by using that of (i) and (ii), Theorem 5.2, and Theorem 5.3. ∎

References

Viral V. Acharya, Lasse H. Pedersen, Thomas Philippon, and Matthew Richardson. Measuring systemic risk. *The Review of Financial Studies*, 30(1):2–47, 2017.

Tobias Adrian and Markus K. Brunnermeier. CoVaR. *American Economic Review*, 106(7):1705–1741, 2016.

Yacine Aït-Sahalia, Julio A. Cacho-Diaz, and Roger J. A. Laeven. Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics*, 117(3):585–606, 2015.

Alexandru V. Asimit and Russell Gerrard. On the worst and least possible asymptotic dependence. *Journal of Multivariate Analysis*, 144:218–234, 2016.

Alejandro Balbás, José Garrido, and Silvia Mayoral. Properties of distortion risk measures. *Methodology and Computing in Applied Probability*, 11(3):385–399, 2009.

Richard E. Barlow and Frank Proschan. *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart, and Winston, New York, 1975.

Francesca Biagini, Jean-Pierre Fouque, Marco Frittelli, and Thilo Meyer-Brandis. A unified approach to systemic risk measures via acceptance sets. *Mathematical Finance*, 29(1):329–367, 2019.

Henry W. Block, Thomas H. Savits, and Moshe Shaked. Some concepts of negative dependence. *The Annals of Probability*, 10(3):765–772, 1982.

Phelim Boyle and Joseph H. T. Kim. Designing a countercyclical insurance program for systemic risk. *Journal of Risk and Insurance*, 79(4):963–993, 2012.

Christian Brownlees and Robert F. Engle. SRISK: A conditional capital shortfall measure of systemic risk. *The Review of Financial Studies*, 30(1):48–79, 2016.

Jun Cai and Wei Wei. On the invariant properties of notions of positive dependence and copulas under increasing transformations. *Insurance: Mathematics and Economics*, 50(1):43–49, 2012.

Alexander Cherny and Dilip Madan. New measures for performance evaluation. *The Review of Financial Studies*, 22(7):2571–2606, 2009.

Dieter Denneberg. Premium calculation: Why standard deviation should be replaced by absolute deviation. *ASTIN Bulletin: The Journal of the IAA*, 20(2):181–190, 1990.

Michel Denuit, Jan Dhaene, Marc J. Goovaerts, and Rob Kaas. *Actuarial Theory for Dependent Risks*. New York: Wiley, 2005.

Michel Denuit, Jan Dhaene, Marc J. Goovaerts, Rob Kaas, and Roger J. A. Laeven. Risk measurement with equivalent utility principles. *Statistics and Decisions, Special Issue on Risk Measures: General Aspects and Applications*, 24(1):1–26, 2006.

Jan Dhaene and Marc J. Goovaerts. Dependency of risks and stop-loss order. *ASTIN Bulletin: The Journal of the IAA*, 26(2):201–212, 1996.

Jan Dhaene and Marc J. Goovaerts. On the dependency of risks in the individual life model. *Insurance: Mathematics and Economics*, 19(3):243–253, 1997.

Jan Dhaene, Steven Vanduffel, Qihe Tang, Marc J. Goovaerts, Rob Kaas, and David Vyncke. Risk measures and comonotonicity: A review. *Stochastic Models*, 22(4):573–606, 2006.

Jan Dhaene, Alexander Kukush, Daniël Linders, and Qihe Tang. Remarks on quantiles and distortion risk measures. *European Actuarial Journal*, 2(2):319–328, 2012.

Jan Dhaene, Roger J. A. Laeven, and Yiying Zhang. Systemic risk: Stochastic orders. *Working Paper*, 2020.

Louis R. Eeckhoudt and Roger J. A. Laeven. Dual moments and risk attitudes. *Operations Research*, 2020. doi: 10.1287/opre.2020.2040.

Louis R. Eeckhoudt, Roger J. A. Laeven, and Harris Schlesinger. Risk apportionment: The dual story. *Journal of Economic Theory*, 185:104971, 2020.

Paul Embrechts, Andrea Höing, and Giovanni Puccetti. Worst VaR scenarios. *Insurance: Mathematics and Economics*, 37(1):115–134, 2005.

Rui Fang and Xiaohu Li. Some results on measures of interaction between paired risks. *Risks*, 6(3):88, 2018.

Hans Föllmer and Alexander Schied. *Stochastic Finance, 3rd ed.* Berlin: De Gruyter, 2011.

Christian Genest, Hans U. Gerber, Marc J. Goovaerts, and Roger J. A. Laeven. Editorial to the special issue on modeling and measurement of multivariate risk in insurance and finance. *Insurance: Mathematics and Economics*, 44(2):143–145, 2009.

Giulio Girardi and A. Tolga Ergün. Systemic risk measurement: Multivariate GARCH estimation of CoVaR. *Journal of Banking & Finance*, 37(8):3169–3180, 2013.

Marc J. Goovaerts and Roger J. A. Laeven. Actuarial risk measures for financial derivative pricing. *Insurance: Mathematics and Economics*, 42(2):540–547, 2008.

Marc J. Goovaerts, Rob Kaas, and Roger J. A. Laeven. Decision principles derived from risk measures. *Insurance: Mathematics and Economics*, 47(2):294–302, 2010.

Marc J. Goovaerts, Rob Kaas, and Roger J. A. Laeven. Worst case risk measurement: Back to the future? *Insurance: Mathematics and Economics*, 49(3):380–392, 2011.

Christian Gourieroux and Alain Monfort. Allocating systemic risk in a regulatory perspective. *International Journal of Theoretical and Applied Finance*, 16(7):1–20, 2013.

Hannes Hoffmann, Thilo Meyer-Brandis, and Gregor Svindland. Risk-consistent conditional systemic risk measures. *Stochastic Processes and their Applications*, 126(7):2014–2037, 2016.

Werner Hürlimann. Distortion risk measures and economic capital. *North American Actuarial Journal*, 8(1):86–95, 2004.

Harry Joe. *Multivariate Models and Multivariate Dependence Concepts*. Chapman and Hall/CRC, 1997.

Rob Kaas, Roger J. A. Laeven, and Roger B. Nelsen. Worst VaR scenarios with given marginals and measures of association. *Insurance: Mathematics and Economics*, 44(2):146–158, 2009.

Emmanouil N. Karimalis and Nikos K. Nomikos. Measuring systemic risk in the European banking sector: A Copula CoVaR approach. *The European Journal of Finance*, 24(11):944–975, 2018.

Jacob Kleinow, Fernando Moreira, Sascha Strobl, and Sami Vähämaa. Measuring systemic risk: A comparison of alternative market-based approaches. *Finance Research Letters*, 21:40–46, 2017.

Coenraad C. A. Labuschagne and Theresa M. Offwood. A note on the connection between the Esscher–Girsanov transform and the Wang transform. *Insurance: Mathematics and Economics*, 47: 385–390, 2010.

Roger J. A. Laeven. Worst VaR scenarios: A remark. *Insurance: Mathematics and Economics*, 44(2): 159–163, 2009.

Roger J. A. Laeven and Mitja Stadje. Entropy coherent and entropy convex measures of risk. *Mathematics of Operations Research*, 38(2):265–293, 2013.

Georg Mainik and Eric Schaanning. On dependence consistency of CoVaR and some other systemic risk measures. *Statistics & Risk Modeling*, 31(1):49–77, 2014.

Albert W. Marshall and Ingram Olkin. *Life Distributions*, volume 13. Springer, 2007.

Alfred Müller and Dietrich Stoyan. *Comparison Methods for Stochastic Models and Risks*, volume 389. Wiley New York, 2002.

Roger B. Nelsen. *An Introduction to Copulas*. Springer Science & Business Media, 2007.

Moshe Shaked and J. George Shanthikumar. *Stochastic Orders*. Springer Science & Business Media, 2007.

Abe Sklar. Fonctions de répartition á n dimensions et leurs marges. *Publications de l'Institut Statistique de l'Université de Paris*, 8:229–231, 1959.

Miguel A. Sordo. Characterizations of classes of risk measures by dispersive orders. *Insurance: Mathematics and Economics*, 42(3):1028–1034, 2008.

Miguel A. Sordo and Héctor M. Ramos. Characterization of stochastic orders by l-functionals. *Statistical Papers*, 48(2):249–263, 2007.

Miguel A. Sordo, Alfonso Suárez-Llorens, and Alfonso J. Bello. Comparison of conditional distributions in portfolios of dependent risks. *Insurance: Mathematics and Economics*, 61:62–69, 2015.

Miguel A. Sordo, Alfonso J. Bello, and Alfonso Suárez-Llorens. Stochastic orders and co-risk measures under positive dependence. *Insurance: Mathematics and Economics*, 78:105–113, 2018.

Shaun S. Wang. Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, 17(1):43–54, 1995.

Julia L. Wirch and Mary R. Hardy. Distortion risk measures: coherence and stochastic dominance. *In International Congress on Insurance: Mathematics and Economics*, pages 15–17, 2001.

Menahem E. Yaari. The dual theory of choice under risk. *Econometrica*, 55(1):95–115, 1987.