Index options

A model-free approach

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Index options: a model-free approach

Main references

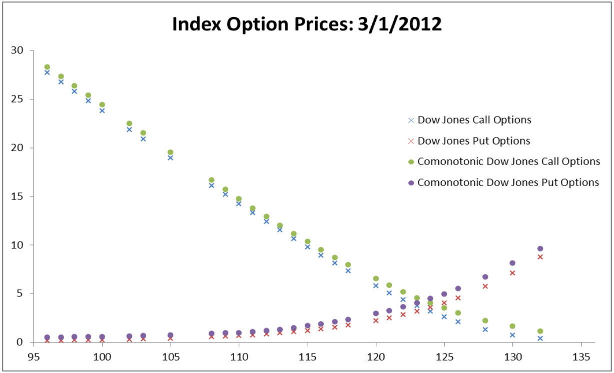
- **Static super-replicating strategies for a class of exotic options**.
  X. Chen, G. Deelstra, J. Dhaene & M. Vanmaele (2008).
  Insurance: Mathematics and Economics, 42(3), 1067-1085.

- **Index options: a model-free approach.**
  D. Linders, J. Dhaene, H. Hounnon, M. Vanmaele (2016).
  Research Report AFI-1265, FBE, KU Leuven.

- **The Herd Behavior Index: a new measure for the implied degree of co-movement in financial markets**.
  J. Dhaene, D. Linders, W. Schoutens & D. Vyncke (2012).
  Insurance: Mathematics and Economics, 50(3), 357-370.

Index options: a model-free approach

Dow Jones option prices and comonotonic upper bounds



Stocks, the market index and options

- The usual set up:

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P}\right)$$

- Current time is denoted by 0.
- The stock market:

$$X_i(t) = \text{price of (dividend paying) stock } i \text{ at time } t$$

  - $i = 1, 2, \dots, n$.
  - $0 \le t \le T$.
  - $X_i(t) \ge 0$ is known at time t.
- European-type call options on stock i:
  - Expiration date: T.
  - Strike price: $K \ge 0$.
  - **Pay-off** at time T:

$$(X_i(T) - K)_+$$

  - $(x)_+ = \max(x, 0)$.
  - **Price** at time 0:

$$C_i[K, T]$$

Stocks, the market index and options

- $S(t)$ = weighted sum of stock prices:

$$S(t) = w_1 \; X_1(t) + \cdots + w_n \; X_n(t)$$

  - w_i = positive weight factors.
- European-type index call options:
  - Expiration date: T.
  - Strike price: $K \geq 0$.
  - **Pay-off** at time T:

$$(S(T) - K)_+$$

  - **Price** at time 0:

$$C\left[K, T\right]$$

- Problem to be solved:
  - Suppose that for each stock i, we observe the prices $C_i\left[K_{ij}, T\right]$ of stock options for different values of j.
  - What can we conclude about the price $C\left[K, T\right]$ of the index option with strike K?

Stocks, the market index and options

- Further assumptions about the market:
  - The market is arbitrage-free.
  - There exists an equivalent martingale measure $\mathbb{Q}$ such that the current price of any traded contingent claim with pay-off $A(T)$ at time T is given by

$$e^{-rT}\ \mathbb{E}^{\mathbb{Q}}\left[A(T)\right]$$

  - $r =$ time-0 risk-free interest rate to expiration T.
- Stock option prices:

$$C_i\left[K, T\right] = e^{-rT}\ \mathbb{E}^{\mathbb{Q}}\left[(X_i(T) - K)_+\right]$$

- Index option prices:

$$C\left[K, T\right] = e^{-rT}\ \mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}(T) - K)_+\right]$$

- Notational conventions:
  - Denote $X_i(T)$ and $S(T)$ by X_i and S, respectively.
  - Denote $C_i\left[K, T\right]$ and $C\left[K, T\right]$ by $C_i\left[K\right]$ and $C\left[K\right]$.
  - Ommit $\mathbb{Q}$ to denote expectations in the $\mathbb{Q}$-world.
  - F_{X_i} and F_S are the cdf's of X_i and S in the $\mathbb{Q}$-world.

Stocks, the market index and options

- <u>**The infinite market**</u>:

  <u>Assumption</u>: For each i, $C_i[K]$ traded for any $K \geq 0$

- <u>**The finite market**</u>:

  <u>Assumption</u> : For each i, $C_i[K]$ only traded
  for strikes $K = K_{i,0}, K_{i,1}, \ldots, K_{i,m_i}$

- <u>*Model-free approach vs. model-based approach*</u>:
  - <u>Model-free</u>: Prices $C_i[K]$ are observed in the market.
  - <u>Model-based</u>: Prices $C_i[K]$ follow from an assumed $\mathbb{Q}$.

Convex order

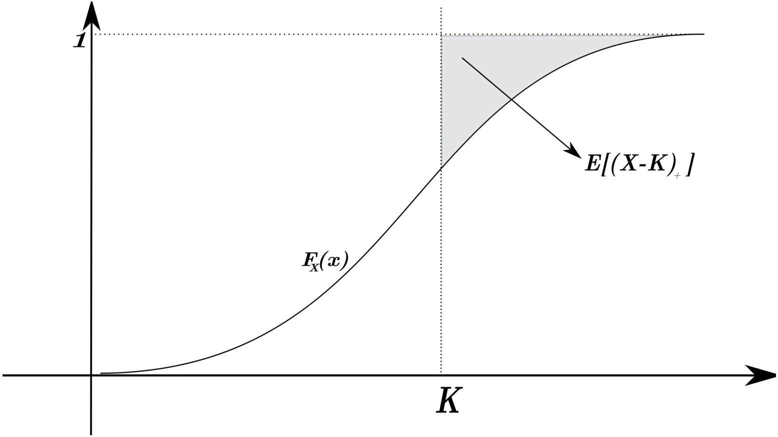
- R.v.'s are assumed to have finite means.
- **Convex order**:
  A r.v. X is said to precede a r.v. Y in *convex order* sense if

  $$\mathbb{E}\left[X\right] = \mathbb{E}\left[Y\right] \text{ and } \mathbb{E}\left[\left(X-K\right)_+\right] \leq \mathbb{E}\left[\left(Y-K\right)_+\right], \quad \text{for all } K$$

  - Notation: $X \leq_{\text{cx}} Y$.
- Other characterization:

  $$X \leq_{\text{cx}} Y \Leftrightarrow \begin{cases} \mathbb{E}\left[\left(X-K\right)_+\right] \leq \mathbb{E}\left[\left(Y-K\right)_+\right] \\ \mathbb{E}\left[\left(K-X\right)_+\right] \leq \mathbb{E}\left[\left(K-Y\right)_+\right] \end{cases}, \qquad \text{for all } K$$

Convex order



Inverse cdf's

- The usual choice:

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}$$

- Alternative choice:

$$F_X^{-1+}(p) = \sup \{x \in \mathbb{R} \mid F_X(x) \leq p\}$$

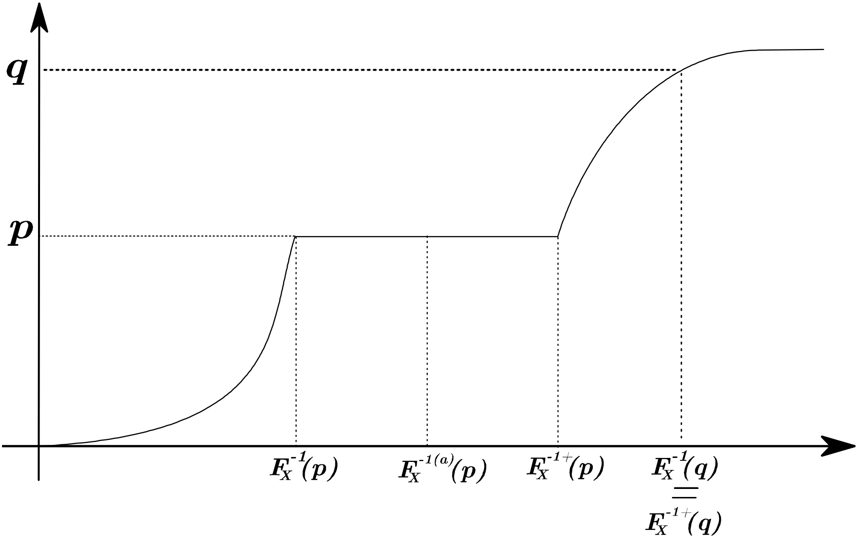
- The α - inverse in case $p \in (0,1)$:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p), \qquad \alpha \in [0,1]$$

- For strictly increasing cdf's:

$$F_X^{-1(\alpha)}(p) = F_X^{-1}(p)$$

Inverse cdf's



Comonotonicity

Definition and notations

- U is a r.v. which is uniformly distributed on $(0, 1)$.
- Definition:
  The random vector $(Y_1, Y_2, \dots, Y_n)$ is *comonotonic* if

$$(Y_1, Y_2, \dots, Y_n) \stackrel{d}{=} \left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \dots, F_{Y_n}^{-1}(U)\right)$$

- Notations:

$$\begin{aligned} S &= w_1 X_1 + \cdots + w_n X_n \\ S^c &= w_1 F_{X_1}^{-1}(U) + \cdots + w_n F_{X_n}^{-1}(U) \end{aligned}$$

  - The weights $w_i \geq 0$.

Comonotonicity

Properties of comonotonic sums

- α-inverses of S^c:

$$F_{S^c}^{-1(\alpha)}(p) = \sum_{i=1}^{n} w_i F_{X_i}^{-1(\alpha)}(p)$$

- Stop-loss premiums of S^c for $K \in \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$:

$$\mathbb{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} w_i \mathbb{E}\left[(X_i - K_i^*))_+\right]$$

  -
$$K_i^* = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right)$$

  -
$$\alpha_K \in [0,1] \text{ is such that } \sum_{i=1}^{n} w_i K_i^* = K$$

  -
$$F_{\mathbb{S}^c}(K) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^{n} w_i F_{X_i}^{-1}(p) \le K\right\}$$

- Convex order relation:

$$\sum_{i=1}^{n} w_i X_i \leq_{\mathrm{cx}} \sum_{i=1}^{n} w_i F_{X_i}^{-1}(U)$$

- Generalized convex order relation:

$$X_i \leq_{\mathrm{cx}} Y_i \text{ for } i = 1, \ldots, n \Rightarrow \sum_{i=1}^{n} w_i X_i \leq_{\mathrm{cx}} \sum_{i=1}^{n} w_i F_{Y_i}^{-1}(U)$$

The infinite market case

From option prices to risk neutral distributions

- Stock i:

$$X_i = \text{price of stock } i \text{ at time } T \geq 0, \qquad i = 1, 2, \ldots, n$$

- The index:

$$S = w_1 X_1 + \cdots + w_n X_n$$

- The comonotonic index:

$$S^c = w_1 F_{X_1}^{-1}(U) + \cdots + w_n F_{X_n}^{-1}(U)$$

The infinite market

From option prices to risk neutral distributions

- Stock option prices:

$$C_i\left[K\right] = e^{-rT}\ \mathbb{E}\left[\left(X_i - K\right)_+\right]$$

- Index option prices:

$$C\left[K\right] = e^{-rT}\ \mathbb{E}\left[\left(S - K\right)_+\right]$$

- From option prices C_i to risk neutral distribution F_{X_i}:

$$F_{X_i}(x) = 1 + e^{rT}\ C_i'[x+]$$

- From risk neutral distribution F_{X_i} to option prices C_i:

$$C_i\left[K\right] = e^{-rT}\int_K^{\infty}\left(1 - F_{X_i}(x)\right)\ dx$$

The infinite market

From option prices to risk neutral distributions

- **The infinite market case**:

  Assumption: For any i, $C_i[K]$ traded for all $K \geq 0$

- **Equivalent characterisation**:

  Assumption: For any i, $F_{X_i}(x)$ known for all $x \geq 0$

- In the infinite market case, we know the cdf of S^c.
- Knowledge of all prices $C_i[K]$ does not allow us to specify the multivariate distribution $F_{\underline{X}}(\underline{x})$ of $\underline{X} = (X_1, X_2, \ldots, X_n)$.
- The put-call parity:

$$C_i[K] + e^{-rT}K = P_i[K] + e^{-rT}\mathbb{E}[X_i]$$

The infinite market

An upper bound for the index option price

- Goal: Determine an upper bound for the index option price $C[K]$ in terms of observed stock option prices.
- **Theorem**:

$$C[K] \leq e^{-rT}\mathbb{E}\left[\left(S^c - K\right)_+\right] \overset{\text{not.}}{=} C^c[K]$$

- When $K \leq F_{S^c}^{-1+}(0)$:

$$C[K] = C^c[K] = \sum_{i=1}^{n} w_i \; C_i[0] - e^{-rT}K$$

- When $K \geq F_{S^c}^{-1}(1)$:

$$C[K] = C^c[K] = 0$$

- In the sequel, we always assume that

$$K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$$

The infinite market

An upper bound for the index option price

- **Theorem**: "$C^c\left[K\right]$ *is a l.c. of stock option prices.*"

$$C^c\left[K\right] = \sum_{i=1}^{n} w_i \; C_i\left[K_i^*\right]$$

  - with

$$K_i^* = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right)$$

  - α_K determined from

$$\sum_{i=1}^{n} w_i K_i^* = K$$

  - and $F_{S^c}(K)$ from

$$F_{S^c}(K) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^{n} w_i F_{X_i}^{-1}(p) \leq K\right\}$$

The infinite market

An upper bound for the index option price

- **Theorem**: "*$C^c[K]$ is the price of a static superhedging strategy for the index option $C[K]$.*"
  - Consider the following strategy:
    - At time 0, for each stock i, buy w_i stock options $C_i[K_i^*]$.
    - Hold these calls until they expire at time T.
  - The pay-off of this strategy super-replicates the pay-off of the index option $C[K]$:

$$\left(\sum_{i=1}^{n} w_i\, X_i - K\right)_+ \leq \sum_{i=1}^{n} w_i \left(X_i - K_i^*\right)_+$$

  - The price of this strategy is given by the comonotonic index option price $C^c[K]$.

The infinite market

The cheapest super-replicating strategy

- Question: Can we improve the upper bound for $C[K]$ by looking for the price of a cheaper super-replicating strategy?
- A general class of investment strategies $\mathcal{I}$:
  - At time 0, for each stock i, calls $C_i[y]$ can be bought or sold for any $y \geq 0$.
  - Hold the taken positions until time T.
  - We describe any such investment strategy by a vector of functions $\underline{\nu} \equiv (\nu_1, \nu_2, \dots, \nu_n)$, with

$$\nu_i(y) = \text{number of purchased calls on } i \text{ with strike} \leq y$$

  - Assumption: Each function ν_i is a r.c. jump function with $\nu_i(y) = 0$ if $y < 0$ and with only a finite number of jumps (upwards or downwards) on $[0, \infty)$.

The infinite market

The cheapest super-replicating strategy

- Price of the investment strategy $\underline{\nu} \in \mathcal{I}$:

$$\text{Price}\ \left[\underline{\nu}\right] = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} C_i\left[y\right]\ \mathrm{d}\nu_i(y)$$

- Pay-off of the investment strategy $\underline{\nu} \in \mathcal{I}$ at time T:

$$\text{Pay-off}\ \left[\underline{\nu}, \underline{X}\right] = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} (X_i - y)_+\ \mathrm{d}\nu_i(y)$$

The infinite market

The cheapest super-replicating strategy

- The investment strategy $\underline{\nu}^*$: For $i = 1, 2, \dots, n$,

$$\nu_i^*(y) = \begin{cases} 0 & y < K_i^* \\ w_i & y \geq K_i^* \end{cases}$$

  - with

$$K_i^* = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right)$$

  - and α_K determined from

$$\sum_{i=1}^{n} w_i K_i^* = K$$

- The price of $\underline{\nu}^*$:

$$\text{Price}\ \left[\underline{\nu}^*\right] = \sum_{i=1}^{n} w_i\ C_i\left[K_i^*\right] = C^c\left[K\right]$$

- The pay-off of $\underline{\nu}^*$:

$$\text{Pay-off}\ \left[\underline{\nu}^*, \underline{X}\right] = \sum_{i=1}^{n} w_i\ \left(X_i - K_i^*\right)_+$$

The infinite market

The cheapest super-replicating strategy

- The class $\mathcal{C}_K$:

$$\mathcal{C}_K = \left\{ \underline{\nu} \in \mathcal{I} \mid \left(\sum_{i=1}^{n} w_i \ x_i - K \right)_+ \leq \text{ Pay-off } [\underline{\nu}, \underline{x}] \text{ for all } \underline{x} \right\}$$

  - 'for all $\underline{x}$' means

$$\text{'for all } \underline{x} \text{ with } x_i \in \text{Support } [X_i] \text{'}$$

- Any $\underline{\nu} \in \mathcal{C}_K$ is a super-replicating strategy:

$$\mathbb{P}\left[(S-K)_+ \leq \text{Pay-off } [\underline{\nu}, \underline{X}] \right] = 1$$

The infinite market

The cheapest super-replicating strategy

- Some elements of $\mathcal{C}_K$:
  - Consider the investment strategy $\underline{\nu}$ given by

$$\nu_i(y) = \begin{cases} 0, & y < K_i \\ w_i & y \geq K_i \end{cases}$$

with

$$\sum_{i=1}^{n} w_i \; K_i \leq K$$

  - This investment strategy is an element of $\mathcal{C}_K$.
- In particular, we find that

$$\underline{\nu}^* \in \mathcal{C}_K$$

The infinite market

The cheapest super-replicating strategy

- **Theorem**:

$$\min_{\underline{\nu} \in \mathcal{C}_K} \text{Price} \ [\underline{\nu}] = \text{Price} \ [\underline{\nu}^*] = C^c [K]$$

- Important remark:
  - Suppose that the index option $C[K]$ is not traded in the market.
  - In case this option is sold over-the-counter, then $C^c [K]$ is a reasonable price:
    - The seller can super-replicate the pay-off by buying $\underline{\nu}^*$.
    - The buyer cannot find a cheaper super-replicating strategy.

The infinite market

The least upper bound

- $\mathcal{D}_n$ = the class of all n - dimensional cdf's F on the non-negative orthant of $\mathbb{R}^n$.
- The marginal cdf's of $F \in \mathcal{D}_n$ are denoted by F_i, $i = 1, 2, \ldots, n$.
- The Fréchet class $\mathcal{R}_n$:

$$\mathcal{R}_n = \{F \in \mathcal{D}_n \mid F_i = F_{X_i},\ i = 1, \ldots, n\}$$

- Equivalent characterization of $\mathcal{R}_n$:

$$\mathcal{R}_n = \left\{F \in \mathcal{D}_n \mid e^{-rT}\mathbb{E}_{F_i}\left[(X_i - y)_+\right] = C_i\left[y\right], \text{ for all } i,\ y\right\}$$

- A comonotonic element in $\mathcal{R}_n$:

The cdf of $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$ belongs to $\mathcal{R}_n$

The infinite market

The least upperbound

- **Theorem**:

$$\max_{F \in \mathcal{R}_n} e^{-rT} \mathbb{E}_F \left[\left(\sum_{i=1}^{n} w_i X_i - K \right)_+ \right] = C^c [K]$$

- Interpretation:
  - $C^c [K]$ is the lowest upper bound for the index option price with pay-off $(\sum_{i=1}^{n} w_i X_i - K)_+$ in the class of all models which are consistent with the observed stock option prices $C_i[y]$, for all $i = 1, 2, \ldots, n$ and $y \geq 0$.
  - $C^c [K]$ is reached when $(X_1, X_2, \ldots, X_n)$ is comonotonic.

The infinite market

Computational aspects

- Suppose that for any stock i, the cdf F_{X_i} is strictly increasing on $(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1))$ and continuous on $\mathbb{R}$.
- The comonotonic index option price $C^c[K]$ is then given by

$$C^c[K] = \sum_{i=1}^{n} w_i \; C_i\left[F_{X_i}^{-1}\left(F_{S^c}(K)\right)\right]$$

- $F_{S^c}(K)$ is the unique solution of

$$\sum_{i=1}^{n} w_i F_{X_i}^{-1}\left(F_{S^c}(K)\right) = K$$

- Example: The Black & Scholes model.

The finite market

Traded options and approximations - Stock options

- Traded calls for stock i, $(i = 1, 2, \ldots n)$:

$$C_i\left[K_{i,0}\right], C_i\left[K_{i,1}\right], \ldots, C_i\left[K_{i,m_i}\right]$$

- The chain of strikes:

$$0 = K_{i,0} < K_{i,1} < K_{i,2} < \cdots < K_{i,m_i} < F_{X_i}^{-1}(1)$$

- We assume that $F_{X_i}^{-1}(1)$ is finite and known:

$$F_{X_i}^{-1}(1) \overset{\text{not.}}{=} K_{i,m_i+1} < \infty$$

- Stock option prices:

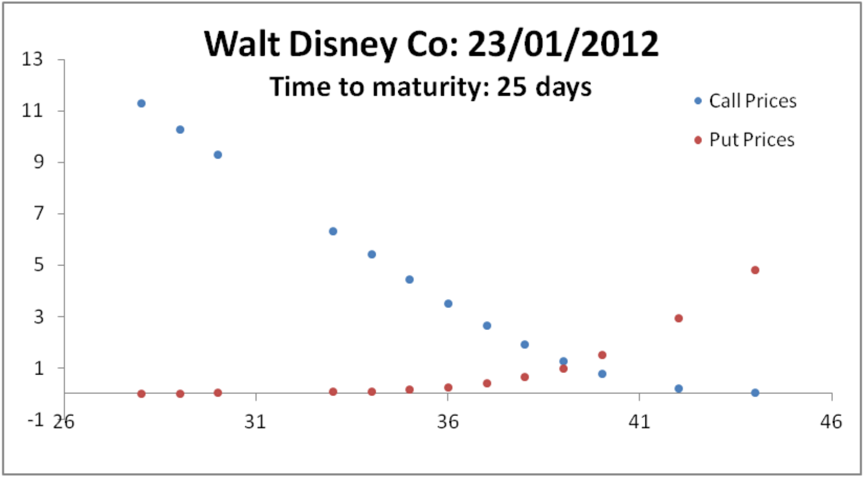
$$C_i\left[K_{i,j}\right] = e^{-rT}\ \mathbb{E}\left[\left(X_i - K_{i,j}\right)_+\right], \qquad j = 0, 1, \ldots, m_i + 1$$

- For each i, define the function $C_i[K]$:

$$C_i[K] = e^{-rT}\ \mathbb{E}\left[\left(X_i - K\right)_+\right], \qquad \text{for all } K \geq 0$$

The finite market

Traded options and approximations - Walt Disney



The finite market

Traded options and approximations - Index options

- Consider the traded index option with pay-off $(S-K)_+$ at time T.
- Index option price:

$$C[K] = e^{-rT}\ \mathbb{E}\left[(S-K)_+\right]$$

- Goal:
  Determine an upper bound for $C[K]$
  in terms of the observed $C_i[K_{i,j}]$.

The finite market

Traded options and approximations - Finite vs. infinite market

- Upper bound for $C[K]$:

$$C[K] \leq \sum_{i=1}^{n} w_i \; C_i[K_i^*]$$

- This bound can be calculated in the infinite market case:

$$C_i[K] \text{ known for all } K \geq 0$$

- This bound cannot be calculated in the finite market case:

$$C_i[K] \text{ only known for } K_{i,0}, \ldots, K_{i,m_i+1}$$

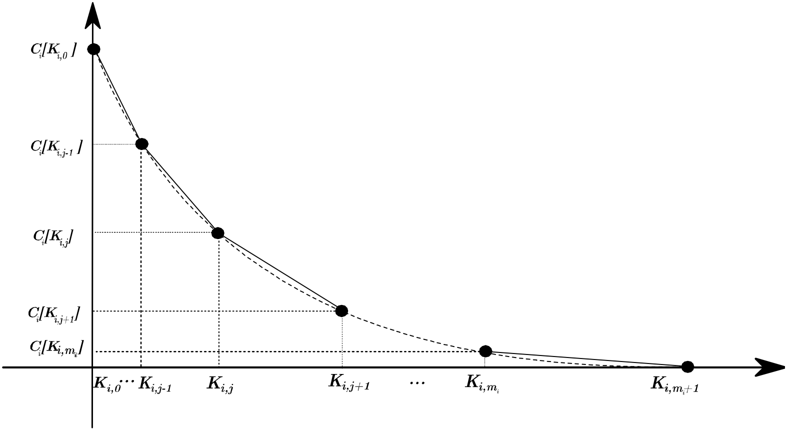
The finite market

Traded options and approximations - The 'artificial infinite market'

- Approximate the function $C_i\left[K\right]$ by the piecewise linear function $\overline{C}_i\left[K\right]$ which connects the $\left(K_{i,j}, C_i\left[K_{i,j}\right]\right)$ and such that
  $\overline{C}_i\left[K\right] = C_i\left[K\right]$ if $K \notin \left(0, K_{i,m_i+1}\right)$.
- Properties of $\overline{C}_i\left[K\right]$:
  - $\overline{C}_i\left[K\right]$ is convex and decreasing.
  - $\overline{C}_i\left[K\right]$ is known for all K:
    - For any $K \geq 0$, we have that $\overline{C}_i\left[K\right]$ can be expressed as a convex combination of known option prices $C_i\left[K_{i,j}\right]$.
  - $\overline{C}_i\left[K\right] \geq C_i\left[K\right]$ for all K.
- Apply the results of the infinite market to the functions $\overline{C}_i\left[K\right]$.
- We end up with an upper bound for the index option price which contains at most two traded strikes per stock.

The finite market

Traded options and approximations - The 'artificial infinite market'



- $C_i\left[K_{i,j}\right]$ for $j = 0, 1, \ldots, m_i + 1$.
- $C_i\left[K\right]$ (dashed line) vs. $\overline{C}_i\left[K\right]$ (solid line).

- **Lemma**:
  - If $K_{i,j} \leq K \leq K_{i,j+1}$, $j = 0, 1, \ldots, m_i$:

$$\overline{C}_i[K] = C_i[K_{i,j}] - \frac{C_i[K_{i,j}] - C_i[K_{i,j+1}]}{K_{i,j+1} - K_{i,j}}(K - K_{i,j})$$

  - Furthermore,

$$\overline{C}_i[K] = C_i[0] - e^{-rT}K \qquad \text{if } K \leq 0$$

and

$$\overline{C}_i[K] = 0 \qquad \text{if } K \geq K_{i,m_i+1}$$

The finite market

Traded options and approximations - The artificial infinite market

- **Lemma**:
  - Let $\overline{F}_{X_i}$ be the cdf of X_i such that

$$e^{-rT}\ \mathbb{E}_{\overline{F}_{X_i}}\left[(X_i - K)_+\right] = \overline{C}_i\left[K\right] \text{ for all } K$$

  - The cdf $\overline{F}_{X_i}$:
    - If $K_{i,j} \leq x < K_{i,j+1},\ \ j = 0, 1, \ldots, m_i$:

$$0 \leq \overline{F}_{X_i}(x) = 1 + e^{rT}\,\frac{C_i\left[K_{i,j+1}\right] - C_i\left[K_{i,j}\right]}{K_{i,j+1} - K_{i,j}} < 1$$

    - Furthermore,

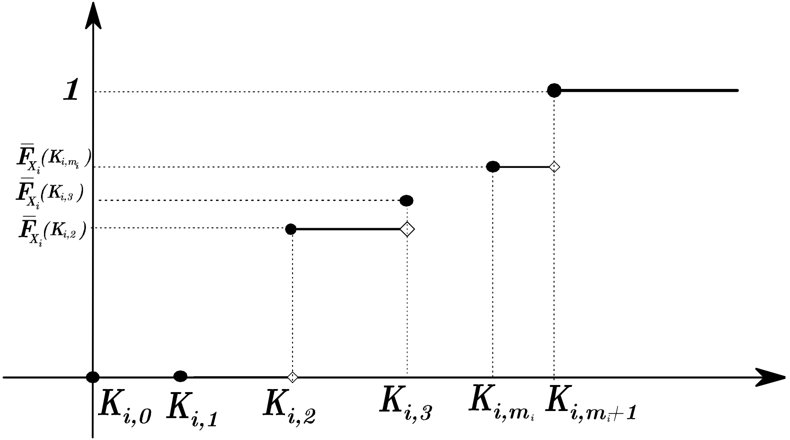
$$\begin{array}{rcll} \overline{F}_{X_i}(x) & = & 0 & \text{if } x < 0 \\ \overline{F}_{X_i}(x) & = & 1 & \text{if } x \geq K_{i,m_i+1} \end{array}$$

- Ordering relation:

$$X_i \overset{d}{=} F_{X_i}^{-1}(U) \leq_{cx} \overline{F}_{X_i}^{-1}(U)$$

The finite market

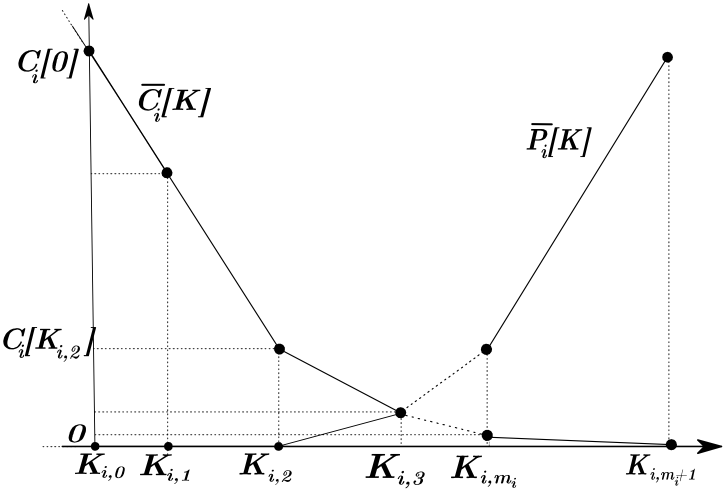
Traded options and approximations - The artificial infinite market



$$\overline{F}_{X_i}^{-1+}(0) = K_{i,2} \quad \text{and} \quad \overline{F}_{X_i}^{-1}(1) = K_{i,m_i+1}$$

The finite market

Traded options and approximations - The artificial infinite market



$$\overline{F}_{X_i}^{-1+}(0) = K_{i,2} \quad \text{and} \quad \overline{F}_{X_i}^{-1}(1) = K_{i,m_i+1}$$

The finite market

Traded options and approximations - The artificial infinite market

- An expression for $\overline{F}_{X_i}(K_{i,j})$, $j = 0, 1, \ldots, m_i$:

$$\overline{F}_{X_i}(K_{i,j}) = \frac{1}{K_{i,j+1} - K_{i,j}} \int_{K_{i,j}}^{K_{i,j+1}} F_{X_i}(x)\ dx$$

- Equivalence relations for $j = 0, 1, \ldots, m_i$:

$$\overline{F}_{X_i}^{-1+}(0) = K_{i,j} \iff K_{i,j} \leq F_{X_i}^{-1+}(0) < K_{i,j+1}$$

The finite market

An upper bound for the index option price

- The comonotonic sum $\overline{S}^c$:

$$\bar{S}^c = w_1 \overline{F}_{X_1}^{-1}(U) + w_2 \overline{F}_{X_2}^{-1}(U) + \cdots + w_n \overline{F}_{X_n}^{-1}(U)$$

- The 'extreme' outcomes of $\overline{S}^c$:

$$F_{\overline{S}^c}^{-1+}(0) = \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1+}(0) \quad \text{and} \quad F_{\overline{S}^c}^{-1}(1) = \sum_{i=1}^{n} w_i K_{i,m_i+1}$$

- **Theorem**:

$$C[K] \leq e^{-rT}\, \mathbb{E}\left[\left(\overline{S}^c - K\right)_+\right] \overset{\text{not.}}{=} \overline{C}^c[K]$$

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An upper bound for the index option price

- In the sequel, we always assume that

$$K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1) \right)$$

- **Theorem**:

$$\overline{C}^c [K] = \sum_{i=1}^{n} w_i \, \overline{C}_i [K_i^*]$$

  - with

$$K_i^* = \overline{F}_{X_i}^{-1(\alpha_K)} \left(F_{\overline{S}^c}(K) \right) \text{ and } \alpha_K \text{ from } \sum_{i=1}^{n} w_i K_i^* = K$$

- Question:
How to determine the $\overline{C}_i [K_i^*]$ from the observed $C_i [K_{i,j}]$?

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An upper bound for the index option price

- The integers j_i :
  - Notation: $K_{i,-1} = -1$. Then

$$\overline{F}_{X_i}(K_{i,-1}) = 0$$

  - j_i is defined as the unique $j \in \{0, 1, \ldots, m_i + 1\}$ that satisfies

$$\overline{F}_{X_i}(K_{i,j-1}) < F_{\bar{S}^c}(K) \leq \overline{F}_{X_i}(K_{i,j})$$

  - Notice that j_i depends on K.
- The set N_K:

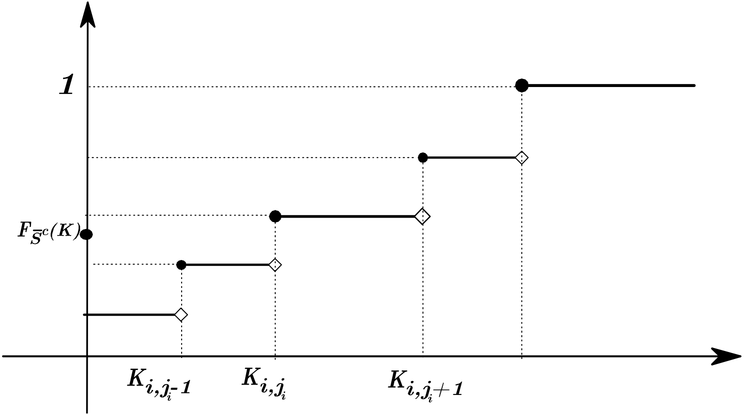
$$N_K = \left\{ i \in \{1, \ldots, n\} \mid \overline{F}_{X_i}(K_{i,j_i-1}) < F_{\bar{S}^c}(K) < \overline{F}_{X_i}(K_{i,j_i}) \right\}$$

- The set $\overline{N}_K$:

$$\overline{N}_K = \{ i \in \{1, 2, \ldots, n\} \mid F_{\bar{S}^c}(K) = \overline{F}_{X_i}(K_{i,j_i}) \}$$

The finite market

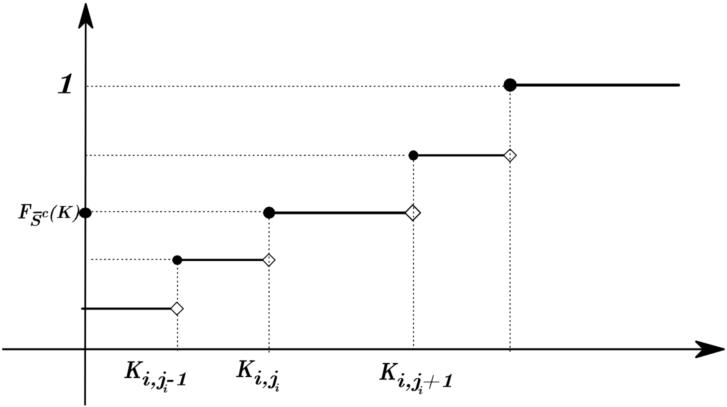
An upper bound for the index option price



- The cdf $\overline{F}_{X_i}(x)$
- $i \in N_K$
- $K_i^* = \overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right) = K_{i,j_i}$

The finite market

An upper bound for the index option price



- The cdf $\overline{F}_{X_i}(x)$
- $i \in \overline{N}_K$
- $K_i^* = \overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right) = \alpha_K\ K_{i,j_i} + (1-\alpha_K)K_{i,j_i+1}$

The finite market

An upper bound for the index option price

- Recall:

$$C[K] \leq \overline{C}^c[K] = \sum_{i=1}^{n} w_i \, \overline{C}_i[K_i^*]$$

- Determining $K_i^* = \overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right)$:

$$K_i^* = \begin{cases} K_{i,j_i} & \text{if } i \in N_K \\ \alpha_K \, K_{i,j_i} + (1-\alpha_K) K_{i,j_i+1} & \text{if } i \in \overline{N}_K \end{cases}$$

- Determining $\overline{C}_i[K_i^*]$:

$$\overline{C}_i[K_i^*] = \begin{cases} C_i[K_{i,j_i}] & \text{if } i \in N_K \\ \alpha_K \, C_i[K_{i,j_i}] + (1-\alpha_K) C_i[K_{i,j_i+1}] & \text{if } i \in \overline{N}_K \end{cases}$$

The finite market

An upper bound for the index option price

- **Theorem**: "$\overline{C}^c[K]$ *is the price of a static superhedging strategy for the index option* $C[K]$."
  - Consider the following strategy:
    - At time 0, for any $i \in N_K$,

      $$\text{buy } w_i \text{ calls } C_i\left[K_{i,j_i}\right]$$

    - At time 0, for any $i \in \overline{N}_K$,

      $$\begin{aligned} &\text{buy } \alpha_K \; w_i \;\; \text{calls } C_i\left[K_{i,j_i}\right] \\ &\text{buy } (1-\alpha_K)w_i \;\; \text{calls } C_i\left[K_{i,j_i+1}\right] \end{aligned}$$

    - Hold each of these calls until time T.
  - The pay-off of this strategy super-replicates the pay-off of the index option $C[K]$.
  - The price of this strategy is given by $\overline{C}^c[K]$.

The finite market

The cheapest super-replicating strategy

- Question: Can we improve the upper bound for $C[K]$ by looking for the price of a cheaper super-replicating strategy?
- A general class of investment strategies $\overline{\mathcal{I}}$:
  - At time 0, for each i, vanilla calls $C_i[y]$ can be bought or sold for any $y \in \{K_{i,0}, K_{i,1}, K_{i,2}, \ldots, K_{i,m_i}\}$.
  - Hold the taken positions until time T.
- We describe any such investment strategy by a vector $\underline{\nu} \equiv (\nu_1, \nu_2, \ldots, \nu_n)$, with

$$\nu_i(y) = \text{number of purchased calls on } i \text{ with strike} \leq y$$

- Each $\nu_i : \mathbb{R} \to \mathbb{R}$ is a r.c. jump function which can only have jumps (upwards or downwards) at $K_{i,0}, K_{i,1}, K_{i,2}, \ldots$ and K_{i,m_i}.

The finite market

The cheapest super-replicating strategy

- Price of the investment strategy $\underline{\nu} \in \overline{\mathcal{I}}$:

$$\text{Price }\left[\underline{\nu}\right] = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} C_i\left[y\right] \, \mathrm{d}\nu_i(y)$$

- Pay-off of the investment strategy $\underline{\nu} \in \overline{\mathcal{I}}$ at time T:

$$\text{Pay-off }\left[\underline{\nu}, \underline{X}\right] = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} (X_i - y)_+ \, \mathrm{d}\nu_i(y)$$

The finite market

The cheapest super-replicating strategy

- The investment strategy $\underline{\nu}^*$:
  - For any $i \in N_K$:

$$\nu_i^*(y) = \begin{cases} 0 & : \quad y < K_{i,j_i} \\ w_i & : \quad y \geq K_{i,j_i} \end{cases}$$

  - For any $i \in \overline{N}_K$:

$$\nu_i^*(y) = \begin{cases} 0 & : \quad y < K_{i,j_i} \\ \alpha_K \; w_i & : \quad K_{i,j_i} \leq y < K_{i,j_i+1} \\ w_i & : \quad y \geq K_{i,j_i+1} \end{cases}$$

- Price $\left[\underline{\nu}^*\right]$:

$$= \sum_{i \in N_K} w_i C_i \left[K_{i,j_i}\right] + \sum_{i \in \overline{N}_K} w_i \left\{\alpha_K C_i \left[K_{i,j_i}\right] + (1 - \alpha_K) C_i \left[K_{i,j_i+1}\right]\right\}$$

$$= \overline{C}^c \left[K\right]$$

The finite market

The cheapest super-replicating strategy

- The class $\overline{\mathcal{C}}_K$ of super-replicating strategies:

$$\overline{\mathcal{C}}_K = \left\{ \underline{\nu} \in \overline{\mathcal{I}} \mid \left(\sum_{i=1}^{n} w_i \; x_i - K \right)_+ \leq \text{ Pay-off } [\underline{\nu}, \underline{x}] \text{ for all } \underline{x} \right\}$$

  - 'for all $\underline{x}$' means

$$\text{'for all } \underline{x} \text{ with } x_i \in \left[\overline{F}_{X_i}^{-1+}(0), K_{i,m_i+1} \right] \text{'}$$

  - $\underline{\nu}^* \in \overline{\mathcal{C}}_K$.

- Any $\underline{\nu} \in \overline{\mathcal{C}}_K$ is a super-replicating strategy:

$$\mathbb{P} \left[(S - K)_+ \leq \text{Pay-off } [\underline{\nu}, \underline{X}] \right] = 1$$

The finite market

The cheapest super-replicating strategy

- <u>A particular element of $\overline{\mathcal{C}}_K$</u> :
  - Consider the investment strategy $\underline{\nu}^\circ$ defined by

$$\nu_i^\circ(y) = \begin{cases} 0 & y < K_{i,j_i} \\ w_i & y \geq K_{i,j_i} \end{cases}$$

    with the K_{i,j_i} as defined above.
  - This investment strategy is an element of $\overline{\mathcal{C}}_K$.
  - It's price is given by

$$\text{Price } [\underline{\nu}^\circ] = \sum_{i=1}^{n} w_i \; C_i \left[K_{i,j_i}\right]$$

The finite market

The cheapest super-replicating strategy

- **Theorem** :

$$\min_{\underline{\nu} \in \overline{\mathcal{C}}_K} \text{Price } [\underline{\nu}] = \text{Price } [\underline{\nu}^*] = \overline{C}^c [K]$$

- Important remark:
  - Suppose that the index option $C[K]$ is not traded in the market.
  - In case this option is sold over-the-counter, then $\overline{C}^c [K]$ is a reasonable price:
    - The seller can super-replicate the pay-off of $C[K]$ by buying $\underline{\nu}^*$.
    - The buyer cannot find a cheaper super-replicating strategy.

The finite market

The least upper bound

- $\mathcal{D}_n$ = the class of all n - dimensional cdf's F on the non-negative orthant of $\mathbb{R}^n$.
- The marginal cdf's of $F \in \mathcal{D}_n$ are denoted by F_i, $i = 1, 2, \ldots, n$.
- The class $\overline{\mathcal{R}}_n$:

$$\overline{\mathcal{R}}_n = \left\{ F \in \mathcal{D}_n \mid e^{-rT} \mathbb{E}_{F_i} \left[\left(X_i - K_{i,j} \right)_+ \right] = C_i \left[K_{i,j} \right] \text{ for all } i, \ j \right\}$$

  - 'for all i, j' means

$$\text{'for all } i = 1, \ldots, n \text{ and } j = 0, \ldots, m_i + 1\text{'}$$

- Comonotonic elements in $\overline{\mathcal{R}}_n$:

The cdf of $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U) \right)$ is an element of $\overline{\mathcal{R}}_n$

The finite market

The least upper bound

- **Theorem**:

$$\max_{F\in\overline{\mathcal{R}}_n} e^{-rT}\ \mathbb{E}_F\left[\left(\sum_{i=1}^{n} w_i X_i - K\right)_+\right] = \overline{C}^c\left[K\right]$$

- Interpretation:
  - $\overline{C}^c\left[K\right]$ is the lowest upper bound for the index option price $C\left[K\right]$ in the class of all models which are consistent with the observed stock option prices $C_i[K_{ij}]$.
  - $\overline{C}^c\left[K\right]$ is reached when $(X_1, X_2, \ldots, X_n) \overset{d}{=} \left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$.

The finite market

Computational aspects

- How to evaluate the upper bound $\overline{C}^c[K]$ for $C[K]$ numerically?
- How to choose the maximal values K_{i,m_i+1}?
- How to determine $C_i[0]$?
- How to express $\overline{C}^c[K]$ in terms of the $\overline{F}_{X_i}^{-1}$?
- What in case of no option data for some stocks in the index?

References

- Albrecher, H., Dhaene, J., Goovaerts, M.J. and Schoutens, W. (2005). Static hedging of Asian options under Lévy models: the comonotonicity approach, *The Journal of Derivatives*, vol. 12(3), 63-72.
- Chen, X.; Deelstra, G., Dhaene, J.; Vanmaele, M. (2008). Static super-replicating strategies for a class of exotic options, Insurance: Mathematics & Economics 42(3), 1067-1085.
- Dhaene, J.; Denuit, M.; Goovaerts, M.; Kaas, R.; Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and finance: theory, Insurance: Mathematics & Economics 34(1), 55-77.
- Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and finance: applications. *Insurance: Mathematics & Economics*, vol. 31(2), 133-161.
- Dhaene, J., Linders, D., Schoutens, W. and Vyncke, D. (2012). The Herd Behavior Index: a new measure for the implied degree of co-movement in financial markets, *Insurance: Mathematics and Economics*, 50(3), 357-370.
- Hobson, D., Laurence, P. and Wang, T.H. (2005). Static-arbitrage upper bounds for the prices of basket options. *Quantitative Finance*, vol. 5(4), 329-342.
- Linders, D., Dhaene, J., Hounnon, H. and Vanmaele, M. (2012). Model-free upper bounds for index call and put options: a unified approach. Research Report, AFI, FEB, KU Leuven.

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