Lower and upper bounds for survival functions of the smallest and largest claim amounts in layer coverages

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Abstract

We consider n risks $X_1, X_2, \ldots, X_n$ insured by a layer coverage with deductibles and limits given by $(d_1, l_1), \ldots, (d_n, l_n)$, respectively. We investigate the optimal allocation of insurance layers from the viewpoint of the insurer. We derive lower and upper bounds for the survival function of the smallest and largest claim amounts using the first stochastic dominance order. We find that assigning a small deductible and a large limit to large risks increases (decreases) stop-loss premiums of the largest (smallest) claim amounts.
Keywords: Deductible, first stochastic dominance order, hazard rate order, policy limit, stop-loss order.

1 Introduction

Let us introduce the risk X as a loss faced by a policyholder which is a non-negative random variable. The insurance layer $X(d, d+l]$ is defined by the pay-off

$$X(d, d+l] = \begin{cases} 0, & \text{if } 0 < X \leq d \\ X - d, & \text{if } d < X \leq d + l \\ l, & \text{if } d + l < X \end{cases}$$

where d and l are pre-specified values called the deductible and policy limit, respectively, while $x_+ = \max\{x, 0\}$ and $x \wedge y = \min\{x, y\}$, see e.g. Wang (1996, 2000).

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It follows from the definition of the layer contract that the risk $X(d, d+l] = (X-d)_+ \wedge l$ is covered by the insurer and the remaining risk, $X - X(d, d+l] = (X \wedge d) + (X-(d+l))_+$, is self-insured by the policyholder. When $d = 0$, then it is equivalent to the policy limit coverage, while if $l = \infty$, it is equivalent to the deductible coverage (cf. Klugman et al., 2004).

The optimality conditions of an insurance layer contract have been studied in various aspects in the literature. For example, Wang (1996) have introduced new premium principles by determining the premium for insurance layers. Goovaerts and Dhaene (1998) have characterized Wang's class of premium principles. Sung et al. (2011) have studied the optimal insurance policy, in which insurers' decision behavior was modeled by Kahneman and Tverskys Cumulative Prospect Theory with convex probability distortions. They have shown that, based on a fixed premium rate, an insurance layer could be an optimal insurance policy. Cheung et al. (2012) have studied the optimal reinsurance policies with an Average Value-at-Risk of the retained risk, under Wang's premiums. They have shown that under the budget constraints of the reinsurance premium, the optimal reinsurance design is an insurance layer. Zhu et al. (2014) have studied reinsurance strategies for an insurer with multiple business lines who buys reinsurance for each business line separately. They have shown that the optimal strategy for the insurer is to purchase a two-layer insurance policy for each business line, if the premium for each business line is allowed to vary by line, but they all satisfy three relatively mild conditions: distribution invariance, risk loading and preserving the convex order. On the other hand, the optimal reinsurance strategy for the insurer will be a one-layer insurance contract for each business line, if the premium satisfies additional mild conditions, which are met by the expected value principle, standard deviation principle and Wang's principle among many others. Some more studies on insurance layers can be found in Cui et al. (2013), Cheung et al. (2014), Zheng and Cui (2014), Assa (2015), Zhang and Liang (2016) and references therein.

Now, consider a situation where a policyholder is facing n risks $X_1, \ldots, X_n$, each of which is insured under an insurance layer coverage. Suppose the amounts d and l are the total deductible and the total policy limit amounts corresponding to the n risks. The policyholder wants to divide d and l into n non-negative values $d_1, \ldots, d_n$ and $l_1, \ldots, l_n$, respectively, such that $\sum_{i=1}^n d_i = d$ and $\sum_{i=1}^n l_i = l$, and for $i = 1, \ldots, n$, d_i and l_i are respectively the deductible and the policy limit corresponding to the risk X_i. Then $(X_1(d_1, d_1+l_1], \ldots, X_n(d_n, d_n+l_n])$ is the pay-off vector where for random variable X_k, $X_k(u, u+v] = (X_k - u)_+ \wedge v$. Let us set $\boldsymbol{d} = (d_1, \ldots, d_n)$ and $\boldsymbol{l} = (l_1, \ldots, l_n)$ and

$$s_n(d, l) = \{(d_1, \ldots, d_n, l_1, \ldots, l_n) \mid \sum_{i=1}^n d_i = d, \sum_{i=1}^n l_i = l, d_i \geq 0,\ l_i \geq 0\}.$$

In view of these considerations, the covered amount by the insurer is given by $\sum_{i=1}^n [(X_i - d_i)_+ \wedge l_i]$ and the retained risk is given by $\sum_{i=1}^n [(X_i \wedge d_i) + (X_i - (l_i + d_i))_+]$.

From the view point of the insurer, it is of interest to study the stochastic properties of some important statistics of $(X_1(d_1, d_1+l_1], \ldots, X_n(d_n, d_n+l_n])$ if the parameters $\boldsymbol{d}$ and $\boldsymbol{l}$ are replaced by another set of parameters $\boldsymbol{d}^*$ and $\boldsymbol{l}^*$, respectively, such that $(\boldsymbol{d}, \boldsymbol{l}), (\boldsymbol{d}^*, \boldsymbol{l}^*) \in s_n(d, l)$. Using the notion of majorization and various kinds of stochastic orderings, the statistic $\sum_{i=1}^n [(X_i - d_i)_+ \wedge l_i]$, i.e., the total risk insured by the insurer, has been studied in the particular cases when $d_1 = \ldots = d_n = 0$ or $l_1 = \ldots = l_n = \infty$ by many researchers. See e.g. Cheung (2007), Lu and Meng (2011), Hu and Wang (2014), Fathimanesh and Khaledi (2015) and Fathimanesh et al. (2016) among others. Amiri et al. (2019) recently investigated stochastic properties of the statistic $\sum_{i=1}^{n}[X_i - (X_i - d_i)_+ \wedge l_i]$ for the case when $X_1, \ldots, X_n$ are i.i.d exponential risks.

This paper is devoted to the ordering properties of the largest and smallest claims in the deductible and limit policies, and generally, in the layer coverage. The insurer is more interested in the aggregate claim than in the largest individual claim. But maybe this statement is a statement which holds after the distribution of the claims (or the d's and the l's are 'distributed') is fixed. The maximum and minimum of the individual risks are important and relevant in terms of portfolio composition. That is the stage where the insurer decides what to insure and what not to insure (or even to reinsure). Hence, when different clients come to the insurer with their full risk, the insurer has to decide what layer he/she is willing to insure for the different clients. Then, the result of the paper gives an indication where the insurer should allow the broadest insurance (in the sense of the smallest d and largest l). Balakrishnan et al. (2018) and Zhang et al. (2019) have notified that the analysis of the smallest and largest claims provides useful information for determining the premium.

Using the notions of stochastic orderings and majorization order, many researchers studied the stochastic properties of the smallest and the largest claims. Barmalzan and Payandeh (2015) recently investigated the ordering properties of the smallest claim amounts from a set of Weibull heterogeneous portfolios in the sense of the convex transform order and the right spread order. Barmalzan et al. (2015) studied the stochastic comparison between the smallest claim amounts from a portfolio risks in a general scale model, in the sense of the usual stochastic and hazard rate orders and between the largest claim amounts in the sense of the usual stochastic order. Barmalzan et al. (2016) discussed the likelihood ratio order and dispersive order between the smallest claim amounts from two sets of independent heterogeneous Weibull claims. Balakrishnan et al. (2018) compared the largest claims from two sets of independent or non-independent portfolio risks in the sense of the usual stochastic order. They also established comparison results on the largest claim amounts in the sense of the reversed hazard rate and hazard rate orders for two batches of heterogeneous independent claims.

To the best of our knowledge, there is no research available in the literature concerning the ordering properties of the largest and the smallest claims in the deductible and limit policies, and generally, in the layer coverage. In this paper, we will try to fill this gap in the literature.

Hereafter, we introduce some definitions which we will use later. Let X and Y be two random variables with distribution functions $F(.)$ and $G(.)$, survival functions $\overline{F}(.)$ and $\overline{G}(.)$, right endpoints of the supports u_X and u_Y and left endpoints l_X and l_Y, respectively.

Definition 1. *X is said to be smaller than Y in the first stochastic dominance order (known as the usual stochastic order and denoted by $X \leq_{st} Y$), if and only if $\overline{F}(t) \leq \overline{G}(t)$ for all $t \in R$.*

Definition 2. *X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\frac{\overline{G}(t)}{\overline{F}(t)}$ is increasing in $t \in (-\infty, \max\{u_X, u_Y\})$. Equivalently, $X \leq_{hr} Y$ if $\overline{F}(y)\overline{G}(x) \leq \overline{F}(x)\overline{G}(y)$ for all $x \leq y$.*

Definition 3. *X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $\frac{G(t)}{F(t)}$ is increasing in $t \in (\min\{l_X, l_Y\}, \infty)$. Equivalently, $X \leq_{rh} Y$ if $F(y)G(x) \leq F(x)G(y)$ for all $x \leq y$.*

Definition 4. *X is said to be smaller than Y in the stop-loss order (denoted by $X \leq_{sl} Y$) if and only if $E(X-d)_+ \leq E(Y-d)_+$ for all $d \in R$.*

It is known that the hazard rate order and the reversed hazard rate order imply the first stochastic dominance order, which in turn, implies the stop-loss order. For more details about stochastic orders, interested readers may refer to Müller and Stoyan (2002), Denuit et al. (2005) or Shaked and Shanthikumar (2007).

For any real vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ denote the increasing arrangement of $x_1, \ldots, x_n$.

2 Main Results

Suppose that $X_1, X_2, \ldots, X_n$ are independent non-negative random variables with distribution functions $F_1, \ldots, F_n$ respectively. It is easy to see that

$$\mathbb{P}(X_i(d_i, d_i + l_i] > t) = \begin{cases} 1 & t < 0, \\ \overline{F}_i(d_i + t) & 0 \leq t < l_i \\ 0 & t \geq l_i, \end{cases}$$

where $\overline{F}_i(.) = 1 - F_i(.)$ is the survival function of X_i, $i = 1, \ldots, n$. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be two permutations of $(1, \ldots, n)$. Then, the survival function of $\min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\}$ and the distribution function of $\max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\}$, for all $t \geq 0$, are given by

$$\begin{aligned} \mathbb{P}(\min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\} > t) &= \prod_{i=1}^{n} \overline{F}_{X_i(d_{\tau_i}, d_{\tau_i}+l_{\pi_i}]}(t) \\ &= \prod_{i=1}^{n} \overline{F}_i(d_{\tau_i} + t) I(0 \leq t < l_{(1)}) \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(\max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\} \leq t) &= \prod_{i=1}^{n} F_{X_i(d_{\tau_i}, d_{\tau_i}+l_{\pi_i}]}(t) \\ &= \prod_{i=1}^{n} [1 - \overline{F}_i(d_{\tau_i} + t) I(0 \leq t < l_{\pi_i})], \end{aligned}$$

respectively, where $I(A) = 1$ if A occurs and $I(A) = 0$ otherwise.

Theorem 5. *Let $X_1, X_2 \ldots, X_n$ be independent random risks such that $X_1 \leq_{rh} X_2 \leq_{rh} \ldots \leq_{rh} X_n$. Then, for two non-negative vectors $(d_1, \ldots, d_n)$ and $(l_1, \ldots, l_n)$ and any permutations $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ of $(1, 2, \ldots, n)$, we have that*

$$\max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\} \leq_{st} \max_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)} + l_{(i)}]\}$$

Proof. We first prove the theorem for $n = 2$. For $n \geq 2$, the proof is given by induction.

For $n = 2$, assume without loss of generality that $d_1 \geq d_2$ and $l_1 \leq l_2$. For the cases $\boldsymbol{\tau} = (2, 1)$, $\boldsymbol{\pi} = (2, 1)$ and $\boldsymbol{\tau} = (2, 1)$, $\boldsymbol{\pi} = (1, 2)$, from the assumption $X_1 \leq_{rh} X_2$, it is easy to see that for all $t \geq 0$,

$$[1-\overline{F}_1(d_1+t)I(t < l_1)][1-\overline{F}_2(d_2+t)I(t < l_2)] \leq [1-\overline{F}_1(d_2+t)I(t < l_{\pi_1})][1-\overline{F}_2(d_1+t)I(t < l_{\pi_2})].$$

For the cases $\boldsymbol{\tau} = (1, 2)$, $\boldsymbol{\pi} = (1, 2)$ and $\boldsymbol{\tau} = (1, 2)$, $\boldsymbol{\pi} = (2, 1)$, the proof is straightforward. This completes the proof of the required result.

We now take the inductive step and assume that the result holds for $n \geq 2$ risks. We show that the result holds for $n + 1$ risks, that is, for any permutations $(\pi_1, \ldots, \pi_{n+1})$ and $(\tau_1, \ldots, \tau_{n+1})$ of $(1, 2, \ldots, n + 1)$,

$$\max_{1 \leq i \leq n+1} \{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\} \leq_{st} \max_{1 \leq i \leq n+1} \{X_i(d_{(n-i+2)}, d_{(n-i+2)} + l_{(i)}]\}.$$

If $d_{\tau_{n+1}} = d_{(1)}, l_{\pi_{n+1}} = l_{(n+1)}$, the result immediately follows from the induction hypothesis and Theorem 1.A.3 in Shaked and Shanthikumar (2007). Suppose that $d_{\tau_i} = d_{(1)}$ and $l_{\pi_j} = l_{(n+1)}$, $i < j$. Then

$$\begin{aligned}
&\max_{1 \leq k \leq n+1} \{X_k(d_{\tau_k}, d_{\tau_k} + l_{\pi_k}]\} \\
&\quad = \max\Big\{ \max_{\substack{1 \leq k \leq n+1 \\ k \neq i,j}} \{X_k(d_{\tau_k}, d_{\tau_k} + l_{\pi_k}]\}, \max\{X_i(d_{(1)}, d_{(1)} + l_{\pi_i}], X_j(d_{\tau_j}, d_{\tau_j} + l_{(n+1)}]\}\Big\} \\
&\quad \leq_{st} \max\Big\{ \max_{\substack{1 \leq k \leq n+1 \\ k \neq i,j}} \{X_k(d_{\tau_k}, d_{\tau_k} + l_{\pi_k}]\}, \max\{X_i(d_{\tau_j}, d_{\tau_j} + l_{\pi_i}], X_j(d_{(1)}, d_{(1)} + l_{(n+1)}]\}\Big\} \\
&\quad = \max\Big\{ \max_{\substack{1 \leq k \leq n \\ k \neq i,j}} \{X_k(d_{\tau_k}, d_{\tau_k} + l_{\pi_k}]\}, X_i(d_{\tau_j}, d_{\tau_j} + l_{\pi_i}], \\
&\qquad\qquad \max\{X_j(d_{(1)}, d_{(1)} + l_{(n+1)}], X_{n+1}(d_{\tau_{n+1}}, d_{\tau_{n+1}} + l_{\pi_{n+1}}]\}\Big\} \\
&\quad \leq_{st} \max\Big\{ \max_{\substack{1 \leq k \leq n \\ k \neq i,j}} \{X_k(d_{\tau_k}, d_{\tau_k} + l_{\pi_k}]\}, X_i(d_{\tau_j}, d_{\tau_j} + l_{\pi_i}], \\
&\qquad\qquad \max\{X_j(d_{\tau_{n+1}}, d_{\tau_{n+1}} + l_{\pi_{n+1}}], X_{n+1}(d_{(1)}, d_{(1)} + l_{(n+1)}]\}\Big\} \\
&\quad = \max\Big\{ \max\{ \max_{\substack{1 \leq k \leq n \\ k \neq i,j}} \{X_k(d_{\tau_k}, d_{\tau_k} + l_{\pi_k}]\}, X_i(d_{\tau_j}, d_{\tau_j} + l_{\pi_i}], X_j(d_{\tau_{n+1}}, d_{\tau_{n+1}} + l_{\pi_{n+1}}]\}, \\
&\qquad\qquad X_{n+1}(d_{(1)}, d_{(1)} + l_{(n+1)}]\Big\} \\
&\quad \leq_{st} \max_{1 \leq i \leq n+1} \{X_i(d_{(n-i+2)}, d_{(n-i+2)} + l_{(i)}]\}
\end{aligned}$$

where the first and the second inequalities above follow from the result for $n = 2$ and Theorem 1.A.3. in Shaked and Shanthikumar (2007) and the third inequality follows from the induction hypothesis and again Theorem 1.A.3. in Shaked and Shanthikumar (2007). For the case when $i \geq j$, the proof is similar and is therefore omitted. Thus the proof is completed. □

From the result of Theorem 5, we conclude that if the size of a risk is large and we assign a smaller deductible and a larger policy limit to the risk, in other words, cover the larger layer of larger risks, then the largest claim paid by the insurance company will be maximized.

This provides a convenient upper bound on the survival function of the largest claim amount given as

$$\mathbb{P}(\max_{1\le i\le n}\{X_i(d_{\tau_i}, d_{\tau_i}+l_{\pi_i}]\} > t) \le 1-\prod_{i=1}^{n}[1-\overline{F}_i(d_{(n-i+1)}+t)I(t<l_{(i)})].$$

From the fact that $\le_{st}$ implies $\le_{sl}$ and Theorem 5, we have the following corollary which enables us to find an upper bound to the stop-loss transform of the largest claim amount in a portfolio.

Corollary 6. *Under the conditions of Theorem 5, we have that*

$$\max_{1\le i\le n}\{X_i(d_{\tau_i}, d_{\tau_i}+l_{\pi_i}]\} \le_{sl} \max_{1\le i\le n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)}+l_{(i)}]\}.$$

Let $H(.)$ be a premium calculation principle (risk measure) which preserves stop-loss order. Then we can conclude from Corollary 6, that

$$H(\max_{1\le i\le n}\{X_i(d_{\tau_i}, d_{\tau_i}+l_{\pi_i}]\}) \le H(\max_{1\le i\le n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)}+l_{(i)}]\}).$$

The next question may arise is whether assigning a smaller deductible and a larger policy limit to the smaller risk provides a lower bound on the survival function of the largest claim amount, that is

$$\max_{1\le i\le n}\{(X_i(d_{(i)}, d_{(i)}+l_{(n-i+1)}]\} \le_{st} \max_{1\le i\le n}\{(X_i(d_{\pi_i}, d_{\pi_i}+l_{\tau_i}]\}.$$

The following example shows that this may not be true.

Example 7. We use the notation $W(\alpha,\lambda)$ for the Weibull distribution with shape parameter α and scale parameter λ. Suppose X_1, X_2 and X_3 are independent Weibull random variables with common shape parameter $\alpha = 0.5$ and the scale parameters $4, 7$ and 10, respectively. We know that $X_1 \le_{rh} X_2 \le_{rh} X_3$. In Figure 1, we plot the survival function of $\max_{1\le i\le n}\{X_i(d_{\pi_i}, d_{\pi_i}+l_{\tau_i}]\}$ where $(l_1,l_2,l_3)=(10,20,30)$, $(d_1,d_2,d_3)=(3,6,9)$ and (π_1,π_2,π_3) and (τ_1,τ_2,τ_3) are permutations of $(1,2,3)$. It is seen that the survival function of $\max_{1\le i\le 3}\{(X_i(d_{(i)}, d_{(i)}+l_{(3-i+1)}]\}$ crosses that of $\max_{1\le i\le 3}\{(X_i(d_{\pi_i}, d_{\pi_i}+l_{\tau_i}]\}$ at some point.

Next, we turn our attention to comparing the largest claim amounts in two cases:(i) $d_1=\ldots=d_n=d'$, (ii) $l_1=\ldots=l_n=l'$. First note that the distribution functions of $\max_{1\le i\le n}\{X_i(d', d'+l_{\pi_i}]\}$ and $\max_{1\le i\le n}\{X_i(d_{\tau_i}, d_{\tau_i}+l'\}$, for $t\ge 0$, are given by

$$\begin{aligned}\mathbb{P}(\max_{1\le i\le n}\{X_i(d', d'+l_{\pi_i}]\} \le t) &= \prod_{i=1}^{n}F_{X_i(d',d'+l_{\pi_i}]}(t)\\ &= \prod_{i=1}^{n}[1-\overline{F}_i(d'+t)I(t<l_{\pi_i})]\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(\max_{1\le i\le n}\{X_i(d_{\tau_i}, d_{\tau_i}+l'\} \le t) &= \prod_{i=1}^{n}F_{X_i(d_{\tau_i},d_{\tau_i}+l']}(t)\\ &= \prod_{i=1}^{n}[1-\overline{F}_i(d_{\tau_i}+t)I(t<l')],\end{aligned}$$

respectively.

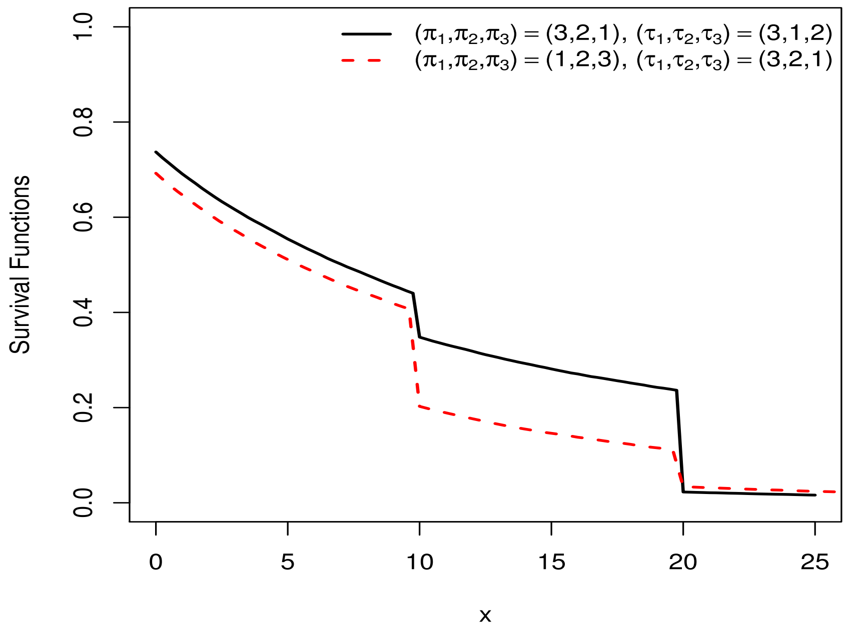

Figure 1: Survival functions of $\max_{1\leq i\leq n}\{(X_i(d_{\pi_i}, d_{\pi_i}+l_{\tau_i}]\}$, in case $X_i \sim W(0.5,\lambda_i), i = 1,2,3$ and $(\lambda_1,\lambda_2,\lambda_3) = (4,7,10)$, $(l_1,l_2,l_3) = (10,20,30)$ and $(d_1,d_2,d_3) = (3,6,9)$.

Theorem 8. *Let $X_1, X_2 \ldots, X_n$ be independent random risks and let $(l_1,\ldots,l_n)$ and $(d_1,\ldots,d_n)$ be two non-negative vectors. Then,*

(i) for any permutation $\boldsymbol{\pi}$ of $(1,\ldots,n)$, if $X_1 \leq_{st} X_2 \leq_{st} \ldots \leq_{st} X_n$, one has that

$$\max_{1\leq i\leq n}\{X_i(d', d'+l_{(n-i+1)}]\} \leq_{st} \max_{1\leq i\leq n}\{X_i(d', d'+l_{\pi_i}]\} \leq_{st} \max_{1\leq i\leq n}\{X_i(d', d'+l_{(i)}]\},$$

(ii) for any permutation $\boldsymbol{\tau}$ of $(1,\ldots,n)$, if $X_1 \leq_{rh} X_2 \leq_{rh} \ldots \leq_{rh} X_n$, one has that

$$\max_{1\leq i\leq n}\{X_i(d_{(i)}, d_{(i)}+l']\} \leq_{st} \max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i}+l']\} \leq_{st} \max_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)}+l']\}.$$

Proof. It is easy to see that the results of (i) and (ii) hold true for the case $n = 2$. Thus, the proof is easily done by induction similar to that of Theorem 5. □

Theorem 8 enables us to find an upper and a lower bound for the distribution function of the largest claim amount for two cases indeed,
(i)

$$\mathbb{P}(\max_{1\leq i\leq n}\{X_i(d', d'+l_{(i)}]\} \leq t) \leq \mathbb{P}(\max_{1\leq i\leq n}\{X_i(d', d'+l_{\pi_i}]\} \leq t) \leq \mathbb{P}(\max_{1\leq i\leq n}\{X_i(d', d'+l_{(n-i+1)}]\} \leq t)$$

$$\Updownarrow$$

$$\prod_{i=1}^{n}[1-\overline{F}_i(d'+t)I(t<l_{(i)})] \leq \prod_{i=1}^{n}[1-\overline{F}_i(d'+t)I(t<l_{\pi_i})] \leq \prod_{i=1}^{n}[1-\overline{F}_i(d'+t)I(t<l_{(n-i+1)})]$$

and (ii)

$$\mathbb{P}(\max_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)}+l']\} \leq t) \leq \mathbb{P}(\max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i}+l']\} \leq t) \leq \mathbb{P}(\max_{1\leq i\leq n}\{X_i(d_{(i)}, d_{(i)}+l']\} \leq t)$$

$$\Updownarrow$$

$$\prod_{i=1}^{n}[1-\overline{F}_i(d_{(n-i+1)}+t)I(t<l')] \leq \prod_{i=1}^{n}[1-\overline{F}_i(d_{\tau_i}+t)I(t<l')] \leq \prod_{i=1}^{n}[1-\overline{F}_i(d_{(i)}+t)I(t<l')]$$

The following corollary is a direct consequence of Theorem 8.

Corollary 9. *Under the conditions of Theorem 8,*

(i) if $X_1 \leq_{st} X_2 \leq_{st} \ldots \leq_{st} X_n$, then for any permutation $\boldsymbol{\pi}$ of $(1,\ldots,n)$,

$$\max_{1\leq i\leq n}\{X_i(d', d'+l_{(n-i+1)}]\} \leq_{sl} \max_{1\leq i\leq n}\{X_i(d', d'+l_{\pi_i}]\} \leq_{sl} \max_{1\leq i\leq n}\{X_i(d', d'+l_{(i)}]\},$$

(ii) if $X_1 \leq_{rh} X_2 \leq_{rh} \ldots \leq_{rh} X_n$, then for any permutation $\boldsymbol{\tau}$ of $(1,\ldots,n)$,

$$\max_{1\leq i\leq n}\{X_i(d_{(i)}, d_{(i)}+l']\} \leq_{sl} \max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i}+l']\} \leq_{sl} \max_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)}+l']\}.$$

The following example numerically illustrates the results of Theorem 8.

Example 10. Suppose X_1, X_2 and X_3 are independent Weibull random variables with the common shape parameter $\alpha = 0.5$ and the scale parameters $\lambda_1 \leq \lambda_2 \leq \lambda_3$. In Figure 2, we graph the survival function of $\max_{1\leq i\leq 3}\{X_i(d', d'+l_{\pi_i}]\}$ where $(l_1, l_2, l_3) = (10, 20, 30)$, $d' = 5$ and $(\lambda_1, \lambda_2, \lambda_3) = (4, 7, 10)$. The figure demonstrates the result of Theorem 8(i). In Figure 3, the survival function of $\max_{1\leq i\leq 3}\{X_i(d_{\tau_i}, d_{\tau_i}+l']\}$ is graphed where $(d_1, d_2, d_3) = (2, 6, 10)$, $l' = 30$ and $(\lambda_1, \lambda_2, \lambda_3) = (0.5, 2, 10)$. It is observed that the figure demonstrates the concept of Theorem 8(ii).

In the following theorem we derive a lower and an upper bound for the survival function of the smallest claim amount in a portfolio. First note that, from (1), it is clear that for any permutation $\boldsymbol{\pi}$,

$$\min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i}+l_{\pi_i}]\} \stackrel{d}{=} \min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i}+l_i]\}.$$

This means that the permutation of the limits does not have effect on the magnitude of the smallest claim.

Theorem 11. *Let $X_1, X_2 \ldots, X_n$ be independent random risks such that $X_1 \leq_{hr} X_2 \leq_{hr} \ldots \leq_{hr} X_n$. Then, for two non-negative vectors $(d_1,\ldots,d_n)$ and $(l_1,\ldots,l_n)$ and any permutation $\boldsymbol{\tau}$ of $(1,2,\ldots,n)$, we have that*

$$\min_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)}+l_i]\} \leq_{st} \min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i}+l_i]\} \leq_{st} \min_{1\leq i\leq n}\{X_i(d_{(i)}, d_{(i)}+l_i]\}.$$

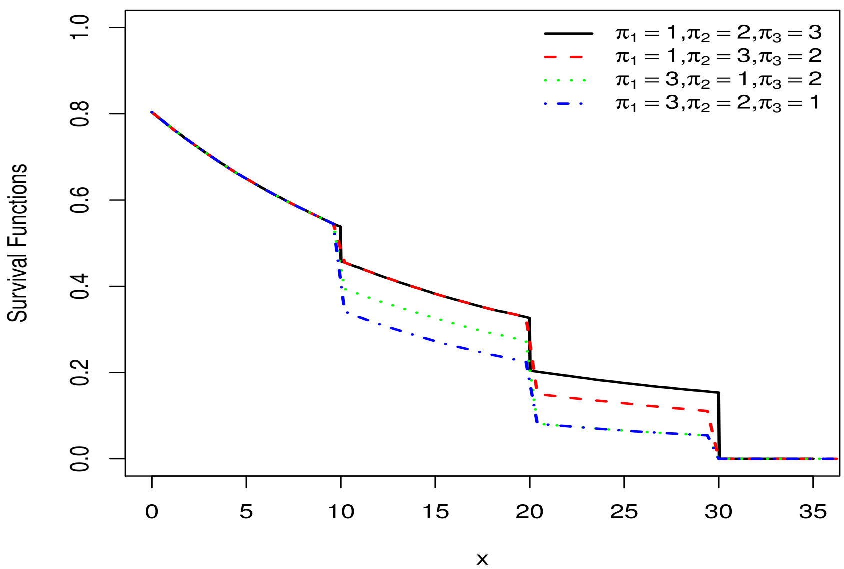

Figure 2: Survival functions of $\max_{1\le i\le 3}\{X_i(5, 5 + l_{\pi_i}]\}$, in case $X_i \sim W(0.5, \lambda_i), i = 1, 2, 3$ and $(\lambda_1, \lambda_2, \lambda_3) = (4, 7, 10)$, $(l_1, l_2, l_3) = (10, 20, 30)$.

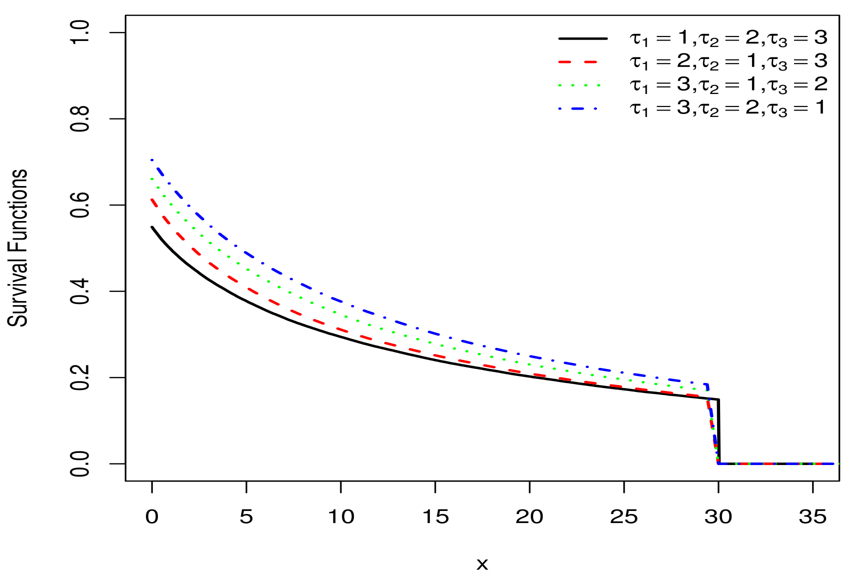

Figure 3: Survival functions of $\max_{1\le i\le 3}\{X_i(d_{\tau_i}, d_{\tau_i}+30]\}$, in case $X_i \sim W(0.5, \lambda_i), i = 1, 2, 3$ and $(\lambda_1, \lambda_2, \lambda_3) = (0.5, 2, 10)$, $(d_1, d_2, d_3) = (2, 6, 10)$.

Proof. For the case $n = 2$, the assumption $X_1 \leq_{hr} X_2$ implies that for $t \geq 0$,

$$\overline{F}_1(d_{(2)} + t)\overline{F}_2(d_{(1)} + t)I(t < l_{(1)}) \leq \overline{F}_1(d_{(1)} + t)\overline{F}_2(d_{(2)} + t)I(t < l_{(1)}) \quad (2)$$

which is the required result. Now, similar to the proof of Theorem 5, for $n \geq 2$, the required result follows by induction. □

From Theorem 11, it follows that a lower and an upper bound for the survival function of the smallest claim amount in a portfolio, for all $t \geq 0$, are given by

$$\mathbb{P}(\min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_i]\} > t) \geq \prod_{i=1}^{n} \overline{F}_i(d_{(n-i+1)} + t)I(t < l_{(1)})$$

and

$$\mathbb{P}(\min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_i]\} > t) \leq \prod_{i=1}^{n} \overline{F}_i(d_{(i)} + t)I(t < l_{(1)}).$$

The following corollary is a direct consequence of Theorem 11, which enables us to compare the smallest claim amounts in the sense of the stop-loss order.

Corollary 12. *Under the conditions of Theorem 11, we have that*

$$\min_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)} + l_i]\} \leq_{sl} \min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_i]\} \leq_{sl} \min_{1\leq i\leq n}\{X_i(d_{(i)}, d_{(i)} + l_i]\}.$$

The stop-loss preserving property, together with Corollary 12, then yields

$$H(\min_{1\leq i\leq n}\{X_i(d_{(n-i+1)}, d_{(n-i+1)} + l_i]\}) \leq H(\min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_i]\}) \leq H(\min_{1\leq i\leq n}\{X_i(d_{(i)}, d_{(i)} + l_i]\}).$$

3 Conclusion

In this paper, we derived lower and upper bounds for survival functions of the smallest and largest claim amounts in a portfolio of n risks which is insured under layer insurance policy. The results of the paper are in line with intuition. This result is a mathematical description of an intuitive fact that attaching the smaller d's and larger l's to the larger risks will lead to the larger value for the maximal claim. Topics for future research are solving a similar problem from the viewpoint of the policyholder. One also could investigate the stochastic comparison for range of claims $\max_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\} - \min_{1\leq i\leq n}\{X_i(d_{\tau_i}, d_{\tau_i} + l_{\pi_i}]\}$.

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