Systemic Risk: Stochastic Orders

Jan Dhaene
Faculty of Business and Economics
Katholieke Universiteit Leuven
Jan.Dhaene@kuleuven.be

Roger J. A. Laeven
Amsterdam School of Economics
University of Amsterdam, CentER
and EURANDOM
R.J.A.Laeven@uva.nl

Yiying Zhang
Department of Mathematics
Southern University of Science and Technology
zhangyy3@sustech.edu.cn

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Abstract

There is pervasive interconnectedness within the increasingly complex system of financial institutions. Due to this interconnectedness, a contagious failure of a singular player can cause a disruptive collapse in part of the system, with a reverberating effect on the system and economy as a whole. A pivotal step in assessing this systemic risk is the development of concepts, tools, and techniques to provide a(n) (partial) ordering of financial institutions or systems and the systemic risk borne and induced by them. In this paper, we introduce some new stochastic orders related to systemic risk—the systemic contribution order, the systemic relevance order, and the systemic aggregation order—and analyze their characterizations and properties. Examples are provided to explain these new concepts and the main results.

Keywords: Systemic risk; Risk measures; Contagion; Financial crisis; Macroprudential regulation; Conditional stochastic ordering; Stop-loss ordering.

JEL Classification: G21, G22, G31.

1 Introduction

Within the financial sector, a “systemic risk” is present when the failure or loss of an individual financial institution, or of a few players, threatens the security and stability of the other players, and as a result of the financial system and economy as a whole. A high level of interconnectedness among financial institutions provides a fertile soil for the contagious transmission and rapid propagation of adverse shocks leading to such a systemic risk.[1] The global financial crisis of 2007-2009 was characterized by spillover effects and pronounced transmissions of a few major adverse events that led to a cascading sequence of many more adverse events with severe financial distress as a consequence. In the United States in 2008, we saw the collapse of a number of financial institutions and corporations following distressed results of subprime mortgage loans and credit default swaps. This rapidly affected financial and investment activities that led to bank failures in other parts of the world and the downturn of the US stock market that spiralled across several other stock markets around the world. It ultimately threatened the collapse of the global financial market to the extent that the system was on the brink of a meltdown.

Not quite unexpectedly, this led to initiatives of several nations to cooperate and develop increased prudential regulation and supervision for financial institutions. Developed since 1974 by a committee of members representing the G-10 countries, the Basel Accords consist of a set of recommended actions to regulate the banking industry. In a similar fashion, though of a binding nature, the EU Solvency Directives codify insurance regulation and supervision within the European Union. Both sets of regulations are primarily motivated to ensure harmonized solvency rules for individual banks and insurance companies by requiring them to hold and maintain a prudent level of capital according to their respective individual risk profiles. Following the global financial crisis, financial regulators and supervisory authorities have increasingly been recognizing the presence of intercon-

[1] As to the definition of what constitutes a systemic risk, Kaufman and Scott (2003) argue that “systemic risk is the risk or probability of breakdowns in an entire system, as opposed to breakdowns in individual parts or components, and is evidenced by co-movements (correlation) among most or all of the parts.” Quite similar to the one suggested by the Group of Ten (2001), Cummins and Weiss (2013) give the following definition: “Systemic risk is the risk that an event will trigger a loss of economic value or confidence in a substantial segment of the financial system that is serious enough to have significant adverse effects on the real economy with a high probability.” Thus, a systemic risk can be viewed as the risk of individual adverse events, which trigger further adverse events, in the financial system, and as a result in the real economy. While this last definition does not directly allude to the idea of “interconnectedness” of the entities in the financial market, it does imply that the system is intertwined to the effect that a systemic event can reverberate to the entire or a substantial part of the financial system.

nectedness of financial institutions in financial markets globally.[2] This interconnectedness creates a form of systemic risk that should be accounted for in the monitoring and supervision of financial institutions. Therefore, the risks borne and induced by financial institutions should not be monitored in isolation. In recent years, this insight is leading to a major shift from purely microprudential regulation and supervision of financial institutions to both micro- and macroprudential regulation and supervision. Macroprudential regulation is now the term being used to regulate and supervise financial institutions to alleviate the consequences of systemic risk.[3] Today we are seeing a growing need and interest among financial regulators and supervisory authorities, not only to determine solvency capital requirements at the micro level, but also to assess the aggregate risk in the financial system from a macro perspective and to identify Systemically Important Financial Institutions (SIFIs) and Global or Domestic Systemically Important Insurers (G-SIIs or D-SIIs).

Over the last few years, the world has witnessed the impact of systemic risk on financial banks, especially in the United States and Switzerland. For example, on March 10, 2023, Silicon Valley Bank announced its failure after experiencing panicky runs, making it the largest bank failure since the 2008 financial crisis and the second largest failure in US history. Shortly after, on March 12, 2023, the Treasury Department, Federal Reserve, and Federal Deposit Insurance Corporation issued a joint statement to announce that Signature Bank of America was closed by the New York State Department of Financial Services due to "systemic risk". This marked the second bank closure in the US within three days, following the bankruptcy of Silicon Valley Bank.

In this paper, we seek to develop the "language of stochastic orders for systemic risk", to bear on the problem of assessing systemic risk. More specifically, we aim to provide methods to establish a partial ordering of financial institutions or financial systems from the perspective of the systemic risk borne and/or induced by them. To this end, we introduce some new stochastic orders related to systemic risk and analyze their characterizations and properties. We introduce the "systemic contribution order" and the "systemic relevance order" to stochastically compare the contributions and relevance of individual financial institutions within a financial system. We also introduce the "sys-

[2]See, e.g., Basel Committee on Banking Supervision (2011).

[3]See, e.g., the initiatives of the Financial Stability Board (FSB), the European Systemic Risk Board (ESRB), and, of a more specific nature, the European Insurance and Occupational Pensions Authority (EIOPA) which recently published a series of papers with the aim of contributing to the debate on systemic risk and macroprudential policy for insurers.

temic aggregation order" to provide a partial ordering of the aggregate risk in financial systems; the ordered systems only differ in terms of the dependence structure among the individual risks.

The *systemic contribution order* introduced in this paper is shown to be intimately related to a conditional stochastic order (Christofides and Hadjikyriakou, 2015). There is a long and rich history in actuarial science and applied probability of partially ordering *univariate* risks; see, e.g., Kaas et al. (1994); Marshall and Olkin (2007) and the references therein. Subsequently, the literature has focused its attention on the analysis of *multivariate* stochastic orders; see, e.g., the monographs Müller and Stoyan (2002); Denuit et al. (2005); Shaked and Shanthikumar (2007). By contrast, *conditional* stochastic ordering and its applications in insurance and finance, as considered in this paper, has so far seen relatively little interest. Revealing the usefulness of conditional stochastic ordering in an insurance and financial context can be viewed as an additional contribution of this paper that is of independent interest.

The *systemic relevance order* introduced in this paper is a particular type of "stop-loss information order" that we formally define in Section 3.2. This order of information compares a given risk conditioned upon two different sets of information, expressed in terms of two different events. The stop-loss information order can be interpreted as an order of relevance: it indicates what information is more relevant, where "more relevant" means making the magnitude and variability of a risk larger (in the stop-loss order sense).

The *systemic aggregation order* introduced in this paper can be viewed as a special case of the existing, general multivariate stop-loss order, confined to the Fréchet space and in which the respective increasing convex functions are restricted to the class of *linear-convex* functions. Interestingly, this situation is complementary to Koshevoy and Mosler (1996, 1997, 1998), where attention is restricted to the special case of *convex-linear* functions.

In addition to ordering systemic risk, it is important to provide a suitable measurement of systemic risk, and to equitably attribute the aggregate risk in the system together with its associated capital to the financial institutions that generate this risk. By "systemic risk allocation", we mean a fair subdivision of the aggregate risk capital in the system across the constituents of the system. The objective of partially ordering systemic risk and comparing the contributions and relevance of individual institutions in the financial system from a macro perspective is the aim of the present paper. In a companion paper (Dhaene et al., 2022), we have proposed conditional distortion risk measures and distortion risk contribution measures to assess systemic risk and present sufficient conditions for two random vectors to be ordered in terms of the proposed measures.

The remainder of this paper is organized as follows: In Section 2, we describe the setting, introduce some notation and provide some relevant definitions. Section 3 recalls the class of conditional stochastic orders, defines a new conditional ordering called the *stop-loss information order* and presents some of its basic properties. In Section 4, we introduce the systemic contribution order. In Section 5, we introduce the systemic relevance order. Section 6 defines the notion of systemic aggregation order. Conclusions are given in Section 7. Some preliminaries for univariate and multivariate stochastic ordering, definitions for risk measures and a generalization of the systemic contribution order are presented in the Appendix.

2 Preliminaries

In this section, we present some definitions and notions used in the sequel. All random variables (r.v.'s) considered hereafter are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All expectations and density functions are assumed to be well-defined when they appear. Two r.v.'s are identified if they are almost surely (a.s.) equal, and we understand throughout equalities and inequalities between r.v.'s in the a.s. sense. We denote by F_X the cumulative distribution function (cdf) of a given r.v. X under the reference probability measure $\mathbb{P}$: $F_X(x) = \mathbb{P}[X \leq x]$. We use the terms "increasing" and "decreasing" in a non-strict sense. Furthermore, we use the notation "$\overset{d}{=}$" for equality in distribution and $x_+ \equiv \max\{x, 0\}$. For an event $A \in \mathcal{F}$ and a r.v. Z, we use "$A \perp \sigma(Z)$" to denote that "A is independent of any event taken from $\sigma(Z)$", that is, $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$, for all $C \in \sigma(Z)$. Furthermore, we adopt "$Z \perp [X \mid A]$" to mean that "the r.v. Z is independent of the conditional random variable $X_A := [X \mid A]$", that is to say, $\mathbb{P}(X_A \leq x, Z \leq z) = \mathbb{P}(X_A \leq x)\mathbb{P}(Z \leq z)$, for all $x, z \in \mathbb{R}$.

Consider a market composed of n (non-collaborating, competitive) financial institutions or financial conglomerates, the stochastic losses (related to their business activities) of which are represented by the r.v.'s $X_1, \ldots, X_n$. Henceforth, we identify a financial system with a random vector $(X_1, \ldots, X_n)$. Suppose that, by microprudential regulation, each individual financial institution is required to hold a certain amount of *microprudential risk capital* equal to R_i, $i = 1, \ldots, n$, where the italic upper case "R" stands for "Required". All of the results developed in this paper can also be applied to the case where the R_i are interpreted as *available capital*. The aggregate amount of microprudential risk capital in the market, denoted by R, thus equals $R = \sum_{i=1}^{n} R_i$. For insurers, we interpret R_i to include both the technical provision and the solvency capital requirement.

We introduce the notations $\underline{x}$, $\underline{X}$ and $\underline{R}$ for the vectors $(x_1, \ldots, x_n)$, $(X_1, \ldots, X_n)$ and $(R_1, \ldots, R_n)$, respectively. Furthermore, the inequality "$\underline{x} > \underline{R}$" is used to denote the componentwise order. The *Fréchet space* $\mathcal{R}(F_1, \ldots, F_n)$ is defined as the class of all n-dimensional random vectors with fixed marginal distributions F_i, for $i = 1, \ldots, n$. In particular, we shall denote $\underline{X} \in \mathcal{R}(F_{X_1}, \ldots, F_{X_n})$ when $F_i = F_{X_i}$, where F_{X_i} is the distribution function of X_i, for $i = 1, \ldots, n$.

Modern regulation and supervision should not be solely concerned with microprudential risk management, but also with macroprudential risk management. From a macroprudential perspective, the regulatory authority is facing, and supposed to also monitor, the random vector $(X_1 - R_1, \ldots, X_n - R_n)$.

3 Conditional orders

In Section 3.1, we recall some useful conditional stochastic orders defined in the literature, whereas in Section 3.2, we introduce a new "conditional information order" referred to as the *stop-loss information order* and present some of its theoretical properties. For ease of presentation and readability of the paper, the definitions and related properties of the canonical univariate and multivariate stochastic orders, of comonotonicity and of other dependence notions used hereafter are summarized in the Appendix.

3.1 Conditional stochastic orders

In this subsection, we provide the definitions of some *conditional stochastic orders*, which were introduced in Christofides and Hadjikyriakou (2015) (some related stochastic orders were introduced earlier in Whitt, 1980; Arjas, 1981; Rüschendorf, 1991). Up to now, these useful notions have rarely been cited or explored in the actuarial and quantitative risk management literature. In Section 4, we shall employ these conditional stochastic orders and compare them with the "systemic contribution order" that we introduce. (Note that the inequalities in the following definition are inequalities in the a.s. sense.)

Definition 1 *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. The r.v. X is said to be smaller than the r.v. Y in the* *(i) $\mathcal{G}$-convex order (denoted by $X \leq_{\mathcal{G}-cx} Y$), if $\mathbb{E}\left[g\left(X\right) \mid \mathcal{G}\right] \leq \mathbb{E}\left[g\left(Y\right) \mid \mathcal{G}\right]$ holds for every convex function $g : \mathbb{R} \to \mathbb{R}$;*

(ii) $\mathcal{G}$-stop-loss order (denoted by $X \leq_{\mathcal{G}-sl} Y$), if $\mathbb{E}\left[g\left(X\right) \mid \mathcal{G}\right] \leq \mathbb{E}\left[g\left(Y\right) \mid \mathcal{G}\right]$ holds for every increasing convex function $g : \mathbb{R} \to \mathbb{R}$;

(iii) $\mathcal{G}$-stochastic dominance order (denoted by $X \leq_{\mathcal{G}-st} Y$), if $\mathbb{E}\left[g\left(X\right) \mid \mathcal{G}\right] \leq \mathbb{E}\left[g\left(Y\right) \mid \mathcal{G}\right]$ holds for every increasing function $g : \mathbb{R} \to \mathbb{R}$.

In case $\mathcal{G} = \{\varnothing, \Omega\}$, the conditional stochastic orders in Definition 1 reduce to the conventional definitions of convex order, stop-loss order, and stochastic dominance order, respectively; see the Appendix. The following implications, which are straightforward to prove, can be found in Christofides and Hadjikyriakou (2015):

$$X \leq_{\mathcal{G}-st\ [\mathcal{G}-sl,\ \mathcal{G}-cx]} Y \Longrightarrow X \leq_{st\ [sl,\ cx]} Y. \tag{1}$$

In addition, it is not hard to verify that the following relation holds:

$$X \leq_{\mathcal{G}-st} Y \Longrightarrow X \leq_{\mathcal{G}-sl} Y,$$

where the implication also holds if "$\leq_{\mathcal{G}-st}$" is replaced by "$\leq_{\mathcal{G}-cx}$".

In the sequel, we will consider conditional stop-loss (and other) orders, based on the sub-σ-algebra $\mathcal{G}$ generated by a r.v. Z. In this case, we will denote the conditional order relation "$\leq_{\mathcal{G}-sl}$" by "$\leq_{Z-sl}$".

3.2 Stop-loss information order

The conditional stochastic orders of Section 3.1 involve two r.v.'s and a common conditioning set, i.e., a common sub-σ-algebra. In this subsection, we introduce the *stop-loss information order*, which compares a single given risk conditioned upon two different sets of information, expressed in terms of two different events. As will be seen further in the paper, the systemic relevance order introduced in Section 5 can be seen as special case of the stop-loss information order.

Definition 2 (Stop-loss information order) *Let A and B be two events taken from the σ-algebra $\mathcal{F}$. We say that for a given r.v. X, A is smaller than B in the stop-loss information order with respect to X (denoted by $A \leq_{sl}^{X} B$), if $[X \mid A] \leq_{sl} [X \mid B]$, i.e., $\mathbb{E}[g(X) \mid A] \leq \mathbb{E}[g(X) \mid B]$ holds for every increasing convex function $g : \mathbb{R} \to \mathbb{R}$.*

Clearly, the stop-loss information order satisfies reflexivity and transitivity, but not the antisymmetry property. For example, consider $A = \{Z > z_1\}$ and $B = \{Z > z_2\}$, with $z_1 \neq z_2$, where Z is a r.v. independent of X. As a result, one has $A \leq_{sl}^X B$ and $B \leq_{sl}^X A$; however, A does not equal to B, containing different information sets.

It is straightforward to verify that the stop-loss information order between A and B with respect to X is preserved under positive affine transformations of X. Indeed, it is preserved under increasing convex transformation, which is proven in the following theorem. We also show that the stop-loss information order is preserved under convolution and mixture.

Theorem 3 *Stop-loss information order is preserved under:*

(i) increasing convex transformations: $A \leq_{sl}^X B$ implies $A \leq_{sl}^{g(X)} B$, for any increasing convex function g;

(ii) convolution with a r.v. Z such that $A \perp \sigma(Z)$, $B \perp \sigma(Z)$, and Z is independent of X given both A and B: $A \leq_{sl}^X B$ implies $A \leq_{sl}^{X+Z} B$, with $A \perp \sigma(Z)$, $B \perp \sigma(Z)$, $Z \perp [X \mid A]$ and $Z \perp [X \mid B]$;

(iii) mixture: for a r.v. X_θ with cdf F_X^θ and a r.v. $Y = X_\Theta$ with cdf $\int_{\mathbb{R}} F_X^\theta(x)\,\mathrm{d}W(\theta)$, where W is the cdf of another r.v. Θ, $A \leq_{sl}^{X_\theta} B$ for all real θ implies $A \leq_{sl}^{Y} B$.

Proof. Proof of (i): Suppose that $A \leq_{X-sl} B$. For any increasing convex functions g, h, it follows that $\mathbb{E}[h(g(X)) \mid A] \leq \mathbb{E}[h(g(X)) \mid B]$ since $h \circ g$ is increasing convex. Thus, the stated result is proved by applying Definition 2.

Proof of (ii): Since $A \perp \sigma(Z)$, one has that for any $z \in \mathbb{R}$, the event $\{Z > z\}$ is independent of A, which implies that $\mathbb{P}(Z > z \mid A) = \mathbb{P}(Z > z)$. Hence $Z \stackrel{d}{=} [Z \mid A]$. Similarly, $Z \stackrel{d}{=} [Z \mid B]$. Therefore, we have $[X + Z \mid A] \stackrel{d}{=} Z + [X \mid A]$ and $[X + Z \mid B] \stackrel{d}{=} Z + [X \mid B]$. By applying the independence conditions $Z \perp [X \mid A]$ and $Z \perp [X \mid B]$, one can derive from Theorem 4.A.8(c) in Shaked and Shanthikumar (2007) that

$$[X + Z \mid A] \stackrel{d}{=} Z + [X \mid A] \leq_{sl} Z + [X \mid B] \stackrel{d}{=} [X + Z \mid B],$$

which proves the desired result.

Proof of (iii): Let $Z_\theta^A = [X_\theta \mid A]$ and $Z_\theta^B = [X_\theta \mid B]$, for any real θ taken from the support of Θ. Suppose that $A \leq_{X_\theta - sl} B$ for any θ. Then it immediately follows that

$Z_\theta^A \leq_{sl} Z_\theta^B$ for all real θ. By applying Theorem 4.A.8(a) in Shaked and Shanthikumar (2007), we have $Z_\Theta^A \leq_{sl} Z_\Theta^B$, i.e., $[X_\Theta \mid A] \leq_{sl} [X_\Theta \mid B]$, implying the desired result. ■

If we define the conditioning events A and B in terms of information on r.v.'s, we have (at least) three options.

- The first option is to define A and B by taking different pieces of information of X itself. For example, $A = \{X \leq x\}$ and $B = \Omega$.
- The second option is to define A and B by considering different information emerging from a single r.v. Y, which may be useful to investigate e.g., the effect on X of different levels of information on Y. For example, $A = \{Y \leq y_1\}$ and $B = \{Y \leq y_2\}$.
- The third option is to define A and B in terms of two different r.v.'s, say Y and Z, which provides a way of investigating the effects of Y and Z on X. For example, replacing X, A, and B with $S_{\underline{R}}$ (see (2′)), $\{X_j > R_j\}$, and $\{X_k > R_k\}$, respectively, in Definition 2, gives rise to the systemic relevance order, which we will introduce in Definition 12.

For the above mentioned ways of defining events A and B, we have the following proposition ordering A and B whenever one event is a subset of the other one.

Proposition 4 *The following statements hold:*

(i) *Let* $A = \{X \leq x\}$ *and* $B = \Omega$, *for* $x \in \mathbb{R}$. *Then* $A \leq_{sl}^X B$.

(ii) *Let* $A = \{X > x\}$ *and* $B = \Omega$, *for* $x \in \mathbb{R}$. *Then* $A \geq_{sl}^X B$.

(iii) *Let* $A = \{Y \leq y\}$ *and* $B = \Omega$, *for* $y \in \mathbb{R}$. *If* (X, Y) *is PQD,*[4] *then* $A \leq_{sl}^X B$.

(iv) *Let* $A = \{Y > y\}$ *and* $B = \Omega$, *for* $y \in \mathbb{R}$. *If* (X, Y) *is PQD, then* $A \geq_{sl}^X B$.

(v) *Let* $A = \{Y \leq y_1\}$ *and* $B = \{Y \leq y_2\}$ *with* $y_1 > y_2$. *If* $X \uparrow_{LTD} Y$,[5] *then* $A \geq_{sl}^X B$.

(vi) *Let* $A = \{Y > y_1\}$ *and* $B = \{Y > y_2\}$ *with* $y_1 > y_2$. *If* $X \uparrow_{RTI} Y$,[6] *then* $A \geq_{sl}^X B$.

[4] See Appendix A.2.1.

[5] For a bivariate random vector (X, Y), X is said to be left tail decreasing in Y (denoted by $X \uparrow_{LTD} Y$) if $\mathbb{P}(X \leq x \mid Y \leq y)$ is decreasing in $y \in \mathbb{R}$, for all $x \in \mathbb{R}$.

[6] For a bivariate random vector (X, Y), X is said to be right tail increasing in Y (denoted by $X \uparrow_{RTI} Y$) if $\mathbb{P}(X > x \mid Y > y)$ is increasing in $y \in \mathbb{R}$, for all $x \in \mathbb{R}$.

Proof. Proof of (i) and (ii). We only prove (i) since the proof for (ii) is similar. Note that we have to show that $[X \mid X \le x] \le_{sl} X$, which holds immediately if we can show that $[X \mid X \le x] \le_{st} X$. Note that $\mathbb{P}(X \le s \mid X \le x) = \mathbb{P}(X \le \min\{s, x\})/\mathbb{P}(X \le x) = \mathbb{P}(X \le s)/\mathbb{P}(X \le x) \ge \mathbb{P}(X \le s)$, for all $s \le x$; otherwise, $\mathbb{P}(X \le s \mid X \le x) = 1 \ge \mathbb{P}(X \le s)$, for all $s > x$. Therefore, $\mathbb{P}(X \le s \mid X \le x) \ge \mathbb{P}(X \le s)$, for all $s \in \mathbb{R}$, which ends the proof of (i).

Proof of (iii) and (iv). We only give the proof of (iii) since the proof of (iv) can be conducted in a similar manner. It suffices to show that $[X \mid Y \le y] \le_{st} X$, or equivalently, $\mathbb{P}(X \le x \mid Y \le y) \ge \mathbb{P}(X \le x)$, for all $x \in \mathbb{R}$, which is $\mathbb{P}(X \le x, Y \le y) \ge \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$. This holds obviously by noting that (X, Y) is PQD.

Proof of (v) and (vi). We only prove (v) since the proof of (vi) is analogous. Since $X \uparrow_{LTD} Y$, it follows that $\mathbb{P}(X \le x \mid Y \le y_1) \le \mathbb{P}(X \le x \mid Y \le y_2)$, implying that $[X \mid Y \le y_1] \ge_{st} [X \mid Y \le y_2]$. Thus, $[X \mid A] \ge_{sl} [X \mid B]$, yielding the desired result. ■

According to Proposition 4(i)-(ii), we find that restricting the value of X within a left[right] half-closed[open] interval will lead to a smaller[greater] conditional r.v. compared with X itself on the whole support in the sense of the stop-loss order. Therefore, "more information" does not necessarily lead to larger conditional r.v.'s in the sense of the stop-loss order. Here, the term "more information" should be understood in the sense that one knows more about the r.v. given that information. For example, as stated in Proposition 4(v), the set $B = \{Y \le y_2\}$ has "more" information than $A = \{Y \le y_1\}$ provided that $y_1 > y_2$, which, however, leads to the conclusion that $A \ge_{X-sl} B$.

Remark 5 *By analogy, one can define other information orders (such as "first-dominance stochastic order" and "convex information order") by invoking other stochastic orders than stop-loss order. We shall pursue findings on these notions and their potential applications in risk management and actuarial science in a future paper.*

4 Systemic contribution order

In this section, we introduce a partial order of financial institutions in terms of their contribution to systemic risk. We consider a market composed of n (non-collaborating, competitive) financial institutions. The potential losses over the coming reference period (of one year, say) are denoted by $\underline{X}$. The respective microprudential risk capitals are denoted by $\underline{R}$.

Throughout, the *aggregate Δ_i-loss* in the financial market is defined as

$$S_{\underline{R}} = \sum_{i=1}^{n} \Delta_i \left(X_i - R_i \right), \tag{2}$$

where $\Delta_i : \mathbb{R} \to \mathbb{R}$ is assumed to be increasing and convex such that $\Delta_i(0) = 0$, for $i = 1, \ldots, n$. Reminiscent of actuarial stop-loss contracts and financial put-option contracts, we often focus on the (more specific) *aggregate Δ_i-residual loss* $S_{\underline{R}}$ defined as

$$S_{\underline{R}} = \sum_{i=1}^{n} \Delta_i \left((X_i - R_i)_+ \right). \tag{2'}$$

In this section, we suppose that, for $i = 1, \ldots, n$, $\Delta_i(x) \equiv \Delta(x_+)$ with increasing convex Δ such that $\Delta(0) = 0$, in which case we shall call $S_{\underline{R}}$ the *aggregate Δ-residual loss*, with its expression given by

$$S_{\underline{R}} = \sum_{i=1}^{n} \Delta \left((X_i - R_i)_+ \right). \tag{2''}$$

Examples of Δ include $\Delta(x) = x$, and the exponential loss function $\Delta(x) = \frac{1}{\alpha}[\exp(\alpha x) - 1]$, for $\alpha > 0$. In particular, if $\Delta_i(x) \equiv x_+$ for $i = 1, \ldots, n$, $S_{\underline{R}}$ is called the *aggregate residual loss*, given by

$$S_{\underline{R}} = \sum_{i=1}^{n} (X_i - R_i)_+ . \tag{2'''}$$

Clearly, the path from (2) to (2‴) corresponds to a decreasing degree of generality. Under (2), both the left and right deviations are considered (and may offset one another), whereas under (2′)–(2‴) only the right deviations are taken into account. The individual residual losses due to the right deviations may be measured differently by (2′), but are measured identically by (2″)–(2‴). Finally, under (2‴), the net (i.e., plain) residual losses are considered, whereas under (2)–(2″) large right deviations may be punished more severely than small right deviations.

Now, we introduce the *systemic contribution order* by conditioning on the event that financial market's aggregate residual loss $S_{\underline{R}}$ given in (2‴) exceeds a minimum aggregate loss level $s \geq 0$, i.e., $S_{\underline{R}} > s$. Thus, we consider the base case $\Delta_i(x) \equiv x_+$, for $i = 1, \ldots, n$, which is very meaningful in practice, and represents the net residual loss beyond the microprudential risk capital.

Definition 6 (Systemic contribution order) *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, the aggregate residual loss $S_{\underline{R}}$ defined in (2‴), and the* *aggregate loss level* $s \in \mathbb{R}_+$. *Individual loss* X_j *is said to be smaller in systemic contribution order than individual loss* X_k *under microprudential regulation* $\underline{R}$ *and aggregate loss level* s, *denoted by* $X_j \leq_{(\underline{R},s)-con} X_k$, *if*

$$\left[(X_j - R_j)_+ \mid S_{\underline{R}} > s\right] \leq_{sl} \left[(X_k - R_k)_+ \mid S_{\underline{R}} > s\right].$$

The definition above can be interpreted as follows. The conditional individual residual loss of institution j, given that the aggregate residual loss exceeds level s, is smaller in terms of the stop-loss order than the corresponding conditional individual residual loss of institution k. In that sense, institution k "contributes more" to the aggregate shortfall in the financial system, under an aggregate residual loss that is strictly larger than s. A generalization of the systemic contribution order defined above can be found in Definition 35 of Appendix B, which is based on the aggregate Δ-residual loss $S_{\underline{R}}$ taking the form (2″).

Let us now introduce the indicator variable $I(\underline{R}, s)$, which equals 1 if the aggregate residual loss $S_{\underline{R}}$ exceeds s and 0 otherwise:

$$I(\underline{R}, s) = \begin{cases} 0, & \text{if } S_{\underline{R}} \leq s, \\ 1, & \text{if } S_{\underline{R}} > s. \end{cases}$$

Based on this indicator variable, we employ the conditional stop-loss order "$\leq_{I(\underline{R},s)-sl}$" as defined in Section 3.1 to establish a relation between the systemic contribution order and the conditional stop-loss order. The next theorem states that the conditional order "$\leq_{I(\underline{R},s)-sl}$" between $(X_j - R_j)_+$ and $(X_k - R_k)_+$ implies the systemic contribution order "$\leq_{(\underline{R},s)-con}$" between X_j and X_k given in Definition 6.

Theorem 7 *Consider the financial system* $\underline{X}$, *the microprudential regulation* $\underline{R}$, *the aggregate residual loss* $S_{\underline{R}}$ *defined in (2‴), and the aggregate loss level* $s \in \mathbb{R}_+$. *Then,*

$$(X_j - R_j)_+ \leq_{I(\underline{R},s)-sl} (X_k - R_k)_+ \Longrightarrow X_j \leq_{(\underline{R},s)-con} X_k.$$

Proof. Suppose that $(X_j - R_j)_+ \leq_{I(\underline{R},s)-sl} (X_k - R_k)_+$. This inequality can be rewritten in terms of the following a.s. inequality:

$$\mathbb{E}\left[g\left((X_j - R_j)_+\right) \mid I(\underline{R}, s)\right] \leq \mathbb{E}\left[g\left((X_k - R_k)_+\right) \mid I(\underline{R}, s)\right],$$

which has to hold for every increasing convex function $g : \mathbb{R} \to \mathbb{R}$. This implies in particular that

$$\mathbb{E}\left[g\left((X_j - R_j)_+\right) \mid I(\underline{R}, s) = 1\right] \leq \mathbb{E}\left[g\left((X_k - R_k)_+\right) \mid I(\underline{R}, s) = 1\right]$$

holds for every increasing convex g. Hence, we can conclude that $X_j \leq_{(\underline{R},s)-con} X_k$. ■

As a special case of Definition 6, we consider the systemic contribution order based on an event that at least one market participant exhibits a shortfall, i.e., $S_{\underline{R}} > 0$.

Definition 8 (Systemic contribution order—special case) *Consider the financial system $\underline{X}$, the corresponding microprudential regulation $\underline{R}$, and the aggregate residual loss $S_{\underline{R}}$ defined in (2‴). Individual loss X_j is said to be smaller in systemic contribution order than individual loss X_k under microprudential regulation $\underline{R}$, denoted by $X_j \leq_{\underline{R}-con} X_k$, if*

$$\left[(X_j - R_j)_+ \mid S_{\underline{R}} > 0\right] \leq_{sl} \left[(X_k - R_k)_+ \mid S_{\underline{R}} > 0\right].$$

Similar to Definition 6, Definition 8 can be given the following interpretation. The conditional individual residual loss of institution j, given that at least one institution in the market is in financial distress, is smaller than the corresponding conditional individual residual loss of institution k in the sense of the stop-loss order. In that sense, institution k "contributes more" to the aggregate shortfall in the market, under a collapse of at least one of its financial entities.

The next two theorems present necessary and sufficient conditions for the systemic contribution order, which strengthens the result of Theorem 7 for the special case that $s = 0$.

Theorem 9 *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, and the aggregate residual loss $S_{\underline{R}}$ defined in (2‴) with $\mathbb{P}[S_{\underline{R}} > 0] > 0$. Then,*

$$X_j \leq_{\underline{R}-con} X_k \iff (X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+. \tag{3}$$

Proof. The contribution order relation $X_j \leq_{\underline{R}-con} X_k$ is equivalent to

$$\mathbb{E}\left[\left((X_j - R_j)_+ - d\right)_+ \mid S_{\underline{R}} > 0\right] \leq \mathbb{E}\left[\left((X_k - R_k)_+ - d\right)_+ \mid S_{\underline{R}} > 0\right],$$

which has to hold for any d. Taking into account the law of total expectations and the fact that $\mathbb{P}[S_{\underline{R}} > 0] > 0$, the inequality above can be rewritten as

$$\mathbb{E}\left[\left((X_j - R_j)_+ - d\right)_+\right] \leq \mathbb{E}\left[\left((X_k - R_k)_+ - d\right)_+\right], \qquad \text{for any } d.$$

This proves the stated result. ■

Remark 10 *Theorem 9 states that the systemic contribution order* $\leq_{\underline{R}-con}$ *between two market participants is equivalent to the stop-loss order of their respective residual losses. Note, however, that in general there exists no equivalence relation between* $X_j \leq_{(\underline{R},s)-con} X_k$ *and* $(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+$ *if* $s > 0$*. This non-equivalence is illustrated in the following example.*

Consider the losses X_1 *and* X_2 *with respective microprudential risk capitals* R_1 *and* R_2*. Suppose that* X_1 *and* X_2 *are mutually independent. Further, suppose that* $(X_1 - R_1)_+$ *is either equal to* 0 *(with probability* $\frac{1}{2}$*) or* 2*, while* $(X_2 - R_2)_+$ *can take the values* $0, 1$ *and* 2*, with respective probabilities* $\frac{1}{4}, \frac{1}{4}$ *and* $\frac{1}{2}$*. It is then straightforward to prove that*

$$(X_1 - R_1)_+ \leq_{sl} (X_2 - R_2)_+ .$$

(In fact, "$\leq_{sl}$*" can even be replaced by "*$\leq_{st}$*".) Let us now assume that* $s = 2$*. We have that* $\left[(X_1 - R_1)_+ \mid S_{\underline{R}} > 2\right]$ *is equal to* 2*. On the other hand,* $\left[(X_2 - R_2)_+ \mid S_{\underline{R}} > 2\right]$ *equals* $\left[(X_2 - R_2)_+ \mid (X_2 - R_2)_+ \neq 0\right]$*, which can take the values* 1 *or* 2 *with positive probabilities. These observations imply that*

$$\left[(X_2 - R_2)_+ \mid S_{\underline{R}} > 2\right] \leq_{sl} \left[(X_1 - R_1)_+ \mid S_{\underline{R}} > 2\right],$$

or equivalently,

$$X_2 \leq_{(\underline{R},2)-con} X_1.$$

We can conclude that for $s > 0$*, the systemic contribution order* $X_j \leq_{(\underline{R},s)-con} X_k$ *does not imply the stop-loss order* $(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+$*, and vice versa.*

From Remark 10 we have that, in general, there exists no equivalence relation between $X_j \leq_{(\underline{R},s)-con} X_k$ and $(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+$, except when $s = 0$, in which case Theorem 9 applies.

In the sequel, we shall denote "$\leq_{I(\underline{R},s)-sl}$" as "$\leq_{I(\underline{R})-sl}$" for the special case $s = 0$. Next, we prove that the conditional stop-loss order "$\leq_{I(\underline{R})-sl}$" is equivalent to the systemic contribution order "$\leq_{\underline{R}-con}$".

Theorem 11 *Consider the financial system* $\underline{X}$*, the microprudential regulation* $\underline{R}$*, and the aggregate residual loss* $S_{\underline{R}}$ *defined in (2'''). Then,*

$$X_j \leq_{\underline{R}-con} X_k \Longleftrightarrow (X_j - R_j)_+ \leq_{I(\underline{R})-sl} (X_k - R_k)_+ . \tag{4}$$

Proof. If $\mathbb{P}[S_{\underline{R}} > 0] > 0$, then the "$\Longleftarrow$" implication follows immediately from (1) and Theorem 9. If $\mathbb{P}[S_{\underline{R}} > 0] = 0$, we must have $(X_j - R_j)_+ = 0$ for all $i = 1, \ldots, n$, and thus the "$\Longleftarrow$" implication holds trivially.

Let us now assume that $X_j \leq_{\underline{R}-con} X_k$. This inequality can be rewritten as

$$\mathbb{E}\left[g\left((X_j - R_j)_+\right) \mid I_{\underline{R}} = 1\right] \leq \mathbb{E}\left[g\left((X_k - R_k)_+\right) \mid I_{\underline{R}} = 1\right],$$

which has to hold for every increasing convex function g. On the other hand, we have that

$$\mathbb{E}\left[g\left((X_j - R_j)_+\right) \mid I_{\underline{R}} = 0\right] = g(0) = \mathbb{E}\left[g\left((X_k - R_k)_+\right) \mid I_{\underline{R}} = 0\right]$$

holds for every increasing convex function g. Hence, we can conclude that $X_j \leq_{\underline{R}-con} X_k$ implies that

$$\mathbb{E}\left[g\left((X_j - R_j)_+\right) \mid I_{\underline{R}}\right] \leq \mathbb{E}\left[g\left((X_k - R_k)_+\right) \mid I_{\underline{R}}\right]$$

holds for every increasing convex function g. This means that also the "$\Longrightarrow$" implication holds. ■

5 Systemic relevance order

In this section, we introduce the notion of *systemic relevance order* as a partial order of financial institutions, characterizing one institution to be more systemically relevant than another one.

The systemic relevance order is defined as follows:

Definition 12 (Systemic relevance order) *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, and the aggregate Δ_i-residual loss $S_{\underline{R}}$ defined in (2'). Individual loss X_j is said to be smaller in systemic relevance order (or less systemically relevant) than individual loss X_k under microprudential regulation $\underline{R}$, denoted by $X_j \leq_{\underline{R}-rel} X_k$, if*

$$[S_{\underline{R}} \mid X_j > R_j] \leq_{sl} [S_{\underline{R}} \mid X_k > R_k].$$

The definition above can be interpreted as follows. The conditional aggregate residual loss in the market, given that institution j is in financial distress, is smaller (in terms of stop-loss order) than the conditional aggregate residual loss in the market, given that institution k is in financial distress. In that sense, institution k is "more relevant" for systemic risk in the financial system $\underline{X}$.

It is readily seen, by taking $X = S_{\underline{R}}$, $A = \{X_j > R_j\}$ and $B = \{X_k > R_k\}$, that the systemic relevance order constitutes a "stop-loss information order" in the sense of Section 3.2.

Important to notice is that the "systemic relevance order" does not order in terms of "contribution" of the institution to the aggregate residual loss. A small institution could, for example, be very "relevant" in terms of systemic risk in the sense that the failure of the institution is highly "connected" to the failure of the big players in the market, but at the same time this small institution itself will not contribute heavily to the aggregate loss. The following theoretical example illustrates this fact.

Example 13 *Let (X_1, X_2) be a discrete random vector such that $p_{00} = \mathbb{P}[X_1 = 0, X_2 = 0] = 0.1$, $p_{01} = \mathbb{P}[X_1 = 0, X_2 = 3] = 0.3$, $p_{10} = \mathbb{P}[X_1 = 1, X_2 = 0] = 0.1$ and $p_{11} = \mathbb{P}[X_1 = 1, X_2 = 3] = 0.5$. Suppose the microprudential risk capitals R_1 and R_2 for X_1 and X_2 are given by $R_1 = 0.9$ and $R_2 = 2.8$. Assume that $s \in [0, 0.1)$ and $\Delta_i(x) = x_+$, for $i = 1, \ldots, n$, which means that $S_{\underline{R}}$ is defined by (2‴). We have that*

$$\mathbb{P}[(X_1 - R_1)_+ > t | S_{\underline{R}} > s] = \begin{cases} \frac{p_{10}+p_{11}}{1-p_{00}}, & \text{for } t \in [0, 0.1]; \\ 0, & \text{for } t \in (0.1, +\infty]. \end{cases}$$

Similarly,

$$\mathbb{P}[(X_2 - R_2)_+ > t | S_{\underline{R}} > s] = \begin{cases} \frac{p_{01}+p_{11}}{1-p_{00}}, & \text{for } t \in [0, 0.2]; \\ 0, & \text{for } t \in (0.2, +\infty]. \end{cases}$$

From the expressions above, we find that

$$\mathbb{P}[(X_1 - R_1)_+ > t | S_{\underline{R}} > s] \le \mathbb{P}[(X_2 - R_2)_+ > t | S_{\underline{R}} > s], \quad \text{for all } t \in \mathbb{R}_+,$$

which is equivalent to

$$[(X_1 - R_1)_+ | S_{\underline{R}} > s] \le_{st} [(X_2 - R_2)_+ | S_{\underline{R}} > s] \, .$$

Since stochastic dominance order implies stop-loss order, we find that $X_1 \le_{(\underline{R},s)-con} X_2$.

On the other hand, denoting the survival functions of $[S_{\underline{R}} | X_1 > R_1]$ and $[S_{\underline{R}} | X_2 > R_2]$ by $\overline{F}_1$ and $\overline{F}_2$, respectively, we find that

$$\overline{F}_1(t) = \begin{cases} 1, & \text{for } t \in [0, 0.1]; \\ \frac{p_{11}}{p_{10}+p_{11}} = \frac{5}{6}, & \text{for } t \in (0.1, 0.3]; \\ 0, & \text{for } t \in (0.3, +\infty), \end{cases}$$

and

$$\overline{F}_2(t) = \begin{cases} 1, & \text{for } t \in [0, 0.2]; \\ \frac{p_{11}}{p_{01}+p_{11}} = \frac{5}{8}, & \text{for } t \in (0.2, 0.3]; \\ 0, & \text{for } t \in (0.3, +\infty). \end{cases}$$

It is straightforward to prove that $\int_t^\infty \overline{F}_1(x)\mathrm{d}x \geq \int_t^\infty \overline{F}_2(x)\mathrm{d}x$*, for any* $t \in \mathbb{R}_+$*, which means that*

$$[S_{\underline{R}}|X_1 > R_1] \geq_{sl} [S_{\underline{R}}|X_2 > R_2] \,.$$

In other words, we have found that $X_2 \leq_{\underline{R}-rel} X_1$*. We can conclude that* X_1 *is smaller than* X_2 *in systemic contribution order, but that at the same time* X_1 *is larger than* X_2 *in systemic relevance order under regulation* $\underline{R}$*. This means that* X_1 *is more systemically relevant, but in case of a market shortfall (*$S_{\underline{R}} > 0$*), it contributes less to the aggregate residual loss.*

Let us now consider some special dependence structures among the losses in the financial system, and analyze how these dependence structures play a role in relation to the systemic relevance order. We will call the event "$X_i > R_i$" ruin and $\mathbb{P}[X_i > R_i]$ the ruin probability, where "ruin" should be understood as "shortfall". Hereafter, we assume that the distributions of losses are continuous and strictly increasing to avoid unnecessary technical complications.

Theorem 14 (Comonotonic losses with identical ruin probabilities) *Consider the financial system* $\underline{X}$*, the microprudential regulation* $\underline{R}$*, and the aggregate* Δ_i*-residual loss* $S_{\underline{R}}$ *defined in (2). Suppose that* $(X_j, X_k) = \left(F_{X_j}^{-1}(U), F_{X_k}^{-1}(U)\right)$ *with* U *a standard uniform r.v., and* $F_{X_j}(R_j) = F_{X_k}(R_k)$*, for some* $1 \leq j \neq k \leq n$*. Then,*

$$[S_{\underline{R}} \mid X_j > R_j] = [S_{\underline{R}} \mid X_k > R_k] \,.$$

Proof. The event that institution j ruins, i.e., $X_j > R_j$, can be equivalently expressed as $U > F_{X_j}(R_j)$. Indeed,

$$X_j > R_j \iff F_{X_j}^{-1}(U) > R_j \iff U > F_{X_j}(R_j).$$

Similarly, we have that $X_k > R_k \iff U > F_{X_k}(R_k)$. Hence, the ruin of institution j is equivalent to the ruin of institution k due to $F_{X_j}(R_j) = F_{X_k}(R_k)$. ■

The theorem above states that in a market with microprudential risk capitals determined as VaRs at the same probability level (thus yielding the same ruin probabilities), any two institutions that are comonotonic are equally relevant in terms of systemic risk. In particular, we have that in a VaR-based comonotonic market, all institutions are equally relevant in terms of systemic risk provided that all microprudential risk capitals have the same confidence levels. This observation is in line with intuition.

Next, we revisit Theorem 14 when the confidence levels for microprudential regulations are different. In the following theorem, we will use the hazard rate order, notation "$\leq_{hr}$", which is defined in the Appendix.

Theorem 15 (Comonotonic losses with different ruin probabilities) *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, and the aggregate Δ_i-residual loss $S_{\underline{R}}$ defined in (2). Suppose that $(X_1, \ldots, X_n) = \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$ with U standard uniform. If $F_{X_j}(R_j) \geq F_{X_k}(R_k)$ for some $1 \leq j \neq k \leq n$, then*

$$[S_{\underline{R}} \mid X_j > R_j] \leq_{hr} [S_{\underline{R}} \mid X_k > R_k] .$$

Proof. Note that $\{X_j > R_j\} = \{F_{X_j}^{-1}(U) > R_j\} = \{U > F_{X_j}(R_j)\}$ and $\{X_k > R_k\} = \{U > F_{X_k}(R_k)\}$. Thus, it follows that $\{X_j > R_j\} \subseteq \{X_k > R_k\}$. According to Theorem 1.B.20 of Shaked and Shanthikumar (2007), it holds that

$$[U \mid X_j > R_j] = [U \mid U > F_{X_j}(R_j)] \leq_{hr} [U \mid U > F_{X_k}(R_k)] = [U \mid X_k > R_k].$$

Since $S_{\underline{R}}$ is an increasing function of U, from Theorem 1.B.2 of Shaked and Shanthikumar (2007) we have

$$[S_{\underline{R}} \mid X_j > R_j] \leq_{hr} [S_{\underline{R}} \mid X_k > R_k] ,$$

which yields the desired result. ■

It should be mentioned that the result of Theorem 15 can be strengthened to the likelihood ratio order (Section 1.C in Shaked and Shanthikumar, 2007) by using Theorems 1.C.8 and 1.C.27 in Shaked and Shanthikumar (2007). Since the hazard rate order implies the stop-loss order, the theorem above implies that in a VaR-based comonotonic market with microprudential risk capitals having different confidence levels, the institution with the lower confidence level is more systemically relevant than the institution with the larger confidence level.

Next, we consider the situation where the market consists of two identically distributed counter-monotonic losses having a common microprudential regulation, and where the regulator uses the same measurement function for each loss.

Theorem 16 (Counter-monotonic losses with same ruin probability) *Consider a market of two losses* $(X_1, X_2) = (F^{-1}(U), F^{-1}(1-U))$ *for some cdf* F *with* U *a standard uniform r.v., with microprudential regulation* $\underline{R} = (R, R)$ *where* $\frac{1}{2} < F(R) < 1$*, and with the aggregate* Δ*-residual loss* $S_{\underline{R}}$ *defined in (2′). Then,*

$$[S_{\underline{R}} \mid X_1 > R] \overset{d}{=} [S_{\underline{R}} \mid X_2 > R] .$$

Proof. The event that institution 1 ruins, i.e., $X_1 > R$, can be equivalently expressed as $U > F(R)$. Indeed,

$$X_1 > R \Longleftrightarrow F^{-1}(U) > R \Longleftrightarrow U > F(R).$$

Similarly, the event that institution 2 fails, i.e., $X_2 > R$, can be equivalently expressed as $U < 1 - F(R)$. Since $\frac{1}{2} < F(R) < 1$, we have $1 - F(R) < F(R)$. Hence, we find that

$$\begin{aligned}
[S_{\underline{R}} \mid X_1 > R] &= [S_{\underline{R}} \mid U > F(R)] \\
&= [\Delta (X_1 - R) \mid U > F(R)] \\
&= \left[\Delta \left(F^{-1}(U) - R\right) \mid U > F(R)\right] .
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
[S_{\underline{R}} \mid X_2 > R] &= [S_{\underline{R}} \mid U < 1 - F(R)] \\
&= [\Delta (X_2 - R) \mid U < 1 - F(R)] \\
&= \left[\Delta \left(F^{-1}(1-U) - R\right) \mid 1 - U > F(R)\right] .
\end{aligned}$$

This proves the stated result. ∎

The theorem above states that in a market consisting of two institutions with counter-monotonic but identically distributed risks and identical microprudential regulation, each institution is equally relevant in terms of systemic risk, which is also in line with intuition.

Let us now consider a market consisting of two institutions with counter-monotonic and identically distributed stochastic losses, but with different microprudential risk capitals. Recall that a r.v. X or its distribution F is said to be IFR [DFR] if $-\log \overline{F}(x)$ is convex [concave] (see p. 31 in Denuit et al., 2005).

Theorem 17 (Counter-monotonic losses with different ruin probabilities) *Consider a market of two losses* $(X_1, X_2) = (F^{-1}(U), F^{-1}(1-U))$ *for some cdf* F *with* U *a standard uniform r.v., with microprudential regulation* $\underline{R} = (R_1, R_2)$ *where* $0 < F(R_1), F(R_2) < 1$

such that $F(R_1) + F(R_2) > 1$ and $R_1 > R_2$. Furthermore, consider the aggregate Δ-residual loss $S_{\underline{R}}$ defined in (2′). If F is IFR [DFR], then

$$[S_{\underline{R}} \mid X_1 > R_1] \leq_{hr} [\geq_{hr}] [S_{\underline{R}} \mid X_2 > R_2] .$$

Proof. Note that $X_1 > R_1 \iff U > F(R_1)$ and $X_2 > R_2 \iff U < 1 - F(R_2)$. Then, from $F(R_1) + F(R_2) > 1$, it follows that

$$\begin{aligned}[S_{\underline{R}} \mid X_1 > R_1] &= [S_{\underline{R}} \mid U > F(R_1)] \\ &= [\Delta (X_1 - R_1) \mid U > F(R_1)] \\ &= \left[\Delta \left(F^{-1}(U) - R_1\right) \mid F^{-1}(U) > R_1\right]\end{aligned}$$

and

$$\begin{aligned}[S_{\underline{R}} \mid X_2 > R_2] &= [S_{\underline{R}} \mid U < 1 - F(R_2)] \\ &= [\Delta (X_2 - R_2) \mid U < 1 - F(R_2)] \\ &= \left[\Delta \left(F^{-1}(1-U) - R_2\right) \mid 1 - U > F(R_2)\right] \\ &\stackrel{d}{=} \left[\Delta \left(F^{-1}(U) - R_2\right) \mid U > F(R_2)\right] \\ &= \left[\Delta \left(F^{-1}(U) - R_2\right) \mid F^{-1}(U) > R_2\right] .\end{aligned}$$

Taking into account that $R_1 > R_2$, the desired result follows from Theorem 1.B.38 and Theorem 1.B.2 in Shaked and Shanthikumar (2007). ■

If F has a log-concave density, the result "$\leq_{hr}$" in Theorem 17 can be strengthened to the likelihood ratio order by using Theorem 1.C.52 of Shaked and Shanthikumar (2007). The theorem above states that in a market of two institutions with counter-monotonic identically IFR-distributed risks with different risk capitals, the institution with the larger microprudential risk capital is less systemically relevant than the other institution. The opposite conclusion holds for DFR-distributed risks.

To conclude this subsection, we present sufficient conditions for independent individual financial institutions to be ordered in the systemic relevance order sense.

Theorem 18 (Independent losses with identical ruin probabilities) *Consider a market of mutually independent losses $\underline{X}$, microprudential regulation $\underline{R}$, and aggregate Δ-residual loss $S_{\underline{R}}$ as defined in (2′). If $F_{X_j}(R_j) = F_{X_k}(R_k)$ for some $1 \leq j \neq k \leq n$, then*

$$(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+ \Longrightarrow X_j \leq_{\underline{R}-rel} X_k. \quad (5)$$

Proof. We introduce the notation

$$S_{R_j,R_k} = \Delta\left((X_j - R_j)_+\right) + \Delta\left((X_k - R_k)_+\right).$$

From the stop-loss ordering relation (5) and the increasingness and convexity of Δ, we find that

$$\Delta\left((X_j - R_j)_+\right) \leq_{sl} \Delta\left((X_k - R_k)_+\right).$$

This observation implies that

$$\begin{aligned}
&\left(1 - F_{X_j}(R_j)\right) \mathbb{E}\left[\left(S_{R_j,R_k} - d\right)_+ \mid X_j > R_j\right] \\
&= \mathbb{E}\left[\left(S_{R_j,R_k} - d\right)_+\right] - F_{X_j}(R_j)\mathbb{E}\left[\left(S_{R_j,R_k} - d\right)_+ \mid X_j \leq R_j\right] \\
&= \mathbb{E}\left[\left(S_{R_j,R_k} - d\right)_+\right] - F_{X_j}(R_j)\mathbb{E}\left[\left(\Delta\left((X_k - R_k)_+\right) - d\right)_+\right] \\
&\leq \mathbb{E}\left[\left(S_{R_j,R_k} - d\right)_+\right] - F_{X_k}(R_k)\mathbb{E}\left[\left(\Delta\left((X_j - R_j)_+\right) - d\right)_+\right] \\
&= \left(1 - F_{X_k}(R_k)\right) \mathbb{E}\left[\left(S_{R_j,R_k} - d\right)_+ \mid X_k > R_k\right].
\end{aligned}$$

Hence, we have proven that $\left[S_{R_j,R_k} \mid X_j > R_j\right] \leq_{sl} \left[S_{R_j,R_k} \mid X_k > R_k\right]$. Taking into account the mutual independence between the components of $\underline{X}$, we find that

$$\begin{aligned}
\left[S_{\underline{R}} \mid X_j > R_j\right] &= \left[\sum_{\substack{i=1 \\ i \neq j,k}}^{n} \Delta\left((X_i - R_i)_+\right) + S_{R_j,R_k} \mid X_j > R_j\right] \\
&\leq_{sl} \left[\sum_{\substack{i=1 \\ i \neq j,k}}^{n} \Delta\left((X_i - R_i)_+\right) + S_{R_j,R_k} \mid X_k > R_k\right] \\
&= \left[S_{\underline{R}} \mid X_k > R_k\right],
\end{aligned}$$

which proves that $X_j \leq_{\underline{R}-rel} X_k$. ■

The theorem above states the intuitive result that in a market of players with mutually independent losses, for any two institutions with microprudential VaR-based risk capitals at the same confidence levels, the institution with the largest individual residual risk (in terms of stop-loss order) is more systemically relevant.

Remark 19 *In Theorems 14-18, we have derived sufficient conditions for the systemic relevance order based on the idea to restate the aggregate market event "$S_{\underline{R}} > s$" in a* *simpler form, making use of the institution-specific conditioning events "$X_j > R_j$", for $j = 1, \ldots, n$. For both orders—systemic contribution order and systemic relevance order—there are a r.v. as well as a conditioning event on both the right-hand side (RHS) and left-hand side (LHS) of the respective inequalities defining these orders. For the systemic relevance order, the r.v.'s are equal on the RHS and the LHS, but the conditioning event is different. For the systemic contribution order, the r.v.'s are different on the LHS and the RHS, but the conditioning event is the same. Notice that a similar restating procedure as for the systemic relevance order is in general not possible for the systemic contribution order, as the individual r.v.'s $(X_j - R_j)_+$ cannot be restated based on the market-based conditioning event "$S_{\underline{R}} > s$".*

6 Systemic aggregation order

In this section, we define and investigate the *systemic aggregation order*. This stochastic order allows us to compare financial systems which only differ in terms of the dependence structure between the stochastic losses.

6.1 Definition

Consider the financial market's aggregate Δ_i-losses defined in (2). We state the following definition.

Definition 20 (Systemic aggregation order) *Consider two financial systems $\underline{X}$ and $\underline{Y}$, both elements of the Fréchet space $\mathcal{R}(F_1, \ldots, F_n)$. Furthermore, consider the microprudential regulation given by $\underline{R}$. Then, $\underline{X}$ is said to be smaller than $\underline{Y}$ in systemic aggregation order under microprudential regulation $\underline{R}$, denoted by $\underline{X} \leq_{\underline{R}-sa} \underline{Y}$, if the aggregate Δ_i-loss of $\underline{X}$ is smaller in stop-loss order than the aggregate Δ_i-loss of $\underline{Y}$:*

$$\underline{X} \leq_{\underline{R}-sa} \underline{Y} \quad \Longleftrightarrow \quad \sum_{i=1}^{n} \Delta_i \left(X_i - R_i\right) \leq_{sl} \sum_{i=1}^{n} \Delta_i \left(Y_i - R_i\right). \tag{6}$$

Furthermore, we say that $\underline{X}$ is smaller than $\underline{Y}$ in systemic aggregation order, denoted by $\underline{X} \leq_{sa} \underline{Y}$, if $\underline{X} \leq_{\underline{R}-sa} \underline{Y}$ for any $\underline{R}$.

The systemic contribution order and the systemic relevance order, which were introduced in Sections 4 and 5, respectively, partially order financial institutions in a given financial system (i.e., a system of n institutions with given individual loss distributions and given dependence structure connecting these individual loss distributions). On the other hand, the systemic aggregation order introduced in Definition 20 considers the Fréchet space of all financial systems with fixed individual loss distributions. This order allows us to partially order financial systems in the given Fréchet space based on the dependence structure among the individual losses.

Under the setup of Definition 20, the microprudential regulation $\underline{R}$ is fixed and $\underline{X}$ and $\underline{Y}$ have the same marginals for the systemic aggregation order "$\leq_{\underline{R}-sa}$". Hence, $\underline{X} \leq_{\underline{R}-sa} \underline{Y}$ means that $\underline{Y}$ is more (positively) dependent and thus leads to a larger aggregate Δ_i-loss in the stop-loss order sense. In the absence of macroprudential regulation, the financial markets $\underline{X}$ and $\underline{Y}$, with $\underline{X} \leq_{\underline{R}-sa} \underline{Y}$, are treated equally, while obviously, the second situation is the more dangerous one. In order to overcome this inconsistency, one should set up a regulation with a microprudential as well as a macroprudential policy.

The definition of systemic aggregation order can be extended to compare random vectors of a broader class in case the micro-level capital requirements R_i's only depend on an upper tail of the distribution. Suppose that R_i only depends on $\left\{F_i^{-1}(q) \mid q \geq p\right\}$. Then we could define the "$\leq_{\underline{R}-sa}$" order between members of the class of all n-dimensional distributions with fixed tails $\left\{\overline{F}_i(x_i) \mid x_i > F_i^{-1}(p)\right\}$, $i = 1, 2, \ldots, n$.

6.2 Some basic properties of the systemic aggregation order

In this subsection, we present some sufficient conditions imposed on the random vectors $\underline{X}$ and $\underline{Y}$ in the same Fréchet space to be ordered in the systemic aggregation order. We first note that $\underline{X} \leq_{sa} \underline{Y}$, that is, $\underline{X} \leq_{\underline{R}-sa} \underline{Y}$ for any $\underline{R}$, is equivalent to requiring that $\sum_{i=1}^n f_i(X_i) \leq_{sl} \sum_{i=1}^n f_i(Y_i)$, for any increasing and convex f_i, which is a natural generalization (strengthening) of $\sum_{i=1}^n X_i \leq_{sl} \sum_{i=1}^n Y_i$. Next, we state the following theorem.

Theorem 21 *Assume that $\underline{X}, \underline{Y} \in \mathcal{R}(F_1, \ldots, F_n)$. Then, we have that:*

(i) Systemic risk is highest when the losses in the market are comonotonic:

$$\underline{X} \leq_{sa} \left(F_1^{-1}(U), \ldots, F_n^{-1}(U)\right).$$

(ii) Supermodular order[7] implies systemic aggregation order:

$$\underline{X} \leq_{sm} \underline{Y} \Longrightarrow \underline{X} \leq_{sa} \underline{Y}.$$

[7] See Definition 6.3.1 in Denuit et al. (2005) and also Definition 32 in the Appendix.

(iii) Multivariate stop-loss order[8] *implies systemic aggregation order:*

$$\underline{X} \leq_{sl} \underline{Y} \Longrightarrow \underline{X} \leq_{sa} \underline{Y}.$$

(iv) A more strict micro-level regulation leads to a smaller aggregate Δ_i-loss:

$$\underline{R} \leq \underline{R}' \Longrightarrow \sum_{i=1}^{n} \Delta_i(X_i - R_i') \leq \sum_{i=1}^{n} \Delta_i \left(X_i - R_i\right).$$

Proof. For (i), it is easy to see that $(F_1^{-1}(U), \ldots, F_n^{-1}(U))$ is a comonotonic vector contained in $\mathcal{R}(F_1, \ldots, F_n)$, which implies that $(\Delta_1(F_1^{-1}(U) - R_1), \ldots, \Delta_n(F_n^{-1}(U) - R_n))$ is the comonotonic counterpart of $(\Delta_1(X_1 - R_1), \ldots, \Delta_n(X_n - R_n))$. Hence, the desired result follows from the fact that the sum of a vector in the Fréchet space is maximized (in terms of the convex order) in the comonotonic case; see Corollary 3.4.30 in Denuit et al. (2005).

By using Definition 20, the proofs of (ii) and (iii) follow from Propositions 6.3.9 and 3.4.65 in Denuit et al. (2005).

The proof of (iv) is easily obtained from the increasingness of Δ_i and the condition that $R_i' \leq R_i$, for $i = 1, \ldots, n$. ■

The systemic aggregation order can be identified as a specific version of the multivariate stop-loss order when confined to the Fréchet space and in which the increasing convex functions $g : \mathbb{R}^n \to \mathbb{R}$ are restricted to the class of linear-convex functions. Interestingly, Koshevoy and Mosler (1996, 1997, 1998) restrict to the complementary case of convex-linear functions in a similar setting.

Recall that $\underline{X}$ is said to be $\underline{R}$-upper comonotonic if $((X_1 - R_1)_+, \ldots, (X_n - R_n)_+)$ is comonotonic. This concept and related properties were introduced and discussed in Cheung (2009), Dong et al. (2010) and Nam et al. (2011). Let us for the moment suppose that $R_i = F_i^{-1}(p)$, for some $p \in (0, 1)$. Then, the most dangerous situation (in terms of systemic risk) occurs when $\underline{X}$ is upper comonotonic at level p. In this case, a shortfall of one of the institutions in the sense that $X_i > F_i^{-1}(p)$, is accompanied by a shortfall of any institution. Again, in existing regulation where only a microprudential capital requirement applies, no distinction is made between this "explosive" situation and the less frightening situation where all tails (beyond $F_i^{-1}(p)$) are independent.

For special loss measurement functions, weaker requirements already yield special partial ordering results. This is shown in the following proposition.

[8] See Definition 3.4.59 in Denuit et al. (2005) and also Definition 33 in the Appendix.

Proposition 22 *Suppose $\Delta_i(x) = \Delta_i(x_+)$, for $i = 1, \ldots, n$, such that (2') occurs and $\underline{X}, \underline{Y} \in \mathcal{R}(F_1, \ldots, F_n)$. Then, for any given microprudential regulation $\underline{R}$, we have that*

(i) Systemic risk is highest when the losses in the market are upper-comonotonic:

$$\sum_{i=1}^{n} \Delta_i \left((X_i - R_i)_+ \right) \leq_{sl} \sum_{i=1}^{n} \Delta_i \left((X_i - R_i)_+^c \right).$$

(ii) "Upper" supermodular order implies systemic aggregation order:

$$\underline{(X - R)_+} \leq_{sm} \underline{(Y - R)_+} \Longrightarrow \underline{X} \leq_{\underline{R}-sa} \underline{Y}.$$

(iii) "Upper" multivariate stop-loss order implies systemic risk order:

$$\underline{(X - R)_+} \leq_{sl} \underline{(Y - R)_+} \Longrightarrow \underline{X} \leq_{\underline{R}-sa} \underline{Y}.$$

Proof. (i) holds as a consequence of Theorem 1 in Dong et al. (2010). (ii) and (iii) follow from Propositions 6.3.9 and 3.4.65 in Denuit et al. (2005), respectively. ■

Note that the systemic aggregation order in Definition 20 is defined under the assumption "$\underline{X}, \underline{Y} \in \mathcal{R}(F_1, \ldots, F_n)$". It is natural to extend this definition to the general case when $\underline{X}$ and $\underline{Y}$ have non-identical marginal distributions. For this generalized definition of systemic aggregation order, Proposition 23 can be derived similarly.

Proposition 23 *For the generalized systemic aggregation order in Definition 20 (that is, $\underline{X}, \underline{Y} \in \mathcal{R}(F_1, \ldots, F_n)$), the following results hold:*

(i) Multivariate stop-loss order implies systemic aggregation order:

$$\underline{X} \leq_{sl} \underline{Y} \Longrightarrow \underline{X} \leq_{sa} \underline{Y}.$$

(ii) For two sequences of independent losses $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ such that $X_i \leq_{sl} Y_i$, for $i = 1, \ldots, n$, we have $\underline{X} \leq_{sa} \underline{Y}$.

(iii) "Upper" multivariate stop-loss order implies systemic aggregation order:

$$\underline{(X - R)_+} \leq_{sl} \underline{(Y - R)_+} \Longrightarrow \underline{X} \leq_{\underline{R}-sa} \underline{Y}.$$

Proof. The proofs of (i) and (ii) follow from Proposition 3.4.65 in Denuit et al. (2005). Now, we prove (ii). Note that $X_i - R_i \leq_{sl} Y_i - R_i$, $i = 1, \ldots, n$, for any $\underline{R}$, which leads to $\Delta_i(X_i - R_i) \leq_{sl} \Delta_i(Y_i - R_i)$, for $i = 1, \ldots, n$, by applying the Theorem 4.A.8(a) in Shaked and Shanthikumar (2007). Since both vectors $(\Delta_1(X_1 - R_1), \ldots, \Delta_n(X_n - R_n))$ and $(\Delta_1(Y_1 - R_1), \ldots, \Delta_n(Y_n - R_n))$ are independent, it immediately follows from Theorem 4.A.8(d) in Shaked and Shanthikumar (2007) that $\sum_{i=1}^n \Delta_i(X_i - R_i) \leq_{sl} \sum_{i=1}^n \Delta_i(Y_i - R_i)$, which yields the desired result. ■

6.3 Majorization, diversity and systemic risk

The notion of majorization, characterizing the diversity of coordinates of real vectors, is useful in establishing various inequalities arising naturally in actuarial science, applied probability as well as reliability theory. Let $x_{1:n} \leq \cdots \leq x_{n:n}$ be the increasing arrangement of the components of the vector $\underline{x} = (x_1, \ldots, x_n)$.

Definition 24 *A vector $\underline{x} \in \mathbb{R}^n$ is said to majorize another vector $\underline{y} \in \mathbb{R}^n$, denoted by $\underline{x} \overset{m}{\succeq} \underline{y}$, if $\sum_{i=1}^j x_{i:n} \leq \sum_{i=1}^j y_{i:n}$ for $j = 1, \ldots, n-1$, and $\sum_{i=1}^n x_{i:n} = \sum_{i=1}^n y_{i:n}$.*

Employing the concept of majorization, Pan et al. (2015) studied stochastic properties of the random sum $\sum_{i=1}^n \phi(X_i, a_i)$ in the sense of the stochastic dominance order and the stop-loss order, under some additional conditions stipulating that the joint density $f_{\underline{X}}(\underline{x})$ of the r.v.'s X_i is arrangement increasing (AI; see A.2.2 in the Appendix), where ϕ is a bivariate function and a_i is an indexing parameter of the r.v. X_i, for $i = 1, \ldots, n$. In particular, Pan et al. (2015) showed that $\underline{a} \overset{m}{\succeq} \underline{b}$ implies $\sum_{i=1}^n \phi(X_i - a_{(n-i+1):n}) \geq_{st} \sum_{i=1}^n \phi(X_i - b_{(n-i+1):n})$ provided that $f_{\underline{X}}(\underline{x})$ is log-concave, arrangement increasing and ϕ is a convex function. If $f_{\underline{X}}(\underline{x})$ is only arrangement increasing (but not necessarily log-concave) and ϕ is convex, then $\underline{a} \overset{m}{\succeq} \underline{b}$ implies $\sum_{i=1}^n \phi(X_i - a_{(n-i+1):n}) \geq_{sl} \sum_{i=1}^n \phi(X_i - b_{(n-i+1):n})$.

The notion of stochastic arrangement increasing (SAI; see Definition 30 in the Appendix) was introduced in Cai and Wei (2014). It depicts not only a positive dependence structure of the components of a random vector, but also orders the components in some stochastic sense. For an absolutely continuous random vector, SAI is equivalent to the statement that the joint density function is arrangement increasing.[9] Right tail weakly

[9]Many multivariate distributions have an AI density, including the multivariate versions of the Dirichlet distribution, the inverted Dirichlet distribution, the F distribution, and the Pareto distribution of type I (see Hollander et al., 1977).

stochastic arrangement increasing (RWSAI; see Definition 30 in the Appendix), which is weaker than SAI, can be viewed as a multivariate generalization of the joint hazard rate order, and was also introduced by Cai and Wei (2014). Both notions are useful to describe dependence structures among random variables arising from various research areas and have attracted considerable attention. For further discussions and applications of these concepts and other related dependence notions, we refer to Cai and Wei (2014, 2015) and Zhang et al. (2018); see also the Appendix. Recently, the above result of Pan et al. (2015) was generalized to the case of RWSAI $\underline{X}$ in Proposition 5.1 of You and Li (2015).

From the point of view of the regulator, it is important to effectively reduce the aggregate Δ-loss and seek to assign or allocate the microprudential risk capital in an efficient manner. In this respect, we next analyze the effect of heterogeneity (i.e., diversity) within various allocation policies on the aggregate Δ-loss. Consider two different configurations of the microprudential regulation, $\underline{R} = (R_1, \ldots, R_n)$ and $\underline{R}' = (R'_1, \ldots, R'_n)$, where $\sum_{i=1}^n R_i = \sum_{i=1}^n R'_i$. Then, by applying Corollary 3.8, Theorem 3.12(ii) in Pan et al. (2015), and Proposition 5.1 of You and Li (2015), the following result is obtained if $\Delta_i \equiv \Delta$ is increasing and convex, for $i = 1, \ldots, n$.

Theorem 25 *Let $\underline{R} = (R_1, \ldots, R_n)$ and $\underline{R}' = (R'_1, \ldots, R'_n)$, while $S_{\underline{R}} = \sum_{i=1}^n \Delta(X_i - R_i)$ and $S_{\underline{R}'} = \sum_{i=1}^n \Delta(X_i - R'_i)$. Assume that $R_1 \geq R_2 \geq \ldots \geq R_n$.*

(i) If $f_{\underline{X}}(\underline{x})$ is log-concave and AI, then $\underline{R} \overset{m}{\succeq} \underline{R}'$ implies $S_{\underline{R}} \geq_{st} S_{\underline{R}'}$.

(ii) If $\underline{X}$ is RWSAI, then $\underline{R} \overset{m}{\succeq} \underline{R}'$ implies $S_{\underline{R}} \geq_{sl} S_{\underline{R}'}$.

Proof. Since Δ is increasing and convex, we have that $\Delta(x - y)$ is submodular in (x, y). By using a similar proof as the one of Proposition 3.7 in Pan et al. (2015), while taking into account the results in Theorem 3.12(ii) in Pan et al. (2015) and Proposition 5.1 of You and Li (2015), the desired result is obtained. ■

The conditions (i) and (ii) in the theorem above mean that the n losses are arrayed in ascending order. If a stochastically larger loss is accompanied by a smaller risk capital, more heterogeneity among the risk capitals leads to greater aggregate Δ-loss in the sense of the stochastic dominance order or the stop-loss order.

6.4 Aggregate loss and conditioning events

In this subsection, we discuss the stochastic properties of the aggregate Δ_i-loss when conditioned on some "systemic risk events". It is common practice to evaluate risks conditionally upon stress scenarios; see also Dhaene et al. (2022). Furthermore, the choice of the risk measure used to evaluate $S_{\underline{R}}$ may include a choice of a conditioning event. From this perspective it is relevant to investigate the behavior of $S_{\underline{R}}$ with respect to conditioning events.

Let $\underline{\Delta(X-R)} = (\Delta_1(X_1 - R_1), \ldots, \Delta_n(X_n - R_n))$. We prove the following theorem.

Theorem 26 *The following statements hold:*

(i) $\underline{X}$ *is associated*[10] $\Longleftrightarrow \underline{\Delta(X-R)}$ *is associated for all* $\underline{\Delta}$.

Let $\underline{X}$ *be associated. Then:*

(ii) $S_{\underline{R}} \leq_{st} [S_{\underline{R}}|A]$, *for aggregate* Δ_i*-loss* $S_{\underline{R}}$ *defined in (2) and systemic risk event* $A = \{\Delta_i (X_i - R_i) > s,$ *for some* $s \geq 0$ *and* $i = 1, 2, \ldots, n\}$.

(iii) $[S_{\underline{R}}|A_1] \leq_{st} [S_{\underline{R}}|A_2]$, *for aggregate* Δ_i*-loss* $S_{\underline{R}}$ *defined in (2) and systemic risk events*

$$A_j = \{\Delta_i (X_i - R_i) > s_j, \textit{for some } s_j \geq 0 \textit{ and } i = 1, 2, \ldots, n\},$$

for $j = 1, 2$ *and* $s_1 < s_2$.

Proof. Proof of (i): If $\underline{X}$ is associated, then $\underline{\Delta(X-R)}$ is also associated since associatedness is preserved under component-wise increasing transformations. The converse follows trivially.

Proof of (ii): Since $\underline{\Delta(X-R)}$ is associated, it follows that $\underline{\Delta(X-R)} \leq_{st} [\underline{\Delta(X-R)}|A]$ for the risk event $A = \{\Delta_i (X_i - R_i) > s, \text{for } i = 1, 2, \ldots, n\}$ by using Theorem 3.1 of Colangelo et al. (2008). Then, the statement is proved by applying Theorem 6.B.16 of Shaked and Shanthikumar (2007).

Proof of (iii): By treating the conditional random vector $[\underline{\Delta(X-R)}|A_1]$ as a new vector, the proof can be established by using a similar argument as in (ii) since $[\underline{\Delta(X-R)}|A_2] = [[\underline{\Delta(X-R)}|A_1]|A_2]$. ■

If we define the conditioning event as "the aggregate Δ_i-loss $S_{\underline{R}}$ exceeds a certain threshold", we obtain the following result.

Theorem 27 *Let* $B = \{S_{\underline{R}} > s, \textit{for some } s \geq 0\}$. *The following statements hold:*

(i) $S_{\underline{R}} \leq_{hr} [S_{\underline{R}}|B]$, *for aggregate* Δ_i*-loss* $S_{\underline{R}}$ *defined in (2).*

[10] See A.2.1 in the Appendix.

(ii) $[S_{\underline{R}}|B_1] \leq_{hr} [S_{\underline{R}}|B_2]$, *for aggregate* Δ_i*-loss* $S_{\underline{R}}$ *defined in (2) and systemic risk events* $B_j = \{S_{\underline{R}} > s_j, \text{for some } s_j \geq 0\}$, *for* $j = 1, 2$ *and* $s_1 < s_2$.

Proof. The proof of (i) follows directly from Theorem 1.B.20 of Shaked and Shanthikumar (2007). For (ii), the proof can be established by using a similar argument as in Theorem 26(iii) according to the result in Theorem 1.B.20 of Shaked and Shanthikumar (2007). ■

The two theorems above introduce additional requirements for systemic risk events such that the aggregate Δ_i-loss increases in the sense of the stochastic dominance order and the hazard rate order, respectively.

7 Conclusions

In this paper, we have introduced the *systemic contribution order* and the *systemic relevance order*, which are useful for stochastically comparing the contributions to, and relevance for, systemic risk of individual financial institutions within a financial system.

The systemic contribution order "$X_j \leq_{(\underline{R},s)-con} X_k$" (base definition) indicates that institution j contributes less systemically than institution k, in case the aggregate residual loss in the financial market, given microprudential regulation $\underline{R}$, exceeds a loss level s. In particular, the systemic contribution order "$\leq_{\underline{R}-con}$" (corresponding to the case $s = 0$) can be characterized in terms of a *conditional stop-loss order* of individual residual risks, as introduced in Christofides and Hadjikyriakou (2015).

On the other hand, the systemic relevance order is an "order of information". Indeed, "$X_j \leq_{\underline{R}-rel} X_k$" means that the information that institution k is in financial distress is more relevant in terms of the aggregate residual loss than the corresponding information about institution j. It occurs as a special case of the new *stop-loss information order* that we introduce in Section 3.2.

We have also introduced the *systemic aggregation order* which can be used to study the effect of the interconnectedness of individual losses in a financial system on the aggregate loss for a given microprudential regulation $\underline{R}$ in terms of the stop-loss order. The systemic aggregation order compares the aggregate loss and involves the dependence structure (copula) for a given Fréchet space, which means that the distributions of the individual losses are fixed.

The three new stochastic orders that we have introduced in this paper may be used to investigate and compare systemic risk in financial institutions and financial systems.

All three stochastic orders invoke the time-honored actuarial stop-loss order but can be straightforwardly adapted to other stochastic orders. The systemic contribution and systemic relevance orders are intimately connected to conditional stochastic orders and stochastic information orders, which, as a contribution of independent interest, are shown to be highly versatile in an insurance and financial context.

As observed from Example 13, a small institution (say, A) contributing less in terms of the systemic contribution order than another one (say, B) may be very relevant to the whole system in the sense of the systemic relevance order. Indeed, this small institution A might be, e.g., highly related to a third big institution, which contributes more than B to the systemic risk in the market. A promising research direction is how to set up both micro- and macroprudential regulations for all individual institutions.

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A Related definitions and useful properties

Throughout this appendix, we adopt the same conventions, setting and notation as in the main text.

A.1 Univariate stochastic ordering

In this subsection, we recall the definitions of some univariate stochastic orders used in this paper.

Definition 28 *Let $\overline{F}_X(x) = 1 - F_X(x)$ and $\overline{F}_Y(y) = 1 - F_Y(y)$ be the survival functions of the r.v.'s X and Y, respectively, and let h_X and h_Y be their hazard rates (i.e., the ratios of the probability density functions to the survival functions). Then, X is said to be smaller than Y in the*

(i) hazard rate order, denoted by $X \leq_{hr} Y$, if $\overline{F}_Y(t)/\overline{F}_X(t)$ is increasing in $t \in \mathbb{R}$, or equivalently, $h_Y(t) \leq h_X(t)$ for all $t \in \mathbb{R}$;

(ii) stochastic dominance order, denoted by $X \leq_{st} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all increasing $\phi : \mathbb{R} \to \mathbb{R}$;

(iii) stop-loss order [or increasing convex order], denoted by $X \leq_{sl} Y$, if $\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$ for all $d \in \mathbb{R}_+$, or equivalently, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all increasing convex $\phi : \mathbb{R} \to \mathbb{R}$;

(iv) convex order, denoted by $X \leq_{cx} Y$, if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $X \leq_{sl} Y$.

It is known that the hazard rate order implies the stochastic dominance order, which in turn implies the stop-loss order. For further details on the properties of the above-mentioned stochastic orders and their applications, we refer to Denuit et al. (2005) and Shaked and Shanthikumar (2007).

A.2 Multivariate stochastic ordering

A.2.1 Measuring dependence

In this subsection, we recall the definitions of some positive dependence notions used in the main text and indicate their connection to other positive dependence notions. A subset $A \subseteq \mathbb{R}^n$ is said to be *comonotonic* if, for any $\underline{x} \in A$ and $\underline{y} \in A$, either $x_i \leq y_i$ for

$i = 1, \ldots, n$ or $x_i \geq y_i$ for $i = 1, \ldots, n$. A random vector $\underline{X}$ is said to be *comonotonic* if there is a comonotonic subset A such that $\mathbb{P}[\underline{X} \in A] = 1$.

Consider n r.v.'s $X_1, \ldots, X_n$. Define $S = \sum_{i=1}^{n} X_i$ and let $S^c = \sum_{i=1}^{n} F_i^{-1}(U)$ be the comonotonic sum, where F_i is the distribution of X_i, for $i = 1, \ldots, n$, and U is uniform on $(0, 1)$. It is known that the comonotonic random vector $(F_1^{-1}(U), \ldots, F_n^{-1}(U))$ is maximal within the corresponding Fréchet space in the sense of the convex order of the sum, i.e., $S \leq_{sl} S^c$. This useful concept has been widely employed in actuarial science to model the strongest positive dependence structure among risks. For comprehensive discussions on comonotonicity and its applications in insurance and finance, readers are referred to Dhaene et al. (2002a,b).

A random vector $\underline{X}$ is said to be positively lower and upper orthant dependent (PLOD and PUOD) if

$$\mathbb{P}\left[\underline{X} \leq \underline{x}\right] \geq \prod_{i=1}^{n} \mathbb{P}\left[X_i \leq x_i\right] \quad \text{and} \quad \mathbb{P}\left[\underline{X} > \underline{x}\right] \geq \prod_{i=1}^{n} \mathbb{P}\left[X_i > x_i\right], \qquad \forall \underline{x} \in \mathbb{R}^n, \tag{7}$$

respectively. The vector $\underline{X}$ is positively orthant dependent (POD) if both inequalities in (7) hold. In the bivariate case, the two inequalities in (7) are equivalent and POD reduces to positive quadrant dependence (PQD). Both PUOD and PLOD (hence, POD) are preserved under component-wise increasing and continuous transformations.

It was shown in Proposition 5.3.9 of Denuit et al. (2005) that $X_1^{\perp} + X_2^{\perp} \leq_{sl} X_1 + X_2$, where $(X_1^{\perp}, X_2^{\perp})$ is an independent version of PQD (X_1, X_2) having the same marginal distributions. This result, however, does not extend to the multivariate case ($n > 2$) of POD.

Next, we say that $\underline{X}$ is *associated* (Esary et al., 1967) if

$$\text{Cov}\left[g(\underline{X}), h(\underline{X})\right] \geq 0, \tag{8}$$

for all increasing functions $g, h : \mathbb{R}^n \to \mathbb{R}$ such that the covariance exists. Associatedness is preserved under component-wise increasing transformations. Association implies POD (Denuit et al. (2005), p. 319). Lindqvist (1988) provided an equivalent representation of associatedness: $\underline{X}$ is associated if

$$\mathbb{P}\left[X \in U \cap V\right] \geq \mathbb{P}\left[X \in U\right] \mathbb{P}\left[X \in V\right],$$

for all upper sets $U, V \subseteq \mathbb{R}^n$.

Furthermore, we define the notions of conditional increasingness (CI) and conditional increasingness in sequence (CIS); see Müller and Scarsini (2001). We say that $\underline{X}$ is CIS if, for all $i = 2, 3, \ldots, n$,

$$\{X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\} \leq_{st} \{X_i | X_1 = y_1, \ldots, X_{i-1} = y_{i-1}\},$$

whenever $y_j \geq x_j$, assumed to be in the support of X_j, $j = 1, \ldots, i-1$. Based on this notion, we say that $\underline{X}$ is CI if $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ is CIS for all permutations π of $\{1, \ldots, n\}$. Of course, CI implies CIS.

Finally, we define the notion of multivariate total positivity of order 2 (MTP_2). Suppose $\underline{X}$ has a continuous or discrete density $f_{\underline{X}}$. Then, $\underline{X}$ is MTP_2 if $\log f_{\underline{X}}$ is supermodular. (A function $g : \mathbb{R}^n \to \mathbb{R}$ is supermodular if $g(\underline{x}) + g(\underline{y}) \leq g(\underline{x} \wedge \underline{y}) + g(\underline{x} \vee \underline{y})$ for all $\underline{x}, \underline{y} \in \mathbb{R}^n$ with the minimum and maximum operators $\wedge$ and $\vee$ applied component-wise.) We note that

$$\mathrm{MTP}_2 \Longrightarrow \mathrm{CI} \Longrightarrow \mathrm{CIS} \Longrightarrow \text{Associatedness} \Longrightarrow \mathrm{POD}.$$

See Joe (1997), Dhaene et al. (2002a), Denuit et al. (2005), Embrechts et al. (2005), Kaas et al. (2009), Laeven (2009) and Goovaerts et al. (2011) for further details on these dependence notions and their connection to VaR and TVaR.

A.2.2 Arrangement increasing

For any vector $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we use $\tau(\underline{x})$ to denote the permuted vector $(x_{\tau(1)}, \ldots, x_{\tau(n)})$, where $\{\tau(1), \ldots, \tau(n)\}$ is any permutation of $\{1, \ldots, n\}$ under the permutation operator τ. Let $\underline{x}_\downarrow$ and $\underline{x}_\uparrow$ denote the decreasing and increasing rearrangement of $\underline{x}$, respectively.

Definition 29 *A real-valued function $g(\underline{x}, \underline{\lambda})$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be arrangement increasing (AI) if g is permutation invariant, i.e., $g(\underline{x}, \underline{\lambda}) = g(\tau(\underline{x}), \tau(\underline{\lambda}))$ for any permutation τ, and g exhibits permutation order, i.e., $g(\underline{x}_\downarrow, \underline{\lambda}_\uparrow) \leq g(\underline{x}_\downarrow, \tau(\underline{\lambda})) \leq g(\underline{x}_\downarrow, \underline{\lambda}_\downarrow)$ for any permutation $\{\tau(1), \ldots, \tau(n)\}$.*

For any (i, j) with $1 \leq i < j \leq n$, let $\tau_{ij}(\underline{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ and

$$\begin{aligned} \mathcal{G}_s^{i,j}(n) &= \{g(\underline{x}) : g(\underline{x}) \geq g(\tau_{ij}(\underline{x})) \text{ for any } x_i \leq x_j\}, \\ \mathcal{G}_r^{i,j}(n) &= \{g(\underline{x}) : g(\underline{x}) - g(\tau_{ij}(\underline{x})) \text{ is increasing in } x_j \geq x_i\}. \end{aligned}$$

Definition 30 *A random vector $\underline{X} = (X_1, \ldots, X_n)$ is said to be*

(i) stochastic arrangement increasing (SAI) if $\mathbb{E}[g(\underline{X})] \geq \mathbb{E}[g(\tau_{ij}(\underline{X}))]$ for any $g \in \mathcal{G}_s^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$;

(ii) right tail weakly stochastic arrangement increasing (RWSAI) if $\mathbb{E}[g(\underline{X})] \geq \mathbb{E}[g(\tau_{ij}(\underline{X}))]$ for any $g \in \mathcal{G}_r^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$.

For further details, we refer to Cai and Wei (2014, 2015).

A.2.3 Comparing dependence structures

The correlation order was introduced in the actuarial literature by Dhaene and Goovaerts (1996) to find an ordering between random couples $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ such that the sums of their components are ordered in the stop-loss (increasing convex order) sense; see also the concordance order in e.g., Nelsen (2007).

Definition 31 *Consider two random couples $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ in $\mathcal{R}(F_1, F_2)$. If $F_{\underline{X}}(x_1, x_2) \leq F_{\underline{Y}}(x_1, x_2)$ for all $\underline{x} \in \mathbb{R}^2$, or equivalently, $\overline{F}_{\underline{X}}(x_1, x_2) \leq \overline{F}_{\underline{Y}}(x_1, x_2)$ for all $\underline{x} \in \mathbb{R}^2$, then $\underline{X}$ is said to be smaller than $\underline{Y}$ in the correlation order (denoted by $\underline{X} \leq_{corr} \underline{Y}$).*

The supermodular order can be seen as a multivariate extension of the correlation order from two dimensions to higher dimensions, based on supermodular functions. (One can easily verify that, for two random vectors to be comparable by the supermodular order, identical marginal distributions are required.)

Definition 32 *Let $\underline{X}$ and $\underline{Y}$ be two n-dimensional random vectors such that $\mathbb{E}[g(\underline{X})] \leq \mathbb{E}[g(\underline{Y})]$ for any supermodular function $g : \mathbb{R}^n \to \mathbb{R}$. Then $\underline{X}$ is said to be smaller than $\underline{Y}$ in the supermodular order, denoted by $\underline{X} \leq_{sm} \underline{Y}$.*

The multivariate stop-loss order is obtained by substituting the cones of the increasing convex functions on $\mathbb{R}^n$ for the corresponding cone of univariate functions.

Definition 33 *For two n-dimensional random vectors $\underline{X}$ and $\underline{Y}$, one says that $\underline{X}$ is smaller than $\underline{Y}$ in the multivariate stop-loss order, denoted by $\underline{X} \leq_{sl} \underline{Y}$, if $\mathbb{E}[g(\underline{X})] \leq \mathbb{E}[g(\underline{Y})]$ for any increasing convex function $g : \mathbb{R}^n \to \mathbb{R}$.*

It is well known that $\underline{X} \leq_{sl} \underline{Y}$ if, and only if, $g(\underline{X}) \leq_{sl} g(\underline{Y})$ for any increasing convex function $g : \mathbb{R}^n \to \mathbb{R}$; see Proposition 3.4.65 in Denuit et al. (2005).

Finally, we recall the definition of the multivariate usual stochastic order.

Definition 34 *For two n-dimensional random vectors $\underline{X}$ and $\underline{Y}$, one says that $\underline{X}$ is smaller than $\underline{Y}$ in the multivariate usual stochastic order, denoted by $\underline{X} \leq_{st} \underline{Y}$, if $\mathbb{E}[g(\underline{X})] \leq \mathbb{E}[g(\underline{Y})]$ for any increasing function $g : \mathbb{R}^n \to \mathbb{R}$.*

It is well known that $\underline{X} \leq_{st} \underline{Y}$ implies that $\sum_{i=1}^n X_i \leq_{st} \sum_{i=1}^n Y_i$ and $X_i \leq_{st} Y_i$, for $i = 1, \ldots, n$. For more detailed properties on the multivariate usual stochastic order, we refer to Shaked and Shanthikumar (2007).

A.3 Risk measure

In full generality, a risk measure is a mapping ρ from a set $\mathcal{X}$ of real-valued r.v.'s to the extended real line, $\overline{\mathbb{R}}$:

$$\rho : \mathcal{X} \to \overline{\mathbb{R}} : X \in \mathcal{X} \mapsto \rho[X].$$

In this paper, we restrict our attention to law-invariant risk measures. We denote by F_X^{-1} the left-continuous generalized inverse distribution function of X:

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in (0, 1), \tag{9}$$

where $\inf\{\emptyset\} = \infty$ by convention. In quantitative risk management, $F_X^{-1}(p)$ is commonly referred to as the Value-at-Risk (VaR) of X at probability level p, denoted by $\mathrm{VaR}_p[X]$. For further details on various risk measures, their appealing and appalling properties, and their modern alternatives, we refer to Denuit et al. (2005, 2006), Dhaene et al. (2006), Goovaerts et al. (2010), Föllmer and Schied (2011), and Laeven and Stadje (2013).

B Generalized definition of the systemic contribution order

In this part of Appendix, we generalize Definition 6 to the case of a general increasing and convex function $\Delta(x_+)$ with $\Delta(0) = 0$ and assume that the regulator applies the same loss measurement function $\Delta(x_+)$ to all institutions in the system. Thus, we consider the aggregate Δ-residual loss $S_{\underline{R}}$ that takes the form ($2''$), instead of ($2'''$) as used in Section 4. Throughout this part, we adopt the exact same notation "$\leq_{(\underline{R},s)-con}$" and "$\leq_{\underline{R}-con}$" as in the Section 4, which should now be understood in the following sense:

Definition 35 *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, the aggregate Δ-residual loss $S_{\underline{R}}$ defined in (2′), and the aggregate loss level $s \in \mathbb{R}_+$. Individual loss X_j is said to be smaller in systemic contribution order than individual loss X_k under microprudential regulation $\underline{R}$ and aggregate loss level s, denoted by $X_j \leq_{(\underline{R},s)-con} X_k$, if*

$$[\Delta((X_j - R_j)_+) \mid S_{\underline{R}} > s] \leq_{sl} [\Delta((X_k - R_k)_+) \mid S_{\underline{R}} > s].$$

In accordance with the generalized Definition 35 and the fact that the conditional stop-loss order is preserved under increasing and convex transformations, we can generalize Theorem 7 as follows.

Theorem 36 *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, the aggregate Δ-residual loss $S_{\underline{R}}$ defined in (2′), and the aggregate loss level s. Then,*

$$(X_j - R_j)_+ \leq_{I(\underline{R},s)-sl} (X_k - R_k)_+ \Longrightarrow X_j \leq_{(\underline{R},s)-con} X_k.$$

The following definition corresponds to the special case of Definition 35 when $s = 0$.

Definition 37 *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, and the aggregate Δ-residual loss $S_{\underline{R}}$ defined in (2′). Individual loss X_j is said to be smaller in systemic contribution order than individual loss X_k under microprudential regulation $\underline{R}$, denoted by $X_j \leq_{\underline{R}-con} X_k$, if*

$$[\Delta((X_j - R_j)_+) \mid S_{\underline{R}} > 0] \leq_{sl} [\Delta((X_k - R_k)_+) \mid S_{\underline{R}} > 0].$$

Next, we partially generalize the results in Theorems 9 and 11 to the case of a general loss measurement function by exploiting the fact that the (unconditional) stop-loss order is (also) preserved under increasing and convex transformations.

Theorem 38 *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, and the aggregate Δ-residual loss $S_{\underline{R}}$ defined in (2′) with $\mathbb{P}[S_{\underline{R}} > 0] > 0$. Then,*

$$(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+ \Longrightarrow X_j \leq_{\underline{R}-con} X_k.$$

Proof. In light of Theorem 4.A.8 in Shaked and Shanthikumar (2007), we know that $(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+$ implies $\Delta\left((X_j - R_j)_+\right) \leq_{sl} \Delta\left((X_k - R_k)_+\right)$ for increasing and convex Δ. Then, the proof of the desired result follows from a similar argument as the one used in the proof of Theorem 9. ∎

Under the setting of Theorem 38, it clearly holds that $X_j \leq_{\underline{R}-con} X_k$ is equivalent to $\Delta\left((X_j - R_j)_+\right) \leq_{sl} \Delta\left((X_k - R_k)_+\right)$, which in general, however, does not imply that $(X_j - R_j)_+ \leq_{sl} (X_k - R_k)_+$.

Theorem 39 *Consider the financial system $\underline{X}$, the microprudential regulation $\underline{R}$, and the aggregate Δ-residual loss $S_{\underline{R}}$ defined in (2′). Then,*

$$(X_j - R_j)_+ \leq_{I(\underline{R})-sl} (X_k - R_k)_+ \Longrightarrow X_j \leq_{\underline{R}-con} X_k.$$

Proof. First, it is easy to show that the conditional stop-loss order is preserved under increasing and convex transformations, which means that $(X_j - R_j)_+ \leq_{I(\underline{R})-sl} (X_k - R_k)_+$ implies $\Delta\left((X_j - R_j)_+\right) \leq_{I(\underline{R})-sl} \Delta\left((X_k - R_k)_+\right)$. Then, the proof follows from a similar argument as the one in Theorem 11. ■