To: Researchers Interested in Longevity Risk Sharing & Pooling

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Subject: *Is there more than one way to skin a tontine cat?*

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We would like to elicit expert opinion on the *non-discriminatory* allocation and *sharing of longevity risk*, and hope you can spare a few minutes to help. Please read the following and let us know your thoughts, if you have any.

A group of n investors (a.k.a. retirees) *pool* together into the following one-period longevity risk-sharing scheme. They each invest or allocate $\pi_j > 0$ dollars at time zero into an account earning a one-plus risk-free rate: $(1 + R) \geq 1$, but they face a $p_j > 0$ probability of surviving to the end of the period. They decide to share the total fund among survivors, which is also known as a one-period tontine. For the sake of a simple numerical example, we will assume $n = 3$, $R = 0$. The first retiree invests $\pi_1 = \$80$, the second $\pi_2 = \$50$ and third $\pi_3 = \$20$. The one-period survival probabilities are: $p_1 = 20\%, p_2 = 50\%, p_3 = 80\%$, which reflect mortality rates over a decade at old, middle and early retirement ages. The table summarizes the inputs.

$\pi_1 =$	\$80.00	$p_1 =$	20.0%
$\pi_2 =$	\$50.00	$p_2 =$	50.0%
$\pi_3 =$	\$20.00	$p_3 =$	80.0%
$\sum \pi_k$	\$150.00	and	$R = 0\%$

Clearly, investor $j = 1$ has placed the most at risk because he/she faces an 80% probability of dying and losing it all <u>and</u> has invested \$80. Think of the function $g(\pi, p) = \pi/p$ as a theoretical measure of "money at risk", although $g(\pi, p)$ could be defined as any function that is increasing in its first argument and decreasing in its second. Regardless of how exactly "money at risk" is measured, investor $i = 3$ risks a mere \$20, and faces an 80% probability of surviving, so their $g(\pi, p) = \pi/p = 25$. Note that the third investor expects to survive since $(1/p) > 0.5$, while the first investor doesn't. Ergo, and perhaps even for ethical reasons, investor $j = 1$, with a $g(\pi, p) = 400$, should be entitled to a larger share of the gains if he/she happens to (get lucky) and survive. That should be obvious, the question is how much more.

There are: $2^n = 2^3 = 8$ different scenarios, the most vexing is the ω in which everyone dies. Now, one can set the rules of this game in many ways – perhaps even by offering refunds to beneficiaries – but we will assume that in the scenario in which everyone dies, which has a $(1 - p_1)(1 - p_2)(1 - p_3) = 8\%$ probability (assuming independence), the \$150 is taxed, taken by Government and lost to participants. Why should Government be entitled to a joint life insurance policy on n heads? Well, perhaps it's compensation for legally enforcing the contract in the other $2^n - 1 = 7$ scenarios. Or, perhaps it's just how things work in the real world for unclaimed money in bank accounts. Either way, that's our assumption for (what we label) $\omega_{(2^n)}$. There are another $n = 3$ scenarios that are trivial, namely when there is only one survivor who takes the entire: $\sum \pi_k = \$150$. This leaves $2^n - n - 1 = 4$ scenarios in which the \$150 fund must be distributed in a non-discriminatory manner, *and, we want your help.*

Our specific question – and the reason for this memo – is that we would like to gauge professional and academic opinion on exactly how the funds should be distributed in each of those 4 non-trivial scenarios, in a manner that is perceived as *non-discriminatory*. We should note that due to the (8%) probability (assuming independence) that everyone dies, the expected investment return to the entire group is strictly negative when $n < \infty$.

Why negative? Well, in 7 scenarios the entire pool shares \$150, but in the vexing 8th one the pool gets nothing. Ergo, the pool's expected payout is 92% times \$150 plus 8% times zero, which is \$138, and the pool's expected investment return is: $(138/150) - 1 = -8\%$, yes minus. Now, if $R > 0$ the pool's expected investment return might be positive, but it would still be less than R itself.

Here is a matrix to visualize the scenarios ω_i and the corresponding payouts W_i, where 1 represents alive and 0 represents dead.

ω_1	W_1	ω_2	W_2	ω_3	W_3	ω_4	W_4	ω_5	W_5	ω_6	W_6	ω_7	W_7	ω_8	W_8
1	**?**	0	\$0	0	\$0	1	**?**	1	**?**	1	\$150	0	\$0	0	\$0
1	**?**	1	**?**	0	\$0	0	\$0	1	**?**	0	\$0	1	\$150	0	\$0
1	**?**	1	**?**	1	\$150	1	**?**	0	\$0	0	\$0	0	\$0	0	\$0
8%		32%		32%		8%		2%		2%		8%		8%	

Each row in the table represents one of the three investors (in the original order) and their life status: dead (=0) or alive (=1). We ask that you complete the above table – the allocations under scenarios $\omega_1, \omega_2, \omega_4$ and ω_5, and send them back to us. And, if you can explain your logic it would be even better, but even a simple email with the (nine) numbers would be appreciated. **If** you only have time (or energy) for one scenario, then please focus on ω_1 when everyone survives. We think they should **not** all get their original money back due to the asymmetric risk exposure, as measured by the $g(\pi, p)$ function. But you might disagree. Likewise, if you feel (strongly) that you can't *solve this problem* without knowing the investor's risk preferences (a.k.a. a utility function), or for some other reason, then that would be a helpful response as well.

We are compiling a summary of responses from scholars and experts, which we will compare against replies from non-experts. We plan to share the results of this work in a presentation (and paper) that will be finalized during the fall of 2023, so we hope to get your reply by September 15th, 2023. The impetus and motivation for this particular (research) question is the nascent literature on *decentralized risk sharing* which suggests a wide assortment of *rules* for allocating insurance risk in the wild, which indicates there isn't a universally accepted way to *skin the tontine cat.*

Thank you in advance and we promise to credit you (and your proposed solution) in the paper, unless you want to be anonymous. Very much appreciated.

Moshe M. & Jan D.

Memo Appendix: You might not agree (and we actually hope you don't!) but this is one possible way or "rule" that can be used to distribute the $150 in the four *non-trivial* scenarios.

ω_1	W_1	ω_2	W_2	ω_3	W_3	ω_4	W_4	ω_5	W_5	ω_6	W_6	ω_7	W_7	ω_8	W_8
1	114.29	0	0	0	0	1	141.18	1	120	1	150	0	0	0	0
1	28.57	1	120	0	0	0	0	1	30	0	0	1	150	0	0
1	7.14	1	30	1	150	1	8.82	0	0	0	0	0	0	0	0
8%		32%		32%		8%		2%		2%		8%		8%	

Our logic is as follows: Start with scenario ω_1, where all three investors are alive, an event with 8% probability. The first ($\pi_1 = \$80, p_1 = 0.20$) investor thinks: *Had I purchased a pure endowment from an insurance company – and survived – my payout would have been:* $\pi_1/p_1 = \$400$. This is an insurance claim, but one in which a limited pool of money is available. The insurance claim is the (above noted) "money at risk" function $g(\pi, p) \times (1 + R)$. Likewise, the second ($\pi_2 = \$50, p_2 = 0.50$) investor is entitled to an insurance claim of: $\pi_2/p_2 = \$100$, and the third investor claims: $\pi_3/p_3 = \$25$, using the same actuarial logic. In total, for the three survivors in scenario ω_1, the aggregate insurance claim is $\mathbf{C}(\omega_1) := \sum_{k=1}^{n}(\pi_k/p_k) = \sum_{k=1}^{n} g(\pi_k, p_k) = \$525$, but alas there is only $(1 + R)\sum_{k=1}^{n} \pi_k = \$150$ available to distribute to the pool.

So...our simple rule is to give investors the relative fraction of their *personal* insurance claim against the *aggregate* insurance claim. The first investor claims 400 out of a total 525, which is 76.19%, or $(0.762)(150) = \$114.3$ from the available $150. This is *more* than individual $i = 1$ invested, but *less* than his *personal* insurance claim. Algebraically this investor takes: $(\pi_1/p_1)/\mathbf{C}(\omega_1)$. The same logic gives the second investor 100 out of 525, or 19.04% of the $150 available, which is $28.57, and less than the $\pi_2 = \$50$ invested. The third and final investor makes a personal claim of 25 from an aggregate claim of 525, a mere 4.76% of the available $150, for a total payout of $7.14. The third investor, like the second, walks away with less than orignally invested, while the (relative) winner is investor number one, who gets more than his original $\pi_1 = \$80$.

The same logic can be applied to the other three scenarios $\omega_2, \omega_4, \omega_5$. While the *personal* insurance claim (π/p) remains the same, the *aggregate* insurance claim is lower due to the smaller number of survivors. For example, the value of $\mathbf{C}(\omega_2) = 125$, $\mathbf{C}(\omega_4) = 425$ and $\mathbf{C}(\omega_5) = 500$. Again, these are the denominators for the fractional allocation of end-of-period available funds, where the numerator is the *personal* claim (π_k/p_k).

In sum, our suggested (general) rule for $W_{(i,j)}$, which represents the payout in scenario ω_i (column) and individual j (row) in the above table, can be written as:

$$W_{(i,j)} = (1 + R)\sum_{k=1}^{n} \pi_k \times \left(\frac{(\pi_j/p_j) \times I_{(i,j)}}{\sum_{k=1}^{n}(\pi_k/p_k) \times I_{(i,k)}} \right), \quad j = 1, 2, 3; \quad i = 1..2^n. \tag{1}$$

where $I_{(i,j)}$ is the (scalar) life status of the j'th investor in the i'th scenario. For example, under ω_4, the indicator variables are: $I_{(4,1)} = 1$, $I_{(4,2)} = 0$, $I_{(4,3)} = 1$, and the denominator is $(80/0.2) \times 1 + (50/0.5) \times 0 + (20/0.8) \times 1$, which is the above noted $\mathbf{C}(\omega_4) = \$425$. Again, the quantity in brackets in equation (1) is the ratio of *personal* to *aggregate* insurance claim, which is then multiplied by the money in the pool. And, when $i = 2^n$, which is the ω scenario in which everyone is dead, we define $W_{(i,j)} := 0$, despite the zero in the denominator. Equation (1) is a *proportional* risk-sharing rule, but other rules could be proposed.

Do you think the above would be appropriate? Do you have an alternate allocation?

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