AN AXIOMATIC THEORY FOR QUANTILE-BASED RISK SHARING

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Abstract

This paper studies the quantile risk-sharing rule introduced in Denuit, Dhaene & Robert (2022). This rule is not actuarially fair, but instead satisfies another type of fairness, which is comparable with "solvency fairness" in classical centralized insurance. New properties are investigated and an axiomatic theory is developed for the quantile risk-sharing rule, which allows for a deeper understanding of its proper use. The axiomatic characterization of the quantile risk-sharing rule is based on aggregate and comonotonicity-related properties of risk-sharing rules.

Keywords: quantile risk-sharing rule, conditional mean risk-sharing rule, pooling, comonotonicity, P2P insurance.

1 Introduction

As pointed out by Borch (1968), economic agents are primarily concerned with their solvency, or survival probabilities. This is in line with the standard approach in insurance risk management which consists in controlling the probability that the aggregate net loss amount (i.e., aggregate claim amount minus net premium income) be less than some specified threshold (corresponding to the assets held by the insurance company). Fixing this probability leads to the classical Value-at-Risk approach. Insurance regulators typically require a uniform survival probability (99.5% under Solvency 2) to all market players to ensure fair competition. This implies that regulators guarantee to all policyholders the same chance to be indemnified for their claims. This approach can thus be seen as fairness in terms of solvency.

Also in a decentralized risk sharing context (as opposed to classical centralized insurance), there may be a need for a way to share losses among market participants so that each entity has always the same "ex-ante" survival probability as the other participants once the individual losses have been observed ex-post. The amount of contribution to be paid ex-post by a participant can be seen as a capital requirement (to be satisfied only ex-post), given the realization of the aggregate losses. Formally, let $X_1, X_2, ..., X_n$ denote insurance loss amounts, modeled as non-negative random variables with 1-to-1 distribution functions F_{X_i} over $(0, \infty)$ and a possible positive probability mass $\mathrm{P}[X_i = 0]$ at 0. The latter is often negligible when participants are insurance companies. Let $S_{\boldsymbol{X}} = \sum_{i=1}^{n} X_i$ be the aggregate loss. When realized losses $x_1, x_2, \ldots, x_n$ are observed, they can be turned into probability levels p_i via the equation $x_i = F_{X_i}^{-1}(p_i)$. This means that participant i staying alone, without engaging in any risk-sharing activities with other participants, would have remained solvent provided his or her available assets were (at least) equal to the Value-at-Risk at level p_i. The randomness of individual losses causes differences in the solvency levels p_i. The idea of joining the pool according to the quantile risk-sharing rule is to replace the possibly different $p_1, \ldots, p_n$ by a unique and uniform probability level p_s corresponding to the realized aggregate loss s. The unique solution is to ask participant i to contribute the amount $F_{X_i}^{-1}(p_s)$ where p_s satisfies

$$\sum_{i=1}^{n} F_{X_i}^{-1}(p_i) = \sum_{i=1}^{n} F_{X_i}^{-1}(p_s) = s.$$

This defines the quantile risk-sharing rule, allocating the ex-post contribution $F_{X_i}^{-1}(p_s)$ to participant i. Quantile risk sharing achieves a kind of fairness which is comparable with fairness in terms of solvency in classical centralized insurance: allocating the amount $F_{X_i}^{-1}(p_s)$ of losses to participant i ex-post ensures that they would all have reached the same probability that losses exceed contributions if they set an ex-ante solvency capital equal to this contribution. Such uniformity is considered as a reasonable requirement from the regulatory perspective. The key argument in the study of the quantile risk-sharing rule is that $\sum_{i=1}^{n} F_{X_i}^{-1}(\cdot)$ defining the common probability level p_s is the quantile function of the sum of the comonotonic modification of the random vector $(X_1, X_2, ..., X_n)$, which leads to many of the important properties of this rule.

This quantile risk-sharing rule has been introduced in Denuit, Dhaene & Robert (2022) where several of its properties have been investigated. This rule is a comonotonic risk-sharing rule in the sense that the contributions are non-decreasing functions of total losses $S_{\boldsymbol{X}}$, which is a desirable property since it ensures that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. This paper further investigates this risk-sharing rule.

Embrechts, Liu & Wang (2018) and Wang and Wei (2020) characterized Pareto-optimal risk-sharing rules, where the Pareto-optimality is expressed in terms of a sum of quantile-based risk measures applied to the individual losses in the pool. The approach in the present paper is different as we investigate some properties that the quantile risk-sharing rule may or may not possess, and we determine the axioms underlying this risk-sharing rule.

The axiomatic theory developed in this paper compares with Jiao, Kou, Liu & Wang (2022) who pioneered the theory on axiomatic characterization of certain classes of "anonymized" risk-sharing rules, i.e. risk-sharing rules that do not require any information on the preferences of the agents, a risk exchange market, or subjective decisions of a central planner. They proved that four axioms characterize the conditional mean risk-sharing rule introduced by Denuit and Dhaene (2012). In this paper, we consider three axioms and prove that these axioms characterize the quantile risk-sharing rule.

The remainder of the paper is organized as follows. Section 2 introduces notation and recalls basic concepts including allocations and risk-sharing rules. Section 3 defines the quantile risk-sharing rule. Several of its properties are considered in Section 4. Section 5 proposes an axiomatic theory for the quantile risk-sharing rule. Technical material about supports of distribution functions is provided in the appendix.

2 Allocations and risk-sharing rules

2.1 Notation

All random variables considered in this paper are defined on a common probability space $(\Omega, \mathcal{G}, \mathrm{P})$. The latter is assumed to contain the random variable U which is uniformly distributed over the unit interval $(0, 1)$. (In-)equalities between random variables are supposed to hold almost surely. Similarly, (in-)equalities between random vectors hold almost surely and component-wise. A random variable will always be denoted by an upper-case letter (e.g. X_i), while its realization (observed ex post) will be denoted by the corresponding lower-case letter (e.g. x_i). A random vector will be denoted by a bold upper-case letter, e.g. $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, while its realization (observed ex post) is denoted by the corresponding bold lower-case small letters, e.g. $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. In this paper, "$\stackrel{\mathrm{d}}{=}$" stands for "equality in distribution".

2.2 Allocations

Let χ be an appropriate set of random variables on the probability space $(\Omega, \mathcal{G}, \mathrm{P})$ under consideration. We interpret χ as the collection of risks (losses) under interest. For particular situations, the set χ could be defined as the set L^q of all random variables X with $\mathrm{E}\left[|X|^q\right] < \infty$, for an appropriate choice of $q \in [0, \infty)$, with $\mathrm{E}[\cdot]$ being the expectation under P. Another possible choice for χ is the set of all (essentially) bounded random variables L^∞.

Also, for any q considered above, the set L^q_+ of all non-negative elements of L^q might be an appropriate choice. More general, χ can be chosen as a convex cone of random variables on the probability space $(\Omega, \mathcal{G}, \mathrm{P})$, which means that for any $X, Y \in \chi$ and any scalars $a > 0$ and $b > 0$, one has that $aX + bY \in \chi$. In this paper, we assume that $\chi = L^q_+$ or $\chi = L^q$ for some q in $[0, \infty]$, appropriate for the situation at hand.

Consider n economic agents, numbered $i = 1, 2, \ldots, n$. Let time 0 be "now". Each agent i faces a loss $X_i \in \chi$ at time 1. Without insurance or pooling, each individual agent bears his or her own loss, i.e. at time 1, agent i suffers loss x_i, which is the realization of X_i.

The n-dimensional random vector of the losses $\boldsymbol{X}$ is called the (initial) loss vector. The joint distribution function of the loss vector $\boldsymbol{X}$ is denoted by $F_{\boldsymbol{X}}$. The marginal distribution functions of the individual losses are denoted by $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$, respectively. As in the introduction, the aggregate loss faced by the n agents with loss vector $\boldsymbol{X}$ is denoted by

$$S_{\boldsymbol{X}} = \sum_{i=1}^{n} X_i.$$

Hereafter, we will often call $\boldsymbol{X}$ the pool and each agent a participant in the pool.

Definition 2.1. *For any random vector $\boldsymbol{X} \in \chi^n$ with aggregate loss $S_{\boldsymbol{X}}$, the set $\mathcal{A}_n(S_{\boldsymbol{X}})$ is defined by:*

$$\mathcal{A}_n(S_{\boldsymbol{X}}) = \left\{ (Y_1, Y_2, \ldots, Y_n) \in \chi^n \middle| \sum_{i=1}^{n} Y_i = S_{\boldsymbol{X}} \right\}.$$

The elements of $\mathcal{A}_n(S_{\boldsymbol{X}})$ are called the n-dimensional allocations of $S_{\boldsymbol{X}}$ in χ^n. Notice that the initial loss vector $\boldsymbol{X}$ is an element of $\mathcal{A}_n(S_{\boldsymbol{X}})$, and that for any $\boldsymbol{Y} \in \mathcal{A}_n(S_{\boldsymbol{X}})$, one has that $\mathcal{A}_n(S_{\boldsymbol{Y}}) = \mathcal{A}_n(S_{\boldsymbol{X}})$.

2.3 Risk sharing

Risk sharing in a pool $\boldsymbol{X} \in \chi^n$ is a two-stage process. In the *ex-ante step* (at time 0), the losses X_i in the pool are re-allocated by transforming $\boldsymbol{X}$ into another random vector $\boldsymbol{H} = (H_1, H_2, \ldots, H_n) \in \mathcal{A}_n(S_{\boldsymbol{X}})$ called the contribution vector. Participants thus exchange their individual risks X_i to the contributions H_i when they join the pool. As $\boldsymbol{H} \in \mathcal{A}_n(S_{\boldsymbol{X}})$, risk-sharing is self-financing in the sense that the identity

$$\sum_{i=1}^{n} H_i = \sum_{i=1}^{n} X_i \tag{2.1}$$

holds true. This self-financing condition (2.1) in risk-sharing is often called the full allocation condition. In the *ex-post step* (at time 1), any participant receives the realization x_i of his initial loss X_i from the pool and pays the realization of H_i to the pool. This leads to the following definition.

Definition 2.2. *A risk-sharing rule is a mapping $\mathbb{H} : \chi^n \to \chi^n$ associating to each pool $\boldsymbol{X} \in \chi^n$ a contribution vector $\boldsymbol{H}$ satisfying $\boldsymbol{H} \in \mathcal{A}_n(S_{\boldsymbol{X}})$.*

In order to be able to determine the contribution H_i for each participant i at time 1, one needs different types of information. The first type of information is of a deterministic nature and is available at time 0, such as certain parameters (e.g. expectations of the X_i) or certain distribution functions (e.g. the possibly unequal distribution functions F_{X_i} of the X_i, or the joint distribution function $F_{\boldsymbol{X}}$ of $\boldsymbol{X}$). A second type of information, which is only available at time 1, is the outcome (realization) of certain random variables and random vectors (e.g. the outcome of $S_{\boldsymbol{X}}$, the outcome of $\boldsymbol{X}$ and eventually also the outcome of other random variables). Hereafter, we assume that the deterministic information is "correct fairness" in the sense that the assumed expectations distributions, etc. are the "solvency right" ones. At time 1, the realization of the contribution vector $\boldsymbol{H}$ is a deterministic vector, as the realization of any random source is assumed to be observable at time 1.

At time 0, the contribution vector $\boldsymbol{H}$ is a random vector, as it depends on $\boldsymbol{X}$, and eventually also on other sources of randomness. This means that knowing the realization of $\boldsymbol{X}$ may not be enough to know the realization of $\boldsymbol{H}$. In other words, in general we do not assume that for any $\boldsymbol{X} \in \chi^n$, there exists a (measurable) function $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X})$. Introducing the notation $\sigma(\boldsymbol{X})$ for the σ-algebra generated by $\boldsymbol{X}$, this means that $\boldsymbol{H}$ is not necessarily $\sigma(\boldsymbol{X})$-measurable. The next example illustrates this fact.

Example 2.3. *At time* 1*, we flip a coin. The random variable* Z *equals* 0 *in case of heads and* 1 *in case of tails. Consider the risk-sharing rule* $\mathbb{H} : \chi^2 \to \chi^2$*, where for any pool* $\boldsymbol{X} = (X_1, X_2)$*, the contribution vector is determined by*

$$\boldsymbol{H} = \begin{cases} \left(X_1 + X_2,\ 0\right) & \textit{if } Z = 0, \\ \\ \left(0,\ X_1 + X_2\right) & \textit{if } Z = 1. \end{cases}$$

Obviously, the knowledge of the realization of the loss vector $\boldsymbol{X}$ *is not sufficient to determine the realization of the contribution vector* $\boldsymbol{H}$*. In other words,* $\boldsymbol{H}$ *is not* $\sigma(\boldsymbol{X})$*-measurable.*

The risk-sharing rule $\mathbb{H}$ considered in Example 2.3 is not "internal" in the sense that it requires information coming from outside the pool $\boldsymbol{X}$ (the result of the coin toss). In practice, participants may let the risk-sharing rule depend on some external event, related to the financial market or to the occurrence of a catastrophe, for instance. In this paper, we focus on internal rules, which are precisely defined next. Let $\mathcal{F}(\chi^n)$ be the set of all n-dimensional distribution functions of elements in χ^n.

Definition 2.4. *A risk-sharing rule* $\mathbb{H} : \chi^n \to \chi^n$ *is said to be internal if there exists a function* $\boldsymbol{h} : \mathbb{R}^n \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ *such that the contribution vector* $\boldsymbol{H}$ *for any pool* $\boldsymbol{X} \in \chi^n$ *with distribution function* $F_{\boldsymbol{X}}$ *can be expressed as*

$$\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X}; F_{\boldsymbol{X}}). \tag{2.2}$$

Henceforth, all risk-sharing rules considered in this paper are assumed to be internal, that is, to possess the representation (2.2), unless explicitly stated otherwise. Moreover, $\boldsymbol{h}$ will be called the "internal function" of the internal risk-sharing rule $\boldsymbol{H}$.

Under a rule which can be expressed in the form (2.2), one has that the realization of $\boldsymbol{H}$ is known once the realization of $\boldsymbol{X}$ is revealed at time 1. In other words, $\boldsymbol{H}$ can be expressed as a function of $\boldsymbol{X}$ and hence is $\sigma(\boldsymbol{X})$-measurable. Furthermore, the argument $F_{\boldsymbol{X}}$ in $\boldsymbol{h}(\boldsymbol{X}; F_{\boldsymbol{X}})$ indicates that the realization of the contribution vector $\boldsymbol{H}$ does not only depend on the realization of $\boldsymbol{X}$, but may also depend on the distribution function of $\boldsymbol{X}$ (which is assumed to be known at time 0).

Example 2.5. *An example of an internal rule is the risk-sharing rule where each participant i contributes his or her own loss, i.e. $\boldsymbol{H} = \boldsymbol{X}$. In this case, the internal function $\boldsymbol{h} : \mathbb{R}^n \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ in (2.2) is given by $\boldsymbol{h}(\boldsymbol{x}; F_{\boldsymbol{X}}) = \boldsymbol{x}$. This rule is referred to as the stand-alone risk sharing rule in Denuit, Dhaene & Robert (2022), and the identity risk-sharing rule in Jiao, Kou, Liu & Wang (2022).*

Example 2.6. *Another example of an internal rule is the risk-sharing rule $\mathbb{H} : \chi^2 \to \chi^2$ defined by*

$$\boldsymbol{H} = \begin{cases} (X_1, X_2), & \text{if } F_{\boldsymbol{X}} = \min\{F_{X_1}, F_{X_2}\} \\ \left(\frac{X_1+X_2}{2}, \frac{X_1+X_2}{2}\right), & \text{otherwise.} \end{cases}$$

Notice that $F_{\boldsymbol{X}} = \min\{F_{X_1}, F_{X_2}\}$ means that X_1 and X_2 are perfectly dependent (comonotonic) in the sense that X_1 and X_2 are non-decreasing functions of the same random variable (see next section for more information). In this case, $\mathbb{H}$ takes into account that risk cannot be diversified so that each participant remains with his or her own loss, as in Example 2.5. In all other cases, participants share equally the total loss of the pool. The function $\boldsymbol{h} : \mathbb{R}^n \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ in (2.2) is thus given by

$$\boldsymbol{h}(\boldsymbol{x}; F_{\boldsymbol{X}}) = \begin{cases} (x_1, x_2), & \text{if } F_{\boldsymbol{X}} = \min\{F_{X_1}, F_{X_2}\} \\ \left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right), & \text{otherwise,} \end{cases}$$

It is important to note that a risk-sharing rule $\mathbb{H}$ under which the contribution vector $\boldsymbol{H}$ is $\sigma(\boldsymbol{X})$-measurable for any $\boldsymbol{X}$ does not necessarily have a representation of the form (2.2) because it might depend on other information. The next example illustrates this point.

Example 2.7. *Suppose that $\boldsymbol{X} \in \chi^n$ is a pool of health-related losses of the n participants, who can be divided into m age categories (e.g. 18-35, 36-50, 51-65 and 65+) denoted by $j = 1, 2, \ldots, m$. The age category of participant i is denoted by A_{X_i}. Consider the risk-sharing rule $\mathbb{H}$ in which losses are uniformly shared within each age category. The contribution vector $\boldsymbol{H} = (H_1, \ldots, H_n)$ is then given by*

$$H_i = \frac{\sum_{i=1}^n X_i 1_{A_{X_i}=j}}{\sum_{i=1}^n 1_{A_{X_i}=j}} \quad \text{if } A_{X_i} = j;\ j = 1, \ldots, m;\ i = 1, \ldots, n.$$

It is clear that although $\boldsymbol{H}$ is $\sigma(\boldsymbol{X})$-measurable, it is not internal because it has no representation of the form (2.2). The contribution vector $\boldsymbol{H}$ depends not only on $\boldsymbol{X}$ but also on the vector of the age categories of all participants $(A_{X_1}, \ldots, A_{X_n})$. This information can in general not be captured by $F_{\boldsymbol{X}}$.

For instance, when $n = 4$, and the realization $(100, 120, 130, 80)$ of $\boldsymbol{X}$ is observed, one still cannot determine the corresponding contribution vector unless the vector of the age category of the four participants is also known. If such vector is $(2, 2, 1, 2)$, then

$$\boldsymbol{H} = \left(\frac{100+120+80}{3}, \frac{100+120+80}{3}, 130, \frac{100+120+80}{3}\right) = (100, 100, 130, 100).$$

Let us now define aggregate risk-sharing rules. Throughout this paper, for any given $\boldsymbol{x} \in \mathbb{R}^n$, we denote by $s_{\boldsymbol{x}}$ the sum of the n components of $\boldsymbol{x}$.

Definition 2.8. *A RS rule $\mathbb{H} : \chi^n \to \chi^n$ is said to be aggregate if there exists a function $\boldsymbol{h}^{aggr} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ such that the contribution vector $\boldsymbol{H}$ for any pool $\boldsymbol{X} \in \chi^n$ can be expressed as*

$$\boldsymbol{H} = \boldsymbol{h}^{aggr}(S_{\boldsymbol{X}}, F_{\boldsymbol{X}}). \tag{2.3}$$

It is clear from the definition that an aggregate risk-sharing rule is internal with internal function $\boldsymbol{h}$ satisfying

$$\boldsymbol{h}(\boldsymbol{X}; F_{\boldsymbol{X}}) = \boldsymbol{h}^{\text{aggr}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})$$

for any $\boldsymbol{X} \in \chi^n$.

Example 2.9. *An example of an aggregate risk-sharing rule is the conditional mean risk-sharing rule introduced in Denuit & Dhaene (2012). In this case, we have that*

$$\boldsymbol{H} = \big(\mathrm{E}[X_1 \mid S_{\boldsymbol{X}}],\ \mathrm{E}[X_2 \mid S_{\boldsymbol{X}}], \ldots,\ \mathrm{E}[X_n \mid S_{\boldsymbol{X}}]\big),$$

which implies that

$$\boldsymbol{h}^{aggr}(s, F_{\boldsymbol{X}}) = \big(\mathrm{E}[X_1 \mid S_{\boldsymbol{X}} = s],\ \mathrm{E}[X_2 \mid S_{\boldsymbol{X}} = s], \ldots,\ \mathrm{E}[X_n \mid S_{\boldsymbol{X}} = s]\big).$$

In case $\mathbb{H}$ is an aggregate risk-sharing rule, for any pool $\boldsymbol{X}$, one has that the realization of the contribution vector $\boldsymbol{H}$ is known once the realization of the aggregate claims $S_{\boldsymbol{X}}$ is known. In other words, $\boldsymbol{H}$ is $\sigma(S_{\boldsymbol{X}})$-measurable. Notice however that a risk-sharing rule $\mathbb{H}$ such that for any $\boldsymbol{X}$, we have that $\boldsymbol{H}$ is $\sigma(S_{\boldsymbol{X}})$-measurable, does not necessarily have a representation of the form (2.3). Considering the health coverage pool described in Example 2.7, with only two participants, and suppose that the risk-sharing rule stipulates that for two risks in the same age category participants adopt the uniform risk-sharing rule, while for two risks in different age categories, they apply the conditional mean risk-sharing rule. In this case, $\boldsymbol{X}$ is $\sigma(S_{\boldsymbol{X}})$-measurable but does not have a presentation of the form (2.3).

3 The quantile risk-sharing rule

3.1 α-quantiles and comonotonicity

For any real-valued random variable X, the left-continuous quantile of order $p \in [0,1]$ is defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\},$$

while its right-continuous quantile of order $p \in [0,1]$ is defined by

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}.$$

In these definitions, we set $\inf\{\emptyset\} = +\infty$ and $\sup\{\emptyset\} = -\infty$, by convention. For any $\alpha \in [0,1]$, the α-quantile of order p is then defined by

$$F_X^{-1(\alpha)}(p) = \begin{cases} F_X^{-1+}(0) & \text{if } p = 0 \\ \alpha\ F_X^{-1}(p) + (1-\alpha)\ F_X^{-1+}(p) & \text{if } p \in (0,1) \\ F_X^{-1}(1) & \text{if } p = 1 \end{cases}.$$

Notice that in this definition for the α-quantiles, we have that $F_X^{-1(\alpha)}(0)$ and $F_X^{-1(\alpha)}(1)$ are both independent of the particular choice of α. They are chosen as the "smallest" and the "largest" value of X, respectively.

The next result is central to the determination of the probability level defining quantile risk sharing. Important there is that we make the convention that the interval of the type $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$ has to be considered as subset of $\mathbb{R}$ rather than as a subset of the extended real line $\mathbb{R} \cup \{\pm\infty\}$. For instance, if $F_X^{-1+}(0) = -\infty$ and $F_X^{-1}(1) = +\infty$, then $\left[F_X^{-1+}(0), F_X^{-1}(1)\right] = \mathbb{R}$ but not $[-\infty, +\infty]$. Similarly, if $F_X^{-1+}(0) = 0$ and $F_X^{-1}(1) = +\infty$, then $\left[F_X^{-1+}(0), F_X^{-1}(1)\right] = [0, +\infty)$. This convention is made throughout this paper.

Proposition 3.1. *For any random variable X and any $x \in \left[F_X^{-1+}(0), F_X^{-1}(1)\right]$, there exists a (not necessary unique) $\alpha_x \in [0,1]$ such that*

$$F_X^{-1(\alpha_x)}\big(F_X(x)\big) = x. \tag{3.1}$$

The proof of Proposition 3.1 is straightforward. Notice that in case $(x, F_X(x))$ lies on a strictly increasing part of F_X, then any element of $[0,1]$ is a possible choice for α_x. On the other hand, when $(x, F_X(x))$ lies on a flat part of F_X, then α_x is uniquely determined. Further, if $F_X(x) = 0$ or $F_X(x) = 1$, then any element of $[0,1]$ is a possible choice for α_x.

Example 3.2. *Consider the random variable X_1 with $\mathrm{P}[X_1 = 0] = p_1 = 1 - \mathrm{P}[X_1 = 1]$. Then we have that*

$$F_{X_1}^{-1}(p) = \begin{cases} -\infty & \text{if } p = 0 \\ 0 & \text{if } 0 < p \leq p_1 \\ 1 & \text{if } p_1 < p \leq 1 \end{cases}$$

and

$$F_{X_1}^{-1+}(p) = \begin{cases} 0 & \text{if } 0 \leq p < p_1 \\ 1 & \text{if } p_1 \leq p < 1 \\ +\infty & \text{if } p = 1 \end{cases}.$$

For any $\alpha \in [0,1]$, one has that

$$F_{X_1}^{-1(\alpha)}(p) = \begin{cases} 0 & \text{if } 0 \leq p < p_1 \\ 1-\alpha & \text{if } p = p_1 \\ 1 & \text{if } p_1 < p \leq 1 \end{cases}$$

Finally, for any $x \in \left[F_{X_1}^{-1+}(0), F_{X_1}^{-1}(1)\right] = \left[0, 1\right]$, the solutions α_x of (3.1) are given by $\alpha_x = 1 - x$ if $0 \leq x < 1$, while $\alpha_x \in [0,1]$ in case $x = 1$.

Comonotonicity is an important dependency structure which is particularly relevant for the study of the quantile risk-sharing rule. For completeness, we repeat its definition hereafter.

Definition 3.3. *A random vector $\boldsymbol{X}$ is comonotonic if there exist non-decreasing functions $g_i : \mathbb{R} \to \mathbb{R}$ such that*

$$\boldsymbol{X} = \big(g_1(S_{\boldsymbol{X}}), \ldots, g_n(S_{\boldsymbol{X}})\big). \tag{3.2}$$

Equivalently, $\boldsymbol{X}$ is comonotonic if for the random variable U which is uniformly distributed over the unit interval $[0,1]$, one has that

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right). \tag{3.3}$$

Comonotonicity and its applications in insurance and finance have been studied in detail in the actuarial literature, see e.g. Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a, 2002b), Deelstra, Dhaene & Vanmaele (2010) and Linders, Dhaene & Schoutens (2015).

To any pool $\boldsymbol{X}$, let us associate its "comonotonic counterpart"

$$\boldsymbol{X}^c = \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{3.4}$$

which is by our earlier convention about U, defined in the original probability space. We introduce the notation $S_{\boldsymbol{X}}^{\mathrm{c}}$ for the sum of the components of $\boldsymbol{X}^c$. For any $\alpha \in [0,1]$ and $p \in [0,1]$, the following additivity property holds:

$$F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1(\alpha)}(p) = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(p)\,. \tag{3.5}$$

In particular, we find that

$$F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1+}(0) = \sum_{i=1}^{n} F_{X_i}^{-1+}(0) \quad \text{and} \quad F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1}(1) = \sum_{i=1}^{n} F_{X_i}^{-1}(1)\,. \tag{3.6}$$

In this paper, we say that a set $C \subseteq \mathbb{R}^n$ is a support of a random vector $\boldsymbol{X}$ if $\mathrm{P}[\boldsymbol{X} \in C] = 1$. One particular choice for the support of $S_{\boldsymbol{X}}$ is given by

$$\text{Support}\,[S_{\boldsymbol{X}}] = \left\{F_{S_{\boldsymbol{X}}}^{-1}(F_{S_{\boldsymbol{X}}}(s)) \mid s \in \left[F_{S_{\boldsymbol{X}}}^{-1+}(0), F_{S_{\boldsymbol{X}}}^{-1}(1)\right]\right\}. \tag{3.7}$$

In particular, one choice of the support of $S_{\boldsymbol{X}}^{\mathrm{c}}$ is

$$\text{Support}\,[S_{\boldsymbol{X}}^{\mathrm{c}}] = \left\{F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1}(F_{S_{\boldsymbol{X}}^{\mathrm{c}}}(s)) \mid s \in \left[F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1+}(0), F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1}(1)\right]\right\}. \tag{3.8}$$

Notice that our definition of supports differs from the usual one, where the support of $\boldsymbol{X}$ is the smallest closed set C such that $\mathrm{P}[\boldsymbol{X} \in C] = 1$. See Appendix A for a discussion on the supports of $S_{\boldsymbol{X}}$ and $S_{\boldsymbol{X}}^{\mathrm{c}}$ defined respectively by (3.7)-(3.8).

3.2 Definition of the quantile risk-sharing rule

We can now define the quantile risk-sharing rule, which was described informally in the introduction to this paper, in a rigorous way.

Definition 3.4. *Under the quantile risk-sharing rule $\mathbb{H}^{\text{quant}} : \chi^n \to \chi^n$, the contribution vector $\boldsymbol{H}^{\text{quant}}$ for a pool $\boldsymbol{X} \in \chi^n$ is given by*

$$\boldsymbol{H}^{\text{quant}} = \boldsymbol{h}^{\text{quant}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})\,, \tag{3.9}$$

where $\boldsymbol{h}^{\text{quant}} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ is defined by

$$\boldsymbol{h}^{\text{quant}}(s; F_{\boldsymbol{X}}) = \left(F_{X_1}^{-1(\alpha_s)}\left(F_{S_{\boldsymbol{X}}^{\mathrm{c}}}(s)\right), \ldots, F_{X_n}^{-1(\alpha_s)}\left(F_{S_{\boldsymbol{X}}^{\mathrm{c}}}(s)\right)\right), \tag{3.10}$$

with α_s following from

$$F_{S_{\boldsymbol{X}}^{\mathrm{c}}}^{-1(\alpha_s)}\left(F_{S_{\boldsymbol{X}}^{\mathrm{c}}}(s)\right) = s. \tag{3.11}$$

For any given $s \in \mathbb{R}$ and $\boldsymbol{X} \in \chi^n$, the additivity property (3.5) combined with (3.11) guarantees that

$$\sum_{i=1}^{n} \boldsymbol{h}_i^{\text{quant}}(s; F_{\boldsymbol{X}}) = s \quad \text{whenever } s \in \Big[F_{S_{\boldsymbol{X}}^c}^{-1+}(0), F_{S_{\boldsymbol{X}}^c}^{-1}(1)\Big], \tag{3.12}$$

and hence $\mathbb{H}^{\text{quant}}$ satisfies the self-financing condition (2.1). Furthermore, $\mathbb{H}^{\text{quant}}$ is an aggregate risk-sharing rule by definition. An important observation is that the quantile risk-sharing rule does not require the knowledge of the dependency structure of the joint distribution function $F_{\boldsymbol{X}}$ of loss vector $\boldsymbol{X}$. It suffices to know the marginal distribution functions F_{X_i} of the individual losses X_i. This property will be formalized in the next section. Furthermore, it can be proven that all $F_{X_i}^{-1(\alpha_s)}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ are non-decreasing and Lipschitz continuous functions in s, see Denuit, Dhaene & Robert (2022). This observation immediately implies that the contribution vector $\boldsymbol{H}^{\text{quant}}$ is a comonotonic random vector, which means that the quantile risk-sharing rule transforms pools into comonotonic contribution vectors.

For any given comonotonic random vector $\boldsymbol{X}^c$, one particular choice of its support is

$$\text{Support}[\boldsymbol{X}^c] = \{\big(F_{X_1}^{-1}(u), \ldots, F_{X_n}^{-1}(u)\big) : 0 \le u \le 1\}.$$

This support is not necessarily a connected curve in $\mathbb{R}^n$ but rather a series of ordered connected curves in general; any horizontal segment of one of the marginal distribution functions F_{X_i} would lead to a discontinuity in Support$[\boldsymbol{X}^c]$. If the endpoints of consecutive curves in Support$[\boldsymbol{X}^c]$ are connected by straight lines, we obtain a comonotonic connected curve in $\mathbb{R}^n$. We will call this set the connected support of $\boldsymbol{X}^c$ and denote it by CSupport$[\boldsymbol{X}^c]$. It can be parameterized as follows:

$$\text{CSupport}[\boldsymbol{X}^c] = \{\big(F_{X_1}^{-1(\alpha)}(u), \ldots, F_{X_n}^{-1(\alpha)}(u)\big) : 0 \le u \le 1, 0 \le \alpha \le 1\}.$$

We make the convention here that both Support$[\boldsymbol{X}^c]$ and CSupport$[\boldsymbol{X}^c]$ have to be seen as subsets of $\mathbb{R}^n$. We refer to Dhaene et al. (2002a) for more discussion on the notion of connected support.

Of course, one may choose to enlarge Support$[\boldsymbol{X}^c]$ to form a comonotonic connected curve by connecting the endpoints in any other way as long as the curve after connection is comonotonic. However, connecting the endpoints by straight lines is probably the most natural and easiest way to do so, and there seems no theoretical reasons to justify other ways.

Example 3.5. *Consider the pool $\boldsymbol{X} = (X_1, X_2, X_3)$ with individual losses X_i such that* $\mathrm{P}[X_i = 0] = p_i = 1 - \mathrm{P}[X_i = 1]$. *Suppose that $0 < p_1 < p_2 < p_3 < 1$. The comonotonic sum $S_{\boldsymbol{X}}^c$ is defined by $S_{\boldsymbol{X}}^c = \sum_{i=1}^{3} F_{X_i}^{-1}(U)$. Taking into account the results in Example 3.2, we find for any $\alpha \in [0,1]$ that*

$$F_{S_{\boldsymbol{X}}^c}^{-1(\alpha)}(p) = \begin{cases} 0 & \textit{if } 0 \le p < p_1 \\ 1-\alpha & \textit{if } p = p_1 \\ 1 & \textit{if } p_1 < p < p_2 \\ 2-\alpha & \textit{if } p = p_2 \\ 2 & \textit{if } p_2 < p < p_3 \\ 3-\alpha & \textit{if } p = p_3 \\ 3 & \textit{if } p_3 < p \le 1 \end{cases}.$$

It follows then that $S^{\mathrm{c}}_{\boldsymbol{X}}$ can be expressed as follows:

$$S^{\mathrm{c}}_{\boldsymbol{X}} = F^{-1}_{S^{\mathrm{c}}_{\boldsymbol{X}}}(U) = \begin{cases} 0 & \text{if } 0 \le U \le p_1 \\ 1 & \text{if } p_1 < U \le p_2 \\ 2 & \text{if } p_2 < U \le p_3 \\ 3 & \text{if } p_3 < U \le 1 \end{cases}.$$

This implies that the cdf of $S^{\mathrm{c}}_{\boldsymbol{X}}$ is given by

$$F_{S^{\mathrm{c}}_{\boldsymbol{X}}}(s) = \begin{cases} p_1 & \text{if } 0 \le s < 1 \\ p_2 & \text{if } 1 \le s < 2 \\ p_3 & \text{if } 2 \le s < 3 \\ 1 & \text{if } s = 3 \end{cases}.$$

For any $s \in [0,3]$, the solutions $\alpha_s \in [0,1]$ of (3.11) are given by

$$\begin{cases} \alpha_s = 1 - s & \text{if } 0 \le s < 1 \\ \alpha_s = 2 - s & \text{if } 1 \le s < 2 \\ \alpha_s = 3 - s & \text{if } 2 \le s < 3 \\ \alpha_s \in [0,1] & \text{if } s = 3 \end{cases}.$$

From the results stated above, we find that the contribution H_i^{quant} for participant $i = 1, 2, 3$ in the pool $\boldsymbol{X}$ is given by

$$H_i^{quant} = \begin{cases} 0 & \text{if } S_{\boldsymbol{X}} < i \\ 1 & \text{if } S_{\boldsymbol{X}} \ge i \end{cases}.$$

This risk-sharing rule can be interpreted as follows. Suppose that p_i is the no-claim probability for participant i. At time 1, the pool pays 1 to any participant with a claim in previous period. In case no claims are reported in the pool, not any participant has to contribute. In case of 1 claim, only the participant with the highest claim probability (i.e. participant 1) pays a contribution of 1. In case of 2 claims, the two participants with highest claim probabilities (i.e. participant 1 and 2) have to contribute an amount of 1. Finally, in case of 3 reported claims, each participant contributes an amount of 1.

4 Properties of risk-sharing rules

Properties that risk-sharing rules may (or may not) satisfy have been studied in detail in Denuit, Dhaene & Robert (2022), as well as in Jiao, Kou, Liu & Wang (2022), who also provide axiomatic characterizations of the conditional mean risk-sharing rule. Hereafter, we repeat the definitions of the "comonotonicity" property and the "stand-alone for comonotonic pools" property of risk-sharing rules. We also introduce two new properties which we call "dependence-free" and "law-invariance".

Definition 4.1 (Comonotonicity property)**.** *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is comonotonic if there exists a function $\boldsymbol{h}^{\mathrm{com}} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ such that the contribution vector $\boldsymbol{H}$ of any $\boldsymbol{X} \in \chi^n$ can be expressed as*

$$\boldsymbol{H} = \boldsymbol{h}^{\mathrm{com}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}}) = \left(h_1^{\mathrm{com}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}}), \ldots, h_n^{\mathrm{com}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})\right), \tag{4.1}$$

where each h_i^{com}, $i = 1, 2, \ldots, n$, is non-decreasing in its first argument.

If $\mathbb{H}$ is comonotonic, then it is an aggregate risk-sharing rule, and for any pool $\boldsymbol{X}$, one has that the contribution vector $\boldsymbol{H}$ is a comonotonic random vector. Notice that the definition here is more restrictive than the one in Denuit, Dhaene & Robert (2022), as here we also require the risk-sharing rule $\mathbb{H}$ to be internal.

Example 4.2. *The quantile risk-sharing rule $\mathbb{H}^{quant} : \chi^n \to \chi^n$ is comonotonic. Indeed, by (3.10) and (3.11), every component in the contribution vector $\boldsymbol{h}^{\text{quant}}(\boldsymbol{X}; F_{\boldsymbol{X}})$ of any given pool $\boldsymbol{X}$ is non-decreasing in $S_{\boldsymbol{X}}$, and hence the contribution vector is a comonotonic random vector.*

Next, we introduce the stand-alone property for comonotonic pools.

Definition 4.3. *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ with internal function $\boldsymbol{h} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ is stand-alone for comonotonic pools if the contribution vector $\boldsymbol{H}$ of any comonotonic pool $\boldsymbol{X}^c \in \chi^n$ is given by*

$$\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X}^c, F_{\boldsymbol{X}^c}) = \boldsymbol{X}^c.$$

In a comonotonic pool $\boldsymbol{X}$, for any realized loss amounts $x_1, \ldots, x_n$, the corresponding probability levels $p_1, \ldots, p_n$ that solve the equations $x_i = F_{X_i}^{-1}(p_i)$ would always be identical. Therefore, it maybe reasonable to require that in such a pool each participant remains with his or her own risk, as stated in the property of stand-alone for comonotonic pools.

Suppose that $\mathbb{H} : \chi^n \to \chi^n$ is a risk-sharing rule with internal function $\boldsymbol{h}$, then for any comonotonic pool $\boldsymbol{X}^c$, by definition, $\boldsymbol{X}^c = \boldsymbol{h}(\boldsymbol{X}^c; F_{\boldsymbol{X}^c})$, which is equivalent to

$$\boldsymbol{h}(\boldsymbol{x}^c; F_{\boldsymbol{X}^c}) = \boldsymbol{x}^c \quad \text{for any } \boldsymbol{x}^c \in \text{Support}[\boldsymbol{X}^c]. \tag{4.2}$$

Now we introduce a slightly more general property, which we call the "generalized stand-alone for comonotonic pools" property.

Definition 4.4. *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is generalized stand-alone for comonotonic pools if it is stand-alone for comonotonic pools, with internal function $\boldsymbol{h} : \mathbb{R}^n \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ satisfying*

$$\boldsymbol{h}(\boldsymbol{x}^c; F_{\boldsymbol{X}^c}) = \boldsymbol{x}^c \quad \textit{for any } \boldsymbol{x}^c \in \textit{CSupport}[\boldsymbol{X}^c] \tag{4.3}$$

for any comonotonic pool $\boldsymbol{X}^c$.

In case the risk-sharing rule is aggregate too, condition (4.3) reads as

$$\boldsymbol{h}(s_{\boldsymbol{x}^c}; F_{\boldsymbol{X}^c}) = \boldsymbol{x}^c \quad \text{for any } \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c].$$

The next result demonstrates that the quantile RS rule satisfies the 'generalized stand-alone for comonotonic pools' property.

Proposition 4.5. *The quantile RS rule satisfies the "generalized stand-alone for comonotonic pools" property.*

Proof. Consider a comonotonic pool $\boldsymbol{X}^c \in \chi^n$ and let $\boldsymbol{x}^c$ be a point in CSupport$[\boldsymbol{X}^c]$ with $s_{\boldsymbol{x}^c} = s$. By construction, both $\boldsymbol{x}^c$ and $\boldsymbol{h}^{\text{quant}}(s_{\boldsymbol{x}^c}, F_{\boldsymbol{X}^c})$ lie in the intersection of the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n \mid s_{\boldsymbol{x}} = s\}$ and CSupport$[\boldsymbol{X}^c]$. As there can be no more than one point in such intersection, we conclude that $\boldsymbol{x}^c = \boldsymbol{h}^{\text{quant}}(s_{\boldsymbol{x}^c}, F_{\boldsymbol{X}^c})$, and hence the stated result holds. □

Proposition 4.6. *For any comonotonic random vector $\boldsymbol{X}^c$, one has that*

$$\{s_{\boldsymbol{x}^c} \mid \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c]\} = \left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0), F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right].$$

Proof. From the definition of CSupport$[\boldsymbol{X}^c]$, one finds that

$$\{s_{\boldsymbol{x}^c} \mid \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c]\} = \{\sum_{i=1}^{n} F^{-1(\alpha)}_{X_i}\left(u\right) \mid 0 \leq u \leq 1, 0 \leq \alpha \leq 1\}.$$

Taking into account the additivity property (3.5) leads to

$$\begin{aligned}\{s_{\boldsymbol{x}^c} \mid \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c]\} &= \{F^{-1(\alpha)}_{S^c_{\boldsymbol{X}}}\left(u\right) \mid 0 \leq u \leq 1, 0 \leq \alpha \leq 1\} \\ &= \left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0), F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right],\end{aligned}$$

which proves the stated result. □

Let us now define the dependence-free property of a risk-sharing rule.

Definition 4.7 (Dependence-free property)**.** *The risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is dependence-free if there exists a function $\boldsymbol{h}^{\text{dep-free}} : \mathbb{R}^n \times (\mathcal{F}(\chi))^n \to \mathbb{R}^n$ such that for any pool $\boldsymbol{X} \in \chi^n$, one has that the contribution vector $\boldsymbol{H}$ is given by*

$$\boldsymbol{H} = \boldsymbol{h}^{\text{dep-free}}\left(\boldsymbol{X}; F_{X_1}, \ldots, F_{X_n}\right).$$

From Definition 4.7, it follows that in order to determine the contribution vector $\boldsymbol{H}$ under a dependence-free risk-sharing rule, we only need to know the outcome of $\boldsymbol{X}$ and the marginal distribution functions of the individual losses X_i, but not the dependency structure of $\boldsymbol{X}$. Given the outcome $\boldsymbol{x}$ of $\boldsymbol{X}$, the contribution vector remains the same, regardless of what the dependence structure of $\boldsymbol{X}$ is. It is clear from the definition that a dependence-free risk-sharing is internal, with internal function $\boldsymbol{h}$ satisfying $\boldsymbol{h}\left(\boldsymbol{X}; F_{\boldsymbol{X}}\right) = \boldsymbol{h}^{\text{dep-free}}\left(\boldsymbol{X}; F_{X_1}, \ldots, F_{X_n}\right)$ for any $\boldsymbol{X} \in \chi^n$.

Example 4.8. *The quantile risk-sharing rule $\mathbb{H}^{quant}$ is dependence-free because for any pool $\boldsymbol{X} \in \chi^n$, the function $\boldsymbol{h}^{quant}\left(s_{\boldsymbol{x}}; F_{\boldsymbol{X}}\right)$ is completely determined by $s_{\boldsymbol{x}}$ and the marginal distributions $F_{X_1}, \ldots, F_{X_n}$, and the knowledge of the dependence structure of $\boldsymbol{X}$ is not required.*

In view of the fact that the quantile risk-sharing rule is dependence-free, we can also write $\boldsymbol{h}^{\text{quant}}(s; F_{\boldsymbol{X}})$ as $\boldsymbol{h}^{\text{quant}}(s; F_{X_1}, \ldots, F_{X_n})$.

Remark 4.9 (Distribution-free property)**.** Dependence-free risk-sharing rules do not use the joint distribution function $F_{\boldsymbol{X}}$, only its marginals $F_1, \ldots, F_n$. Let us mention that some rules do not use the joint distribution function at all so that they could be called distribution-free. Formally, a risk-sharing rule is distribution-free if there exists a function $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that the contribution vector $\boldsymbol{H}$ for any pool $\boldsymbol{X}$ is given by $\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X})$. Examples of distribution-free risk-sharing rules are the stand-alone and the uniform risk-sharing rules.

Finally, inspired by the property of law-invariance for risk measures, we introduce the property of law-invariance for risk-sharing rules.

Definition 4.10. *A risk-sharing rule* $\mathbb{H} : \chi^n \to \chi^n$ *is law-invariant if the contribution vectors for equally distributed pools* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, i.e.* $\boldsymbol{X} \overset{d}{=} \boldsymbol{Y}$ *are identically distributed.*

Law-invariance seems to be a reasonable property, in the sense that if pools are "equal in distribution", then it reasonable to require that also their contribution vectors should be "equal in distribution".

Example 4.11. *A straightforward example of a law-invariant risk-sharing rule is the uniform risks-sharing rule under which the contribution vector* $\boldsymbol{H}$ *for any pool* $\boldsymbol{X}$ *is given by*

$$\boldsymbol{H} = \left(\bar{X}_n, \ldots, \bar{X}_n\right) \text{ with } \bar{X}_n = \frac{X_1 + \ldots + X_n}{n}.$$

Indeed, if $\boldsymbol{X} \overset{d}{=} \boldsymbol{Y}$*, then also*

$$\left(\bar{X}_n, \ldots, \bar{X}_n\right) \overset{d}{=} \left(\bar{Y}_n, \ldots, \bar{Y}_n\right)$$

where $\bar{Y}_n = \frac{Y_1 + \ldots + Y_n}{n}$.

Example 4.12. *It is also easy to verify that the conditional mean risk-sharing rule is law-invariant: If* $\boldsymbol{X} \overset{d}{=} \boldsymbol{Y}$*, then*

$$\big(\mathrm{E}\left[X_1 \mid S_{\boldsymbol{X}}\right], \ldots, \mathrm{E}\left[X_n \mid S_{\boldsymbol{X}}\right]\big) \overset{d}{=} \big(\mathrm{E}\left[Y_1 \mid S_{\boldsymbol{Y}}\right], \ldots, \mathrm{E}\left[Y_n \mid S_{\boldsymbol{Y}}\right]\big).$$

These two examples are just particular cases of the following general statement.

Proposition 4.13. *Any internal risk-sharing rule is law-invariant.*

Proof. Consider the internal risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ defined with the help of the function $\boldsymbol{h} : \mathbb{R}^n \times \mathcal{F}\left(\chi^n\right) \to \mathbb{R}^n$ in (2.2). For any pool $\boldsymbol{X}$ the contribution vector is given by $\boldsymbol{H} = \boldsymbol{h}\left(\boldsymbol{X}; F_{\boldsymbol{X}}\right)$. It follows then immediately that $\boldsymbol{X} \overset{\mathrm{d}}{=} \boldsymbol{Y}$ implies $\boldsymbol{h}\left(\boldsymbol{X}; F_{\boldsymbol{X}}\right) \overset{\mathrm{d}}{=} \boldsymbol{h}\left(\boldsymbol{Y}; F_{\boldsymbol{Y}}\right)$. □

Corollary 4.14. *The quantile risk-sharing rule is law-invariant.*

Proof. The proof follows immediately from Proposition 4.13 and the fact that the quantile risk-sharing rule is an aggregate (and hence, also an internal) risk-sharing rule. □

The following example gives a risk-sharing rule which is law-invariant but not internal.

Example 4.15. *Consider the risk-sharing rule* $\mathbb{H} : \chi^2 \to \chi^2$ *introduced in Example 2.3. Assume in addition that the random variable* Z *is independent of any random vector in* χ^2*. It is obvious that* $\mathbb{H}$ *is law-invariant, but not an internal risk-sharing rule.*

5 Axiomatic characterization of the quantile risk-sharing rule

In this section, we give an axiomatic characterization of the quantile risk-sharing rule.

Theorem 5.1. *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is the quantile risk-sharing rule if, and only if, it satisfies the following axioms:*

Axiom 1 $\mathbb{H}$ *is aggregate.*

Axiom 2 $\mathbb{H}$ *is dependence-free.*

Axiom 3 $\mathbb{H}$ *is generalized stand-alone for comonotonic pools.*

Proof. As $\mathbb{H}$ satisfies Axioms 1 and 2, there exists a function $\boldsymbol{h}^{\text{aggr}} : \mathbb{R} \times (\mathcal{F}(\chi))^n \to \mathbb{R}^n$ such that the contribution vector of any pool $\boldsymbol{X} \in \chi^n$ is given by $\boldsymbol{h}^{\text{aggr}}\left(S_{\boldsymbol{X}}; F_{X_1}, \ldots, F_{X_n}\right)$. Let $\boldsymbol{X}^c = \left(F^{-1}_{X_1}(U), F^{-1}_{X_2}(U), \ldots, F^{-1}_{X_n}(U)\right)$ be the comonotonic counterpart of $\boldsymbol{X}$. From Axiom 3, we have that

$$\boldsymbol{h}^{\text{aggr}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{x}^c, \qquad \text{for any } \boldsymbol{x}^c \in \text{CSupport}\left[\boldsymbol{X}^c\right].$$

Since the quantile allocation rule is also dependence-free and generalized stand-alone for comonotonic risks by Proposition 4.5, that is, $\boldsymbol{h}^{\text{quant}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{x}^c$, we find that

$$\boldsymbol{h}^{\text{aggr}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{h}^{\text{quant}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) \text{ for any } \boldsymbol{x}^c \in \text{CSupport}\left[\boldsymbol{X}^c\right].$$

From Proposition 4.6, we find that

$$\boldsymbol{h}^{\text{aggr}}\left(s, F_{X_1}; \ldots, F_{X_n}\right) = \boldsymbol{h}^{\text{quant}}\left(s; F_{X_1}, \ldots, F_{X_n}\right), \qquad \text{for any } s \in \left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0), F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right].$$

As $\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0), F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right]$ by (A.1), the above equation implies that

$$\boldsymbol{h}^{\text{aggr}}\left(S_{\boldsymbol{X}}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{h}^{\text{quant}}\left(S_{\boldsymbol{X}}; F_{X_1}, \ldots, F_{X_n}\right), \tag{5.1}$$

which proves the "$\Longleftarrow$" part of the theorem.

□

In Figure 1, we give a graphical interpretation of the proof of the characterization theorem in the bivariate case. Consider the risk-sharing rule $\mathbb{H} : \chi^2 \to \chi^2$ which satisfies the 3 axioms of the theorem and a pool $\boldsymbol{X} = (X_1, X_2) \in \chi^2$. Let (X_1^c, X_2^c) be its comonotonic counterpart. Suppose that the time-1 observable outcome of (X_1, X_2) is given by (x_1^*, x_2^*), with $x_1^* + x_2^* = s$.

First, suppose that the marginal cdf's F_{X_i} are strictly increasing in x_i^*, $i = 1, 2$. Taking into account Axiom 1 ($\mathbb{H}$ is aggregate) we have that the contribution vector of the pool (X_1, X_2) is given by $\boldsymbol{h}\left(x_1^* + x_2^*; F_{\boldsymbol{X}}\right)$ for some function $\boldsymbol{h} : \mathbb{R} \times \mathcal{F}\left(\chi^2\right) \to \mathbb{R}^2$. Let (x_1^c, x_2^c) be the unique point on the intersection of the line $x_1 + x_2 = s$ and $\text{Support}[(X_1^c, X_2^c)]$. We know that (x_1^c, x_2^c) is given by $(x_1^c, x_2^c) = \left(F^{-1}_{X_1}(F_{S^c_{\boldsymbol{X}}}(s), F^{-1}_{X2}(F_{S^c_{\boldsymbol{X}}}(s)\right)$. From Axiom 1 ($\mathbb{H}$ is aggregate), we find that

$$\boldsymbol{h}\left(x_1^* + x_2^*; F_{\boldsymbol{X}}\right) = \boldsymbol{h}\left(x_1^c + x_2^c; F_{\boldsymbol{X}}\right).$$

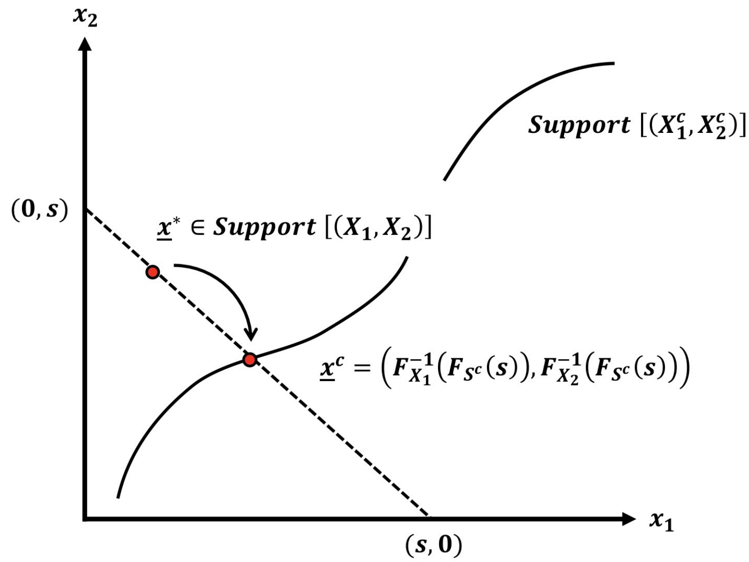

Figure 1: Graphical interpretation of the quantile risk-sharing rule, bivariate case (part I).

From Axiom 2 ($\mathbb{H}$ is dependence-free), it follows that

$$\boldsymbol{h}\left(x_1^c + x_2^c; F_{\boldsymbol{X}}\right) = \boldsymbol{h}\left(x_1^c + x_2^c; F_{\boldsymbol{X}^c}\right).$$

Axiom 3 ($\mathbb{H}$ is generalized stand-alone for comonotonic pools) leads to

$$\boldsymbol{h}\left(x_1^c + x_2^c; F_{\boldsymbol{X}^c}\right) = \left(x_1^c, x_2^c\right).$$

Summarizing, when the realization of (X_1, X_2) equals (x_1^*, x_2^*), then we have that the realization of the contribution vector is given by

$$\boldsymbol{h}\left(x_1^* + x_2^*; F_{\boldsymbol{X}}\right) = \left(F_{X_1}^{-1}(F_{S_{\boldsymbol{X}}^c}(s), F_{X_2}^{-1}(F_{S_{\boldsymbol{X}}^c}(s)\right).$$

Next, suppose that the the marginal cdf's F_{X_i} are not both strictly increasing in x_i^*, $i = 1, 2$. In this case, the line $x_1 + x_2 = s$ has no intersection with Support$[(X_1^c, X_2^c)]$, and we introduce the connected support of this comonotonic random vector. The graphical interpretation of the characterization theorem in this case follows then in a similar way as before, see Figure 2.

Let us now show that the three axioms in Theorem 5.1 are independent, which means that none of them can be removed to characterize the quantile risk-sharing rule. In other words, any combination of two of these axioms does not imply the remaining third axiom.

Proposition 5.2. *Axioms 1-3 in Theorem 5.1 are independent.*

Proof. For each of the three axioms, we have to provide an example of a risk-sharing rule which is different from the quantile risk-sharing rule, and which does not satisfy this axiom, while it satisfies the two other axioms.

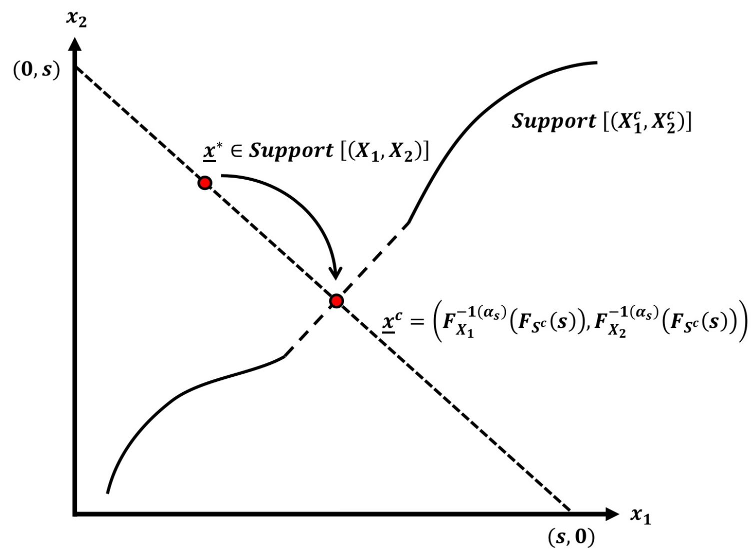

Figure 2: Graphical interpretation of the quantile risk-sharing rule, bivariate case (part II).

Axioms 2 and 3, but not Axiom 1: Consider the stand-alone risk-sharing rule $\mathbb{H}^{\text{sa}} : \chi^n \to \chi^n$ with contribution vector $\boldsymbol{H} = \boldsymbol{X}$ for any pool $\boldsymbol{X} \in \chi^n$ and with internal function $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^n$, see Example 2.5. It is straightforward to prove that $\mathbb{H}^{\text{sa}}$ is "dependence-free" and "generalized stand-alone for comonotonic pools", but does not satisfy the "aggregate" axiom.

Axioms 1 and 3, but not Axiom 2: Let us extend Example 2.6 to any dimension n and define the risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ with contribution vector

$$\boldsymbol{H} = \left\{ \begin{array}{cc} \boldsymbol{X}, & \text{if } \boldsymbol{X} \text{ is comonotonic,} \\ \left(\bar{X}_n, \bar{X}_n, \ldots, \bar{X}_n\right), & \text{otherwise,} \end{array} \right)$$

and internal function satisfying $\boldsymbol{h}\left(\boldsymbol{x}; F_{\boldsymbol{X}^c}\right) = \boldsymbol{x}$ for any $\boldsymbol{x}$ and any $\boldsymbol{X^c}$.

Under this rule, participants are left with their own risk in any comonotonic pool (since there is no diversification in that case) while total losses are distributed uniformly among participants in all other cases. It is straightforward to prove that $\mathbb{H}$ is an "aggregate" risk sharing rule and is "stand-alone for comonotonic pools", but does not satisfy the "dependence-free" axiom.

Axioms 1 and 2, but not Axiom 3: Referring to Example 4.11, the uniform risk-sharing rule $\mathbb{H}^{\text{uni}} : \chi^n \to \chi^n$ is defined for any pool $\boldsymbol{X} \in \chi^n$ by contribution vector

$$\boldsymbol{H}^{\text{uni}} = \left(\bar{X}_n, \bar{X}_n, \ldots, \bar{X}_n\right).$$

It is straightforward to prove that $\mathbb{H}^{\text{uni}}$ satisfies the "aggregate" and "dependence-free" axioms, but not the "generalized stand-alone for comonotonic pools" axiom.

□

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Appendix

A Comonotonicity and supports of distributions

Consider a pool $\boldsymbol{X}$ and its comonotonic counterpart $\boldsymbol{X}^c$, which is defined in the original probability space (see (3.4)). As before, $S^c_{\boldsymbol{X}}$ stands for the sum of the components of $\boldsymbol{X}^c$.

The support of the aggregate claims $S_{\boldsymbol{X}}$ of the pool $\boldsymbol{X}$ is defined by (3.7). Recall that throughout the paper, we make the convention that the interval $\left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]$ has to be replaced by $\left(F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right)$ in case $F^{-1+}_{S_{\boldsymbol{X}}}(0) = -\infty$ and $F^{-1}_{S_{\boldsymbol{X}}}(1) = +\infty$. Similar conventions are made in case only one of the endpoints of the interval $\left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]$ is infinite. One can easily verify that

$$\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]. \tag{A.1}$$

The support of the aggregate claims $S^c_{\boldsymbol{X}}$ of the comonotonic pool $\boldsymbol{X}^c$ is defined by (3.8) where we make a similar convention as before concerning the endpoints of the interval $\left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right]$. In this case, we have that

$$\text{Support}\left[S^c_{\boldsymbol{X}}\right] \subseteq \left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{c}_{S^c_{\boldsymbol{X}}}(1)\right]. \tag{A.2}$$

Remark that the endpoints of the intervals $\left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]$ and $\left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{c}_{S^c_{\boldsymbol{X}}}(1)\right]$ always satisfy the following inequalities:

$$F^{-1+}_{S^c_{\boldsymbol{X}}}(0) \leq F^{-1+}_{S_{\boldsymbol{X}}}(0) \leq F^{-1}_{S_{\boldsymbol{X}}}(1) \leq F^{-1}_{S^c_{\boldsymbol{X}}}(1). \tag{A.3}$$

Example A.1. *It is obvious that* Support $[S^c_{\boldsymbol{X}}]$ *is not always a subset of* Support $[S_{\boldsymbol{X}}]$. *A simple example illustrating this fact is the bivariate random vector* (X_1, X_2), *with* $X_1 = U$ *and* $X_2 = 1 - U$. *In this case, we have that* $S_{\boldsymbol{X}} = 1$, *and hence,*

$$\text{Support}\left[S_{\boldsymbol{X}}\right] = \{1\},$$

while taking into account that $S^c_{\boldsymbol{X}} \stackrel{d}{=} 2U$ *leads to*

$$\text{Support}\left[S^c_{\boldsymbol{X}}\right] = [0, 2].$$

A somewhat less obvious fact is that Support $[S_{\boldsymbol{X}}]$ *is not always a subset of* Support $[S^c_{\boldsymbol{X}}]$. *In order to illustrate this statement, consider the mutually independent random variables* X_1 *and* X_2, *which are both uniformly distributed over* $[0, 1] \cup [2, 3]$. *Then we have that*

$$\text{Support}\left[S_{\boldsymbol{X}}\right] = [0, 6] \qquad \textit{and} \qquad \text{Support}\left[S^c_{\boldsymbol{X}}\right] = [0, 2] \cup [4, 6].$$

The following result gives conditions under which Support $[S_{\boldsymbol{X}}] \subseteq$ Support $[S^c_{\boldsymbol{X}}]$. Remark that we will say that a distribution function F_X is strictly increasing if it is strictly increasing over the interval $\left[F^{-1+}_X(0), F^{-1}_X(1)\right]$.

Proposition A.2. *If F_{X_i} is strictly increasing, $i = 1, 2, \ldots, n$, then*

$$\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \text{Support}\left[S_{\boldsymbol{X}}^c\right] = \left[F_{S_{\boldsymbol{X}}^c}^{-1+}(0), F_{S_{\boldsymbol{X}}^c}^{-1}(1)\right].$$

Proof. If all F_{X_i} are strictly increasing, then all $F_{X_i}^{-1}$ are continuous. Taking into account the additivity property (3.5), this implies that $F_{S_{\boldsymbol{X}}^c}^{-1}$ is continuous, and hence, F_{S^c} is strictly increasing, which implies that $\text{Support}\left[S_{\boldsymbol{X}}^c\right] = \left[F_{S_{\boldsymbol{X}}^c}^{-1+}(0), F_{S_{\boldsymbol{X}}^c}^{-1}(1)\right]$. From (A.1), (A.2) and (A.3), it follows then that

$$\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \left[F_{S_{\boldsymbol{X}}}^{-1+}(0), F_{S_{\boldsymbol{X}}}^{-1}(1)\right] \subseteq \left[F_{S_{\boldsymbol{X}}^c}^{-1+}(0), F_{S_{\boldsymbol{X}}^c}^{-1}(1)\right] = \text{Support}\left[S_{\boldsymbol{X}}^c\right].$$

This ends the proof. □