COMONOTONICITY AND PARETO OPTIMALITY, WITH APPLICATION TO COLLABORATIVE INSURANCE

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Abstract. Two by-now folkloric results in the theory of risk sharing are that (i) any feasible allocation is convex-order-dominated by a comonotonic allocation; and (ii) an allocation is Pareto optimal for the convex order if and only if it is comonotonic. Here, comonotonicity corresponds to the so-called *no-sabotage condition*, which aligns the interests of all parties involved. Several proofs of these two results have been provided in the literature, all based on a version of the comonotonic improvement algorithm of Landsberger and Meilijson (1994) and a limit argument based on the Martingale Convergence Theorem. However, no proof of (i) is explicit enough to allow for an easy algorithmic implementation in practice; and no proof of (ii) provides a closed-form characterization of Pareto optima. In addition, while all of the existing proofs of (i) are provided only for the case of a two-agent economy with the observation that they can be easily extended beyond two agents, such an extension is far from being trivial in the context of the algorithm of Landsberger and Meilijson (1994) and it has never been explicitly implemented. In this paper, we provide novel proofs of these foundational results. Our proof of (i) is based on the theory of majorization and an extension of a result of Lorentz and Shimogaki (1968), which allows us to provide an explicit algorithmic construction that can be easily implemented beyond the case of two agents. In addition, our proof of (ii) leads to a crisp closed-form characterization of Pareto-optimal allocations in terms of α-quantiles (mixed quantiles). An application to peer-to-peer insurance, or collaborative insurance, illustrates the relevance of these results.

1. Introduction and Motivation

Risk-sharing mechanisms have been studied for decades in the actuarial literature. The pioneering work of Borch (1960, 1962) considered equilibrium in a reinsurance market. Under appropriate conditions, this author established that any Pareto-optimal allocation is equivalent to a pooling arrangement, i.e., all the agents hand their individual losses over to a pool and agree on some rule as to how the total pooled loss would be divided among agents. This corresponds to aggregate risk-sharing rules, meaning that individual contributions only depend on the total losses of the pool. Landsberger and Meilijson (1994) then showed that Pareto optima are comonotonic (and are hence pooling arrangements) if agents' preferences agree with the convex order. These authors provided an algorithm to construct a comonotonic convex-order improvement over any non-comonotonic risk allocation in the discrete case, which has since been extended to more general cases. See, e.g., Ludkovski and Rüschendorf (2008) for continuous, unbounded risks, and Carlier et al. (2012) for bounded risks, as well as the references therein. We provide a short overview of the relevant literature at the beginning of Section 3. We believe that it is important to stress from the outset that all of the existing proofs of (i) are provided only for the case of a two-agent economy, with the observation that they can be easily extended beyond two agents. However, such an extension is far from being trivial in the context of the algorithm of Landsberger and Meilijson (1994), and it has never been explicitly implemented in the aforementioned literature.

The so-called *no-sabotage condition*, as referred to by Carlier and Dana (2003), or the comonotonicity property in Denuit et al. (2022) imposes that no individual contribution decreases when total losses increase. Stated otherwise, this means that individual contributions are comonotonic since each component of a comonotonic random vector is almost surely equal to a non-decreasing function of the sum of all of its components. In the context of risk sharing, no-sabotage or comonotonicity seems to be a desirable property since it ensures that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible.

In this paper, we contribute to the literature on risk sharing under the convex order in two directions. First, we revisit the classical result that Pareto optimality for the convex order and comonotonicity are equivalent (Theorem 3.1). Under the assumption that all random variables involved have finite second moments[1], we provide a novel proof, based on some basic properties of comonotonic sums. Our proof of this equivalence is somewhat simpler than the proofs proposed so far in the literature. In addition, our approach provides a crisp closed-form characterization of Pareto-optimal allocations in terms of α-quantiles, also known as mixed quantiles (Theorem 2.7). Closed-form characterizations of Pareto optima are typically lacking in the related literature, and our approach fills this gap.

Second, all extensions of the classical result that any feasible allocation is convex-order-dominated by a comonotonic allocation rely, in one way or another, on the comonotonic improvement algorithm of Landsberger and Meilijson (1994) and a limit argument based on the Martingale Convergence Theorem. None of the existing proofs are explicit enough to allow for an easy algorithmic implementation in practice. Moreover, all of the existing proofs are provided only for the case of a two-agent economy, and an extension to more than two agents has never been explicitly implemented. In this paper, we suggest an alternative approach, based on the theory of majorization and an extension of a result of Lorentz and Shimogaki (1968), which allows us to provide an explicit algorithmic construction that can be easily implemented beyond the case of two agents. We provide numerical illustrations thereof, and discuss how our approach compares with the Landsberger-Meilijson algorithm in Section 3.2.

As an application, we consider collaborative insurance, also referred to as Peer-to-Peer (P2P) insurance, or crowdsurance. This corresponds to emerging, technology-based risk-sharing networks where a group of individuals (e.g., friends, family members, affinity groups, or individuals with similar interests, such as patients suffering some disease or farmers in the same geographical area) pool their resources together in order to insure against a given peril. Rooted in the sharing economy, P2P insurance revives early forms of mutual insurance. Actuaries started to investigate the mathematics supporting this new insurance paradigm quite recently (e.g., Denuit and Dhaene (2012), Denuit (2019), and Abdikerimova and Feng (2022)). To avoid counterparty risk and to be able to deal with larger sums insured, Denuit (2020) replaced unlimited *ex post* contributions characterizing pure risk-sharing solutions with a deposit paid in advance, combined with an *ex post* bonus mechanism restoring fairness, with the guarantee that the final amount due never exceeds this down payment. Part of the deposit feeds a common fund, while the remaining part is paid to a partnering insurance company. If the common fund is insufficient to pay for the claims, then the insurance carrier pays the excess amount. Conversely, if the pool has few claims then the surplus is given back to the participants, or to a cause that the pool members care about.

Here, we adopt the opposite approach to Denuit (2020), and we extend the approach to any allocation rule satisfying the comonotonicity property. Specifically, we propose an hybrid scheme where participants retain the lower layer through individual deductibles (depending on their own risk appetite), the community keeps the intermediate layer (which can be seen as a pooled deductible), and the upper layer is ceded to an insurance company. Participants are free to select their maximum contribution to pooled losses, and an excess-of-loss risk transfer to a partnering insurance company operates beyond. The amount to be paid in advance by participants is obtained by adding the price of the excess-of-loss protection and a deposit securing the contribution to the coverage of the layer mutualized inside the P2P community. The system is thus fully funded, and a cash-back or give-back mechanism operates *ex post* to restore fairness. Provided individual contributions are comonotonic, we show that the excess-of-loss protection reduces to a stop-loss protection limiting the community's total payout under an appropriate choice of initial deposits (Proposition 4.1), thereby extending a result of Chen et al. (2023).

[1] This assumption can be dropped if the aggregate risk is a continuous random variable.

The remainder of this paper is organized as follows. Section 2 considers an insurance pool where economic agents have convex-order preferences. The equivalence between comonotonicity and Pareto optimality is then established therein using simple arguments, under the assumption that risks have finite variances. The key argument is the representation of the components of a comonotonic random vector as specific functions of their sum, which are characterized in closed form. Section 3 provides an alternative approach to the convex-order improvement via comonotonic allocations, using the theory of majorization instead of the usual approach that relies on the algorithm of Landsberger and Meilijson (1994) and a limit argument based on the Martingale Convergence Theorem. The main result therein is our Theorem 3.1, the proof of which explicitly provides the relevant convex-order improvement algorithm that can be implemented in practice. Section 4 applies our results to P2P insurance. New schemes are proposed, combining risk retention at the individual level, risk transfer for losses that are too costly, and risk sharing for the middle layer. Section 5 concludes. All proofs are given in the appendices, with the exception of the proof of Theorem 3.1, which is in the main text, because it provides the different steps for the implementation of our comonotonic allocation improvement algorithm.

2. Comonotonicity and Pareto Optimality for the Convex Order

2.1. **Allocations.** Consider a group of individuals exposed to some peril causing a random non-negative monetary loss at the end of a given observation period, taken to be the time interval $(0,1)$. These losses are defined on a common nonatomic[2] probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{X} \subset L^2_+(\Omega, \mathcal{F}, \mathbb{P})$ denote an *ex-ante* given collection of non-negative random variables with finite second moments. We interpret $\mathcal{X}$ as the collection of potential risks under interest. We assume that $\mathcal{X}$ is rich enough to contain all the random variables mentioned throughout the text. In particular, we assume that $\mathcal{X}$ contains unit uniform random variables.

In the remainder of the text, we consider n agents in a risk pool, each subject to an insurable risk modeled as a random variable belonging to $\mathcal{X}$. The initial endowment $\boldsymbol{X}_0 = (X_{0,1}, \ldots, X_{0,n}) \in \mathcal{X}^n$ represents the risks faced by the n agents individually, before any risk sharing takes place. The pool's aggregate risk is the sum of all n individual risks comprised into the pool. Total losses are henceforth denoted as $S := \sum_{i=1}^n X_{0,i}$ and S also belongs to $\mathcal{X}$. An allocation of the aggregate risk S is a random vector $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$ such that $\sum_{i=1}^n X_i = S$. Let $\mathcal{A}$ denote the collection of all allocations:

$$\mathcal{A} := \left\{ \boldsymbol{X} \in \mathcal{X}^n \;\middle|\; \sum_{i=1}^n X_i = S \right\}. \tag{1}$$

Obviously, the initial endowment $\boldsymbol{X}_0$ is itself an allocation in $\mathcal{A}$.

In insurance applications, individual allocations within insurance pools are generally obtained by applying risk-sharing rules. Formally, a risk-sharing rule is a mapping which transforms any pool $\boldsymbol{X} \in \mathcal{A}$ into another random vector $\boldsymbol{Z} \in \mathcal{A}$. The results derived in this paper can be equivalently stated in terms of risk-sharing rules or allocations.

[2] The assumption of nonatomicity is without loss of generality, since any finite measure space can be embedded in a non-atomic measure space. See, for instance, Luxemburg (1967) or Chong and Rice (1971) for a description of this classical procedure. This embedding is also covered in Bennett and Sharpley (1988), where it is referred to as the "method of retracts" on page 54.

2.2. **Pareto Optimality for the Convex Order.** Each agent ranks elements of $\mathcal{X}$ weakly according to the convex order $\preccurlyeq_{\text{CVX}}$, and strictly according to the strict convex order $\prec_{\text{CVX}}$, whose definition is recalled next.

Definition 2.1. *For all $Y, Z \in \mathcal{X}$,*

$$Y \preccurlyeq_{CVX} Z \iff \mathbb{E}[Y] = \mathbb{E}[Z] \text{ and } \mathbb{E}\left[(Y-d)_+\right] \leq \mathbb{E}\left[(Z-d)_+\right], \ \forall d \in \mathbb{R}^+.$$

If, in addition, the inequality is strict for some $d^* \in \mathbb{R}^+$, we then write $Y \prec_{CVX} Z$.

Hence, $Y \preccurlyeq_{\text{CVX}} Z$ can be understood as Y having the same expectation as Z, but Y being "less variable" than Z is some sense. In particular, $Y \preccurlyeq_{\text{CVX}} Z \Rightarrow \text{Var}[Y] \leq \text{Var}[Z]$. We refer the reader to Denuit et al. (2005) or Shaked and Shanthikumar (2007) for an extensive presentation of the convex order and its applications.

We are now ready to define the concept of Pareto optimality with respect to the convex order.

Definition 2.2. *An allocation $\boldsymbol{Y} = (Y_1, \ldots, Y_n) \in \mathcal{A}$ is said to be:*

(1) A Pareto Improvement over an allocation $\boldsymbol{Z} = (Z_1, \ldots, Z_n) \in \mathcal{A}$ if

$$\begin{cases} Y_i \preccurlyeq_{CVX} Z_i, \text{ for all } i \in \{1, \ldots, n\}; \text{ and} \\ \exists\, j \in \{1, \ldots, n\},\, Y_j \prec_{CVX} Z_j. \end{cases}$$

(2) Pareto Optimal (PO) if there is no allocation $\boldsymbol{Z} \in \mathcal{A}$ that is a Pareto Improvement over $\boldsymbol{Y}$.

(3) Fair (or actuarially fair) if $\mathbb{E}[Y_i] = \mathbb{E}[X_{0,i}]$, for all $i \in \{1, \ldots, n\}$.

It is well-known that for the convex order, conditional expectations provide an improvement. This is formally recalled below.

Proposition 2.3. *For any fair allocation $\boldsymbol{Y} \in \mathcal{A}$, the random vector $\big(\mathbb{E}[Y_1|S], \ldots, \mathbb{E}[Y_n|S]\big) \in \mathcal{A}$ is a fair allocation that satisfies $\mathbb{E}[Y_i|S] \preccurlyeq_{CVX} Y_i$, for $i \in \{1, \ldots, n\}$.*

When applied to the initial endowment, conditional expectations given the sum define the Conditional Mean Risk-Sharing (CMRS) rule, as referred to after Denuit and Dhaene (2012). Jiao et al. (2022) provide an axiomatic characterization of the CMRS.

Note that conditional expectations $\mathbb{E}[Y_i|S]$ are not always increasing in S, and so the CMRS rule does not necessarily possess the comonotonicity property. An example of such a situation is given in Section 3.2. When $\boldsymbol{Y}$ has independent components, this can be related to a problem investigated by Efron (1965) who established that log-concavity is a sufficient condition for one term to be stochastically increasing in a sum of independent random variables. This problem has attracted a lot of attention in the literature. General conditions on $\boldsymbol{Y}$ ensuring that $\mathbb{E}[Y_i|S]$ is non-decreasing in S are difficult to establish. When $n = 2$, Saumard and Wellner (2018) extended Efron's monotonicity property to the case of general measures on $\mathbb{R}^2$. Denuit et al. (2021) adopted an asymptotic point of view and studied Efron's monotonicity property for distributions which are not log-concave but have density functions with bounded second derivatives and satisfy a central-limit theorem. This approach is in contrast with sums comprising a limited number of terms where restrictive conditions must be imposed on the distribution of each term, such as log-concavity for instance. By letting the number of random variables in the sum increase, the distribution of the sum will get closer and closer to the standard Gaussian distribution which is log-concave, thereby linking the two approaches.

Remark 2.4. *Note that Proposition 2.3 does not necessarily hold for heavy-tailed risks. Let $Y_1, Y_2, \ldots, Y_n$ be independent risks with common distribution function*

$$F_Y(x) = 1 - x^{-\xi}, \text{ with } 0 < \xi \leq 1.$$

Recall that for a tail index ξ less than, or equal to 1, the mean value is infinite so that these risks do not belong to $\mathcal{X}$. For any $n \geq 1$, Chen et al. (2022) established that

$$\mathbb{P}[Y_1 > t] \leq \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i > t\right], \text{ for all } t \geq 0.$$

Under the expected utility paradigm for choice under risk, this means that every economic agent prefers the stand-alone loss Y_1 over the allocation $\mathbb{E}[Y_1|S] = \frac{1}{n}\sum_{i=1}^{n} Y_i$, whatever the number n of risks.

2.3. A Characterization of Aggregate Risk Allocations.

Some allocations depend on individual losses only through the aggregate loss S. This is the case with the CMRS rule, for instance. The only relevant information that is not known at time 0 is thus the outcome of S, while in general, the information not known at time 0 is the outcomes of the individual losses. Aggregate risk allocations are elements of $\mathcal{A}$ such that each component is a function of S. The next result usefully characterizes such allocations.

Property 2.5. *For any allocation $\boldsymbol{Y} \in \mathcal{A}$ and any aggregate risk allocation $\boldsymbol{Z} = \left(f_1(S), \ldots, f_n(S)\right) \in \mathcal{A}$, $\boldsymbol{Y} \overset{d}{=} \boldsymbol{Z}$ if and only if $\boldsymbol{Y} = \boldsymbol{Z}$.*

The proof of the above result is given in Appendix B. It follows from Property 2.5 that in order to show that an allocation is an aggregate risk allocation, it suffices to show that it is distributed as an aggregate risk allocation. Additionally, the following result shows that PO allocations are aggregate risk allocations. The proof is provided in Appendix C.

Property 2.6. *Any PO allocation is an aggregate risk allocation.*

2.4. Pareto Optimality vs. Comonotonicity.

For each $X \in \mathcal{X}$, let F_X denote the distribution function of X. The left- and right-continuous inverses of F_X are given by

$$F_X^{-1}(p) := \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq p\right\}, \ \forall p \in [0,1], \tag{2}$$

and

$$F_X^{-1+}(p) := \sup\left\{x \in \mathbb{R} \mid F_X(x) \leq p\right\}, \ \forall p \in [0,1], \tag{3}$$

respectively, with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Consider a random vector $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$, with respective marginal distribution functions $F_{X_1}, \ldots, F_{X_n}$, and let $S_{\boldsymbol{X}} := \sum_{i=1}^{n} X_i$. The random vector $\boldsymbol{X}$ is said to be comonotonic if

$$\boldsymbol{X} \overset{\mathrm{d}}{=} \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{4}$$

for any random variable U that is uniformly distributed over the unit interval. Comonotonicity is an important dependency structure, with many applications in insurance and finance. See, for instance, Dhaene et al. (2002a,b, 2006), or Deelstra et al. (2010). In order to indicate that $\boldsymbol{X}$ is a comonotonic random vector, we will use the notation $\boldsymbol{X}^c$. We also introduce the notation $S^c_{\boldsymbol{X}}$ for the related comonotonic sum:

$$S^c_{\boldsymbol{X}} := \sum_{i=1}^{n} X_i^c \overset{\mathrm{d}}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U) . \tag{5}$$

As in Dhaene et al. (2002a), for any $\alpha \in [0,1]$, we define the α-quantile of X (or the α-inverse of F_X) as the following convex combination of F_X^{-1} and F_X^{-1+}:

$$F_X^{-1(\alpha)}(p) := \alpha\, F_X^{-1}(p) + (1-\alpha)\, F_X^{-1+}(p),\ \forall p \in (0,1) . \tag{6}$$

Moreover, for any $\alpha \in [0,1]$,

$$F_X^{-1(\alpha)}(0) := F_X^{-1+}(0) \text{ and } F_X^{-1(\alpha)}(1) := F_X^{-1}(1) . \tag{7}$$

For any x in $\left(F_X^{-1+}(0), F_X^{-1}(1)\right)$, there exists a (not necessary unique) $\alpha_x \in [0,1]$ such that

$$F_X^{-1(\alpha_x)}\left(F_X(x)\right) = x. \tag{8}$$

Indeed, defining α_x as

$$\alpha_x := \begin{cases} \frac{F_X^{-1+}\left(F_X(x)\right) - x}{F_X^{-1+}\left(F_X(x)\right) - F_X^{-1}\left(F_X(x)\right)} & \text{if } F_X^{-1+}\left(F_X(x)\right) \neq F_X^{-1}\left(F_X(x)\right), \\ 1 & \text{otherwise,} \end{cases} \tag{9}$$

leads to eq. (8). By convention, we set

$$\alpha_x := \begin{cases} 0, & \text{if } x \leq F_X^{-1+}(0), \\ 1, & \text{if } x \geq F_X^{-1}(1). \end{cases} \tag{10}$$

This implies that

$$F_X^{-1(\alpha_x)}\left(F_X(x)\right) = \begin{cases} F_X^{-1+}(0), & \text{if } x \leq F_X^{-1+}(0), \\ F_X^{-1}(1), & \text{if } x \geq F_X^{-1}(1). \end{cases} \tag{11}$$

Based on the characterization of comonotonicity given in Denuit et al. (2022), we obtain the following generalization of a result of Denneberg (1994). Notice that hereafter, equalities between random vectors are to be understood as a.s. equalities. The proof of Theorem 2.7 is given in Appendix D.

Theorem 2.7. *A random vector* $\boldsymbol{X}$ *is comonotonic if and only if*

$$\boldsymbol{X} = \left(h_1(S), \ldots, h_n(S)\right),$$

where the functions h_i *are the non-decreasing and* 1*-Lipschitz functions given by*

$$h_i(s) := F_{X_i}^{-1(\alpha_s)}\left(F_{S^c_{\boldsymbol{X}}}(s)\right),\ \forall s \in \mathbb{R},\ \forall i \in \{1, \ldots, n\}, \tag{12}$$

with α_s *as in eq. (8).*

In Proposition 4.5 of Denneberg (1994) it is shown that a random vector $\boldsymbol{X}$ with aggregate sum $S_{\boldsymbol{X}} := \sum_{i=1}^{n} X_i$ is comonotonic if and only if there exist non-decreasing and continuous functions f_i with $\sum_{i=1}^{n} f_i(s) = s$, such that

$$\boldsymbol{X} = \left(f_1\left(S_{\boldsymbol{X}}\right), f_2\left(S_{\boldsymbol{X}}\right), \ldots, f_n\left(S_{\boldsymbol{X}}\right)\right). \tag{13}$$

In fact, Denneberg (1994) gives a proof for the case $n = 2$, but the proof can easily be generalized for any $n > 2$. Theorem 2.7 proves the stronger result that the functions h_i are not only Lipschitz continuous (and hence, also continuous), but they also are of the form given in eq. (12).

The following result is the well-known equivalence between comonotonicity of allocations and Pareto-optimality for the convex order. We provide here a novel proof that, unlike the constructive approaches in the vein of Landsberger and Meilijson (1994), provides a direct and crisp closed-form characterization of Pareto optima. Namely, for convex order preferences, Pareto optima are α-quantile risk-sharing rules, and vice versa.

Before stating the main result of this section, we need to introduce the following concepts.

Definition 2.8. *Let $\boldsymbol{X} := (X_1, \ldots, X_n) \in \mathcal{A}$ be a given allocation of the initial aggregate risk S.*

1. *A suballocation of $\boldsymbol{X}$ is any element $\boldsymbol{Y} := (Y_1, \ldots, Y_m) \in \mathcal{X}^m$, for some $m \in \{1, \ldots, n\}$, such that for each $j \in \{1, \ldots, m\}$, $Y_j = X_i$, for some unique $i \in \{1, \ldots, n\}$.*
2. *For a given suballocation $\boldsymbol{Y} := (Y_1, \ldots, Y_m)$ of $\boldsymbol{X}$, let*
$$\mathcal{A}_{\boldsymbol{Y}} := \left\{ (Z_1, \ldots, Z_m) \in \mathcal{X}^m \;\middle|\; \sum_{j=1}^{m} Z_j = \sum_{j=1}^{m} Y_j \right\}$$
denote the set of all possible reallocations of the aggregate risk $\sum_{j=1}^{m} Y_j$ of the suballocation $\boldsymbol{Y}$.
3. *For a given suballocation $\boldsymbol{Y} := (Y_1, \ldots, Y_m)$ of $\boldsymbol{X}$, an allocation $(Z_1, \ldots, Z_m) \in \mathcal{A}_{\boldsymbol{Y}}$ is said to be $\mathcal{A}_{\boldsymbol{Y}}$-PO if there is no other allocation $(W_1, \ldots, W_m) \in \mathcal{A}_{\boldsymbol{Y}}$ such that*
$$\begin{cases} W_j \preccurlyeq_{CVX} Z_j, \text{ for all } j \in \{1, \ldots, m\}; \text{ and} \\ \exists\, j^* \in \{1, \ldots, m\},\, W_{j^*} \prec_{CVX} Z_{j^*}. \end{cases}$$
4. *For a given $m \in \{1, \ldots, n\}$, an m-reallocation of the aggregate risk S is a vector $(Z_1, \ldots, Z_m) \in \mathcal{X}^m$ such that $\sum_{k=1}^{m} Z_k = S$. Let $\mathcal{A}^{(m)}$ denote the collection of all m-reallocations of the initial aggregate risk:*
$$\mathcal{A}^{(m)} := \left\{ (Z_1, \ldots, Z_m) \in \mathcal{X}^m \;\middle|\; \sum_{k=1}^{m} Z_k = S \right\}.$$
5. *For a given $m \in \{1, \ldots, n\}$, an m-reallocation $\boldsymbol{Z} := (Z_1, \ldots, Z_m)$ of the aggregate risk S is said to be $\mathcal{A}^{(m)}$-PO if there is no other allocation $(W_1, \ldots, W_m) \in \mathcal{A}^{(m)}$ such that*
$$\begin{cases} W_j \preccurlyeq_{CVX} Z_j, \text{ for all } j \in \{1, \ldots, m\}; \text{ and} \\ \exists\, j^* \in \{1, \ldots, m\},\, W_{j^*} \prec_{CVX} Z_{j^*}. \end{cases}$$

The following result shows the equivalence between comonotonicity and Pareto optimality. Its proof is given in Appendix E.

Theorem 2.9. *For any allocation $\boldsymbol{X} := (X_1, \ldots, X_n) \in \mathcal{A}$, the following are equivalent:*

(1) $\boldsymbol{X}$ *is PO.*

(2) *For any $i \neq j \in \{1, \ldots, n\}$, the suballocation $\boldsymbol{Y} := \big(X_i, X_j\big)$ of $\boldsymbol{X}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO.*

(3) *For any $i \neq j \in \{1, \ldots, n\}$, the suballocation $\big(X_i, X_j\big)$ of $\boldsymbol{X}$ is comonotonic.*

(4) $\boldsymbol{X}$ *is comonotonic.*

(5) $\boldsymbol{X} = \big(h_1(S), \ldots, h_n(S)\big)$, *where the functions h_i are the non-decreasing and 1-Lipschitz functions given by (12) with α_s as in eq. (8).*

(6) *Each suballocation $\boldsymbol{Y}$ of $\boldsymbol{X}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO.*

The following result shows that PO essentially reduces to pairwise PO, with pairs involving one participant and merging all other ones. Its proof is given in Appendix F.

Proposition 2.10. *Let $\boldsymbol{X} := (X_1, \ldots, X_n) \in \mathcal{A}$ be a given allocation, and consider the suballocation $\boldsymbol{Y} := (X_2, \ldots, X_n) \in \mathcal{X}^{n-1}$. Then $\boldsymbol{Y}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO and the 2-reallocation $\big(X_1, \sum_{i=2}^n X_i\big) = (X_1, S - X_1)$ is $\mathcal{A}^{(2)}$-PO if and only if $\boldsymbol{X}$ is PO.*

The reduction to 2-reallocations is central to *Operational Anonymity*, as defined in Jiao et al. (2022). Notice that operational anonymity is closely related to fair merging and is a special case of fair bilateral redistribution as defined in Denuit et al. (2022).

Remark 2.11. *Proposition 2.10 suggests an iterative approach to Pareto optimality in the n-agent case:*

(1) *First find an $\mathcal{A}^{(2)}$-PO allocation $(X_1^*, S - X_1^*)$, and let $S^{(2,*)} := S - X_1^*$.*

(2) *Then find a PO allocation of $S^{(2,*)}$ of dimension 2. Denote it by $(X_2^*, S^{(2,*)} - X_2^*)$.*

(3) *Let $S^{(3,*)} := S^{(2,*)} - X_2^*$, and find a PO allocation of $S^{(3,*)}$ of dimension 2. Denote it by $(X_3^*, S^{(3,*)} - X_3^*)$.*

(4) *Continue this process and until X_n^* is determined. Then the resulting allocation $\big(X_1^*, X_2^*, \ldots, X_n^*\big)$ is PO, by Proposition 2.10.*

3. Convex-Order Improvements: An Algorithmic Approach

A classical and foundational result in the literature on Pareto efficient allocations is that any allocation is dominated in the convex order by a comonotonic allocation. Proposition 1 of Landsberger and Meilijson (1994) provides an explicit construction of the dominating allocation for the case of two-dimensional allocations that are supported by a finite set. The extension beyond that case is not entirely trivial. Dana and Meilijson (2023) provides such an extension to the case of more general random variables, based on a limit argument applied to the construction of Landsberger and Meilijson (1994). However, they assume that the extension of the algorithm of Landsberger and Meilijson (1994) to n-dimensional discrete allocations holds, without providing a proof thereof for $n > 2$. Proposition 5.1 of Filipović and Svindland (2008) provides an extension of the approach of Landsberger and Meilijson (1994) to general random variables but remains in the case of two-dimensional allocations. Theorem 1 of Ludkovski and Rüschendorf (2008) uses the same approach as Landsberger and Meilijson (1994) and provides an extension to the case of n-dimensional discrete allocations (but the full details are only given for the case $n = 2$), while their Theorem 2 uses a limit argument similar to that of Dana and Meilijson (2023) to extend their Theorem 1 to the case of general random variables defined on a nonatomic space. Theorems 10.47 and 10.50 of Rüschendorf (2013) provides a unifying proof of this result in the n-dimensional case and for general random variables, again based on the algorithm of Landsberger and Meilijson (1994) and a limit argument, but details are only given for the case $n = 2$.

The difficulty with the aforementioned approach is that it is hard to implement in practice, because of the nature of the limit argument involved. Here, we provide a constructive proof of this comonotonic convex-order-improvement result that suggests an algorithmic approach that can be easily implemented in practice and is detailed for all n. Our approach is related to the literature based on the construction of Landsberger and Meilijson (1994), but it takes a different route and is based instead on an extension of a result of Lorentz and Shimogaki (1968). Specifically, Proposition 2 of Lorentz and Shimogaki (1968), while stated in a different setting and framework than the risk sharing problem we examine here, is of direct relevance. Appendix A provides some background material related to the majorization order of Hardy et al. (1929, 1952).

3.1. **Convex-Order Improvement.** We are now ready to state the main result of this section.

Theorem 3.1. *For each $\boldsymbol{X} := (X_1, \ldots, X_n) \in \mathcal{A}$, there exists $\boldsymbol{Y} := (Y_1, \ldots, Y_n) \in \mathcal{A}$ such that $\boldsymbol{Y}$ is comonotonic and satisfies*

$$Y_i \preccurlyeq_{CVX} X_i, \ \ \forall\, i \in \{1, \ldots, n\}.$$

Proof. Fix $\boldsymbol{X} := (X_1, \ldots, X_n) \in \mathcal{A}$, and let $\boldsymbol{X}^0 := \left(X_1^0, \ldots, X_n^0\right)$ where $X_i^0 := \mathbb{E}\left[X_i \mid S\right]$, for each $i \in \{1, \ldots, n\}$. Then there are Borel-measurable functions $g_i : \mathbb{R} \to \mathbb{R}$ such that $X_i^0 = g_i(S)$, for each $i \in \{1, \ldots, n\}$. Moreover, by Proposition 2.3, $\boldsymbol{X}^0 \in \mathcal{A}$ and for any $i \in \{1, \ldots, n\}$ we have $g_i(S) \preccurlyeq_{\text{CVX}} X_i$.

Since the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, there exists a random variable U on $(\Omega, \mathcal{F}, \mathbb{P})$ with a uniform distribution on $(0, 1)$, such that $S = F_S^{-1}(1 - U)$, a.s. (e.g., Lemma A.32 in Föllmer and Schied (2016)). Therefore, $F_S^{-1}(1 - U) = \sum_{i=1}^n g_i\left(F_S^{-1}(1 - U)\right)$, a.s., or equivalently

$$f(u) = \sum_{i=1}^{n} f_i(u), \quad \text{for } a.e.\ u \in [0, 1],$$

where for all $u \in [0, 1]$,

$$f(u) := F_S^{-1}(1 - u) \quad \text{and} \quad f_i(u) := g_i\left(f(u)\right).$$

Then f is a nonnegative and nonincreasing function on $[0, 1]$, and for each $i \in \{1, \ldots, n\}$, f_i is a nonnegative function on $[0, 1]$. Moreover, $S = f(U)$, a.s., and $f_i(U) = X_i^0$, a.s., for each $i \in \{1, \ldots, n\}$.

We wish to find nonnegative and nonincreasing functions $\{\widetilde{f}_i\}_{i=1}^n$ such that $f = \sum_{i=1}^n \widetilde{f}_i$ and, for each $i \in \{1, \ldots, n\}$,

$$\widetilde{f}_i(U) \preccurlyeq_{\text{CVX}} f_i(U) \ \ (\preccurlyeq_{\text{CVX}} X_i),$$

since this would then imply that $\boldsymbol{Y} := \left(\widetilde{f}_1(U), \ldots, \widetilde{f}_n(U)\right)$ is the desired allocation. It is therefore enough to find functions $\{\widetilde{f}_i\}_{i=1}^n$ such that $\sum_{i=1}^n \widetilde{f}_i = \sum_{i=1}^n f_i = f$ and $\widetilde{f}_i \prec f_i$, for all $i \in \{1, \ldots, n\}$, where $\prec$ is the majorization preorder relation (see Definition A.1) given by:

$$\widetilde{f}_i \prec f_i \iff \int_0^u \widetilde{f}_i^* (t)\, dt \le \int_0^u f_i^* (t)\, dt,\ \forall u \in [0,1],$$

and where $\widetilde{f}_i^*$ and f_i^* denote the nonincreasing rearrangements of $\widetilde{f}_i$ and f_i (see eq. (28)), respectively, since then we have $\widetilde{f}_i (U) \preccurlyeq_{\text{CVX}} f_i (U)$, for all $i \in \{1,\dots,n\}$, and $\sum_{i=1}^n \widetilde{f}_i (U) = f (U)$, by Lemma A.6. We do this in two steps:

(a) Assume first that $f, f_1, \dots, f_n$ are step functions with common intervals of constancy, (a_{k-1}, a_k), for $k = 1, ..., p$, with $a_0 = 0$ and $a_p = 1$. Let $f_i (t) = l_k^{(i)}$ on (a_{k-1}, a_k), for each $i \in \{1,\dots,n\}$ and $k \in \{1,\dots,p\}$. If all functions $f_1,\dots,f_n$ are nonincreasing on $[0,1]$, then the result follows immediately. Assume, therefore, that not all of the functions $f_1,\dots,f_n$ are nonincreasing on $[0,1]$.

Note that (as in Lorentz and Shimogaki (1968)) it is enough to show (by induction) that if, for some $1 \le m \le p-1$, the functions $\{f_i\}_{i=1}^n$ are nonincreasing on (a_0, a_m), then there exist functions $\{\widetilde{f}_i\}_{i=1}^n$ that are nonincreasing on (a_0, a_{m+1}) and satisfy

$$\sum_{i=1}^n \widetilde{f}_i = \sum_{i=1}^n f_i = f \quad \text{and} \quad \widetilde{f}_i \prec f_i,\ \forall i \in \{1,\dots,n\}.$$

Assume, without loss of generality, that the functions $f_1,\dots,f_k$, for $1 \le k \le n-1$, have larger values on (a_m, a_{m+1}) than on (a_{m-1}, a_m), while the other functions are nonincreasing on (a_0, a_{m+1}). For each $i \in \{1,\dots,k\}$, let $p^{(i)}$ be the smallest integer in $[0,m)$ that satisfies

$$\lambda^{(i)} := \frac{1}{a_{m+1} - a_{p^{(i)}}} \int_{a_{p^{(i)}}}^{a_{m+1}} f_i(t)\, dt \ge l_{p^{(i)}+1}^{(i)} \ge \dots \ge l_m^{(i)}.$$

Then $\lambda^{(i)} < l_{p^{(i)}}^{(i)}$ (unless $p^{(i)} = 0$) and $\lambda^{(i)} \le l_{m+1}^{(i)}$. We may assume without loss of generality that

$$p^{(1)} \le p^{(2)} \le \dots \le p^{(k)} < m.$$

We construct the functions $\left\{\widetilde{f}_i\right\}_{i=1}^n$ as follows:

- For $i \in \{1,\dots,k\}$, let

$$\widetilde{f}_i (t) := \begin{cases} \lambda^{(i)}, & \text{if } t \in \left(a_{p^{(i)}}, a_{m+1}\right); \\ f_i (t), & \text{otherwise.} \end{cases} \tag{14}$$

  For $i \in \{1,\dots,k\}$, the step function $\widetilde{f}_i$ is nonincreasing on (a_0, a_{m+1}), nonnegative, and satisfies $\widetilde{f}_i \prec f_i$ by averaging (Proposition A.2).

- For $i \in \{k+1,\dots,n\}$, let

$$\alpha_i := \frac{l_m^{(i)} - l_{m+1}^{(i)}}{\sum_{j=k+1}^n \left(l_m^{(j)} - l_{m+1}^{(j)}\right)}.$$

  Then the constants α_i are nonnegative and sum to 1. For each $j \in \{p^{(1)},\dots,m\}$, let

$$\delta_j := \sum_{h=1}^k \left(\widetilde{f}_h (t) - f_h (t)\right),\ \text{for any } t \in (a_j, a_{j+1}).$$

For $j \in \{p^{(1)}, \ldots, m-1\}$, we have $\delta_j > 0$. Moreover, $\delta_m < 0$ and

$$\sum_{j=p^{(1)}}^{m} \delta_j \left(a_{j+1} - a_j\right) = 0. \tag{15}$$

Now, for $i \in \{k+1, \ldots, n\}$ and $j \in \{p^{(1)}, \ldots, m\}$, let

$$\widetilde{f}_i(t) := \begin{cases} f_i(t) - \delta_j \, \alpha_i, \text{ for all } t \in \left(a_j, a_{j+1}\right); \\ f_i(t), \text{ outside of the interval } \left(a_{p^{(1)}}, a_{m+1}\right). \end{cases} \tag{16}$$

First note that, for each $j \in \{p^{(1)}, \ldots, m\}$ and each $t \in \left(a_j, a_{j+1}\right)$, we have

$$\begin{aligned} \sum_{i=1}^{n} \widetilde{f}_i(t) &= \sum_{i=1}^{k} \widetilde{f}_i(t) + \sum_{i=k+1}^{n} \widetilde{f}_i(t) \\ &= \sum_{i=1}^{k} \widetilde{f}_i(t) + \sum_{i=k+1}^{n} \left(f_i(t) - \sum_{h=1}^{k} \left(\widetilde{f}_h(t) - f_h(t) \right) \alpha_i \right) = \sum_{i=1}^{n} f_i(t). \end{aligned}$$

Moreover, for each $i \in \{k+1, \ldots, n\}$, the function $\widetilde{f}_i$ is nonincreasing on $\left(a_{p^{(1)}}, a_m\right)$ and satisfies

$$\widetilde{f}_i\left(\frac{a_{m-1}+a_m}{2}\right) = l_m^{(i)} - \delta_{m-1}\alpha_i \geq l_{m+1}^{(i)} - \delta_m \alpha_i = \widetilde{f}_i\left(\frac{a_m + a_{m+1}}{2}\right),$$

since f is nonincreasing. Additionally, $\widetilde{f}_i$ is nonnegative on $\left(a_{p^{(1)}}, a_{m+1}\right)$, since $\widetilde{f}_i\left(\frac{a_m+a_{m+1}}{2}\right)$ is nonnegative. Furthermore, $\sum_{i=k+1}^{n}\left(l_m^{(i)} - l_{m+1}^{(i)}\right) \geq \sum_{i=1}^{k}\left(l_{m+1}^{(i)} - l_m^{(i)}\right)$, implying that $\widetilde{f}_i$ is also nonincreasing on (a_m, a_{m+1}).

Hence, for each $i \in \{1, \ldots, n\}$, the function $\widetilde{f}_i$ constructed above satisfies

$$\widetilde{f}_i \begin{cases} \leq f_i \quad \text{on} \quad (a_0, a_m); \\ \geq f_i \quad \text{on} \quad (a_m, a_{m+1}). \end{cases}$$

However, by (14), (15), and (16),

$$\int_0^{a_{m+1}} \widetilde{f}_i(t)\, dt = \int_0^{a_{m+1}} f_i(t)\, dt, \quad \forall\, i \in \{1, \ldots, n\}.$$

Additionally, for each $i \in \{1, \ldots, n\}$,

$$\int_0^{x} \widetilde{f}_i(t)\, dt \leq \int_0^{x} f_i(t)\, dt, \quad \forall x \in \left(a_0, a_{m+1}\right).$$

Since all of these functions are nonincreasing on the intervals considered, it follows that

$$\widetilde{f}_i \prec f_i, \quad \forall\, i \in \{1, \ldots, n\}.$$

(b) Suppose now that the functions $f, f_1, \ldots, f_n$ are any nonnegative functions on $[0,1]$. Then for each $i \in \{1, \ldots, n\}$, there exists a sequence $\left\{f_i^{(k)}\right\}_{k=1}^{+\infty}$ of nonnegative simple functions on $[0,1]$ that converges pointwise and monotonically upward to f_i, namely:

$$f_i^{(k)}(u) := \sum_{j=1}^{k\,2^k} \left(\frac{j-1}{2^k}\right) \mathbb{1}_{\left\{u \in [0,1]:\, \frac{j-1}{2^k} \leq f_i(u) < \frac{j}{2^k}\right\}} + k\,\mathbb{1}_{\left\{u \in [0,1]:\, f_i(u) \geq k\right\}}, \ \forall\, k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, let $f^{(k)} := \sum_{i=1}^n f_i^{(k)}$. Then $\left\{f^{(k)}\right\}_{k=1}^{+\infty}$ is a sequence of nonnegative simple functions on $[0,1]$ that converges pointwise and monotonically upward to f.

Now, it follows from part (a) above that, for each $k \in \mathbb{N}$, there are functions $\left\{\widetilde{f}_i^{(k)}\right\}_{i=1}^n$ such that $\sum_{i=1}^n \widetilde{f}_i^{(k)} = \sum_{i=1}^n f_i^{(k)} = f^{(k)}$ and $\widetilde{f}_i^{(k)} \prec f_i^{(k)}$, for all $i \in \{1,\ldots,n\}$. Then, letting $\widetilde{f}_i := \lim\limits_{k\to\infty} \widetilde{f}_i^{(k)}$, for each $i \in \{1,\ldots,n\}$, it follows that $\sum_{i=1}^n \widetilde{f}_i = f$. The rest follows from Proposition A.3. □

The proof of Theorem 3.1 suggests a novel algorithm that actuaries can implement in order to build a comonotonic improvement over any given allocation. We briefly summarize below the different steps involved in this construction:

Step 1. Each component X_i is replaced by its conditional expectation given the sum S, so that we can work with an initial allocation in which each individual contribution is a measurable function g_i of the aggregate risk S. If all functions g_i are nondecreasing, then the conditional expectations correspond to the comonotonic improvement, and there is no need to perform the next steps.

Step 2. A change of variable is performed so that each function g_i is viewed as a function f_i of the complementary probability level of quantile of S. The condition that all functions g_i are nondecreasing is therefore equivalent to the condition that all functions f_i are nonincreasing. The convex-order improvement of the allocation is then equivalent to obtaining nondecreasing functions $\widetilde{f}_1,\ldots,\widetilde{f}_n$ such that each function $\widetilde{f}_i$ is smaller than f_i with respect to the majorization preorder relation.

The next steps consider the case where f_i are step functions with common intervals of constancy, the general case following by a standard limit argument.

Step 3. As long as one can find an interval of increase, the functions $f_1,\ldots,f_n$ are replaced by monotonic functions $\widetilde{f}_1,\ldots,\widetilde{f}_n$ on that interval, such that f_i and $\widetilde{f}_i$ only cross once, are ordered in the majorization preorder, and the sum of the new functions $\widetilde{f}_1,\ldots,\widetilde{f}_n$ coincides with the sum of the original functions $f_1,\ldots,f_n$. By interval of increase, we mean an interval corresponding to two consecutive plateaus where the value assumed by some of the functions f_i increases.

Step 4. If no such interval exists then the comonotonic improvement is obtained and the algorithm stops.

Note that this algorithm directly applies to any number n of participants, without reducing the problem to $n = 2$.

It is interesting to compare the proposed algorithm to the one described in Ludkovski and Rüschendorf (2008), which is a slightly modified variant of the Landsberger and Meilijson (1994) algorithm. Their approach works for $n = 2$ and first considers discrete losses X_1 and X_2. The universe is partitioned so that (X_1, X_2) remains constant on each element of the partition and these elements are ordered according to the values of the sum $X_1 + X_2$. If a decrease is detected in the ordered sequence of values of X_1 or of X_2, then all intermediary values are modified to restore increasingness with the help of a componentwise mean-preserving spread (keeping mean values unchanged but resulting in a single crossing between each of the marginal distribution functions). The procedure is applied repeatedly and is shown to converge to the desired comonotonic improvement. The general case (i.e., for not discrete losses X_1 and X_2) then follows from a limit argument, except that the convergence is almost sure and in L^1, whereas it follows from the standard approximation of monotonic functions by simple step functions in the algorithm proposed in this paper.

The advantage of our approach is that it applies to any $n \geq 2$ and is easily illustrated graphically, since we work with functions of S. The numerical examples in the next section make this statement clear.

3.2. **Examples.** Here we provide two examples that illustrate the algorithmic approach suggested in the proof of Theorem 3.1. A first very simple example for $n = 2$ agents with a discretely-valued initial random endowment is considered. We detail the calculations that are performed iteratively. These calculations are compared with those obtained using the Landsberger-Meilijson method or a modified version proposed in Li (2013). A second example for $n = 3$ agents with an absolutely continuous initial random endowment is considered. It is not possible to describe the various calculation steps, and we have essentially provided figures showing the density function of the aggregated risk S and the agents' allocations. As the Landsberger-Meilijson method is not explicitly described for 3 agents, we do not propose a comparison.

3.2.1. *Example with $n = 2$ agents.* Let the initial random endowment $\left(X_{0,1}, X_{0,2}\right)$ be given by

$$\begin{pmatrix} 0 & 2 & 3 & 6 \\ 0 & 2 & 3 & 1 \end{pmatrix},$$

with probability 0.25 for each entry. This initial random endowment has been proposed in Example 4.2.1 in Li (2013). Note that the distribution of $\left(X_{0,1}, X_{0,2}\right)$ implies that $X_{0,i} = \mathbb{E}[X_{0,i}|S]$, for $i = 1, 2,$ so that the first step in the algorithm used in the proof of Theorem 3.1 has already been performed. Let $a_0 = 0$, $a_1 = 1/4$, $a_2 = 1/2$, $a_3 = 3/4$, and $a_4 = 1$. The function f_i introduced in the second step of the algorithm (as described in the summary of the algorithm after Theorem 3.1) are characterized as follows:

u	$[0, 1/4)$	$[1/4, 1/2)$	$[1/2, 3/4)$	$[3/4, 1]$
$f_1(u)$	6	3	2	0
$f_2(u)$	1	3	2	0

Therefore, f_2 is not a nonincreasing function. For the first iteration in the third step of the algorithm (as described in the summary of the algorithm after Theorem 3.1), we obtain $m = 2$ and $\lambda = 2$, and we derive the function $\widetilde{f}_i$:

u	$[0, 1/4)$	$[1/4, 1/2)$	$[1/2, 3/4)$	$[3/4, 1]$
$\widetilde{f}_1(u)$	5	4	2	0
$\widetilde{f}_2(u)$	2	2	2	0

Hence, $\widetilde{f}_2$ is now a nonincreasing function and the algorithm stops since the condition in the last step of the algorithm (as described in the summary of the algorithm after Theorem 3.1) is fulfilled.

We finally obtain the following comonotonic reallocation (with probability 0.25 for each entry) as follows:

$$\begin{pmatrix} 0 & 2 & 3 & 6 \\ 0 & 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 4 & 5 \\ 0 & 2 & 2 & 2 \end{pmatrix}.$$

We now compare this procedure with the Landsberger-Meilijson method. The algorithm version of this method proposed in Ludkovski and Rüschendorf (2008) requires two iterations to obtain a comonotonic reallocation (with probability 0.25 for each entry):

$$\begin{pmatrix} 0 & 2 & 3 & 6 \\ 0 & 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 7/3 & 10/3 & 16/3 \\ 0 & 5/3 & 8/3 & 5/3 \end{pmatrix} \to \begin{pmatrix} 0 & 7/3 & 23/6 & 29/6 \\ 0 & 5/3 & 13/6 & 13/6 \end{pmatrix}.$$

A modified version of the Landsberger-Meilijson method was proposed in Li (2013). It requires two additional parameters, δ_1 and δ_2, that must satisfy some constraints. A reallocation for the considered example has the following form:

$$\begin{pmatrix} 0 & 2+\delta_2 & 3+\delta_2 & 6-\delta_1 \\ 0 & 2-\delta_2 & 3-\delta_2 & 1+\delta_2, \end{pmatrix}$$

with $\delta_2 \in [2/3;1]$ and $\delta_1 = 2\delta_2$. If $\delta_2 = 1$, then the reallocation (with probability 0.25 for each entry) is given by

$$\begin{pmatrix} 0 & 2 & 3 & 6 \\ 0 & 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 4 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

while, if $\delta_2 = 2/3$, then the reallocation (with probability 0.25 for each entry) is given by

$$\begin{pmatrix} 0 & 2 & 3 & 6 \\ 0 & 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 8/3 & 11/3 & 14/3 \\ 0 & 4/3 & 7/3 & 7/3 \end{pmatrix}.$$

We observe that reallocations differ according to the methods proposed. They all require a reduced number of calculations, which means that, numerically, the results are obtained almost instantaneously.

3.2.2. *Example with $n = 3$ agents.* In this example, the initial random endowment is the vector $\mathbf{X}_0 = \big(X_{0,1}, X_{0,2}, X_{0,3}\big)$. We assume, for the sake of illustration, that the components of $\mathbf{X}_0$ are independent. $X_{0,1}$ and $X_{0,2}$ follow a truncated exponential distribution with parameter $\beta > 0$ on the interval $[0, M]$, for some given $M < +\infty$, with a probability density function f given by $f(x) := \frac{\beta e^{-\beta x}}{1-e^{-\beta M}}$, for $x \in [0, M]$. For some given $N < +\infty$, $X_{0,3}$ follows on the interval $[0, N]$ a truncated mixture of an exponential distribution with parameter $\beta > 0$ and a gamma distribution with parameters $\alpha > 1$ and $\beta > 0$, with equal mixing weights 0.5. The probability density function g of $X_{0,3}$ is then given by $g(x) := \frac{(\beta+\beta^\alpha x^{\alpha-1}/\Gamma(\alpha))e^{-\beta x}}{\int_0^N (\beta+\beta^\alpha x^{\alpha-1}/\Gamma(\alpha))e^{-\beta x}dx}$, for $x \in [0, N]$. For the numerical illustration, we take $\beta = 1/2$, $M = 10$, $\alpha = 8$, and $N = 30$.

Figure 1 provides the probability density function of the aggregate risk S, as well as a plot of the agents' allocations. The initial allocations considered in the first step of the algorithm (as described in the summary of the algorithm after Theorem 3.1) are the CMRS allocations $\mathbf{X}^0$ characterized by the functions $s \mapsto \mathbb{E}[X_{0,i}|S = s]$, for agent $i \in \{1, 2, 3\}$ (black lines). We note that the allocation functions of agents 1 and 2 are not increasing in s, and so the CMRS allocations are not comonotonic, and *a fortiori* not Pareto optimal. We therefore implement the algorithm presented in the previous section in order to obtain a comonotonic improvement. The final allocation functions (red lines) become non-decreasing functions in s, thereby leading to a comonotonic and hence Pareto-optimal allocation vector. Iterating the third step, the algorithm (as described in the summary of the algorithm after Theorem 3.1) replaced, on an interval including the decreasing parts of agents 1 and 2, the initial functions by constant functions, while preserving the expectations of the allocations and guaranteeing the component-wise convex-order improvement of the allocation vector. Figure 2 provides plots of the initial (corresponding to the second step of the algorithm, as described in the summary of the algorithm after Theorem 3.1) and final (once the condition in the last step of the algorithm as described in the summary of the algorithm after Theorem 3.1 is fulfilled) functions f_i, for $i \in \{1, 2, 3\}$, used in the algorithm.

4. An Application to P2P Insurance

4.1. **Risk Retention, Risk Sharing, and Risk Transfer.** Under pure P2P insurance, the participants' contributions are theoretically unlimited. This requires confidence among the participants since the level of the actual contributions to be paid *ex post* remains unknown until the end of the period, and some participants may be unwilling or unable to pay their contributions at that time. These thoughts lead to the need of an adapted approach combining self-insurance or risk retention by the individual participants, risk pooling at the level of the community, and risk transfer to an insurance company.

As above, we denote the loss of individual $i \in \{1, \dots, n\}$ by $X_{0,i} \in \mathcal{X}$ before any risk sharing operates. Some participants may consider retaining some risk in order to reduce their contribution. In practice, this is often achieved by applying a deductible or a quota-share arrangement. In the former case, participant i retains $\min\{X_{0,i}, l_i\}$ for some positive deductible l_i, while in the latter case, participant i retains $(1 - \alpha_i)X_{0,i}$ for some percentage $\alpha_i \in [0, 1]$. Henceforth, we only consider the part of the loss that is neither retained by participant i through deductible or quota share, nor transferred to the insurer through an excess-of-loss cover. Let X_i be the corresponding random variable with $X_1, X_2, \dots, X_n$ summing to S. Clearly $\boldsymbol{X} := (X_1, X_2, \dots, X_n) \in \mathcal{A}$.

4.2. **Aggregate PO Risk Allocations.** In this section, we will only consider comonotonic risk allocations of which the *ex post* contributions of the participants can be expressed as non-decreasing functions of the aggregate loss S of the pool (see eq. (13)). In other words, we examine Pareto optimal allocations, and we will propose some P2P insurance schemes that will operate on these Pareto optima. Note that, in light of the convex-order improvement result of Theorem 3.1, this is not a limiting assumption.

Then, suppose that at time 0 the individual random losses in the pool $\boldsymbol{X}$ are re-allocated by transforming $\boldsymbol{X}$ into a contribution vector $\boldsymbol{Z} \in \mathcal{A}$ of the form $Z_i = f_i(S)$, $i = 1, 2, \dots, n$, where the functions $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ are non-decreasing and sum to identity. Note that Theorem 3.1 and its proof explain how the transformation from $\boldsymbol{X}$ to $\boldsymbol{Z}$ can be performed. Theorem 2.7 allows us to represent $\boldsymbol{Z}$ with functions f_i given by eq. (12). Moreover, we know from Theorem 2.9 that PO allocations are of this form.

The aim of this section is to propose an effective P2P insurance scheme applying to any PO allocation. Precisely, we extend the approach proposed in Denuit (2020) under the CMRS rule to any aggregate risk-sharing rule satisfying the comonotonicity property, that is, leading to comonotonic individual contributions. Theorem 2.7 allows us to represent the resulting individual contributions.

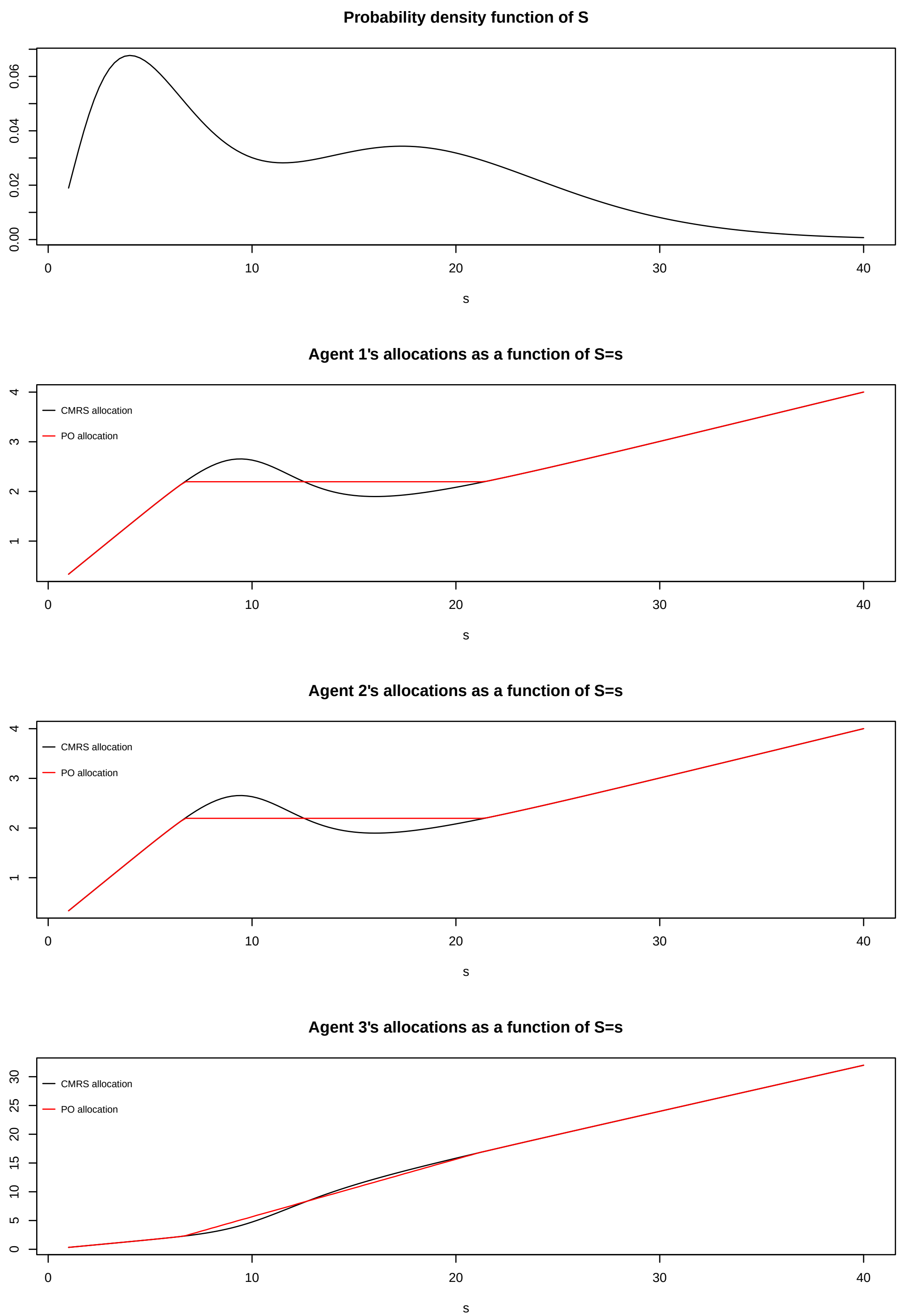

Figure 1. Plots of the probability density function of S and of the agents' allocations.

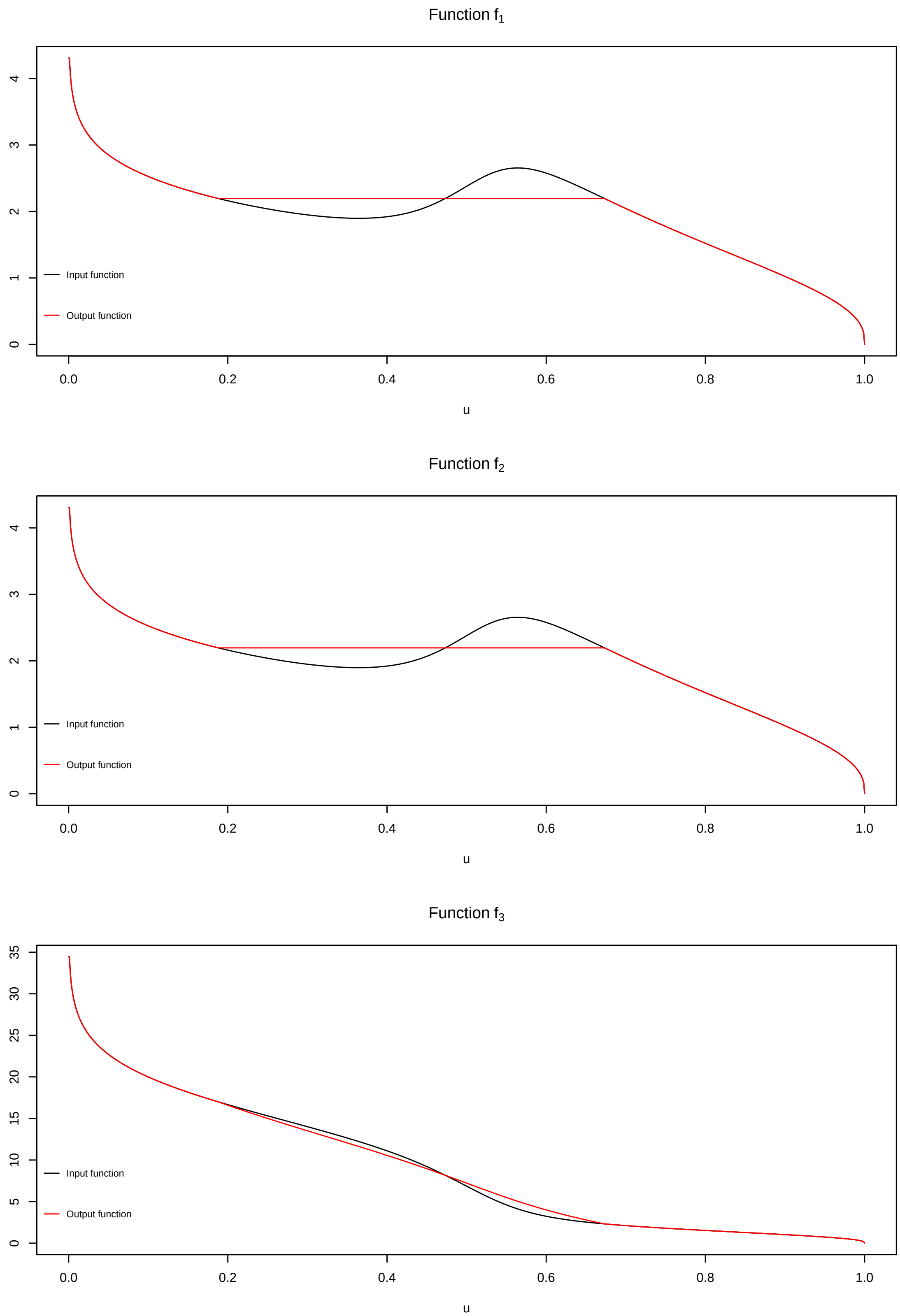

FIGURE 2. Plots of the functions f_i, for $i \in \{1, 2, 3\}$.

4.3. **Excess-of-Loss Risk Transfer under General Deductibles for the Contributions.**

4.3.1. *Limiting each participant's contribution.* Suppose that at the individual level, the contribution of each participant $i \in \{1, \ldots, n\}$ in the pool is limited to the lower layer $[0, d_i]$ of $f_i(S)$, for some deductible $d_i \geq 0$, whereas the upper layer (d_i, ∞) of $f_i(S)$ is transferred to an insurer. In other words, the contribution to be made by each participant i is restricted via an excess-of-loss cover. The full contribution $f_i(S)$ to be paid to the pool is split as follows:

$$f_i(S) = \min\left\{f_i(S), d_i\right\} + \left(f_i(S) - d_i\right)_+, \tag{17}$$

where the contribution of participant i is given by $\min\left\{f_i(S), d_i\right\}$, whereas the excess-of-loss payment to be made by the insurer to the pool equals $\left(f_i(S) - d_i\right)_+$.

Note that the first layer corresponding to individual retention is not visible in (17). As explained in Section 4.1, each participant retains part of his or her initial loss $X_{0,i}$ and only brings the remaining portion X_i to the pool. The sum S of $X_1, \ldots, X_n$ is then decomposed into a lower layer and an upper layer according to (17), resulting in the 3-layer structure referred to in Section 4.1 and in the Introduction.

4.3.2. *Initial deposit and cash-back mechanism.* In order to guarantee that each participant i contributes as promised, the pool may require each of them to pay the deposit $e^{-r}\, d_i$ *ex ante* (where r is a deterministic discount rate), with the guarantee that the final amount to be paid by participant i at time 1 will never exceed the time-1 value d_i of this up-front payment. The surplusses $\left(d_i - f_i(S)\right)_+$, which will be observed at time 1, are then returned *ex post* to the respective members of the pool. Taking into account this cash-back operation, we can rewrite (17) as

$$f_i(S) = d_i - \left(d_i - f_i(S)\right)_+ + \left(f_i(S) - d_i\right)_+, \tag{18}$$

which decomposes the contribution $f_i(S)$ to the pool into three parts: the time-1 value of the deposit paid by participant i to the pool at time 0, the time-1 cash-back value paid by the pool to participant i, and the time-1 excess-of-loss claim payment made by the insurer to the pool, respectively.

4.4. **The Case of Uniform Quantile Deductibles for the Participant's Excess-of-Loss Covers.** Let us now state the main result of this section, which shows that the collection of the participant's individual excess-of-loss covers is equivalent to a stop-loss cover for the pool when d_i are quantiles at the same probability level of $f_i(S)$.

Proposition 4.1. *Assume the deductibles d_i are determined by*

$$d_i = F^{-1}_{f_i(S)}(p), \tag{19}$$

for a given probability level $p \in (0,1)$. Then,

(i) The sum of all retained contributions $\min\left\{f_i(S), F^{-1}_{f_i(S)}(p)\right\}$ paid by the individual participants is equal to the aggregate losses observed in the lower layer $(0, F_S^{-1}(p))$ of S, i.e.,

$$\sum_{i=1}^{n} \min\left\{f_i(S), F^{-1}_{f_i(S)}(p)\right\} = \min\left\{S, F_S^{-1}(p)\right\}. \tag{20}$$

(ii) The sum of all claim payments made by the insurer is equal to the part of the aggregate claims S situated in the upper layer $(F_S^{-1}(p), \infty)$, i.e.,

$$\sum_{i=1}^{n} \left(f_i(S) - F^{-1}_{f_i(S)}(p)\right)_+ = \left(S - F_S^{-1}(p)\right)_+. \tag{21}$$

(iii) The sum of all ex post cash-back payments from the pool to the participants is given by

$$\sum_{i=1}^{n}\left(F^{-1}_{f_i(S)}(p)-f_i(S)\right)_+=\left(F^{-1}_S(p)-S\right)_+. \tag{22}$$

The proof of Proposition 4.1 is given in Appendix G. Let us mention that this method of applying an excess-of-loss cover to an aggregate risk sharing is in line with the unified framework of decentralized insurance proposed by Feng et al. (2024). The equivalence between the collection of individual excess-of-loss insurance contracts and a single aggregate stop-loss insurance contract limiting the community's aggregate loss established in Proposition 4.1 has been obtained by Chen et al. (2023) in a more restrictive setting. These authors refer to these two systems as individual-covered P2P model and group-covered P2P model, respectively.

4.5. **Pricing the Excess-of-Loss Covers.** The total amount to be paid *ex ante* by participant i is given by $e^{-r}d_i+\pi_i$, where $e^{-r}\,d_i$ is the deposit for the cover of the lower layer $(0_i,d_i)$ of $f_i(S)$, while π_i is the insurance premium for the excess-of-loss protection for the upper layer (d_i,∞) of $f_i(S)$. In exchange, the pool pays for the loss X_i and offers a non-guaranteed cash-back $(d_i-f_i(S))_+$ in case of favorable experience. Let us now discuss two ways to compute π_i.

4.5.1. *Mean-value premium calculation principle.* When the individual excess-of-loss insurance premia π_i are determined according to the mean-value premium calculation principle, we have that

$$\pi_i=e^{-r}\,(1+\theta)\,\mathbb{E}\left[\left(f_i(S)-d_i\right)_+\right], \tag{23}$$

for some non-negative loading parameter θ. Under eq. (19), the total premium collected by the insurer is then given by

$$\pi=\sum_{i=1}^{n}\pi_i=e^{-r}\,(1+\theta)\,\mathbb{E}\left[\sum_{i=1}^{n}\left(f_i(S)-d_i\right)_+\right]=e^{-r}(1+\theta)\,\mathbb{E}[(S-F^{-1}_S(p))_+]. \tag{24}$$

If the insurer charges a positive loading θ in addition to the pure premium for the excess-of-loss covers, the system cannot be fair as a whole in the sense that the expected total payments made by participant i will be strictly larger than the corresponding expected claim $\mathbb{E}[X_i]$. This is the price to pay for transferring the upper layer to the insurer. However, in case the allocation satisfies the actuarial fairness property (Definition 2.2), that is $\mathbb{E}\left[f_i(S)\right]=\mathbb{E}[X_i]$, and if we set the loading parameter θ equal to 0, we find that the system is fair.

4.5.2. *Risk-based premium calculation principle.* Let us now suppose that the individual insurance premia are determined by

$$\pi_i=\rho\left(e^{-r}\left(f_i(S)-d_i\right)_+\right), \tag{25}$$

for a given risk measure ρ, which is assumed to be additive for comonotonic risks, such as a distortion risk measure (e.g., Denuit et al. (2005) or Dhaene et al. (2006)). In this case, the total premium collected by the insurer is given by

$$\pi=\sum_{i=1}^{n}\rho\left(\left(f_i(S)-d_i\right)_+\right)=\rho\left(\sum_{i=1}^{n}\left(f_i(S)-d_i\right)_+\right)=\rho\left(\left(S-F^{-1}_S(p)\right)_+\right), \tag{26}$$

because $\left(\left(f_1(S)-d_1\right)_+,\left(f_2(S_n)-d_2\right)_+,\ldots,\left(f_n(S_n)-d_n\right)_+\right)$ is a comonotonic random vector, while ρ is assumed to be additive for comonotonic risks.

Example 4.2. *Suppose that ρ is the Value-at-Risk at probability level $q \in (0,1)$. Hence, $\rho[X] = F_X^{-1}(q)$ for any random variable X. Taking into account that the quantile of a non-decreasing and left-continuous function of a random variable is equal to that function evaluated at the same quantile of the random variable (e.g., Theorem 1 in* Dhaene *et al.* *(2002a)), we find that*

$$\pi = \sum_{i=1}^{n} F^{-1}_{e^{-r}\left(f_i(S)-d_i\right)_+}(q) = e^{-r} \sum_{i=1}^{n} \left(f_i(F_S^{-1}(q)) - d_i \right)_+ , \tag{27}$$

where we assumed that the functions f_i, which are non-decreasing, are also left-continuous.

5. Conclusion

Two important results in the theory of n-person risk sharing with convex-order preferences are that any feasible allocation is convex-order-dominated by a comonotonic allocation, and an allocation is Pareto optimal for the convex order if and only if it is comonotonic. While several proofs of these results have been provided in the literature, none gives a closed-form characterization of Pareto optima, and none provides an algorithm for the convex-order improvement result that can be easily implemented in practice. Indeed, existing proofs rely, in one way or another, on the comonotonic improvement algorithm of Landsberger and Meilijson (1994), which is only provided for the case of 2-agent risk sharing, and a limit argument based on the Martingale Convergence Theorem. This leads to neither closed-form expressions for Pareto optima, nor an easy algorithmic implementation of the convex-order improvement mechanism in the general n-person case.

In this paper, we provide novel proofs of these foundational results that alleviate the concerns raised above. Our proof of the equivalence between convex-order Pareto optimality and comonotonicity uses simple arguments and leads to a crisp closed-form characterization of Pareto-optimal allocations in terms of α-quantiles (mixed quantiles). Our proof of the convex-order improvement result via comonotonic allocation is based on the theory of majorization and an extension of a classic result of Lorentz and Shimogaki (1968), which allows us to provide an explicit algorithmic construction in the general n-person case that can be easily implemented in practice. To illustrate this implementation, we provide a numerical illustration.

Finally, we consider an application to P2P insurance, or decentralized risk sharing, to illustrate the relevance of these results. Specifically, we propose new risk sharing schemes, combining risk retention at the individual level, risk transfer for losses that are too costly, and risk sharing for the middle layer. Participants are free to select their maximum contribution to pooled losses, and an excess-of-loss risk transfer to a partnering insurance company operates beyond. The amount to be paid in advance by participants is obtained by adding the price of the excess-of-loss protection and a deposit securing the contribution to the coverage of the layer mutualized inside the P2P community. The system is thus fully funded, and a cash-back or give-back mechanism operates *ex post* to restore fairness. Provided individual contributions are comonotonic, we show that the excess-of-loss protection reduces to a stop-loss protection limiting the community's total payout under an appropriate choice of initial deposits.

Appendix A. Nonincreasing Rearrangements and Majorization

We first provide some background on the majorization order of Hardy et al. (1929, 1952). We refer to Luxemburg (1967), Chong and Rice (1971), or Bennett and Sharpley (1988) for a detailed treatment, as well as Ghossoub (2015) for additional references.

A.1. **Rearrangements and a Preorder Relation for L^1.** Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ denote the set of extended real-valued measurable functions on given probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, respectively. Two random variables $(X, Y) \in \mathcal{M} \times \widetilde{\mathcal{M}}$ are said to be equimeasurable (and we write $X \sim Y$) if they have the same law, that is,

$$\mathbb{P}\left(\left\{\omega \in \Omega : X(\omega) > t\right\}\right) = \widetilde{\mathbb{P}}\left(\left\{\widetilde{\omega} \in \widetilde{\Omega} : Y(\widetilde{\omega}) > t\right\}\right), \quad \forall t \in \mathbb{R}.$$

For each $X \in \mathcal{M}$ there exists a unique right-continuous, nonincreasing, and Borel-measurable function δ_X on $\Big([0,1], \mathcal{B}\left([0,1]\right), \mathcal{L}\Big)$, where $\mathcal{L}$ denotes Lebesgue measure, such that $X \sim \delta_X$. The random variable δ_X is called the nonincreasing rearrangement of X, and it is given by

$$\begin{aligned}\delta_X(t) :&= \inf\left\{s \in \mathbb{R} : \ \mathbb{P}\left(\left\{\omega \in \Omega : X(\omega) > s\right\}\right) \le t\right\} \\ &= \sup\left\{s \in \mathbb{R} : \mathbb{P}\left(\left\{\omega \in \Omega : X(\omega) > s\right\}\right) > t\right\}, \quad \forall t \in [0,1].\end{aligned} \tag{28}$$

Note that if $F^{-1}_{X,\mathbb{P}}$ denotes the quantile function of $X \in \mathcal{M}$ w.r.t. $\mathbb{P}$, i.e., the left-continuous inverse of the cumulative distribution function $F_{X,\mathbb{P}}$ of X, then

$$\delta_X(t) = F^{-1}_{X,\mathbb{P}}(1-t), \quad \forall t \in [0,1].$$

Definition A.1. *For $(X, Y) \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \times L^1(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, we say that Y majorizes X, and we write $X \prec Y$ whenever*

$$\int_0^1 \delta_X(t)\, dt = \int_0^1 \delta_Y(t)\, dt \quad \textit{and} \quad \int_0^u \delta_X(t)\, dt \le \int_0^u \delta_Y(t)\, dt, \ \forall u \in [0,1]. \tag{29}$$

Equivalently, $X \prec Y$ if and only if

$$\int_0^1 F^{-1}_{X,\mathbb{P}}(t)\, dt = \int_0^1 F^{-1}_{Y,\widetilde{\mathbb{P}}}(t)\, dt \quad \textit{and} \quad \int_u^1 F^{-1}_{X,\mathbb{P}}(s)\, ds \le \int_u^1 F^{-1}_{Y,\widetilde{\mathbb{P}}}(s)\, ds, \ \forall u \in [0,1].$$

Hence, in particular, for $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, it follows that (e.g., Chong (1974) or (Shaked and Shanthikumar, 2007, Theorem 3.A.5))

$$X \prec Y \iff X \preccurlyeq_{\mathrm{CVX}} Y.$$

The following result (e.g., Luxemburg (1967) or Chong (1974)) will be useful.

Proposition A.2. *Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then for each $E \in \mathcal{F}$,*

$$\left[\frac{1}{\mathbb{P}(E)} \int_E X\, d\mathbb{P}\right] \mathbb{1}_E \prec X \mathbb{1}_E.$$

Finally, majorization is preserved under dominated convergence:

Proposition A.3 (Proposition 10.2 of Chong and Rice (1971))**.** *Consider two probability spaces* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$*. If:*

(1) $X, X_n \in L^1_+ (\Omega, \mathcal{F}, \mathbb{P})$ *and* $Y, Y_n \in L^1_+(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$*, for each* $n \in \mathbb{N}$*;*

(2) There exists $(Z, \widetilde{Z}) \in L^1_+ (\Omega, \mathcal{F}, \mathbb{P}) \times L^1_+(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *such that* $X_n \leq Z$ *and* $Y_n \leq \widetilde{Z}$*, for each* $n \in \mathbb{N}$*;*

(3) $X_n \prec Y_n$*, for each* $n \in \mathbb{N}$*; and,*

(4) $X_n \to X$*,* $\mathbb{P}$*-a.s., and* $Y_n \to Y$*,* $\widetilde{\mathbb{P}}$*-a.s.,*

Then $X \prec Y$*.*

A.2. **Measure-Preserving Transformations.**

Definition A.4. *Given two finite measure spaces* (S_1, Σ_1, μ_1) *and* (S_2, Σ_2, μ_2)*, a mapping* $\tau : S_1 \to S_2$ *is said to be a measure-preserving transformation if it is measurable and satisfies* $\mu_1 \circ \tau^{-1} (E) = \mu_2 (E)$*, for all* $E \in \Sigma_2$*.*

In particular, a mapping $\tau : \Omega \to [0, 1]$ is said to be a measure-preserving transformation if $\tau \in \mathcal{M}$ and $\mathbb{P} \circ \tau^{-1} (E) = \mathcal{L} (E)$, for all $E \in \mathcal{B}\big([0, 1]\big)$. For example, if $U \in \mathcal{M}$ has a uniform distribution over $[0, 1]$, then it is a measure-preserving transformation.

Proposition A.5 (Theorem 6.2 of Chong and Rice (1971), Ryff (1965))**.** *If* $(\Omega, \mathcal{F}, \mathbb{P})$ *is nonatomic, then for any* $X \in \mathcal{M}$ *there exists a measure-preserving transformation* $\tau_X : \Omega \to [0, 1]$ *such that* $X = \delta_X \circ \tau_X$*, a.s.*

Note that if $f \in L^1 \Big([0, 1], \mathcal{B} \left([0, 1]\right), \mathcal{L}\Big)$ and τ is a measure-preserving transformation on $(\Omega, \mathcal{F}, \mathbb{P})$, then $f \circ \tau \sim f$. Hence, for all $f, g \in L^1 \Big([0, 1], \mathcal{B} \left([0, 1]\right), \mathcal{L}\Big)$,

$$f \prec g \implies f \circ \tau \prec g \circ \tau.$$

Consequently, we obtain the following result.

Lemma A.6. *If* $U \in \mathcal{M}$ *is uniformly distributed on* $[0, 1]$*, then for all* $f, g \in L^1 \Big([0, 1], \mathcal{B} \left([0, 1]\right), \mathcal{L}\Big)$*,*

$$f \prec g \implies f (U) \prec g (U) \iff f (U) \preccurlyeq_{CVX} g (U) .$$

Appendix B. Proof of Property 2.5

First note that the implication $\mathbf{Y} = \mathbf{Z} \implies \boldsymbol{Y} \overset{\mathrm{d}}{=} \boldsymbol{Z}$ is straightforward. Suppose now that $\boldsymbol{Y} \overset{\mathrm{d}}{=} \boldsymbol{Z}$. This then implies that

$$Y_1 - f_1 \left(\sum_{i=1}^{n} Y_i\right) \overset{\mathrm{d}}{=} f_1 (S) - f_1 \left(\sum_{i=1}^{n} f_i (S)\right).$$

Since $S = \sum_{i=1}^n Y_i = \sum_{i=1}^n f_i (S)$, we can rewrite the equality in distribution as $Y_1 - f_1 (S) \overset{\mathrm{d}}{=} 0$, which is equivalent to $\mathbb{P} [Y_1 = Z_1] = 1$. Hence, $Y_1 = Z_1$. A similar argument can be used to show the a.s. equalities between any Y_i and the corresponding Z_i, for $i \in \{2, \ldots, n\}$.

Appendix C. Proof of Property 2.6

Suppose that $\boldsymbol{X} := (X_1, \ldots, X_n)$ is PO, and let

$$\widetilde{\boldsymbol{X}} := \Big(\mathbb{E}\left[X_1 \mid S\right], \ldots, \mathbb{E}\left[X_n \mid S\right]\Big).$$

Then $\widetilde{\boldsymbol{X}}$ is an agregate risk allocation. Moreover, by Proposition 2.3, $\widetilde{\boldsymbol{X}} \in \mathcal{A}$ and for any $i \in \{1, \ldots, n\}$ we have $\mathbb{E}\left[X_i \mid S\right] \preccurlyeq_{\mathrm{CVX}} X_i$. Since $\boldsymbol{X}$ is PO, there is no $j \in \{1, \ldots, n\}$ such that $\mathbb{E}\left[X_j \mid S\right] \prec_{\mathrm{CVX}} X_j$. Thus, $X_i \overset{\mathrm{d}}{=} \mathbb{E}\left[X_i \mid S\right]$, for all $i \in \{1, \ldots, n\}$. In particular, it follows that for all $i \in \{1, \ldots, n\}$, $\mathrm{Var}\left[X_i\right] = \mathrm{Var}\Big[\mathbb{E}\left[X_i \mid S\right]\Big]$, and therefore $\mathbb{E}\Big[\mathrm{Var}\left[X_i \mid S\right]\Big] = 0$. Consequently,

$$\mathrm{Var}\Big[\mathbb{E}\left[X_i \mid S\right] - X_i\Big] = \mathrm{Var}\Big(\mathbb{E}\Big[\mathbb{E}\left[X_i \mid S\right] - X_i \mid S\Big]\Big) + \mathbb{E}\Big(\mathrm{Var}\Big[\mathbb{E}\left[X_i \mid S\right] - X_i \mid S\Big]\Big) = 0.$$

Hence, since $\mathbb{E}\left[X_i\right] = \mathbb{E}\Big[\mathbb{E}\left[X_i \mid S\right]\Big]$, it follows that

$$\mathbb{E}\Big[\Big(\mathbb{E}\left[X_i \mid S\right] - X_i\Big)^2\Big] = \mathrm{Var}\Big[\mathbb{E}\left[X_i \mid S\right] - X_i\Big] = 0.$$

Since $\Big(\mathbb{E}\left[X_i \mid S\right] - X_i\Big)^2 \geq 0$, it then follows that $\mathbb{E}\left[X_i \mid S\right] = X_i$, a.s., for all $i \in \{1, \ldots, n\}$. Consequently, $\boldsymbol{X} = \widetilde{\boldsymbol{X}}$, a.s.

Appendix D. Proof of Theorem 2.7

By Proposition 5.9 of Denuit et al. (2022), $\boldsymbol{X}$ is comonotonic if and only if

$$\boldsymbol{X} = \big(h_1(S_{\boldsymbol{X}}), \ldots, h_n(S_{\boldsymbol{X}})\big), \tag{30}$$

with the non-decreasing functions $h_i, i = 1, 2, \ldots, n$, given by eq. (12), where α_s is defined in eq. (8). We provide a self-contained proof hereafter, for the sake of completeness. First, suppose that $\boldsymbol{X}$ is comonotonic. We define the connected support of $\boldsymbol{X}$ as follows:

$$\Big\{ \Big(F_{X_1}^{-1(\alpha)}(p), \ldots, F_{X_n}^{-1(\alpha)}(p)\Big) \mid p \in [0,1] \text{ and } \alpha \in [0,1] \Big\}, \tag{31}$$

as in Dhaene et al. (2002). The connected support of $\boldsymbol{X}$ is indeed a connected curve. Moreover, this curve is a comonotonic set, meaning that it is is simultaneously nondecreasing in each component. Let $\boldsymbol{x} := (x_1, \ldots, x_n)$ be an element of this connected support, and let $s := \sum_{i=1}^n x_i$. Following a reasoning similar to the one of the proof of Dhaene et al. (2002, Theorem 7), we find that $\boldsymbol{x}$ is the unique point of the intersection of the connected support and the hyperplane $\big\{(y_1, \ldots, y_n) \mid \sum_{i=1}^n y_i = s\big\}$. The point $\big(h_1(s), \ldots, h_n(s)\big)$, with the non-decreasing functions h_i defined in eq. (12), is an element of the connected support of $\boldsymbol{X}$. Note that for any α in $[0,1]$,

$$\Big(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\Big) \overset{\mathrm{d}}{=} \Big(F_{X_1}^{-1(\alpha)}(U), \ldots, F_{X_n}^{-1(\alpha)}(U)\Big), \tag{32}$$

and

$$F_{S_{\boldsymbol{X}}^c}^{-1(\alpha)}(p) = \sum_{i=1}^n F_{X_i}^{-1(\alpha)}(p), \ \forall p \in [0,1]. \tag{33}$$

Moreover, taking into account eq. (8) and eq. (33), we find that

$$\sum_{i=1}^n h_i(s) = s, \tag{34}$$

meaning that $\left(h_1(s), \ldots, h_n(s)\right)$ is also a point of the hyperplane considered above. From these observations, we find that

$$\boldsymbol{x} = \left(h_1(s), \ldots, h_n(s)\right). \tag{35}$$

As this expression holds for any point $\boldsymbol{x}$ of the connected support of $\boldsymbol{X}$, we can conclude that eq. (30) holds.

Conversely, assume that $\boldsymbol{X}$ is given by eq. (30), with the functions h_i defined in eq. (12). As the functions h_i are all non-decreasing, it follows immediately that $\boldsymbol{X}$ is comonotonic (e.g., Dhaene et al. (2002a, Theorem 3).

It remains to show that the the functions h_i are 1-Lipschitz. First, note that for the functions h_i defined in eq. (12), we have:

$$h_i(s) = F_{X_i}^{-1(\alpha_s)}\left(F_{S^c_{\boldsymbol{X}}}(s)\right) = \begin{cases} F_{X_i}^{-1+}(0) & \text{if } s \leq F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\,, \\ F_{X_i}^{-1}(1) & \text{if } s \geq F_{S^c_{\boldsymbol{X}}}^{-1}(1)\,. \end{cases} \tag{36}$$

Indeed, taking into account our previous conventions eq. (7), we immediately find the following expressions for the functions h_i defined in eq. (12):

$$h_i(s) = F_{X_i}^{-1+}(0) \quad \text{if } s < F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\,,$$

and

$$h_i(s) = F_{X_i}^{-1}(1) \quad \text{if } s \geq F_{S^c_{\boldsymbol{X}}}^{-1}(1)\,.$$

Moreover,

$$F_{S^c_{\boldsymbol{X}}}\left(F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\right) = \mathbb{P}\left[S^c_{\boldsymbol{X}} = F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\right] = \mathbb{P}\left[X_1 = F_{X_1}^{-1+}(0), \ldots, X_n = F_{X_n}^{-1+}(0)\right] \leq F_i\left(F_{X_i}^{-1+}(0)\right).$$

Combining this inequality with the the fact that the following holds for any $\alpha \in [0,1]$:

$$0 \leq p \leq F_X\left(F_X^{-1+}(0)\right) \implies F_X^{-1(\alpha)}(p) = F_X^{-1+}(0)\,, \tag{37}$$

$$\mathbb{P}\left[X < F_X\left(F_X^{-1}(1)\right)\right] \leq p \leq 1 \implies F_X^{-1(\alpha)}(p) = F_X^{-1}(1)\,,$$

we find that

$$h_i\left(F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\right) = F_{X_i}^{-1+}(0) \quad \text{if } s = F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\,.$$

Indeed, apply eq. (37) for $X = X_i$ and $p = F_{S^c_{\boldsymbol{X}}}\left(F_{S^c_{\boldsymbol{X}}}^{-1+}(0)\right)$.

We now show that the functions h_i defined in eq. (12) are 1-Lipschitz continuous on $\mathbb{R}$. First, we prove that the functions h_i defined in eq. (12) are Lipschitz continuous on $\left[F_{S^c_{\boldsymbol{X}}}^{-1+}(0), F_{S^c_{\boldsymbol{X}}}^{-1}(1)\right]$. From the first part of this proof, we find that the connected support of $\boldsymbol{X}$ can be expressed as follows:

$$\left\{\left(\left(h_1(s), \ldots, h_n(s)\right)\right) \middle| s \in \mathbb{R}\right\}. \tag{38}$$

Consider s_1 and s_2 in $\left[F_{S^c_{\boldsymbol{X}}}^{-1+}(0), F_{S^c_{\boldsymbol{X}}}^{-1}(1)\right]$. Without loss of generality, we assume that $s_1 \leq s_2$. For any j, the functions h_j are non-decreasing, implying that $h_j(s_1) \leq h_j(s_2)$. In particular, for the function h_i, we find that

$$\left|h_i(s_2) - h_i(s_1)\right| = h_i(s_2) - h_i(s_1) \leq \sum_{j=1}^{n}\left(h_j(s_2) - h_j(s_1)\right).$$

Taking into account the additivity property eq. (34), this inequality leads to

$$\left|h_i\left(s_2\right)-h_i\left(s_1\right)\right| \leq \left|s_2-s_1\right|,$$

which means that the function h_i is 1-Lipschitz continuous on $\left[F_{S^c_{\boldsymbol{X}}}^{-1+}(0), F_{S^c_{\boldsymbol{X}}}^{-1}(1)\right]$.

The conventions eq. (7) allow for a continuous extension of h_i outside the interval $\left[F_{S^c_{\boldsymbol{X}}}^{-1+}(0), F_{S^c_{\boldsymbol{X}}}^{-1}(1)\right]$ (see eq. (36)). It is easy to verify that this extended function, defined on $\mathbb{R}$, is 1-Lipschitz continuous on $\mathbb{R}$. Indeed, consider e.g., the case where $s_1 < F_{S^c_{\boldsymbol{X}}}^{-1+}(0)$ and $s_2 > F_{S^c_{\boldsymbol{X}}}^{-1}(1)$. Then we find that

$$\left|h_i\left(s_2\right)-h_i\left(s_1\right)\right| = F_i^{-1}\left(1\right) - F_i^{-1+}\left(0\right) \leq \left|s_2-s_1\right|.$$

The other cases follow in a similar way.

Appendix E. Proof of Theorem 2.9

We first state some preliminary results.

E.1. **Preliminary Results.**

Proposition E.1. *If $\boldsymbol{X} := (X_1, \ldots, X_n)$ is comonotonic and $\boldsymbol{Y} := (Y_1, \ldots, Y_n)$ is such that*

$$\begin{cases} Y_i \preccurlyeq_{CVX} X_i, \ \text{for all } i \in \{1, \ldots, n\}; \text{ and} \\ \exists\, j \in \{1, \ldots, n\},\, Y_j \prec_{CVX} X_j, \end{cases}$$

then

$$\sum_{i=1}^{n} Y_i \prec_{CVX} \sum_{i=1}^{n} X_i.$$

Proof. First note that it follows in particular that for all $i \in \{1, \ldots, n\}$, $\mathbb{E}\left[(Y_i-d)_+\right] \leq \mathbb{E}\left[(X_i-d)_+\right]$, for all $d \in \mathbb{R}$. Moreover, the inequality is strict for some $j \in \{1, \ldots, n\}$. Suppose, without loss of generality, that $j=1$. Then there exists $d_1^* \in \left(F_{X_1}^{-1+}(0), F_{X_1}^{-1}(1)\right)$ such that

$$\mathbb{E}\left[\left(Y_1-d_1^*\right)_+\right] < \mathbb{E}\left[\left(X_1-d_1^*\right)_+\right].$$

The connected support of the comonotonic random vector $\boldsymbol{X}$ can be characterized as follows:

$$\left\{ \left(F_{X_1}^{-1(\alpha_s)}\left(F_S(s)\right), \ldots, F_{X_n}^{-1(\alpha_s)}\left(F_S(s)\right)\right) \,\middle|\, s \in \left[F_S^{-1+}(0), F_S^{-1}(1)\right] \right\}.$$

This implies that for d_1^*, there exists some $d^* \in \left[F_S^{-1+}(0), F_S^{-1}(1)\right]$ such that

$$d_1^* = F_{X_1}^{-1(\alpha_{d^*})}\left(F_S(d^*)\right).$$

Now, for $i \in \{2, \ldots, n\}$, let $d_i^* := F_{X_i}^{-1(\alpha_{d^*})}\left(F_S(d^*)\right)$. Since $\boldsymbol{X}$ is comonotonic and $\sum_{i=1}^n d_i^* = d^*$, it follows that

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} Y_i - d^*\right)_+\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[\left(Y_i-d_i^*\right)_+\right] < \sum_{i=1}^{n} \mathbb{E}\left[\left(X_i-d_i^*\right)_+\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i - d^*\right)_+\right].$$

It remains to show that for any $d \in \mathbb{R}$,

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} Y_i - d\right)_+\right] \leq \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)_+\right],$$

but this follows from an argument similar to Corollary 1 in Dhaene et al. (2002a). □

Proposition E.2. *If $\boldsymbol{Y} \in \mathcal{A}$ is comonotonic then it is PO.*

Proof. Suppose that $\boldsymbol{Y}$ is comonotonic but not PO. Then there exists an allocation $\boldsymbol{Z} \in \mathcal{A}$ such that $Z_i \preccurlyeq_{\text{CVX}} Y_i$ for all $i \in \{1, \ldots, n\}$, with at least one strict improvement. Therefore, by Proposition E.1,

$$\sum_{i=1}^{n} Z_i \prec_{\text{CVX}} \sum_{i=1}^{n} Y_i,$$

which contradicts the fact that $\sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} Y_i = S$. Hence, a comonotonic allocation is PO. □

Proposition E.3. *Consider the random vectors (X_1, X_2) and (Y_1, Y_2) with equal marginal distributions, i.e., $Y_i \stackrel{d}{=} X_i$, for $i = 1, 2$. If (X_1, X_2) is comonotonic and (Y_1, Y_2) is not comonotonic, then $X_1 - X_2 \prec_{CVX} Y_1 - Y_2$.*

Proof. Since $(X_1, -X_2)$ and $(Y_1, -Y_2)$ have equal marginal distributions, while the first couple is countermonotonic, it follows from Theorem 3 in Dhaene et al. (1996) that

$$X_1 - X_2 \preccurlyeq_{\text{CVX}} Y_1 - Y_2.$$

Suppose, by way of contradiction, that the above convex order inequality is not strict. Then

$$X_1 - X_2 \stackrel{\text{d}}{=} Y_1 - Y_2.$$

Taking into account the equality in distribution of the respective marginal distributions, this implies that

$$\text{Cov}[X_1, X_2] = \text{Cov}[Y_1, Y_2].$$

It then follows from Proposition 2.3 in Denuit and Dhaene (2003) that (Y_1, Y_2) is comonotonic, a contradiction. Hence, $X_1 - X_2 \prec_{\text{CVX}} Y_1 - Y_2$. □

We first show Pareto optimality and comonotonicity are equivalent in the bivariate case.

Theorem E.4. *A 2-reallocation $(X_1, X_2) \in \mathcal{A}$ of the aggregate risk S is comonotonic if and only if it is $\mathcal{A}^{(2)}$-PO.*

Proof. Suppose first that $(X_1, X_2) \in \mathcal{A}$ is comonotonic. Assume, by way of contradiction, that (X_1, X_2) is not $\mathcal{A}^{(2)}$-PO. Then there exists a 2-reallocation $(Y_1, Y_2) \in \mathcal{A}$ such that $Y_i \preccurlyeq_{\text{CVX}} X_i$, for $i \in \{1, 2\}$, with at at least one strict inequality. Then, by Proposition E.1,

$$Y_1 + Y_2 \prec_{\text{CVX}} X_1 + X_2,$$

contradicting the fact that $Y_1 + Y_2 = X_1 + X_2$. Hence, a bivariate comonotonic allocation is Pareto optimal.

Now, suppose that (X_1, X_2) is $\mathcal{A}^{(2)}$-PO. Assume, by way of contradiction, that (X_1, X_2) is not comonotonic. By Lemma 2.6, (X_1, X_2) is an aggregate risk allocation. That is, there exist functions $f_i : \mathbb{R} \to \mathbb{R}$, for $i = 1, 2$, such that

$$(X_1, X_2) = \big(f_1(X_1 + X_2), f_2(X_1 + X_2)\big). \tag{39}$$

Since (X_1, X_2) is not comonotonic, at least one of the functions f_i is not a nondecrasing function of $X_1 + X_2$. Without loss of generality, assume that f_1 is not a nondecreasing function. Since the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, there exists a random variable U on $(\Omega, \mathcal{F}, \mathbb{P})$ with a uniform distribution on $(0, 1)$, such that $X_1 + X_2 = F^{-1}_{X_1+X_2}(U)$, a.s. (e.g., Lemma A.32 in Föllmer and Schied (2016). Consider the random couple $(Y_1, Y_2) = \left(F^{-1}_{X_1}(U),\, F^{-1}_{X_1+X_2}(U) - F^{-1}_{X_1}(U)\right)$. Then $(Y_1, Y_2) \in \mathcal{A}$, and $Y_1 \overset{\mathrm{d}}{=} X_1$. In particular,

$$Y_1 \preccurlyeq_{\mathrm{CVX}} X_1. \tag{40}$$

Moreover, the random couples $\left(F^{-1}_{X_1+X_2}(U),\, F^{-1}_{X_1}(U)\right)$ and $\left(X_1 + X_2,\, f_1\left(X_1 + X_2\right)\right)$ have equal marginal distributions, with the first couple being comonotonic and the second not comonotonic. Therefore, by Proposition E.3 and eq. (39), it follows that

$$Y_2 = F^{-1}_{X_1+X_2}(U) - F^{-1}_{X_1}(U) \prec_{\mathrm{CVX}} X_1 + X_2 - f_1\left(X_1 + X_2\right) = X_2. \tag{41}$$

Hence, eq. (40) and eq. (41) imply that (Y_1, Y_2) is a Pareto improvement over (X_1, X_2), contradicting the Pareto optimality of the latter. Consequently, (X_1, X_2) is comonotonic. □

E.2. **Proof of Theorem 2.9.** We are now ready to provide a proof of Theorem 2.9.

- (1) $\Longrightarrow$ (2): Let $\boldsymbol{X}$ be PO. Assume, by way of contradiction, that there exists $i \neq j \in \{1, \ldots, n\}$ such that the suballocation $\boldsymbol{Y} := \left(X_i, X_j\right)$ of $\boldsymbol{X}$ is not $\mathcal{A}_{\boldsymbol{Y}}$-PO. Without loss of generality, assume that $i = 1$ and $j = 2$. Then there exists $(Y_1, Y_2) \in \mathcal{A}_{\boldsymbol{Y}}$ such that
$$Y_i \preccurlyeq_{\mathrm{CVX}} X_i,\ \ i \in \{1, 2\},$$
where at least one of the convex order inequalities is strict. Consider now the random vector
$$\boldsymbol{Y} := \left(Y_1, Y_2, X_3, X_4, \ldots, X_n\right).$$
Then $\boldsymbol{Y} \in \mathcal{A}$, and $Y_i \preccurlyeq_{\mathrm{CVX}} X_i$, for $i \in \{1, 2, \ldots, n\}$, where at least one of the convex order inequalities is strict, thereby contradicting the Pareto optimality of $\boldsymbol{X}$. Hence, $\boldsymbol{Y}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO.
- (2) $\Longleftrightarrow$ (3): This follows from Theorem E.4.
- (3) $\Longrightarrow$ (4): This follows from the fact that comonotonicity is equivalent to pairwise comonotonicity (e.g., Theorem 4 in Dhaene et al. (2002a)).
- (4) $\Longrightarrow$ (1): The proof of this implication is a straightforward multivariate generalization of the bivariate case. See the first part of the proof of Theorem E.4.
- (4) $\Longrightarrow$ (5): This has been established in Theorem 2.7.
- (4) $\Longleftrightarrow$ (6): By the above, (1) $\Longleftrightarrow$ (4). Hence, by a similar argument, it follows that for a given suballocation $\boldsymbol{Y}$ of $\boldsymbol{X}$, $\boldsymbol{Y}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO if and only if $\boldsymbol{Y}$ is comonotonic. Now, suppose that $\boldsymbol{X}$ is comonotonic. Then each suballocation $\boldsymbol{Y}$ of $\boldsymbol{X}$ is also comonotonic. Thus, each suballocation $\boldsymbol{Y}$ of $\boldsymbol{X}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO. Conversely, suppose that each suballocation $\boldsymbol{Y}$ of $\boldsymbol{X}$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO. Then each suballocation $\boldsymbol{Y}$ of $\boldsymbol{X}$ is comonotonic. Therefore, $\boldsymbol{X}$ is comonotonic. □

Appendix F. Proof of Proposition 2.10

First, suppose that $\boldsymbol{X}$ is PO. Then by Theorem 2.9 it is comonotonic, and hence by Theorem 2.7,

$$X_i = h_i(S), \ \forall i \in \{1, \ldots, n\},$$

for non-decreasing Lipschitz functions h_i given by (12) where α_s is defined in eq. (8). In particular, the functions $x \mapsto h_i(x)$ and $x \mapsto x - h_i(x)$ are both nondecreasing. Therefore, the 2-reallocation $(X_1, S - X_1) = \left(X_1, \sum_{i=2}^n X_i\right)$ is comonotonic, and hence $\mathcal{A}^{(2)}$-PO, by Theorem 2.9. Moreover, the suballocation $\boldsymbol{Y} := (X_2, \ldots, X_n) \in \mathcal{X}^{n-1}$ is comonotonic. Therefore, it is $\mathcal{A}_{\boldsymbol{Y}}$-PO, by Theorem 2.9.

Conversely, suppose that $\boldsymbol{Y} = (X_2, \ldots, X_n)$ is $\mathcal{A}_{\boldsymbol{Y}}$-PO and the 2-reallocation $(X_1, S - X_1)$ is $\mathcal{A}^{(2)}$-PO. Then it follows from Theorem 2.9 that both $\boldsymbol{Y} = (X_2, \ldots, X_n)$ and $(X_1, S - X_1)$ are comonotonic vectors. Hence, there are nondecreasing functions f_1, f_2 summing to the identity, such that $X_1 = f_1(X_1 + S - X_1) = f_1(S)$, and $S - X_1 = \sum_{i=2}^n X_i = f_2(X_1 + S - X_1) = f_2(S)$. Moreover, there are nondecreasing functions $g_2, \ldots, g_n$ summing to the identity, such that $X_i = g_i\left(\sum_{i=2}^n X_i\right) = g_i(S - X_1) = g_i \circ f_2(S)$, for all $i \in \{2, \ldots, n\}$. Therefore, X_1 and $S - X_1$ are comonotonic, and X_i and $S - X_1$ are comonotonic for each $i \in \{2, \ldots, n\}$. Consequently, the allocation $\boldsymbol{X} := (X_1, \ldots, X_n) \in \mathcal{A}$ is comonotonic, and hence PO, by Theorem 2.9. □

Appendix G. Proof of Proposition 4.1

Taking into account the additivity property for the quantiles of a comonotonic sum, we find from (19) that

$$\sum_{i=1}^n d_i = \sum_{i=1}^n F^{-1}_{f_i(S)}(p) = F^{-1}_S(p), \tag{42}$$

which means that the sum of all deposits paid *ex ante* by the participants accumulates to the quantile at probability level p of the aggregate claims S of the pool.

The following result shows that, under mild condition imposed to the allocation, the choice (19) ensures a form of solidarity among participants, in that either everyone or no one receives cash-back or give-back at the end of the period.

Property G.1. *Suppose that all f_i are non-decreasing. In this case,*

$$f_i(S) > F^{-1}_{f_i(S)}(p) \iff S > F^{-1}_S(p).$$

Proof. Consider the support of the comonotonic contribution vector $\boldsymbol{Y} = \big(f_1(S), f_2(S), \ldots, f_n(S)\big)$, given by

$$\mathcal{S} = \left\{ \left(F^{-1}_{f_1(S)}(q), F^{-1}_{f_2(S)}(q), \ldots, F^{-1}_{f_n(S)}(q)\right) \middle| q \in (0,1) \right\}.$$

Consider two elements $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)$ of $\mathcal{S}$. One has that either $\mathbf{y} \leq \mathbf{z}$ or $\mathbf{y} \geq \mathbf{z}$ must hold, where the inequality is meant componentwise. This observation implies that for any i, one has

$$y_i > z_i \iff \sum_{j=1}^n y_j > \sum_{j=1}^n z_j.$$

Now, chosing $\boldsymbol{z} = \left(F^{-1}_{f_1(S)}(p), F^{-1}_{f_2(S)}(p), \ldots, F^{-1}_{f_n(S)}(p)\right)$, one finds for any i that

$$y_i > F^{-1}_{f_i(S)}(p) \iff \sum_{j=1}^n y_j > \sum_{j=1}^n F^{-1}_{f_j(S)}(p).$$

Taking into account the additivity property of the quantiles of a comonotonic sum, one finds for any i that

$$y_i > F^{-1}_{f_i(S)}(p) \Longleftrightarrow \sum_{j=1}^{n} y_j > F^{-1}_{S}(p)\,.$$

These equivalence relations hold for any element $\boldsymbol{y}$ of the support S of $\boldsymbol{Y}$. Hence, for any i, one has

$$f_i(S) > F^{-1}_{f_i(S)}(p) \Longleftrightarrow \sum_{j=1}^{n} f_j(S) > F^{-1}_{S}(p)\,.$$

Taking into account the full allocation condition, this corresponds with the stated result and ends the proof. □

We are now ready to prove item (i) in Proposition 4.1. From Property G.1, we deduce that in case $S \leq F^{-1}_{S}(p)$, one has

$$\sum_{i=1}^{n} \min\left\{f_i(S), F^{-1}_{f_i(S)}(p)\right\} = \sum_{i=1}^{n} f_i(S) = S,$$

while in case $S > F^{-1}_{S}(p)$, one has

$$\sum_{i=1}^{n} \min\left\{f_i(S), F^{-1}_{f_i(S)}(p)\right\} = \sum_{i=1}^{n} F^{-1}_{f_i(S)}(p) = F^{-1}_{S}(p)\,.$$

We can conclude that (20) holds.

Let us now turn to item (ii) in Proposition 4.1. Taking into account the fact that $(x-y)_+ = x - \min\{x,y\}$ holds for any real x and y, the full allocation condition, as well as (20), we find that

$$\begin{aligned}\sum_{i=1}^{n}\left(f_i(S) - F^{-1}_{f_i(S)}(p)\right)_+ &= S - \sum_{i=1}^{n}\min\left\{f_i(S), F^{-1}_{f_i(S)}(p)\right\}\\ &= S - \min\left\{S, F^{-1}_{S}(p)\right\} = \left(S - F^{-1}_{S}(p)\right)_+,\end{aligned}$$

which completes the proof of (ii).

Finally, item (iii) in Proposition 4.1 follows from item (i) in a similar way.

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