Egalitarian Pooling and Sharing of Longevity Risk
a.k.a.
Can an administrator help skin the tontine cat?

ONLINE APPENDIX

Jan Dhaene
Actuarial Research Group, AFI
Faculty of Business and Economics
KU Leuven
Leuven, Belgium

Moshe A. Milevsky
Finance Area
Schulich School of Business
York University
Toronto, Canada

September 26, 2024

1 Introduction

This online appendix is devoted to comparing classical (centralized) insurance with decentralized risk-sharing. Section 2 describes (single period) classical insurance and the inherent solvency issue in such a centralized approach. In Section 3, we describe (single period) *compensation-based risk-sharing*, as a generalization of risk-sharing via a tontine fund as considered in the paper. A short Section 4 describes another approach of decentralized risk-sharing, which we call *contribution-based risk-sharing*. In Section 5, we present a simple and theoretical example which clearly illustrates how compensation-based systems operate in the presence of an administrator.

2 Classical insurance

Consider a pool of $n > 1$ agents (policyholders). Each one of them purchases an insurance contract at time $t = 0$, which entitles him to a non-negative random claim amount X_i, payable at time $T = 1$. This claim can be a random loss related to a well-defined peril (e.g. hospitalization-related expenses) or it can be a deterministic benefit contingent on the occurrence of a well-defined event (e.g. a pure endowment). Suppose that each policyholder i pays a premium π_i at time $t = 0$, to acquire said protection or benefit. Further, suppose that the aggregate premiums $\sum_{i=1}^{n} \pi_i$ is invested risk-free, at R over the period $[0, 1]$.

The job of an actuary is to ensure the aggregate claims $\sum_{i=1}^{n} X_i$ due at time 1 are not larger than the available assets at time 1,with a sufficiently large probability. Here, the available assets consist of the time-1 value of the premiums $(1 + R) \sum_{i=1}^{n} \pi_i$ and the time-1 value $(1+R)SC$ of the solvency capital SC to be set up at time 0. In particular, the insurer will avoid bankruptcy and remain solvent at the end of the period if the following event occurs:

$$\textbf{Insurance solvency condition:} \qquad \sum_{i=1}^{n} X_i \leq (1 + R) \left(\sum_{i=1}^{n} \pi_i + SC \right). \tag{1}$$

The intuition for this condition is straightforward. The insurer (i.e. guarantor) can fulfill her liabilities *if-and-only-if* the total claims to be paid at time 1 do not exceed the accumulated value of the premiums and solvency capital.

Classical insurance is a form of "centralized" (versus decentralized) risk-sharing, meaning that it is a risk-sharing mechanism under which individual losses faced by policyholders are transferred to a central insurer. Every policyholder is compensated *ex-post* from the insurer for his experienced loss. In return for that total coverage, the insurer charges each policyholder an insurance premium *ex-ante*. Premiums and solvency capital are chosen such that the probability of the event that the sum of all accumulated premiums and solvency capital exceeds the aggregate loss of the insurance portfolio is sufficiently large (e.g. 99.5%). The key is that the centralized approach with *ex-ante* premiums requires capital to be set up by the insurer to be able to meet his *ex-post* obligations with a high probability.

3 Decentralized risk-sharing via compensations

Consider a pool of n participants, each of them suffering a random non-negative claim due at time 1. The claim of participant i is denoted by X_i. Apart from the n participants, there is an administrator (agent $n+1$), who is the person responsible for the management of the fund. Furthermore, we introduce the non-negative and non-degenerate dummy random variable X_{n+1} and we assume that the random couple $\left(X_{n+1}, \sum_{j=1}^{n} X_j\right)$ is mutually exclusive, which means that

$$X_{n+1} > 0 \Leftrightarrow \sum_{j=1}^{n} X_j = 0, \tag{2}$$

which also means that $X_{n+1} > 0$ if and only if every participant has a zero claim. The vector or pool of all claims (benefits or losses) is denoted by $\mathbf{X}$:

$$\mathbf{X} = (X_1, X_2, \ldots, X_{n+1}) .$$

We assume that at time 0, each participant pays a premium (or invests an amount) of size $\pi_i > 0$. Also the administrator pays a premium at time 0, which we denote by π_{n+1}. For that premium she will receive a compensation at time 1, see further. The vector of all premiums is labeled the **premium vector**:

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{n+1}) .$$

Decentralized risk-sharing (henceforth, DRS) refers to a mechanism under which the participants in the pool share the risks among each other in such a way that no solvency risk occurs. To achieve that objective and figure out the allotment, in compensation-based risk-sharing, premium payments are made at time 0. In return, each participant i is partially compensated *ex-post* for his loss X_i. In other words, he does not necessarily get reimbursed for the total amount X_i. At time 1, the pool – or perhaps better labeled the community, managed by the administrator – will pay the non-negative amount W_i to participant i, as a compensation for his loss X_i, taking into account the premium π_i he paid at time 0. Also the administrator will receive a compensation W_{n+1} at time 1, taking into account the premium π_{n+1} she paid at time 0. The random vector of all compensations is called the **compensation vector**:

$$\mathbf{W} = (W_1, W_2, \ldots, W_{n+1}) .$$

We make the following assumption about these compensations:

$$X_i = 0 \Rightarrow W_i = 0, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{3}$$

Typically, $\mathbf{W}$ is a random vector which depends on $\boldsymbol{\pi}$, $\mathbf{X}$ and other deterministic (i.e. time-0 observable) and random (i.e. time-1 observable) information. We will denote the compensation-based DRS mechanism described above by $(\boldsymbol{\pi}, \mathbf{X}, \mathbf{W})$.

In order to avoid solvency risk, the risk-sharing rule is set such that the following condition is fulfilled:

$$\textbf{Compensation-based solvency condition}: \sum_{i=1}^{n+1} W_i = (1+R) \sum_{i=1}^{n+1} \pi_i, \tag{4}$$

which means that the sum of all compensations paid at time 1 is exactly equal to the time-1 value of the accumulated premiums paid by the participants and the administrator at time 0, where the accumulation factor $(1+R)$ is assumed to be deterministic.

Taking into account solvency condition (4), our assumption (3) implies that in case all participants have a zero claim, they will receive a zero compensation and the administrator will receive $(1+R)\sum_{i=1}^{n+1} \pi_i$. In the other case, i.e. when at least one participant has a non-zero claim, then by the mutual exclusivity asumption, we have that $X_{n+1} = 0$ and the administrator will receive zero compensation.

As an example of a compensation-based risk-sharing rule, consider the case where for each participant i and the administrator, the compensation function W_i follows from

$$W_i^{\text{prop}} = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{X_i}{\sum_{j=1}^{n+1} X_j} \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{5}$$

We call this risk-sharing scheme the **proportional risk-sharing scheme**: Given the aggregate claims $\sum_{j=1}^{n+1} X_j$, each participant i receives a compensation proportional to his observed claim X_i, where the proportional factors are determined such that the full compensation condition (4) is satisfied. This risk-sharing rule satisfies the assumptions (3). In general, there is non-zero probability that $\sum_{j=1}^{n} X_j = 0$, which is the rational for introducting the administrator. As mentioned before, we assume that X_{n+1} is mutually exclusive with $\sum_{j=1}^{n} X_j$, implying that W_i^{prop} is always well-defined, as the denominator in (5) is always strictly positive. Notice that any (non-negative, non-degenerate) choice of X_{n+1} is allowed, provided $\left(X_{n+1}, \sum_{j=1}^{n} X_j\right)$ is mutually exclusive. Indeed, the particular choice of X_{n+1} does not impact the compensations to the participants and the administrator. In case the claims X_i of the participants $1, 2, \ldots, n$ are all zero, the administrator will receive the accumulated value of all premiums. On the other hand, in case at least one participant has a strictly positive claim, we have that $X_{n+1} = 0$, implying that $W_{n+1}^{\text{prop}} = 0$, while the compensations of the participants follow from (5).

The tontine fund described in the paper arises as a special case of the compensation-based DRS scheme (5) by choosing $\mathbf{X}$ as follows:

$$\mathbf{X} = (f_1 \times I_1,\ f_2 \times I_2, \ldots, f_{n+1} \times I_{n+1}),$$

with the survival indicator and share allocation vectors $\mathbf{I}$ and $\mathbf{f}$ as defined in the paper. In this case, (5) transforms in

$$W_i^{\text{prop}} = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{f_i}{\sum_{j=1}^{n+1} f_j \times I_j} \times I_i \qquad \text{for } i = 1, 2, \ldots, n+1, \tag{6}$$

which is expression (11) in the main paper.

4 Decentralized risk-sharing via contributions

In contrast to a system with premiums being transformed into compensations, this section describes a *contribution*-based system for DRS. Consider again a pool of n participants with non-negative random end-of-period claims X_i, related to a well-defined peril or contingent benefit. The combined vector of all dollar value claims is again called the claims vector and denoted by the bold $\mathbf{X}$:

$$\mathbf{X} = (X_1, X_2, \ldots, X_n)$$

As alluded to earlier, DRS refers to risk-sharing mechanisms under which the participants in the pool share their risks without generating or creating any solvency risk. To achieve that objective, in this particular section, we assume that each of the n participants in the risk-sharing pool is fully compensated *ex-post* for his claim X_i. But, in return each participant pays an *ex-post* contribution C_i to the pool, which is managed by the insurance *community,* overseen by an administrator. The vector of all contributions is called the **contribution vector**:

$$\mathbf{C} = (C_1, C_2, \ldots, C_n)$$

We assume that $\mathbf{C}$ is random vector which depends on $\mathbf{X}$ and other deterministic (time-0 observable) and random (time-1 observable) information. We denote the contribution-based DRS described above by $(\mathbf{X}, \mathbf{C})$.

To avoid solvency risk – since there is no guarantor company, we assume that the risk-sharing rule is such that the following condition is fulfilled:

$$\textbf{Contribution-based solvency condition}: \sum_{j=1}^{n} X_j = \sum_{j=1}^{n} C_j, \tag{7}$$

This condition means that the sum of all contributions paid by the participants matches the sum of all losses the pool covers.

Notice that for a compensation-based risk-sharing rule, we needed an administrator who pays a premium and receives a compensation in order to guarantee that the solvency condition (4) also holds in case all participants have a zero claim. In a contribution-based risk-sharing scheme as introduced in this section, we do not need the administrator to pay a contribution for the contribution-based solvency condition (7) to hold in all cases. This is due to the fact that the contributions are fixed ex post, after the claims have been observed.

An example of a contribution-based risk-sharing rule is the conditional-mean risk-sharing rule $(\mathbf{X}, \mathbf{C}^{\mathrm{cm}})$, introduced in the actuarial literature in Denuit & Dhaene (2012), with contribution vector defined by

$$\mathbf{C}^{\mathrm{cm}} = \left(\mathbb{E}\left[X_1 \mid \sum_{j=1}^{n} X_j \right],\ \mathbb{E}\left[X_2 \mid \sum_{j=1}^{n} X_j \right], \ldots,\ \mathbb{E}\left[X_n \mid \sum_{j=1}^{n} X_j \right] \right) \tag{8}$$

The properties of these and other risk-sharing rules have been investigated in detail by Denuit, Dhaene & Robert (2022) and Denuit, Dhaene, Ghossoub & Robert (2023) among others. An axiomatic characterization of the conditional mean risk-sharing rule is given in Jiao, Kou, Liu & Wang (2023).

5 A simple example

Consider two agents, denoted as person 1 and person 2, who participate in a game of chance. To enter the game, person 1 pays an amount π_1, while person 2 pays an amount π_2. Person 1 tosses a two-sided coin, and person 2 rolls a six-sided die. In this game, person 1 is successful if he tosses heads, while person 2 is successful if he rolls a 1.

In addition to the two participants, also an administrator is involved. She is also allowed to contribute to the prize pool by paying an amount π_3. According to predetermined rules, the administrator is responsible for collecting the money and distributing it.

For simplicity, let us assume a zero return over the observation period. Furthermore, if the coin lands on heads and the die does not land on 1, the total amount of $\pi_1+\pi_2+\pi_3$ is awarded to person 1 at time 1. Similarly, if the coin does not land on heads but the die lands on 1, the total amount of $\pi_1+\pi_2+\pi_3$ is awarded to person 2 at time 1. If both participants are successful (i.e., heads and 1 appear after the respective throws), the total proceeds of $\pi_1+\pi_2+\pi_3$ are shared by person 1 and person 2 in a well-defined manner. Finally, if both participants are not successful (i.e., neither heads nor 1 appear after their respective throws), the total proceeds of $\pi_1+\pi_2+\pi_3$ go to the administrator. At first glance, it may seem unusual that the administrator also contributes an amount of money π_3 to the prize pool. However, in our example, the probability that the administrator will receive the entire prize pool is $\frac{5}{12}$ assuming independence, so it seems reasonable the administrator should contribute to the prize pool for her chance of winning.

If at most one of the two participants is successful, the rules for paying out the total proceeds are clear. However, in this section, we seek to answer the following question: What is a reasonable way to allocate the total proceeds $(\pi_1+\pi_2+\pi_3)$ in case both participants have a successful throw? A uniform allocation where each participant receives $\frac{\pi_1+\pi_2+\pi_3}{2}$ may be seen as 'unfair' because it does not consider that the chances for success are much larger for participant 1 than for participant 2. To address this, we will denote the payouts to participants 1 and 2 in case of a successful throw by $\beta\times(\pi_1+\pi_2+\pi_3)$ and $(1-\beta)\times(\pi_1+\pi_2+\pi_3)$, respectively, where $0<\beta<1$.

Introducing the indicator variables I_1 and I_2 where I_i is 1 for a successful participant i and 0 otherwise, the payouts W_1 and W_2 can be expressed as follows:

$$(W_1,W_2)=\begin{cases}(\pi_1+\pi_2+\pi_3)\times(1,0) & : \text{if } I_1=1 \text{ and } I_2=0\\ (\pi_1+\pi_2+\pi_3)\times(0,1) & : \text{if } I_1=0 \text{ and } I_2=1\\ (\pi_1+\pi_2+\pi_3)\times(0,0) & : \text{if } I_1=0 \text{ and } I_2=0\\ (\pi_1+\pi_2+\pi_3)\times(\beta,1-\beta) & : \text{if } I_1=1 \text{ and } I_2=1\end{cases} \tag{9}$$

where $(\pi_1 + \pi_2 + \pi_3) \times (a, b)$ represents $((\pi_1 + \pi_2 + \pi_3) \times a, \ (\pi_1 + \pi_2 + \pi_3) \times b)$.

In order to write down the payout W_3 to the administrator, we introduce the indicator variable I_3 which is defined by

$$I_3 = (1 - I_1) \times (1 - I_2) \,. \tag{10}$$

This indicator variable equals 1 when neither participant is successful (i.e., $I_1 = I_2 = 0$) and 0 otherwise (i.e., $I_1 = 1$ or $I_2 = 1$). The administrator's payoff can be expressed as:

$$W_3 = \begin{cases} 0 & : \text{if } I_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 & : \text{if } I_3 = 1 \end{cases}$$

Before playing, the participants and the administrator must agree on the payments π_1, π_2, and π_3, as well as on the value of β.

Let us now assume that the payouts of the two participants are mutually independent. We apply the results of Section 4 of the paper to determine the investment π_1 and π_2 such that the payouts as described above are actuarial fair. Let us denote the total investment effort of the participants by π:

$$\pi = \pi_1 + \pi_2.$$

In a first step, we determine the contribution π_3 of the administrator such that the tontine fund is collective actuarial fair. From expression (23) in the paper, we find that

$$\pi_3 = \pi \times \frac{\Pr[I_3 = 1]}{\Pr[I_3 = 0]} = \frac{5}{7}\pi. \tag{11}$$

In a second step, we determine the investments π_i of the two participants by equation (19) of the paper, guaranteeing that the game is actuarial fair for the two participants. Applying this approach leads to

$$\pi_1 = E[W_1] = \frac{5 + \beta}{7}\pi \tag{12}$$

and

$$\pi_2 = E[W_2] = \frac{2 - \beta}{7}\pi. \tag{13}$$

Note that π_1, π_2 and π_3 are proportional to the total investment π of the two participants. One can chose the magnitude of π first , and then determine the investment efforts of the participants and the administrator by (11), (12) and (13), respectively. Finally, notice that one could also first choose π_1 and π_2, and then determine π_3 and β such that the game with payouts given by (9) is fair.