Egalitarian Pooling and Sharing of Longevity Risk a.k.a. *Can an administrator help skin the tontine cat?*

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Abstract

This paper is concerned with the "problem" of allocating tontine fund winnings in a pool where participants differ in wealth (contributions) and health (longevity), and particularly when the pools are relatively small in size. In other words, we offer a modelling framework for distributing longevity-risk pools' income and benefits (or tontine winnings) when participants are heterogeneous. Similar to the nascent literature on decentralized risk sharing (DRS), there are several equally plausible arrangements for sharing benefits (a.k.a. "skinning the cat") among survivors. Indeed, the selected rule may depend on the extent of social cohesion within the longevity risk pool, ranging from solidarity and altruism to pure individualism. And, if fairness is a concern, we suggest introducing an administrator – which differs from a guarantor – to make the tontine pool payouts collectively actuarial fair in the sense that the group of participants will on average receive the same benefits as what they collectively invested, and we provide the mathematical framework to implement that suggestion. One thing is for certain: actuarial science cannot really offer uniqueness; it is only a methodology.

1 Introduction

1.1 Motivation

One of the hallmarks of a developed country is the existence of a national pension scheme, which forces all working citizens to contribute savings to a retirement collective, which is then used to pay retirement annuities. National pension schemes are distinct from corporate retirement plans, which (arguably) involve more homogenous groups whose financial generosity is (arguably) at the discretion of employers.

For example, in a stylized national pension scheme, all workers might contribute 10,000 (real, inflation-adjusted) euros per year in mandatory premiums in exchange for a benefit of 27,000 (real, inflation-adjusted) euros per year beginning at retirement age 65. Therefore, a citizen who makes these contributions during 40 working years, for example, from age 25 until age 65, and then lives to (and then dies exactly at) age 85, will earn an internal rate of return of 1% per year, in real terms.

This compares favourably with other real risk-free investments around the world and is effectively guaranteed by the national government. The paternalistic calculus is to force all citizens to participate since many would be unlikely to do so on their own or be able to generate these investment returns themselves.

Alas, the challenge – and impetus for this investigation – is what happens to those unhealthy retirees who don't spend 20 years in retirement and do not live to age 85 in the above example. Those who are unfortunate to live 10 years to age 75, for example, will actually earn a negative internal rate of return of -1.6% per year in real (inflation-adjusted) terms. Indeed, contributing a (non-PV adjusted) total of 400,000 euros over the entire life in exchange for only 270,000 euros is not a good investment, especially considering they were forced to participate.

Now, in defence of this stylized national pension scheme, the conventional and centuries-old response by pension economists and insurance actuaries is that for every unlucky person who only lives to age 75 there is another lucky one who lives to a ripe old age 95. They would receive 810,000 = 30 x 27,000 euro worth of payments during retirement and thus earn an even better 2% inflation-adjusted return. Moreover, these defenders argue, that is the nature of longevity risk pooling. Winners and losers are only known at the end, *ex post*, but everyone benefits from longevity pooling, *ex ante*.

Unfortunately, there is a well-known and alarming body of evidence that survival in a group of equally aged persons is not analogous to a series of i.i.d. coin tosses. Whether due to genetics, environment or even lifestyle choices, longevity prospects are heterogeneous for individuals at the same chronological age. For these less fortunate groups, there is little chance they will live to an advanced age and benefit from the insurance aspects of longevity risk pooling. Within society, these groups have a legitimate claim that national pension schemes aren't fair or equitable. The most widely cited article linking income to life expectancy (in the US) is the study by Chetty et al. (2016). They conclude that "higher income was associated with greater longevity throughout the income distribution" and "the gap in life expectancy between the richest 1% and poorest 1% of individuals was 14.6

years for men and 10.1 years for women." Other researchers have looked at non-financial factors for the so-called *stochastic longevity gap* (a.k.a. the non-homogeneity) vis a vis the implications for pension plans, both from a theoretical and empirical perspective.

This problem is more than an academic exercise in probability or a theoretical dilemma. In early 2023, a group representing Aboriginal Australians filed suit claiming the state pension discriminates against them because their life expectancy is much lower than non-aboriginals. Most live to their mid 70s, while the rest of Australia live into their 80s and 90s. Although stochastic, their internal rate of return will fall far short of the safest risk-free alternative. The case has garnered much international attention and is pending before their Federal Court, and various groups are expressing similar concerns worldwide. This is the impetus for our paper: going back to the very first principles and try to find an answer to the question *how longevity risk should be shared in a fair way?* Another case in which a similar problem arises – and perhaps a more controversial position – is that unhealthy males with much lower longevity prospects should be considered for similar treatment; namely, receive higher payments for the same level of contributions.

The literature – on the heterogeneity of longevity and the impact on pension fairness – is vast and growing.[1]

And, while a footnote list is not a proper review of the unique contribution of every paper in the literature, their underlying messages are identical. Namely, the most crucial empirical takeaway is the existence of an identifiable group within society that will not live as long as the fortunate ones. Yet, most national and corporate pension schemes are all pooled together in one sizeable longevity-risk-sharing fund or pool.

With the motivation and background out of the way, this paper offers a modelling framework for distributing longevity-risk pools' income and benefits when participants are heterogeneous. Our central insight is that – similar to the nascent literature on decentralized risk sharing – there are several equally plausible rules for sharing benefits (a.k.a. "skinning the cat") among survivors. Moreover, the selected rule may depend on the extent of social cohesion within the longevity risk pool, ranging from solidarity and altruism to pure individualism. The vehicle we choose to use for analyzing longevity risk pooling and sharing is the simple one-period tontine.[2] Our aim is to demonstrate that there are a multiplicity of feasible arrangements for sharing gains in a one-period model.

[1]See, Ayuso et al. (2017), Bravo et al. (2023), Coppola et al. (2022), Couillard et al. (2021), Dudel and van Raalte (2023), Finegood et al. (2021), Himmelstein et al. (2022), Kinge et al. (2019), Li and Hyndman (2021), Lin et al. (2017), Milligan and Schirle (2021), Mackenbach et al. (2019), Pitacco (2019), Perez-Salamero et al. (2022), Sanzenbacher et al. (2019), Shi and Kolk (2022), Simonovitz and Lacko (2023), Sloan et al. (2010), Strozza (2022) and Woolf et al. (2023).

[2]There is a very long history of tontines in France, and for the most recent laws and regulations relating to these products, see Code des Assurances, ReplierPartie rÃ©glementaire (Articles R111-1 Ã R541-1), Section VII : Tontines. (Articles R322-139 Ã R322-159), which can be accessed from this URL: $https://www.legifrance.gouv.fr/codes/article_lc/LEGIARTI$000006815350

1.2 Setting the Stage

In this subsection, we "set the stage" for our paper's main contributions by illustrating the multitude of ways that *in theory* could allocate gains from longevity. A more formal and general model will be presented in subsequent sections. For now, imagine the following situation. A group of n investors *pool* together into the following one-period longevity risk-sharing scheme. They each invest or allocate $\pi_j > 0$ dollars at time zero into an account earning a one-plus risk-free rate: $(1 + R) \geq 1$, while they face a $p_j > 0$ probability of surviving to the end of the period. They decide to share the total fund among survivors, which is also known as a one-period tontine. For the sake of a simple numerical example, we will assume $n = 3$ and $R = 0$, but our findings can easily be extended to the case of more participants and a positive risk-free return. The first participant invests $\pi_1 = \$80$, the second $\pi_2 = \$50$ and third $\pi_3 = \$20$. Their respective one-period survival probabilities are $p_1 = 20\%, p_2 = 50\%, p_3 = 80\%$, which reflect mortality rates over a decade at old, middle and early retirement ages. The two tables in subsections 1.2 and 1.3 summarize the in- and outputs for the numerical example under consideration.

$\pi_1 =$	\$80	$p_1 =$	20%
$\pi_2 =$	\$50	$p_2 =$	50%
$\pi_3 =$	\$20	$p_3 =$	80%
$\sum \pi_k$	\$150.00	and	$R = 0\%$

Clearly, investor 1 has placed the most at risk because he/she faces an 80% probability of dying and losing it all <u>and</u> has invested \$80. Think of the function $g(\pi, p) = \pi/p$ as a theoretical measure of "money at risk", although $g(\pi, p)$ could be defined as any function that is increasing in its first argument and decreasing in its second. Regardless of how exactly "money at risk" is measured, investor 3 risks a mere \$20, and faces an 80% probability of surviving, so their $g(\pi, p) = \pi/p = 25$. Ergo, and perhaps even for ethical reasons, investor 1, with a $g(\pi, p) = 400$, should be entitled to a larger share of the gains if he/she happens to (get lucky) and survive. That should be obvious, the question is how much more. There are: $2^3 = 8$ different scenarios, the most vexing is the scenario in which everyone dies. Now, one can set the rules of this game in many ways – perhaps even by offering refunds to beneficiaries – but we will assume that in the scenario in which everyone dies, which has a $(1 - p_1)(1 - p_2)(1 - p_3) = 8\%$ probability (assuming independence), the \$150 is lost to participants and donated to charity or simply taken by the Government, as stipulated in the contract. There are another 3 scenarios that are trivial, namely when there is only one survivor who takes the entire: $\sum \pi_k = \$150$. This leaves 4 scenarios in which the fund must be distributed in a non-discriminatory manner. An informal discussion with colleagues indicates *the lack of any clear consensus* on exactly how the funds should be distributed in each of those 4 non-trivial scenarios in a manner that is perceived as *non-discriminatory.*

1.3 One Possible Allocation

We sent the above query to a number of specialists (who are noted and thanked at the end of this paper) and received a variety of replies. In the following table we offer one possible way or "rule" that can be used to distribute the \$150 in the four *non-trivial* scenarios. This solution should help set the stage for a more general discussion later.

Payouts to Participants: $W(\omega)$															
	ω_1		ω_2		ω_3		ω_4		ω_5		ω_6		ω_7		ω_8
1	114.29	0	0	0	0	1	141.18	1	120	1	150	0	0	0	0
1	28.57	1	120	0	0	0	0	1	30	0	0	1	150	0	0
1	7.14	1	30	1	150	1	8.82	0	0	0	0	0	0	0	0
Probability of Scenario (x100)															
8		32		32		8		2		2		8		8	

Let us start with scenario ω_1, where all three investors stay alive (corresponding to a column with all input set equal to 1), an event with 8% probability. The first investor ($\pi_1 = \$80, p_1 = 0.20$) thinks to him or herself: Had I used π_1 to purchase *a pure endowment from an insurance company – my payout would have been:* $\pi_1/p_1 = \$400$, assuming a technical interest or valuation rate equal to $r = 0$ and ignoring loadings for costs. This is an insurance claim, but one in which a limited pool of money is available. The insurance claim is $g(\pi, p)$, which is the (above noted) "money at risk" function. Likewise, the second investor ($\pi_2 = \$50, p_2 = 0.50$) is entitled to an insurance claim of: $\pi_2/p_2 = \$100$, and the third investor ($\pi_3 = \$20, p_3 = 0.80$) claims: $\pi_3/p_3 = \$25$, using the same actuarial logic. In total, for the three survivors in scenario ω_1, the aggregate insurance claim is $\mathbf{C}(\omega_1) := \sum_{k=1}^{3}(\pi_k/p_k) = \$525$, but alas there is only $\sum_{k=1}^{3}\pi_k = \$150$ available to distribute to the pool.

So, our proposed rule is to give investors the relative fraction, i.e. their *personal* insurance claim against the *aggregate* insurance claim of the available funds. The first investor claims 400 out of a total 525, which is 76.19%, from the available \$150. This is *more* than individual $i = 1$ invested, but *less* than his *personal* insurance claim. Algebraically this investor takes: $(\sum_{k=1}^{n}\pi_k) \times (\pi_1/p_1)/\mathbf{C}(\omega_1)$. The same logic gives the second investor 100 out of 525, or 19.04% of the \$150 available, which is \$28.57, and this is less than the $\pi_2 = \$50$ invested. The third and final investor makes a personal claim of 25 from an aggregate claim of 525, a mere 4.76% of the available \$150, for a total payout of \$7.14. The third investor, like the second, walks away with less than originally invested, while the (relative) winner is investor number one, who gets more than his original $\pi_1 = \$80$.

The same logic can be applied to the other three non-trivial scenarios $\omega_2, \omega_4, \omega_5$. While the *personal* insurance claim of amount π_j/p_j remains the same for each, the *aggregate* insurance claim paid to survivors is lower due to the smaller number. For example, the value of the aggregate insurance claim in ω_2, where only the second and third participants survive, is given by $\mathbf{C}(\omega_2) = 125$. Similarly, $\mathbf{C}(\omega_4) = 425$ and $\mathbf{C}(\omega_5) = 500$. Again, these are the denominators for the fractional allocation of end-of-period available funds, where the numerator is the *personal* insurance claim π_k/p_k.

In sum, while there are many different ways to *skin the tontine cat* our suggested (general) rule for $W_{(i,j)}$, which represents the payout in scenario ω_i (column) to individual j (row) in the above table, can be written as:

$$W_{(i,j)} \ = \sum_{k=1}^{3} \pi_k \times \left(\frac{(\pi_j/p_j) \times I_{(i,j)}}{\sum_{k=1}^{3} (\pi_k/p_k) \times I_{(i,k)}} \right), \quad j = 1,2,3 \text{ and } i = 1,\ldots,7. \tag{1}$$

where $I_{(i,j)}$ is the (scalar) life status of the j'th investor in the i'th scenario. The quantity in brackets in equation (1) is the ratio of *personal* to *aggregate* insurance claim, which is then multiplied by the money available in the pool. Remark that in the equation above, only the scenarios $i = 1, 2, \ldots, 7$ are defined. When $i = 8$, which is the ω scenario in which there are no survivors, we cannot apply the equation above as there appears a zero in the denominator. Therefore, we define $W_{(8,j)} := 0$ for $j = 1, 2, 3$. Equation (1), which will reappear in many guises and incarnations over the paper, is a *proportional* risk-sharing rule, but other rules will be proposed and analyzed in due time. Note (once again) that our numerical example assumed investment return $R = 0$ and technical interest rate $r = 0$, but nothing stops us from using the same rule for more general cases of $R \geq 0$ and $r \geq 0$. In this case, we only have to multiply the right-hand side of the above formula by $(1 + R)$.

1.4 The Tontine Fund

A single period tontine fund, as the one considered in the above example, is an investment made by a group of people (participants), who each invest a certain amount in the fund. At the end of the observation period, surviving participants share the proceeds of the tontine fund, while non-surviving participants do not receive anything. If no participant survives, the proceeds are donated to a charity or to another party, as stipulated in the contract. The tontine fund is self-financing, meaning that only the available proceeds are distributed, and insolvency or default is not possible. Therefore, no solvency capital needs to be set up at the beginning of the contract. We refer to the Online Appendix, which provides a comparison between classical 'centralized' insurance (which requires setting a solvency capital) and 'decentralized' pooling mechanisms (which do not require a solvency capital). The literature of multi-period tontines, as well as group self-annuitzation schemes, more generally, are large and continue to grow.[3]. Denuit, Hieber & Robert (2022) and Denuit & Robert (2023) studied single-period tontine funds, also known as longevity funds or endowment contingency funds.

In this paper, we also consider single-period tontine funds where only the surviving participants receive the proceeds at the end of the observation period. However, it is possible to distribute the proceeds among participants who meet objective criteria other than survival, such as a pre-defined health event, hospitalization, etc. The mathematical

[3]Key papers in that literature include, alphabetically listed, Bernhardt & Donelly (2019), Bernhardt and Qu (2023), Blake, Boardman & Cairns (2014), Chen, Chen & Xu (2022), Donnelly (2018), Donnelly, Guillien & Nielsen (2014), Forman and Sabin (2015), McKeever (2009), Piggott, Valdez & Detzel (2005), in particular Sabin (2010), as well as Stamos (2008), Weinert and Grundel (2021).

description is similar, but in this paper, we focus on survival as a trigger for participants to be entitled to proceeds.

We will discuss the situation where initial investments (wealth) and survival probabilities (health) vary among participants, which we call a heterogeneous case. As a special case, we will also examine the situation where all participants invest the same amount and have the same survival probabilities, which we refer to as the homogeneous case.

One concern with (re-introducing) tontines is their actuarial fairness. A single-period tontine, where the probability that all members die before the end date is positive, is clearly (mathematically) unfair in the sense that the expected return for the group, after accounting for any investment gains, is less than the return obtained on the investments. This is due to the fact that in case of survival of at least one participant, the investment gains are distributed among all survivors, but in case no participant survives (which happens with a positive probability) not any investment return is distributed. Previous research has also addressed this problem. To resolve this issue, some researchers have suggested adding an insurance benefit for beneficiaries of deceased members. Most papers on tontines written in the last few years have added this element to repair expectations.

While the above-mentioned approach resolves the mathematical problem, we believe that this isn't why people buy tontines. Indeed, it violates the spirit of the (historical) tontine in which all rights and ownership benefits are lost at death. Furthermore, some members may not have any beneficiaries, leading to yet another unintended redistribution of wealth. In extreme cases, when there is only one person surviving, this may create a moral hazard. In other words, and for many reasons, while adding a death benefit refund or payout "solves" the math, it "ruins" the elegance of the tontine ideal. As far as we are aware, the alternative approach we will present in this paper is new or at least different from the recent literature. We introduce a *tontine administrator* as both a technical and real-world solution to some of these issues, instead of artificially adding legacy or bequest payouts. The same administrator could be invoked within when this problem is examined through the prism of decentralized risk sharing (DRS) with a posteriori contributions to be paid by the participants, although in that context, this "new player" would serve as a legal enforcer more than a mechanism for creating actuarial fairness. More on this DRS aspect is discussed in the Online Appendix.

Why an administrator?

The (modern) tontine scheme is designed to eliminate the need for guarantees, capital, and solvency requirements. However, to ensure that all participants in the scheme behave appropriately, an "authority" must monitor and enforce the "rules of the game". This is not just a real-world friction but a critical aspect of the tontine scheme, as it creates the necessary legal and administrative confidence that payouts will be shared according to pre-specified rules. The tontine administrator, who could be a government agency or regulator, is thus, in our view, a key participant in the scheme and must be provided with compensation for their services. This compensation is the extra leftovers noted above, allocated or bequeathed to the tontine administrator when everyone dies. As we will show, if the administrator contributes to the initial investments, this approach may make the scheme collectively actuarial fair and more realistic for implementation. [4]

In sum, this paper introduces a new player into the (modern) tontine literature, who is called the administrator, and shows how he interacts and engages with the group, as well as whether or not he might be asked to pay (which means he also contributes to the fund) for the 'right to administer' if indeed, he is going to benefit from the tontine leftovers.

With some of the introductory concepts and notation behind us, the structure of what follows in this paper is organized as follows. Section 2 models and discusses the process of allocating tontine share. The subsequent Section 3 examines a multitude of expressions for the payout of a tontine fund. Then, Section 4 moves on to matters of actuarial fairness, while Section 5 links the tontine fund to (classical) pure endowment insurance. Section 6 looks at internal share allocation schemes, and Section 7 concludes the paper. The Online Appendix flushes out the connection between tontine funds more generally and decentralized risk-sharing rules. This appendix ends with a simple theoretical example illustrating the role of an administrator in a tontine fund.

2 Tontine funds and tontine shares

Let us consider a group of n individuals who decide to set up a one-period tontine fund. These individuals are referred to as 'participants'. At the beginning of the investment period, each participant i makes an initial (strictly positive) investment π_i in the fund. Our objective is to establish a fair and practical method for the surviving participants to divide the total investment among themselves if one or more of them survives. There is also a possibility that all participants may pass away, in which case, we need to determine what happens to the fund's proceeds. We have an administrator (party $n+1$) to manage the fund. The administrator's role is to collect investments at the beginning of the investment period, invest them, and distribute the proceeds (initial investments and returns) to the surviving participants. If all participants pass away, the administrator receives the full proceeds of the fund. The administrator also contributes an initial (non-negative) investment π_{n+1} to the fund to receive these funds in case of no survivals. To make things simpler, we introduce the vector $\boldsymbol{\pi}$, which is defined by

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1}),$$

and will call the *investment vector*.

The sum of all the investments made by the participants and the administrator, i.e. $\sum_{j=1}^{n+1} \pi_j$, equals the total value of the fund at the time 0. Each participant invests π_i to

[4] As a point of interest and disclosure, one of the co-authors of this paper was involved in the introduction of a tontine scheme in Canada and can attest to the fact that participants were extremely concerned about who would monitor and oversee the tontine, since the traditional insurance regulators, who demand capital, were absent. Thus, while a utopian version of longevity risk sharing assumes everyone behaves appropriately and discloses the required information concerning survival probabilities correctly, we argue that an administrator is needed to keep the deal fair.

buy shares or units in the fund. Any participant i who survives until time 1 will cash in exchange for his/her shares. This paper aims to determine a reasonable and acceptable number of units each participant should receive at time 0 for their initial investment of π_i. We will consider both the chance of inheriting part of the tontine fund and the initial investment amount while answering this question.

Let us denote the (strictly positive) number of shares of the tontine fund received by participant i by f_i. The vector $\mathbf{f}$ defined by

$$\mathbf{f} = (f_1, f_2, \ldots, f_n)$$

will be called the *(tontine) share allocation vector*.

The total number of shares issued at time 0 is given by $\sum_{j=1}^{n} f_j$. It's important to note that the administrator does not receive any shares, but in case no participant survives, all proceeds from the fund will belong to the administrator.

We define the time-0 value $S(0)$ of a tontine share as follows:

$$S(0) = \frac{\sum_{j=1}^{n+1} \pi_j}{\sum_{j=1}^{n} f_j}. \tag{2}$$

At an individual level, the participant's initial investment is not necessarily equal to the time-0 value of his allocated tontine shares. Indeed,

$$\pi_i \neq S(0) \times f_i, \qquad \text{for } i = 1, 2, \ldots, n,$$

where we make the convention that the symbol $\neq$ means "not necessarily equal". This 'non-equality' is because, in certain situations, two participants with the same investment π_i might require different rewards. For example, suppose the first person has a lower survival probability due to a higher risk profile (older age). In that case, they might need to be compensated for the extra risk they're taking by receiving more shares than the second person. In other situations, giving more to those with higher survival probabilities could be more appropriate, as they are expected to live longer and will need more financial support. We'll explore this issue further in this text.

It is also important to note that the shares or units are assumed to be personalized in the sense that each unit appointed at time 0 is linked to a particular individual participant in the fund. Moreover, the allocated shares of can only be exchanged for cash at the end of the observation period by its owner and only if this owner survives. The units of any participant who dies during the observation period become worthless, and we will say that the participants' tontine shares 'die' in that case.

The number of 'surviving' shares (i.e. shares of which the owner is still alive at time 1) is given by

$$\sum_{j=1}^{n} f_j \times I_j, \tag{3}$$

where I_j stands for the indicator variable (Bernouilli r.v.) which equals 1 in case participant j survives and equals 0 otherwise. On the other hand, the number of shares of which the owner has passed away is given by

$$\sum_{j=1}^{n} f_j \times (1 - I_j) \,. \tag{4}$$

Notice that (3) and (4) may be equal to 0 and $\sum_{j=1}^{n} f_j$, respectively, which will happen in case all participants die.

Apart from the survival indicator variables related to the n participants, we also introduce an indicator variable I_{n+1}, that is related to the payoff that the administrator will receive. Specifically, $I_{n+1} = 1$ if all participants die and the administrator receives all the fund's proceeds. Conversely, $I_{n+1} = 0$ if at least one participant survives and the administrator does not receive any proceeds from the fund. Hence,

$$I_{n+1} = \prod_{j=1}^{n} (1 - I_j) \,. \tag{5}$$

Hereafter, we will always assume that

$$0 < \Pr[I_{n+1} = 1] < 1. \tag{6}$$

This assumption means that the probability that all participant die is strictly positive and strictly smaller than 1.

To differentiate the shares owned by participants of the tontine fund from regular, anonymous shares, we refer to them as 'tontine shares'. These are individualized shares belonging to a specific person that become worthless in the event of their death.

At time 1, the total investment in the tontine fund has grown to

$$(1 + R) \times \left(\sum_{j=1}^{n+1} \pi_j \right),$$

where R is the return over the observation period. We assume that R is deterministic. Notice that we can generalize all coming results to the case where R is random, by replacing R by $E[R]$ in all formulas, provided we assume that R and the I_i are mutually independent.

As previously discussed, we calculate share allocations in a manner such that if no participants survive, the administrator receives the entire time - 1 value of the fund. However, if at least one participant survives, the time-1 value of the fund is distributed among the surviving participants, with each surviving share having a value of $S(1)$ which is defined by the following expression:

$$S(1) = (1 + R) \times \frac{\left(\sum_{j=1}^{n+1} \pi_j \right)}{\sum_{j=1}^{n} f_j \times I_j}, \qquad \text{if } I_{n+1} = 0. \tag{7}$$

In case no participants survive, there are no surviving shares left and hence, we don't have to define $S(1)$ in that case.

Let us denote the time - 1 payouts to the participants and the administrator by W_i, for $i = 1, 2, \ldots, n+1$. To define these payouts, we have to consider the cases $I_{n+1} = 0$ (i.e. at least one participant survives) and $I_{n+1} = 1$ (i.e. not any participant survives). We will introduce the notations $(W_i \mid I_{n+1} = 0)$ and $(W_i \mid I_{n+1} = 1)$ to distinguish between these two cases.

Conditional on $I_{n+1} = 0$, i.e. at least one participant survives, we have that the payouts to the participant and the administrator are given by

$$(W_i \mid I_{n+1} = 0) = \begin{cases} S(1) \times f_i \times I_i, & \text{for } i = 1, 2, \ldots, n, \\ 0, & \text{for } i = n+1. \end{cases} \tag{8}$$

Taking into account the expression (7) of $S(1)$, the conditional payouts for the participants can be expressed as follows:

$$(W_i \mid I_{n+1} = 0) = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{f_i}{\sum_{j=1}^{n} f_j \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n. \tag{9}$$

Hence, in case at least one participant survives, the total proceeds of the fund, that is $(1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right)$, are shared among all surviving participants, where any survivor receives a part of the total funds, which is proportional to the number of tontine shares f_i which were allocated to him at the set-up time of the fund. In this case, the administrator does not receive any payment.

On the other hand, in case $I_{n+1} = 1$, i.e. not any participant survives, the payouts to all parties involved are defined by

$$(W_i \mid I_{n+1} = 1) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, n, \\ (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right), & \text{for } i = n+1. \end{cases} \tag{10}$$

Hence, in case at not any participant survives, the total proceeds of the fund are owned by the administrator, while the (heirs of the) participants do not receive anything.

Remark that the r.v.'s $\sum_{j=1}^{n} f_j \times I_j$ and I_{n+1} are 'mutually exclusive', which is a special kind of countermonotonicity. This means that $\sum_{j=1}^{n} f_j \times I_j$ and I_{n+1} are both non-negative, while the one being strictly positive implies the other being equal to zero. Hence, the realization of $\sum_{j=1}^{n+1} f_j \times I_j$, where f_{n+1} is an arbitrarily chosen strictly positive number can never be equal to 0. Some relevant actuarial papers considering the concept of 'mutual exclusivity' are Dhaene & Denuit (1999), Cheung and Lo (2014) and Lauzier, Lin and Wang (2024).

Taking into account this observation and the expressions (9) and (10), we can express the payouts W_i to the participants and the administrator in the following way:

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{f_i}{\sum_{j=1}^{n+1} f_j \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1, \tag{11}$$

Notice that any strictly positive value of f_{n+1} is allowed, as the particular choice does not influence the payouts W_i of the participants and the administrator. Our proposed rule is to give each surviving investor a fraction of the available funds, where each survivor's fraction is defined as the number of his personally appointed tontine shares against the number of tontine shares that were appointed to all surviving participants. In case no participants survive, the administrator receives all available funds.

It is a straightforward exercise to verify that the payouts to the participants can also be written as follows:

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{f_i}{f_i + \sum_{j \neq i}^{n} f_j \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n,$$

where in $\sum_{j \neq i}^{n} f_j \times I_j$, the sum is taken over all values j from 1 to n, except for $j = i$. This expression is used in Denuit & Robert (2023) in the special case that $\pi_{n+1} = 0$ and all f_i are equal to 1.

From (11), we immediately find that

$$\sum_{i=1}^{n+1} W_i = (1+R) \sum_{i=1}^{n+1} \pi_i, \tag{12}$$

which means that the sum of all payments to the participants and the administrator is equal to the total proceeds of the fund. Hence, there is no default risk. For obvious reasons, we call this property (12) the '*self-financing property*' of the tontine fund.

Let us now introduce the notation $\mathbf{I}$ for the random vector consisting of all the survival indicator variables of the participants:

$$\mathbf{I} = (I_1, I_2, \ldots, I_n)\,.$$

A tontine fund may be set up if the n participants with survival indicator vector $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ and the administrator agree on the vector of investments $\boldsymbol{\pi}$ and the share allocation vector $\mathbf{f}$. Setting up a tontine fund only requires a group of participants and an administrator, as well as agreement between them on the vectors $\boldsymbol{\pi}$ and $\mathbf{f}$. Stated differently, the payout vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_{n+1})$, of the single period tontine fund is fully characterized by $\mathbf{I}, \boldsymbol{\pi}$ and $\mathbf{f}$. Therefore, we will often identify the tontine fund with the triplet $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$. Notice that no probabilities have to be assumed to make the tontine fund operational. There must only be an agreement on the vectors $\boldsymbol{\pi}$ and $\mathbf{f}$. Of course, typically $\mathbf{f}$ may depend on $\boldsymbol{\pi}$ and eventually also on the participants' agreed set of survival probabilities. Specific choices of the share allocation vector $\mathbf{f}$ will be considered hereafter.

3 Other expressions for the payouts of a tontine fund.

Taking into account (2) we can rewrite the payouts W_i of the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ defined in (11) as follows:

$$W_i = S(0) \times (1+R) \times f_i \times \left(1 + \frac{\sum_{j=1}^{n+1} f_j \times (1 - I_j) - f_{n+1}}{\sum_{j=1}^{n+1} f_j \times I_j}\right) \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{13}$$

The expression (13) of the payout W_i to participant i has a straightforward interpretation: In case participant i survives, then $I_{n+1} = 0$, and he will receive two payments at time 1. The first one is the time-1 value $S(0) \times (1+R) \times f_i$ of the tontine shares he was allocated at time 0, where at time 1 each share is valuated by $S(0) \times (1+R)$. In addition, the shares of the persons who did not survive, that is $\sum_{j=1}^{n} f_j \times (1 - I_j)$, are distributed among the survivors, where each survivor gets a part of it proportional to the shares he was allocated at time 0. Hence, person i receives in addition $\sum_{j=1}^{n} f_j \times (1 - I_j) \times \frac{f_i}{\sum_{j=1}^{n} f_j \times I_j}$ shares, where each additional share is also evaluated by $S(0) \times (1+R)$. The value of these extra shares corresponds to the second payment at time 1.

Notice that (13) implies that

$$W_i \geq S(0) \times (1+R) \times f_i \times I_i, \qquad \text{for } i = 1, 2, \ldots, n, \tag{14}$$

which means that in case participant i survives, he will always receive at least the time-1 value of the tontine shares that were allocated to him at time 0, where accumulation is performed with the tontine fund return R. Notice that (14) does not mean that upon survival, the participant receives at least the accumulated value of his initial investment π_i. Hence, upon survival,

$$W_i \ngeqq \pi_i \times (1+R) \times I_i, \qquad \text{for } i = 1, 2, \ldots, n, \tag{15}$$

where $\ngeqq$ has to be interpreted as 'not necessarily larger than or equal to'. Remark that $W_i \geq \pi_i \times (1+R) \times I_i$ will hold for each participant in case $f_i = \pi_i$ for all participants i. We will come back to this particular tontine share allocation rule in a further section.

For any i, we can rewrite formula (13) as follows:

$$W_i = \pi_i \times (1 + R_i') \times (1+R) \times (1 + R'') \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1, \tag{16}$$

with

$$1 + R_i' = \frac{S(0) \times f_i}{\pi_i} \tag{17}$$

and

$$1 + R'' = \left(1 + \frac{\sum_{j=1}^{n+1} f_j \times (1 - I_j) - f_{n+1}}{\sum_{j=1}^{n+1} f_j \times I_j}\right). \tag{18}$$

This means that the return that participant i receives on his initial investment π_i upon survival is composed of 3 parts: a risk adjustment return R_i' (because at time 0, the investment π_i is used to buy shares, where the number of allocated shares to each participant in one way or another reflects his risk profile), the investment return R of the fund, and finally the return R'' which is caused by the mortality credits, as the investments of the participants who died are shared among the surviving participants. Notice that R and R'' are non-negative and independent of i, whereas the risk adjustment return R'_i is participant-specific and may be negative. Further, we point out that R and R'_i are deterministic, whereas R'' is stochastic.

Remark 1 *Suppose that $\pi_1 = \pi_2 = \ldots = \pi_n = \pi$, and also that the participants are ordered in such a way that*

$$f_1 \leq f_2 \leq \ldots \leq f_n.$$

Intuitively, participant 1 is the one who gets the least amount of shares (e.g. because he is the youngest, implying that his investment is least at risk), while participant n is the one who gets the most shares (e.g. because he is the oldest participant). Then we find from (2) that

$$S(0) \times f_1 \leq \pi + \frac{\pi_{n+1}}{n}.$$

This observation and (17) lead to

$$R'_1 \leq \frac{1}{n}\frac{\pi_{n+1}}{\pi}.$$

In case $\pi_{n+1} = 0$, which means that the administrator does not pay any contribution, the person who is allocated the least amount of tontine shares receives a negative adjustment return $R'_1 \leq 0$. On the other hand, one has that

$$S(0) \times f_n \geq \pi + \frac{\pi_{n+1}}{n}$$

and hence from (17), we find that

$$R'_n \geq \frac{1}{n}\frac{\pi_{n+1}}{\pi}.$$

This means that the 'person who gets the most shares' receives a positive risk adjustment return $R'_n \geq 0$.

4 Actuarial fairness of a tontine fund

Following Bernard, Feliciangeli & Vanduffel (2022) and others in the next definition, we say that a tontine fund is 'actuarially fair' for the participants in case it is an actuarial fair deal for each participant. This means that the time 1 value of each participant's initial investment π_i is equal to his expected payoff $E[W_i]$ at time 1.

Definition 1 *The tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ is actuarial fair for each of its participants in case*

$$\pi_i \times (1 + R) = E[W_i], \qquad \text{for } i = 1, 2, \ldots, n. \tag{19}$$

Remark that one could differentiate between a technical (valuation) interest rate, which is used to *discount* expected cash flows, and the deterministic return R of the fund itself, that is, the rate by which the fund grows. Carrying those two Rs wouldn't add much to the underlying longevity risk-sharing insights and would (only) add clutter to the equations. For this reason, we assume that the technical interest rate is equal to the return of the fund.

As the payouts for the participants are zero in case no person survives, we have the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ is actuarially fair in the case

$$\pi_i \times (1+R) = E\left[W_i \mid I_{n+1} = 0\right] \times \Pr\left[I_{n+1} = 0\right], \qquad \text{for } i = 1, 2, \ldots, n. \tag{20}$$

Taking into account (11), the n fairness conditions (20) can be written as follows:

$$\pi_i = \left(\sum_{j=1}^{n+1} \pi_j\right) \times E\left[\frac{f_i \times I_i}{\sum_{j=1}^{n} f_j \times I_j} \mid I_{n+1} = 0\right] \times \Pr\left[I_{n+1} = 0\right], \qquad \text{for } i = 1, 2, \ldots, n. \tag{21}$$

We leave for future work, or perhaps to an enterprising student, a formal proof that – under some appropriate and suitable conditions – *at least* one solution $\boldsymbol{\pi}$ exists to the above set of equations. Also, on the topic of future work, in the event the survival probabilities are themselves stochastic (or entirely unknown), one could devise an *ex ante* agreement for sharing the proceeds of the fund, notwithstanding the fact it might not be "actuarially fair".

Theorem 1 *If the tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ *is actuarial fair for each of its participants, then for any* $\alpha > 0$ *and* $\beta > 0$ *also the tontine fund* $(\mathbf{I}, \alpha \times \boldsymbol{\pi}, \beta \times \mathbf{f})$ *is actuarially fair for all participants.*

Proof: The proof follows immediately from the fairness conditions (21). ■

The theorem above implies that if a tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ is actuarially fair for its participants, then the tontine fund $(\mathbf{I}, \alpha \times \boldsymbol{\pi}, \mathbf{f})$, where we multiply all the investments of all participants and the administrator by a uniform positive factor α, is also actuarially fair for these same participants. In other words, for a given group of participants with given survival index vector $\mathbf{I}$ and given tontine share allocation vector $\mathbf{f}$ not depending on $\boldsymbol{\pi}$, the set of n equations (21) with unknown $\boldsymbol{\pi}$ can never have a single solution: if $\boldsymbol{\pi}$ is a solution of (21), then for any $\alpha > 0$ also $\alpha \times \boldsymbol{\pi}$ is a solution.

Definition 2 *The tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ *is actuarial fair for the administrator in case*

$$\pi_{n+1} \times (1+R) = E\left[W_{n+1}\right]. \tag{22}$$

Taking into account that the payout to the administrator is zero in case at least one participant survives, we find that the tontine fund is actuarially fair for the administrator in case

$$\pi_{n+1} \times (1+R) = E\left[W_{n+1} \mid I_{n+1} = 1\right] \times \Pr\left[I_{n+1} = 1\right],$$

or equivalently, taking into account (11), the tontine fund is actuarial fair for the administrator if and only if

$$\pi_{n+1} = \left(\sum_{j=1}^{n} \pi_j \right) \times \frac{\Pr\left[I_{n+1} = 1\right]}{\Pr\left[I_{n+1} = 0\right]}. \tag{23}$$

Notice that in case the number of participants n is large, we will typically have that the probability that at least one participant survives, i.e. $\Pr\left[I_{n+1} = 0\right]$, will be close to 1. That means that in this case, we will have that

$$\pi_{n+1} \approx 0.$$

In the special case that all I_i are i.i.d. with $\Pr\left[I_i = 0\right] = q$, we have that $\Pr\left[I_{n+1} = 1\right] = q^n$, and (23) transforms into

$$\pi_{n+1} = \left(\sum_{j=1}^{n} \pi_j \right) \times \frac{q^n}{1 - q^n}.$$

A question that we will consider in the following theorem is whether a tontine fund which is fair for all participants is also fair for the administrator.

Theorem 2 *A tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ *that is actuarial fair for each of its participants is also actuarial fair for the administrator, i.e. the conditions (19) imply that* π_{n+1} *is given by (23).*

Proof: Suppose that the tontine fund is actuarial fair for each participant. This means that the conditions (21) hold for all participants. Summing these n actuarial fairness conditions,

$$\sum_{j=1}^{n} \pi_j = \left(\sum_{j=1}^{n+1} \pi_j \right) \times \Pr\left[I_{n+1} = 0\right],$$

implies that π_{n+1} is given by (23), and hence, the tontine fund is actuarial fair for the administrator. ■

From the theorem above, we conclude that a necessary condition for a tontine fund to be actuarially fair for each of its participants is that it is actuarial fair for the administrator. In other words, in case a tontine fund is not actuarial fair for its administrator, it cannot be actuarial fair for all its participants.

In the literature, usually the investment π_{n+1} of the administrator is set equal to 0, which means that the tontine fund is not actuarial fair for the administrator, which in turn implies that it can also not be actuarial fair for all its participants. This observation can also be seen as follows. In case $\pi_{n+1} = 0$, we find from (11) that

$$\begin{aligned} \sum_{i=1}^{n} E\left[W_i\right] &= (1 + R) \left(\sum_{j=1}^{n} \pi_j \right) \times \Pr\left[I_{n+1} = 0\right] \\ &< (1 + R) \left(\sum_{j=1}^{n} \pi_j \right) \end{aligned}$$

This inequality implies that it is impossible that the tontine fund is actuarial fair for each participant, i.e. $E[W_i] = (1+R) \times \pi_i$ for all i. For at least one participant i, one must have that $E[W_i] < \pi_i \times (1+R)$.

Milevsky and Salisbury (2016) investigate tontines where the expected present value of income is always less than the amount contributed or invested in the tontine. They introduce 'equitable' rules, as rules where no specific or identifiable member is disadvantaged in time-zero expectations. More specifically, in that paper, they investigate how to construct a multi-age tontine scheme and "determine the proper share prices to charge participants so that it is equitable and doesn't discriminate against any age or any group." The tontine they propose is a closed pool that does not allow anyone to enter or exit after the initial set-up. To quote from Milevsky & Salisbury (2016):

> "... a heterogeneous tontine scheme can often (though not always) be made equitable by ensuring that the present value of income (although less than the amount contributed) is the same for all participants in the scheme, regardless of age. This scheme will not discriminate against any one cohort, although it won't be fair..."

We should note that they (too) discuss the challenges in designing longevity-risk sharing rules that work for small groups, and they conclude:

> "...We have proved that it is possible to mix cohorts without discriminating provided the diversity of the pool satisfies certain dispersion conditions and we propose a specific design that appears to work well in practice..."

Their conclusions are consistent with the main tenor of this paper, that there are an assortment or multitude of methods in which longevity risk can be shared − the many ways to *skin a cat* − and that *a priori* one isn't necessarily better or worse than the other.

Theorem 3 *The tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ *is actuarial fair for each participant if and only if the following conditions are satisfied:*

$$\frac{\pi_i}{\pi_{n+1}} = E\left[\frac{f_i \times I_i}{\sum_{j=1}^{n} f_j \times I_j} \mid I_{n+1} = 0\right] \times \frac{\Pr[I_{n+1} = 0]}{\Pr[I_{n+1} = 1]}, \qquad \text{for } i = 1, 2, \ldots, n. \tag{24}$$

Proof: (a) Let us first assume that the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ is actuarial fair for each participant. Then we have from Theorem 2 that the tontine fund is also actuarial fair for the administrator and his investment π_{n+1} follows from (23). The actuarial fairness for the participants means that (21) holds for any $i = 1, 2, \ldots, n$. Combining these n expressions with (23) leads to the stated expressions (24) for the participant's investments. (b) Next, we assume the conditions (24) are satisfied. Summing these n equations leads to (23), which is the actuarial fairness condition for the administrator. The conditions (24) can be rewritten as

$$\frac{\pi_i}{\pi_{n+1}} = \left(\frac{\Pr\left[I_{n+1}=0\right]}{\Pr\left[I_{n+1}=1\right]}+1\right) \times E\left[\frac{f_i \times I_i}{\sum_{j=1}^{n} f_j \times I_j} \mid I_{n+1}=0\right] \times \Pr\left[I_{n+1}=0\right].$$

Taking into account the expression (23) for π_{n+1} leads to the actuarial fairness conditions (21) for the participants. ■

A possible application to the above theorem is as follows. A group of people who wants to construct a tontine fund or scheme which is actuarially fair could reach that goal by first choosing the administrator's investment π_{n+1} and the share allocation vector $\mathbf{f}$. The individual investments so that the scheme is actuarially fair, follow then from (24). Of course, in practice, this order is often reversed when the investment vector $\boldsymbol{\pi}$ is chosen first, and the share allocation vector $\mathbf{f}$ is an afterthought, depending on the choice of π.

From Theorem 2, we know that if a tontine fund is actuarially fair for each of its participants, then it is also fair for the administrator. However, the opposite implication does not hold: Actuarial fairness for the administrator is not sufficient to have actuarial fairness for each of its participants. Let us now introduce a weaker form of actuarial fairness, which we baptize "collective actuarial fairness", equally described as "social justness" to avoid the overused and rather loaded term, fair.

Definition 3 *The tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ *is collective actuarial fair for its participants in case*

$$\left(\sum_{j=1}^{n} \pi_j\right) \times (1+R) = E\left[\sum_{j=1}^{n} W_j\right]. \tag{25}$$

Collective actuarial fairness (a.k.a. social justness) means that the time 1 value of the sum of all participant's initial investments $\sum_{j=1}^{n} \pi_j$ is equal to the expected value of the sum of all their payoffs $\sum_{j=1}^{n} W_j$ at time 1.

Theorem 4 *A tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ *is collective actuarial fair for its participants if and only if it is actuarial fair for the administrator, i.e. the conditions (25) and (23) are equivalent.*

Proof: From (11) it follows that the expected value of the total payouts to all participants is given by

$$E\left[\sum_{j=1}^{n} W_j\right] = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times \Pr\left[I_{n+1}=0\right]. \tag{26}$$

This means that the condition (25) for collective actuarial fairness can be rewritten as follows:

$$\left(\sum_{j=1}^{n} \pi_j\right) = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \Pr\left[I_{n+1}=0\right],$$

which is equivalent with the condition (23) of actuarial fairness for the administrator. ■

In the following theorem, we consider the situation where the participants are indistinguishable in the sense that the random vector $(I_1, I_2, \ldots, I_n)$ is exchangeable. A special case of the exchangeability assumption is that all I_i are i.i.d.

Theorem 5 *Consider the tontine fund* $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$*. Suppose that the indicator vector* $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ *is exchangeable and assume that the fund applies a uniform tontine share allocation vector* $\mathbf{f} = (f, f, \ldots, f)$*. Then we have that the tontine fund is actuarial fair for any of its participants if and only if the following condition is satisfied: All participants pay the same initial investment, that is* $\pi_i = \pi$*, for* $i = 1, 2, \ldots, n$*, with:*

$$\pi = \frac{\pi_{n+1}}{n} \times \frac{\Pr\left[I_{n+1} = 0\right]}{\Pr\left[I_{n+1} = 1\right]} \tag{27}$$

Proof: Consider the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$, where $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ is exchangeable and $\mathbf{f} = (f, f, \ldots, f)$. From (24), we know that the tontine fund is actuarial fair for any of its participants if and only if

$$\pi_i = \pi_{n+1} \times E\left[\frac{I_i}{\sum_{j=1}^n I_j} \mid I_{n+1} = 0\right] \times \frac{\Pr\left[I_{n+1} = 0\right]}{\Pr\left[I_{n+1} = 1\right]}, \qquad \text{for } i = 1, 2, \ldots, n. \tag{28}$$

Taking into account the exchangeability of $(I_1, I_2, \ldots, I_n)$, a symmetry argument leads to the conclusion that $E\left[\frac{I_i}{\sum_{j=1}^n I_j} \mid I_{n+1} = 0\right]$ is equal for all i. Further, as

$$\sum_{i=1}^n E\left[\frac{I_i}{\sum_{j=1}^n I_j} \mid I_{n+1} = 0\right] = 1,$$

we find that

$$E\left[\frac{I_i}{\sum_{j=1}^n I_j} \mid I_{n+1} = 0\right] = \frac{1}{n}, \qquad \text{for } i = 1, 2, \ldots, n.$$

We can conclude that the n actuarial fairness conditions for the participants are equivalent with $\pi_i = \pi$, for $i = 1, 2, \ldots, n$, where π is given by (27). ■

5 Single period tontine vs. classical pure endowment

Consider n persons with survival indicator vector $\mathbf{I}$, who want to set up a one-period tontine fund and start negotiations about how much everyone should invest and how the tontine shares should be allocated. To come up with a reasonable tontine fund structure characterized by $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$, they start by deciding on the vector $\boldsymbol{\pi}$. Once this vector is specified, the participants observe the insurance market to find out what kind of pure endowment insurance each could buy for a premium equal to his tontine fund investment. Suppose that person i can buy a pure endowment with survival benefit L_i for a premium equal to π_i. We do not require any particular premium principle to determine the π_i. In other words, we assume the π_i to be chosen and the corresponding L_i to be observed in the market.

In case the n persons buy the insurance from a particular insurer, this insurer faces a possibility of insolvency, that is a possibility that the event

$$\sum_{j=1}^{n} L_j \times I_j - (1+R) \times \sum_{j=1}^{n} \pi_j > 0$$

might occur.

In traditional life insurance, the insurer "solves" the insolvency issue by charging sufficiently high premiums and setting up a solvency capital.

To solve this issue for the tontine fund under construction, the n persons appoint an external administrator, who is assumed to contribute $\pi_{n+1} \geq 0$. As before, we introduce the Bernoulli r.v., defined as follows:

$$I_{n+1} = \prod_{j=1}^{n} (1 - I_j) \,. \tag{29}$$

Furthermore, let L_{n+1} be an arbitrarily chosen strictly positive number. Then, for each participant, the 'insurance payout' $L_i \times I_i$ is replaced by the 'tontine payout'

$$W_i = \alpha(\mathbf{I}) \times L_i \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1, \tag{30}$$

where $\alpha(\mathbf{I})$ follows from

$$\alpha(\mathbf{I}) \times \left(\sum_{j=1}^{n+1} L_j \times I_j \right) - (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) = 0,$$

or, equivalently,

$$\alpha(\mathbf{I}) = (1+R) \times \frac{\sum_{j=1}^{n+1} \pi_j}{\sum_{j=1}^{n+1} L_j \times I_j}. \tag{31}$$

Hence, the random coefficient $\alpha(\mathbf{I})$ is chosen such that the benefits $\alpha(\mathbf{I}) \times L_i \times I_i$ satisfy the full allocation condition.

Notice that $\alpha(\mathbf{I})$ is identical for any particular participant and the administrator, but it is only observable at time 1. It is straightforward to verify that the particular choice of L_{n+1} does not influence the payouts W_i.

Furthermore, from (30) and (31), we find that

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{L_i}{\sum_{j=1}^{n+1} L_j \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{32}$$

We find that these payouts are exactly equal to the payouts of the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ with payouts W_i defined in (11), provided the allocated shares are given by

$$\mathbf{f} = \mathbf{L},$$

where

$$\mathbf{L} = (L_1, L_2, \ldots, L_n).$$

This rule pays the survivors the relative fraction, i.e. their personal insurance claim against the aggregate insurance claims of the survivors, of the available funds.

In order to be able to apply (32), the participants and the administrator only have to agree on the vectors $\boldsymbol{\pi}$ and $\mathbf{L}$. This means that they only have to decide and agree on what everyone invests at time 0 and on what the participants would receive as survival benefit in a classical pure endowment insurance environment for their investment used as a premium. The choice of the premium principle or a mortality table is not required.

So far, we did not consider the choice of π_{n+1}. A possible choice for the administrator's contribution is given by (23),

$$\pi_{n+1} = \left(\sum_{j=1}^{n} \pi_j \right) \times \frac{\Pr[I_{n+1} = 1]}{\Pr[I_{n+1} = 0]}.$$

This choice of π_{n+1} makes the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{L})$ fair for the administrator, and hence, also collective fair for the group of participants.

A possible way to fix $\mathbf{L}$ is chosing the L_i such that

$$\pi_i = \frac{1}{1+r} \times L_i \times p_i, \qquad \text{for } i = 1, 2, \ldots, n$$

for given (agreed) survival probabilities p_i and technical interest r. This means that the participants agree on a lifetable and choose the amounts L_i as the survival benefit corresponding to the net premium in a pure endowment insurance with net premium π_i. Under this choice, we find that (32) reduces to

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{\frac{\pi_i}{p_i}}{\sum_{j=1}^{n+1} \frac{\pi_j}{p_j} \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{33}$$

In the following section, we will come back to the particular payout scheme defined in (33). Remark that the payouts (1) of the example in Section 1 are a translation of (33) to the particular situation in that example.

6 Tontine funds with an internal share allocation scheme.

Let us consider a group of n persons with a survival indicator vector $\mathbf{I} = (I_1, I_2, \ldots, I_n)$. As mentioned above, a tontine fund for this group is characterized by $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$. Let us now assume that the n participants and the administrator agree on a probability vector

$$\mathbf{p} = (p_1,\ p_2, \ldots,\ p_n),$$

where $p_i, i = 1, 2, \ldots, n$ is the survival probability of participant i, i.e. $p_i = P[I_i = 1]$. The vector $\mathbf{p}$ of the survival probabilities have to be interpreted as an 'agreed vector', which may be different from the 'real vector' of the survival probabilities of the participants. We also introduce the notation $p_{n+1} = P[I_{n+1} = 1]$ for the probability that not any participant survives.

In this section, we assume that the tontine share allocation vector $\mathbf{f}$ is a function of the contribution vector $\boldsymbol{\pi}$ and the probability vector $\mathbf{p}$:

$$\mathbf{f} = \mathbf{f}(\boldsymbol{\pi}, \mathbf{p}) = (f_1(\boldsymbol{\pi}, \mathbf{p}), f_2(\boldsymbol{\pi}, \mathbf{p}), \ldots, f_n(\boldsymbol{\pi}, \mathbf{p})),$$

where the value of $f_i(\boldsymbol{\pi}, \mathbf{p})$ corresponds to the number of tontine shares received by person i in the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$. In other words, $\mathbf{f} : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}^n$ maps any couple $(\boldsymbol{\pi}, \mathbf{p})$ consisting of a contribution vector and a survival probability vector into a tontine share allocation vector $\mathbf{f}(\boldsymbol{\pi}, \mathbf{p})$.

Each $f_i(\boldsymbol{\pi}, \mathbf{p})$, $i = 1, 2, \ldots, n$, can be interpreted as a measure of the 'risk exposure' of the corresponding participant, taking into account the information on initial investments and survival probabilities of all participants. We call the function $\mathbf{f}$ an *internal share allocation scheme* in the sense that the allocated number of shares only depends on internal information of the pool, i.e. on the vectors $\boldsymbol{\pi}$ and $\mathbf{p}$. More generally, one could also consider more complex share allocation schemes, where the number of allocated shares does not only depend on $\boldsymbol{\pi}$ and $\mathbf{p}$, but also on other deterministic information and/or on time 1 observable random variables, such as the state of the economy at the end of the year, the occurrence (or not) of a pandemic over the coming year, the precise magnitude of medical inflation over the coming year, etc.

From (11), we find that the payouts W_i of the tontine fund $(\mathbf{I}, \boldsymbol{\pi}, \mathbf{f})$ can be expressed as follows:

$$W_i = (1 + R) \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times \frac{f_i(\boldsymbol{\pi}, \mathbf{p})}{\sum_{j=1}^{n+1} f_j(\boldsymbol{\pi}, \mathbf{p}) \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1. \quad (34)$$

At the set-up of the tontine fund with an internal share allocation rule, an agreement on the investments and an assumption about (or agreement on) the survival probabilities of the participants are required to be able to define the payouts. Once the tontine fund is launched, it only needs an administrator who collects info about the survival or death of the participants, from which he can then determine each payout W_i via formula (34). Notice that any strictly positive choice for $f_{n+1}(\boldsymbol{\pi}, \mathbf{p})$ can be made as the payouts do not depend on this choice.

In certain situations, it may be reasonable to assume that

$$f_i(\boldsymbol{\pi}, \mathbf{p}) = \pi_i \times g(p_i), \qquad \text{for } i = 1, 2, \ldots, n, \quad (35)$$

where g is strictly positive and decreasing (or increasing, or something else) in the survival probability p_i. That means that it may be appropriate to assume a linear behaviour between the number of allocated tontine shares f_i and the initial investment π_i, when the survival probability p_i is fixed. A decreasing g corresponds to the mathematical translation of the fact that 'a participant with a smaller survival probability receives a larger number of tontine shares than a person with the same initial investment but higher survival probability.' Such an approach is inspired by the idea that the person with a smaller survival probability has a higher chance of losing his initial investment. On the other side, in case one imposes an increasing g that means that the allocation rule is such that it favours participants with higher survival probabilities. This might be a desirable property in a closely connected social group, and would be in the hands of the scheme's architects. Finally, notice that we assume here that g is not participant-specific and hence, does not depend on i. More generally, one could introduce a participant-specific function g_i.

In case (35) holds, we have that the payouts W_i in (34) transform into

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{\pi_i \times g(p_i)}{\sum_{j=1}^{n+1} \pi_j \times g(p_j) \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1. \quad (36)$$

Hereafter, we introduce some important special cases of the share allocation scheme defined in (34).

Example 1 The **DM** allocation scheme.
Let us assume the internal tontine share allocation scheme (35), where $g(p_i) = 1/p_i$. In other words, we consider the following tontine share allocation scheme:

$$f_i^{\mathrm{DM}}(\boldsymbol{\pi}, \mathbf{p}) = \frac{\pi_i}{p_i}, \qquad i = 1, 2, \ldots, n+1. \quad (37)$$

In this case, the payouts (36) of the participants are given by

$$W_i^{\mathrm{DM}} = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{\frac{\pi_i}{p_i}}{\sum_{j=1}^{n+1} \frac{\pi_j}{p_j} \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1, \quad (38)$$

which corresponds with the tontine fund payouts W_i that we introduced in (33). Remark that a special case of this allocation scheme was considered in (1). Notice that we have chosen $f_{n+1}^{\mathrm{DM}}(\boldsymbol{\pi}, \mathbf{p}) = \frac{\pi_{n+1}}{p_{n+1}}$, but any other strictly positive value of $f_{n+1}^{\mathrm{DM}}(\boldsymbol{\pi}, \mathbf{p})$ will lead to the same payouts.
A motivation for this allocation (37) in terms of traditional insurance benefits was given in the previous section.

It's interesting to note that a rather special case of the above will arise if all participants are required to have the same risk exposure in the sense that $\frac{\pi_i}{p_i} = c$, where c, is a given constant. In that special case, for a given (or assumed) vector of survival probabilities, the investments are then given by:

$$\pi_i = c \times p_i, \qquad i = 1, 2, \ldots, n+1,$$

and (38) reduces to:

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j \right) \times \frac{I_i}{\sum_{j=1}^{n+1} I_j}, \qquad \text{for } i = 1, 2, \ldots, n+1$$

Example 2 The **T** allocation scheme.
Consider the internal share allocation scheme (35) with $g(p) \equiv 1$:

$$f_i^{\mathrm{T}}(\boldsymbol{\pi}, \mathbf{p}) = \pi_i, \qquad \text{for } i = 1, 2, \ldots, n+1.$$

Then (36) becomes

$$W_i^{\mathrm{T}} = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times \frac{\pi_i}{\sum_{j=1}^{n+1} \pi_j \times I_j} \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{39}$$

Notice that from (2), it follows that the time - 0 value of a tontine share, notation $S^{\mathrm{T}}(0)$ is now given by

$$S^{\mathrm{T}}(0) = \frac{\sum_{j=1}^{n+1} \pi_j}{\sum_{j=1}^{n} \pi_j} = 1 + \frac{\pi_{n+1}}{\sum_{j=1}^{n} \pi_j}.$$

From (13) it follows then that (39) can be rewritten as follows:

$$W_i^{\mathrm{T}} = \left(1 + \frac{\pi_{n+1}}{\sum_{j=1}^{n} \pi_j}\right) \times (1+R) \times \pi_i \times \left(1 + \frac{\sum_{j=1}^{n+1} \pi_j \times (1 - I_j) - \pi_{n+1}}{\sum_{j=1}^{n+1} \pi_j \times I_j}\right) \times I_i,$$
$$\text{for } i = 1, 2, \ldots, n+1. \tag{40}$$

When $\pi_{n+1} = 0$, formula (40) remains to hold, provided we replace π_{n+1} in $\frac{\sum_{j=1}^{n+1} \pi_j \times (1-I_j) - \pi_{n+1}}{\sum_{j=1}^{n+1} \pi_j \times I_j}$ by a strictly positive value f_{n+1}. In this case, we find that

$$W_i^{\mathrm{T}} = (1+R) \times \pi_i \times \left(1 + \frac{\sum_{j=1}^{n} \pi_j \times (1 - I_j) - f_{n+1} \times I_{n+1}}{\sum_{j=1}^{n} \pi_j \times I_j + f_{n+1} \times I_{n+1}}\right) \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1.$$

In case at least one participant survives, i.e. $I_{n+1} = 0$, we have that

$$\left(W_i^{\mathrm{T}} \mid I_{n+1} = 0\right) = (1+R) \times \pi_i \times \left(1 + \frac{\sum_{j=1}^{n} \pi_j \times (1 - I_j)}{\sum_{j=1}^{n} \pi_j \times I_j}\right) \times I_i, \qquad \text{for } i = 1, 2, \ldots, n.$$

In the questionnaire survey that we noted in the early part of the paper, one of the replies[5] that we received was the above-noted formula, and which we denote as Tavin allocation scheme. That scheme favours younger participants, individuals with higher survival probabilities. Indeed, consider two participants, i and j, who invest the same amount $\pi_i = \pi_j$, but the first one is younger than the second one in the sense that $p_i > p_j$. Then obviously, the younger person is favoured as in the case of survival, both receive the same amount, although the younger one has a higher survival probability. To paraphrase Tavin (2023):

[5]Private communication from Bertrand Tavin.

"... In this allocation, the recorded amount upon survival is only driven by the agent's initial stake compared to the others' stakes. This system plays a role in terms of the welfare of the social group. Namely, there is a reallocation of wealth (the total amount in the fund) that is favourable to those who are likely to survive long after the liquidation of the tontine, compared to the risk-return-based allocation, which favours the agents who are likely not to survive long after the liquidation of the tontine. This system increases the group's welfare if we look at the welfare obtained by the surviving agents after time 1. Conditional on survival at time 1, those likely to live long after time 1 need more (because they probably need to take care of an elder parent or children) and are more likely to have projects that benefit the social group (e.g. opening or financing a business). On the other hand, the agent who is not likely to survive long after time 1 will probably not have enough time to enjoy the received amount ..."

Example 3 Consider the share allocation rule with:

$$f_i\left(\boldsymbol{\pi}, \mathbf{p}\right) = \frac{1}{p_i}, \qquad i = 1, 2, \ldots, n.$$

In this case, we find from (34) that

$$W_i = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times \frac{\frac{1}{p_i}}{\sum_{j=1}^{n+1} \frac{I_j}{p_j}} \times \frac{I_i}{p_i}, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{41}$$

This rule favors 'poorer' participants (i.e. participants who invest less). Indeed, consider two persons i and j with initial investments $\pi_i < \pi_j$. Suppose that both have the same survival probability. Then in case of survival both receive the same amount, whereas person i invested less.

Example 4 The **DR** allocation scheme, following the work of Denuit & Robert (2023). Consider the uniform rule with

$$f_i^{\text{DR}} = 1, \qquad i = 1, 2, \ldots, n+1.$$

In this case, we find from (34) that

$$W_i^{\text{DR}} = (1+R) \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times \frac{I_i}{\sum_{j=1}^{n+1} I_j}, \qquad \text{for } i = 1, 2, \ldots, n+1. \tag{42}$$

From (2), we find that $S^{\text{DR}}(0) = \frac{\sum_{j=1}^{n+1} \pi_j}{n}$. Hence, from (13), we find that

$$W_i^{\text{DR}} = (1+R) \times \frac{\sum_{j=1}^{n+1} \pi_j}{n} \times \left(1 + \frac{\sum_{j=1}^{n+1} (1-I_j) - 1}{\sum_{j=1}^{n+1} I_j}\right) \times I_i, \qquad \text{for } i = 1, 2, \ldots, n+1.$$

This scheme is advantageous for individuals who are younger and poorer. Suppose there are two people, i and j. If i is younger and has a higher chance of survival, but is poorer and pays less to the tontine fund than j, he will receive the same payout money if they both survive. Therefore, the tontine arrangement benefits the younger and less affluent person i. A similar scheme can be found in Denuit & Robert (2023), with the difference being that they make their rule fair by returning the initial investments if all participants pass away and defining the initial investments so that the allocations are actuarially fair.

We end this section by a note about the (rather loaded term) "actuarial fairness". Taking into account Theorem 4, the above-discussed allocation schemes or arrangements - although in general not actuarially fair to any given individual - can be made collective actuarial fair (in the sense of Definition 3), or perhaps the proper word is actuarially "just" to add another term to the growing lexicon, by introducing the administrator and making the arrangement actuarially fair for him.

6.1 Back to the Original Motivation

We conclude here, returning to the example from Section 1.2 ("Setting the Stage"), with which we motivated the paper. In particular, we now add the tontine share allocations (f_1, f_2, f_3) and the contribution from the administrator (agent 4) to illustrate how these tontine shares would be (re)valued and how their introduction would make the scheme actuarially fair *collectively*. Recall there were three participants or agents whose contributions π, and survival probabilities p were: $(80, 0.2)$, $(50, 0.5)$ and $(20, 0.8)$ respectively. The total gross fund contribution from the three participants was 150 dollars. The probability of aggregate death was $p^* = 0.08$, all of which was explained in the introductory sections.

6.1.1 Tontine Share Allocations

We (arbitrarily) set the total number of *tontine shares* issued by the scheme to be $1,000$ at time zero, and then allocate those shares to the three participants using the proportional share allocation rule summarized by the following expression:

$$f_i \;=\frac{(\pi_i/p_i)}{\sum_{k=1}^{n}(\pi_k/p_k)} \times 1000, \quad i = 1, 2, 3.$$

The number of tontine shares each participant receives is summarized by the following table:

The key (practical) assumption from implementing tontine shares is that these units are *cancelled* when their owner dies, so the value of the remaining shares increases, no different from the impact of corporate stock-buybacks (and cancellations.) This is yet another way to think about mortality credits from a financial perspective.

$f_1 =$	$\frac{(80/0.2)1000}{\left(\frac{80}{0.2}\right)+\left(\frac{50}{0.5}\right)+\left(\frac{20}{0.8}\right)}$	$= 761.90$ *shares*
$f_2 =$	$\frac{(50/0.5)1000}{\left(\frac{80}{0.2}\right)+\left(\frac{50}{0.5}\right)+\left(\frac{20}{0.8}\right)}$	$= 190.48$ *shares*
$f_3 =$	$\frac{(20/0.8)1000}{\left(\frac{80}{0.2}\right)+\left(\frac{50}{0.5}\right)+\left(\frac{20}{0.8}\right)}$	$= 47.62$ *shares*
Total	$f_1 + f_2 + f_3$	$= 1{,}000$ *shares*

6.1.2 No Administrator

The total contribution or size of the fund was $\sum \pi = 150$ units of currency, so with exactly $1,000$ tontine shares outstanding at time zero, the initial value per share is $150/1000 = 0.15$ units of currency. Then, at the end of the period, if everyone is alive (which is scenario ω_1), the first participant receives a cash payout of 761.90 shares, times 0.15 per share, which is 114.285 dollars. The second participant receives a cash payout of 190.48 shares, times the same 0.15 per share, which is 28.571, and the third and final participant's 47.62 shares entitle him to 7.143 dollars. Under scenario ω_1, the per share tontine value did not change during the period because nobody died *and* we assumed an investment rate of $R = 0$. Needless to point out, the cash payouts are precisely the numbers reported in the solution we offered in Section 1.3, albeit using a separate (actuarial) argument.

To see the mechanics of cancelled tontine shares in action, let's examine the fourth (ω_4) scenario (of eight) in which participant 2 dies, and his 190.48 shares are cancelled before the end of the period payout. In that case, the 150 value of the fund is re-proportioned across the remaining and reduced ($1000 - 190.48 = 809.52$) shares, leading to a revised time $t = 1$ value per tontine share of $150/809.52 = 0.1852$ dollars. The tontine share value has jumped from 0.15 at times zero to 0.1852 at the end of the period due to the death of participant 2. In the fourth scenario, participant 1 receives 761.90 shares $\times$ 0.1852 per share, which is 141.10 dollars. The only other surviving participant, number 3, receives 47.62 shares times 0.1852, which is 8.82 dollars. Again, this is identical to the numerical solution provided in Section 1.3, but the explanation and rationale are grounded in the ownership of tontine shares. To sum up: (i.) death cancels tontine shares, which then (ii) increases the value per remaining shares, which then (iii.) leads to an appropriate cash payout based on the new value per share. All that matters is the number of shares the surviving participant owns (at the end of the period) and the revised value per share. Now, as explained earlier, this tontine scheme is actuarially unfair in the aggregate, a.k.a. collectively due to the $p^* = (0.08)$ probability of aggregate death. Indeed, individual participants 2 and 3 are expected to receive more than they contributed, a.k.a. the tontine is individually generous to them, but not participant 1 and not in aggregate. The expected payout to the participants are: $E[W_1] = 25.84$ for the first one, $E[W_2] = 53.29$ and $E[W_3] = 58.88$ for the third one. Clearly, the second and third agents are expected to receive more than they invested (a.k.a. the scheme is generous to them), and the first agent expects to receive less than he invested.

6.1.3 Enter the Administrator

To make the tontine actuarially fair to the administrator – and by construction for the entire group – the expected payout $E[W_4]$ must equal the to-be-determined contribution: π_4. That number must satisfy the following relationship:

$$\pi_4 \ = \ p^*(\pi_1 + \pi_2 + \pi_3 + \pi_4),$$

From which we find $\pi_4 = (0.08)(150 + \pi_4)$, so the administrator contributes $\pi_4 = 13.04$ at time zero, increasing the size of the tontine fund from 150 to 163.04 dollars. The administrator receives no tontine shares (ever) but is entitled to the entire 163.04 value of the fund *if-and-only-if* everyone dies, a.k.a. aggregate death, which occurs with 0.08 probability. However, this added 13.04, which is now part of the tontine fund, increasing the time $t = 0$ value per share from 0.15 to 163.04/1000 = 0.16304 dollars. The number of shares held by the three participants remains the same as in the no-administrator case. Still, the cash flow to survivors is adjusted by the new share price. The following table is a revised version of the table displayed in section 1.3, with the adjusted cashflows to survivors. These cashflows can also directly be determined from formula (38) of the DM allocation scheme. Compare the two tables to see how the administrator's money is distributed to survivors.

	$W(\omega_1)$		$W(\omega_2)$		$W(\omega_3)$		$W(\omega_4)$		$W(\omega_5)$		$W(\omega_6)$		$W(\omega_7)$
1	124.22	0	0	0	0	1	153.45	1	130.43	1	163.04	0	0
1	31.06	1	130.43	0	0	0	0	1	32.61	0	0	1	163.04
1	7.76	1	32.61	1	163.04	1	9.59	0	0	0	0	0	0

Notice how the payout has increased to all participants and in all states because the value of their tontine shares has increased. For example, under ω_1, the share price of 0.16304 multiplied by the (same) number of tontine shares $(761.90, 190.48, 47.62)$ leads to the first column, etc. With the administrator, the expected payout to the participants are: $E[W_1] = 28.08$ for the first one, $E[W_2] = 57.92$ and $E[W_3] = 64.00$ for the third one and $E[W_4] = 13.04$ to the administrator. Once again, the second and third agents are expected to receive more than they invested, and the first agent (still) expects to receive less than he invested – but it is slightly more favourable to everyone individually, actuarially fair to the administrator and the collective group as a whole.

7 Summary and Conclusion

The motivation for this paper – both conceptually and in practice – revolves around the many justifiable ways in which a group of heterogeneous individuals could, in theory, share the proceeds of a (longevity) risk-pooling agreement. We offered the example and began with a one-period tontine, a product that is enjoying a resurgence of interest worldwide, both in academia and industry. Our (small pool) numerical example where the members of a heterogeneous group invested unequal amounts into the tontine pool made the multiplicity of possible solutions evident. Therefore, one of the contributions of this paper is to argue that the payout structure for a tontine fund can be quite comprehensive, catering to a broad range of groups wishing to share longevity risks without the interference of an external entity assuming the risk of insolvency.

These insights are particularly beneficial for closely-knit smaller groups aiming to redistribute wealth from their older, wealthier members to their younger, less prosperous counterparts. In such scenarios, the emphasis isn't on actuarial fairness or the magnitude of the administrator's contribution. Instead, the focus is on the collective benefit of the group, as the administrator embodies the group's interests, and their contributions directly benefit the group.

Our methodology accommodates larger groups, even when participants do not share social connections or interpersonal ties. In these situations surviving members ought to be compensated based on the actuarial risks they've accepted and been exposed to. Individuals with a lower likelihood of survival should be entitled to a more substantial reward. An external entity, like an insurance company or a government regulator, takes the role of administrator in these contexts. They could also contribute to the fund in case there is a significant likelihood that none of the participants will survive.

So, while the objective of modern tontines, and more generally, uninsured decumulation products (UDP),[6] is to eliminate the costly capital associated with insurance **guarantees**, we are not advocating the elimination of insurers**.** Rather, under these arrangements, the role of the insurer would be to administer the fund – in exchange for "a piece of the action" – which would serve two distinct roles. First, oversight. They would ensure all participants in the scheme were abiding by their obligations and commitments. Second, and just as importantly, administrators in the scheme would make it collectively actuarially fair, that is, socially just.

The next step is to extend this one-period framework to a multiperiod tontine fund, which would be constructed as a sequence of linked one-period funds. Defining the relations between indicator vectors, premium vectors, share allocation vectors, and the all-important payout vectors in consecutive periods is left for future research. In the same category of plans for future research, we leave the discussion of allowing π_i and even f_i to equal zero, allowing certain groups to avoid paying (and still benefiting) or not benefiting (and still paying.) Examples would be targetted demographic groups such as the young and old, or the poor and the rich, respectively.

We conclude by noting that the single-period tontine fund, which is described within the body of this paper can be treated or viewed as a special case of (what we call) *compensation-based* decentralized risk-sharing (DRS) arrangements, where at time $t = 0$ one contributes (deterministic) premiums (or investments) and at time $t = 1$ one receives (random) compensations, which are set such that the risk-sharing scheme is self-financing. Like the literature on tontines, there is a growing literature on this type of DRS. Apart

[6]This is the term recently introduced by Canadian regulators to describe the arrangements of this sort. See: *https://www.fsrao.ca/regulation/guidance/understanding-decumulation-products*.

from compensation-based DRS arrangements, there exist also (what we call) *contribution-based* DRS schemes, which are characterized by time 1 (random) contributions and time 1 (random) benefits or claims, and where the contributions are determined, such that the risk-sharing scheme is again self-financing. There is also a growing literature on this type of DRS. For an overview of a unified theory of DRS, we refer to Feng (2023) and the references in that book. The above-mentioned observations are further explored in some detail in the Online Appendix.

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