AN AXIOMATIC CHARACTERIZATION OF THE QUANTILE RISK-SHARING RULE

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Abstract

This paper studies the quantile risk-sharing rule introduced in Denuit, Dhaene & Robert (2022). New properties are investigated and an axiomatic theory is developed. The axiomatic characterization of this risk-sharing rule is based on aggregate and comonotonicity-related properties of risk-sharing rules. Numerical examples illustrate how this rule allocates losses among participants.

Keywords: quantile risk-sharing rule, pooling, comonotonicity, insurance.

1 Introduction

Consider n economic agents with respective losses $X_1, X_2, ..., X_n$ who decide to form a pool. If the agents presume that $X_1, X_2, ..., X_n$ are identically distributed and if no knowledge about their dependency structure is available, or they do not want to use this information, then they often share losses equally. This means that they adopt the uniform risk-sharing rule allocating to each agent the average loss $\overline{X}_n = \frac{X_1+X_2+...+X_n}{n}$ within the pool. The uniform risk-sharing rule possesses many attractive properties. See e.g. Denuit, Dhaene & Robert (2022).

Now, assume that participants become aware that their respective losses are not identically distributed. Participants bringing comparatively "smaller" losses to the pool may then not be willing to contribute the same amount $\overline{X}_n$ anymore. When the participants know the respective distribution functions of their losses $X_1, X_2, ..., X_n$, the quantile risk-sharing rule may provide them with an acceptable solution. Under this rule, the aggregate loss $S_{\mathbf{X}} = X_1+X_2+...+X_n$ of the entire pool is transformed into a probability level and all participants contribute an amount equal to the quantile of their loss at that probability level. In this way, individual characteristics are accounted for in the allocation. If all distribution functions are identical then the uniform risk-sharing rule is recovered as a particular case.

Formally, let $X_1, X_2, ..., X_n$ denote insurance loss amounts, modeled as non-negative random variables with 1-to-1 distribution functions F_{X_i} over $(0, \infty)$ and a possible positive probability mass $\mathrm{P}[X_i = 0]$ at 0. When realized losses $x_1, x_2, \ldots, x_n$ are observed, they can be turned into probability levels p_i via the equation $x_i = F_{X_i}^{-1}(p_i)$, where $F_{X_i}^{-1}$ is the inverse function of F_{X_i}, called the quantile function. This means that participant i staying alone, without engaging in any risk-sharing activities with other participants, would have remained solvent provided his or her available assets were (at least) equal to the Value-at-Risk $F_{X_i}^{-1}(p_i)$ at level p_i. The randomness of individual losses causes differences in the solvency levels p_i. The idea of joining the pool according to the quantile risk-sharing rule is to replace the possibly different $p_1, \dots, p_n$ by a unique and uniform probability level $p(s)$ corresponding to the realized aggregate loss s. The unique solution is to ask participant i to contribute the amount $F_{X_i}^{-1}(p(s))$ where $p(s)$ satisfies

$$\sum_{i=1}^{n} F_{X_i}^{-1}(p_i) = \sum_{i=1}^{n} F_{X_i}^{-1}(p(s)) = s.$$

This defines the quantile risk-sharing rule, allocating the ex-post contribution $F_{X_i}^{-1}(p(s))$ to participant i. The key argument in the study of the quantile risk-sharing rule is that $\sum_{i=1}^{n} F_{X_i}^{-1}(\cdot)$ defining the common probability level $p(s)$ is the quantile function of the sum of the comonotonic modification of the random vector $\boldsymbol{X} = (X_1, X_2, ..., X_n)$, which leads to many of the important properties of this rule.

Individual contributions under the quantile risk-sharing rule can be expressed as non-linear transformations of the equal allocation $\overline{X}_n$ under the uniform risk-sharing rule. The identity above can indeed be rewritten as

$$\frac{1}{n}\sum_{i=1}^{n} F_{X_i}^{-1}(p(s)) = \overline{x}_n,$$

where $\overline{x}_n = \frac{s}{n}$ is the ex-post contribution under the uniform risk-sharing rule. Defining the function g mapping the unit interval to the half positive real line as $g(q) = \frac{1}{n}\sum_{i=1}^{n} F_{X_i}^{-1}(q)$, we finally obtain the common probability level $p(s) = g^{-1}(\overline{x}_n)$ while the individual contributions are equal to $F_{X_i}^{-1}(g^{-1}(\overline{x}_n))$, $i = 1, 2, ..., n$. This shows that the quantile risk-sharing rule allocates to participant i a non-linear transformation $F_{X_i}^{-1} \circ g^{-1}$ of the average loss $\overline{x}_n$ at pool level, accounting for individual characteristics. The uniformity is transposed here to the common probability level $p(s)$, as all participants contribute an amount equal to the quantile of their respective losses at this same probability level. If individual losses are identically distributed then $F_{X_i}^{-1} \circ g^{-1}$ is just the identity function and the uniform risk-sharing rule is recovered, as mentioned before.

This quantile risk-sharing rule has been introduced in Denuit, Dhaene & Robert (2022) where several of its properties have been investigated. This rule is a comonotonic risk-sharing rule in the sense that the contributions are non-decreasing functions of total losses $S_{\boldsymbol{X}}$, which is a desirable property since it ensures that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. This paper further investigates this risk-sharing rule.

Embrechts, Liu & Wang (2018) and Wang and Wei (2020) characterized Pareto-optimal risk-sharing rules, where the Pareto-optimality is expressed in terms of a sum of quantile-based risk measures applied to the individual losses in the pool. The approach in the present paper is different as we investigate some properties that the quantile risk-sharing rule may or may not possess, and we determine the defining axioms underlying this risk-sharing rule.

The axiomatic theory developed in this paper compares with Jiao, Kou, Liu & Wang (2022) who pioneered the theory on axiomatic characterization of certain classes of “anonymized” risk-sharing rules, i.e. risk-sharing rules that do not require any information on the preferences of the agents, a risk exchange market, or subjective decisions of a central planner. They proved that four axioms characterize the conditional mean risk-sharing rule introduced by Denuit and Dhaene (2012). In this paper, we consider three axioms and prove that these axioms characterize the quantile risk-sharing rule.

Let us discuss these three axioms in an informal way. The first of them requires that participants adopt an aggregate risk-sharing rule, in the sense that individual contributions depend on losses $X_1, X_2, ..., X_n$, only through their sum $S_{\boldsymbol{X}}$. This condition is standard in the literature. It is for instance referred to as "risk anonymity" in the axiomatics supporting the conditional mean risk-sharing rule proposed by Jiao, Kou, Liu & Wang (2022).

The second axiom imposes that the risk-sharing rule adopted by participants is dependence-free, in the sense that individual contributions only depend on the marginal distribution functions $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$ and not on the dependence structure of the random vector $\boldsymbol{X}$. This property is certainly debatable but turns out to be reasonable in certain situations. Assume for instance that the dependence among individual losses results from the agents' exposition to a common environment, making $X_1, X_2, ..., X_n$ positively related, in the sense that large (or small) values of these random variables tend to occur simultaneously. Participants may then be willing to let their contributions only depend on the marginal distribution functions of their losses, especially if the dependence structure is hard to model. This common environment may be geographic when participants live in the same area and wish to share losses due to storms for instance. It may also be economic when participants work in the same sector of activities. Being equally impacted by their common environment, participants may only want to adjust their contributions to the magnitude of the risks they bring to the pool, as reflected in the quantile functions.

The third and last axiom is an extreme case of the second one, in that the risk-sharing rule must leave every participant with his of her own loss when $X_1, X_2, ..., X_n$ become maximally positively dependent, or comonotonic. This means that when the impact of the common environment is so strong that individual losses become functionally related, so that knowing the value of one loss gives the values of all other ones, then there is no diversification possible anymore and agents may wish to only pay for their own losses as if they stood alone.

We show that when participants agree about these three axioms then they must resort to the quantile risk-sharing rule to allocate their respective losses. Additional discussion about the three axioms can be found in the next sections.

The remainder of the paper is organized as follows. Section 2 introduces notation and recalls basic concepts including allocations and risk-sharing rules. Section 3 defines the quantile risk-sharing rule. Several of its properties are considered in Section 4. Section 5 proposes an axiomatic theory for the quantile risk-sharing rule. Numerical illustrations are provided in Section 6. Technical material about supports of distribution functions is provided in the appendix. Interested readers are referred to Dhaene, Robert, Cheung & Denuit (2023) for more pedagogical examples and additional technical material.

2 Allocations and risk-sharing rules

2.1 Notation

All random variables considered in this paper are defined on a common probability space $(\Omega, \mathcal{G}, \mathrm{P})$. The latter is assumed to contain the random variable U which is uniformly distributed over the unit interval $(0,1)$. (In-)equalities between random variables are supposed to hold almost surely. Similarly, (in-)equalities between random vectors hold almost surely and component-wise. A random variable will always be denoted by an upper-case letter (e.g. X_i), while its realization (observed ex post) will be denoted by the corresponding lower-case letter (e.g. x_i). A random vector will be denoted by a bold upper-case letter, e.g. $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, while its realization (observed ex post) is denoted by the corresponding bold lower-case small letters, e.g. $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. In this paper, "$\stackrel{\mathrm{d}}{=}$" stands for "equality in distribution".

2.2 Allocations

Let χ be an appropriate set of random variables on the probability space $(\Omega, \mathcal{G}, \mathrm{P})$ under consideration. We interpret χ as the collection of risks (losses) under interest. For particular situations, the set χ could be defined as the set L^q of all random variables X with $\mathrm{E}\left[|X|^q\right] < \infty$, for an appropriate choice of $q \in [0, \infty)$, with $\mathrm{E}[\cdot]$ being the expectation under P. Another possible choice for χ is the set of all (essentially) bounded random variables L^∞.

Also, for any q considered above, the set L^q_+ of all non-negative elements of L^q might be an appropriate choice. More generally, χ can be chosen as a convex cone of random variables on the probability space $(\Omega, \mathcal{G}, \mathrm{P})$, which means that for any X, $Y \in \chi$ and any scalars $a > 0$ and $b > 0$, one has that $aX + bY \in \chi$. In this paper, we assume that $\chi = L^q_+$ or $\chi = L^q$ for some q in $[0, \infty]$, appropriate for the situation at hand.

Consider n economic agents, numbered $i = 1, 2, \ldots, n$. Let time 0 be "now". Each agent i faces a loss $X_i \in \chi$ at time 1. Without insurance or pooling, each individual agent bears his or her own loss, i.e. at time 1, agent i suffers loss x_i, which is the realization of X_i.

The n-dimensional random vector of the losses $\boldsymbol{X}$ is called the (initial) loss vector. The joint distribution function of the loss vector $\boldsymbol{X}$ is denoted by $F_{\boldsymbol{X}}$. The marginal distribution functions of the individual losses are denoted by $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$, respectively. As in the introduction, the aggregate loss faced by the n agents with loss vector $\boldsymbol{X}$ is denoted by

$$S_{\boldsymbol{X}} = \sum_{i=1}^{n} X_i.$$

Hereafter, we will often call $\boldsymbol{X}$ the pool and each agent a participant in the pool.

Definition 2.1. *For any random vector $\boldsymbol{X} \in \chi^n$ with aggregate loss $S_{\boldsymbol{X}}$, the set $\mathcal{A}_n(S_{\boldsymbol{X}})$ is defined* *by:*

$$\mathcal{A}_n(S_{\boldsymbol{X}}) = \left\{ (Y_1, Y_2, \ldots, Y_n) \in \chi^n \middle| \sum_{i=1}^{n} Y_i = S_{\boldsymbol{X}} \right\}.$$

The elements of $\mathcal{A}_n(S_{\boldsymbol{X}})$ are called the n-dimensional allocations of $S_{\boldsymbol{X}}$ in χ^n. Notice that the initial loss vector $\boldsymbol{X}$ is an element of $\mathcal{A}_n(S_{\boldsymbol{X}})$, and that for any $\boldsymbol{Y} \in \mathcal{A}_n(S_{\boldsymbol{X}})$, one has that $\mathcal{A}_n(S_{\boldsymbol{Y}}) = \mathcal{A}_n(S_{\boldsymbol{X}})$.

2.3 Risk sharing

Risk sharing in a pool $\boldsymbol{X} \in \chi^n$ is a two-stage process. In the *ex-ante step* (at time 0), the losses X_i in the pool are re-allocated by transforming $\boldsymbol{X}$ into another random vector $\boldsymbol{H} = (H_1, H_2, \ldots, H_n) \in \mathcal{A}_n(S_{\boldsymbol{X}})$ called the contribution vector. Participants thus exchange their individual risks X_i to the contributions H_i when they join the pool. As $\boldsymbol{H} \in \mathcal{A}_n(S_{\boldsymbol{X}})$, risk-sharing is self-financing in the sense that the identity

$$\sum_{i=1}^{n} H_i = \sum_{i=1}^{n} X_i \tag{2.1}$$

holds true. This self-financing condition (2.1) in risk-sharing is often called the full allocation condition. In the *ex-post step* (at time 1), any participant receives the realization x_i of his or her initial loss X_i from the pool and pays the realization of H_i to the pool. This leads to the following definition.

Definition 2.2. *A risk-sharing rule is a mapping $\mathbb{H} : \chi^n \to \chi^n$ associating to each pool $\boldsymbol{X} \in \chi^n$ a contribution vector $\boldsymbol{H}$ satisfying $\boldsymbol{H} \in \mathcal{A}_n(S_{\boldsymbol{X}})$.*

In this paper, we only consider internal risk-sharing rules, in the sense that individual contributions are functions of the random vector $\boldsymbol{X}$ gathering individual losses and its joint distribution function $F_{\boldsymbol{X}}$, assumed to be known at time 0. In order to be able to define internal risk-sharing rules, we introduce the notation $\mathcal{F}(\chi^n)$ for the set of all n-dimensional distribution functions of the elements in χ^n.

Definition 2.3. *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is said to be internal if there exists a function $\boldsymbol{h} : \mathbb{R}^n \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ such that the contribution vector $\boldsymbol{H}$ for any pool $\boldsymbol{X} \in \chi^n$ with distribution function $F_{\boldsymbol{X}}$ can be expressed as*

$$\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X}; F_{\boldsymbol{X}}). \tag{2.2}$$

Under a rule which can be expressed in the form (2.2), one has that the realization of $\boldsymbol{H}$ is known once the realization of $\boldsymbol{X}$ is revealed at time 1. In other words, $\boldsymbol{H}$ can be expressed as a function of $\boldsymbol{X}$ and hence is $\sigma(\boldsymbol{X})$-measurable. Furthermore, the argument $F_{\boldsymbol{X}}$ in $\boldsymbol{h}(\boldsymbol{X}; F_{\boldsymbol{X}})$ indicates that the realization of the contribution vector $\boldsymbol{H}$ does not only depend on the realization of $\boldsymbol{X}$, but may also depend on the distribution function of $\boldsymbol{X}$ (which is assumed to be known at time 0).

Let us now define aggregate risk-sharing rules.

Definition 2.4. *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is said to be aggregate if there exists a function $\boldsymbol{h}^{aggr} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ such that the contribution vector $\boldsymbol{H}$ for any pool $\boldsymbol{X} \in \chi^n$ can be expressed as*

$$\boldsymbol{H} = \boldsymbol{h}^{aggr}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}}). \tag{2.3}$$

It is clear from the definition that an aggregate risk-sharing rule is internal with internal function $\boldsymbol{h}$ satisfying

$$\boldsymbol{h}(\boldsymbol{X}; F_{\boldsymbol{X}}) = \boldsymbol{h}^{\text{aggr}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})$$

for any $\boldsymbol{X} \in \chi^n$.

An example of an aggregate risk-sharing rule is the conditional mean risk-sharing rule introduced in Denuit & Dhaene (2012). In this case, we have that

$$\boldsymbol{H} = \big(\mathrm{E}\left[X_1 \mid S_{\boldsymbol{X}}\right],\ \mathrm{E}\left[X_2 \mid S_{\boldsymbol{X}}\right], \ldots,\ \mathrm{E}\left[X_n \mid S_{\boldsymbol{X}}\right]\big),$$

which implies that

$$\boldsymbol{h}^{\mathrm{aggr}}\left(s; F_{\boldsymbol{X}}\right) = \big(\mathrm{E}\left[X_1 \mid S_{\boldsymbol{X}} = s\right],\ \mathrm{E}\left[X_2 \mid S_{\boldsymbol{X}} = s\right], \ldots,\ \mathrm{E}\left[X_n \mid S_{\boldsymbol{X}} = s\right]\big).$$

In case $\mathbb{H}$ is an aggregate risk-sharing rule, for any pool $\boldsymbol{X}$, one has that the realization of the contribution vector $\boldsymbol{H}$ is known once the realization of the aggregate claims $S_{\boldsymbol{X}}$ is known. In other words, $\boldsymbol{H}$ is $\sigma\left(S_{\boldsymbol{X}}\right)$-measurable. Aggregate risk-sharing rules are appropriate in case the contributions of the participants should not take into account the composition (origin) of the aggregate claims. For this reason, Jiao, Kou, Liu & Wang (2022) call this property "risk anonymity".

3 The quantile risk-sharing rule

3.1 α-quantiles and comonotonicity

For any real-valued random variable X, the left-continuous quantile of order $p \in [0,1]$ is defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\},$$

while its right-continuous quantile of order $p \in [0,1]$ is defined by

$$F_X^{-1+}(p) = \sup\left\{x \in \mathbb{R} \mid F_X(x) \leq p\right\}.$$

In these definitions, we set $\inf\{\emptyset\} = +\infty$ and $\sup\{\emptyset\} = -\infty$, by convention. For any $\alpha \in [0,1]$, the α-quantile of order p is then defined by

$$F_X^{-1(\alpha)}(p) = \begin{cases} F_X^{-1+}(0) & \text{if } p = 0 \\ \alpha\ F_X^{-1}(p) + (1-\alpha)\ F_X^{-1+}(p) & \text{if } p \in (0,1) \\ F_X^{-1}(1) & \text{if } p = 1 \end{cases}.$$

Notice that in this definition, we have that $F_X^{-1(\alpha)}(0)$ and $F_X^{-1(\alpha)}(1)$ are both independent of the particular choice of α. They are chosen as the "smallest" and the "largest" value of X, respectively.

The next result is central to the determination of the probability level defining quantile risk sharing. We make the following important convention that the interval of the type $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$ has to be considered as a subset of $\mathbb{R}$ rather than as a subset of the extended real line $\mathbb{R}\cup\{\pm\infty\}$. For instance, if $F_X^{-1+}(0) = -\infty$ and $F_X^{-1}(1) = +\infty$, then $\left[F_X^{-1+}(0), F_X^{-1}(1)\right] = \mathbb{R}$ but not $[-\infty, +\infty]$. Similarly, if $F_X^{-1+}(0) = 0$ and $F_X^{-1}(1) = +\infty$, then $\left[F_X^{-1+}(0), F_X^{-1}(1)\right] = [0, +\infty)$. This convention is made throughout this paper.

Proposition 3.1. *For any random variable* X *and any* $x \in \left[F_X^{-1+}(0), F_X^{-1}(1)\right]$, *there exists a (not necessary unique)* $\alpha_x \in [0,1]$ *such that*

$$F_X^{-1(\alpha_x)}\big(F_X(x)\big) = x. \tag{3.1}$$

The proof of Proposition 3.1 is straightforward, see Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a), who also discussed the possible non-uniqueness of the solution α_x of (3.1). Notice that in case $(x, F_X(x))$ lies on a strictly increasing part of the graph of F_X, then any element of $[0,1]$ is a possible choice for α_x. On the other hand, when $(x, F_X(x))$ lies on a flat part of the graph of F_X, then α_x is uniquely determined. Furthermore, if $F_X(x) = 0$ or $F_X(x) = 1$, then any element of $[0,1]$ is a possible choice for α_x.

Comonotonicity is an important dependency structure which is particularly relevant for the study of the quantile risk-sharing rule. For completeness, we repeat its definition hereafter.

Definition 3.2. *A random vector $\boldsymbol{X}$ is comonotonic if there exist non-decreasing functions g_i : $\mathbb{R} \to \mathbb{R}$ such that*

$$\boldsymbol{X} = \big(g_1(S_{\boldsymbol{X}}), \ldots, g_n(S_{\boldsymbol{X}})\big). \tag{3.2}$$

Equivalently, $\boldsymbol{X}$ is comonotonic if for the random variable U which is uniformly distributed over the unit interval $[0, 1]$, one has that

$$\boldsymbol{X} \overset{\mathrm{d}}{=} \big(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\big). \tag{3.3}$$

Comonotonicity and its applications in insurance and finance have been studied in detail in the actuarial literature, see e.g. Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a, 2002b), Deelstra, Dhaene & Vanmaele (2010) and Linders, Dhaene & Schoutens (2015).

To any pool $\boldsymbol{X}$, let us associate its "comonotonic counterpart"

$$\boldsymbol{X}^c = \big(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\big), \tag{3.4}$$

which is by our earlier convention about U, defined on the original probability space. We introduce the notation $S_{\boldsymbol{X}}^c$ for the sum of the components of $\boldsymbol{X}^c$. For any $\alpha \in [0, 1]$ and $p \in [0, 1]$, the following additivity property holds:

$$F_{S_{\boldsymbol{X}}^c}^{-1(\alpha)}(p) = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(p)\,. \tag{3.5}$$

In particular, we find that

$$F_{S_{\boldsymbol{X}}^c}^{-1+}(0) = \sum_{i=1}^{n} F_{X_i}^{-1+}(0) \quad \text{and} \quad F_{S_{\boldsymbol{X}}^c}^{-1}(1) = \sum_{i=1}^{n} F_{X_i}^{-1}(1)\,. \tag{3.6}$$

In this paper, we say that a set $C \subseteq \mathbb{R}^n$ is a support of a random vector $\boldsymbol{X}$ if $\mathrm{P}[\boldsymbol{X} \in C] = 1$. One particular choice for the support of $S_{\boldsymbol{X}}$ is given by

$$\text{Support}\left[S_{\boldsymbol{X}}\right] = \left\{ F_{S_{\boldsymbol{X}}}^{-1}(F_{S_{\boldsymbol{X}}}(s)) \mid s \in \left[F_{S_{\boldsymbol{X}}}^{-1+}(0), F_{S_{\boldsymbol{X}}}^{-1}(1) \right] \right\}. \tag{3.7}$$

Similarly, one particular choice of the support of $S_{\boldsymbol{X}}^{c}$ is

$$\text{Support}\left[S_{\boldsymbol{X}}^{c}\right] = \left\{ F_{S_{\boldsymbol{X}}^{c}}^{-1}(F_{S_{\boldsymbol{X}}^{c}}(s)) \mid s \in \left[F_{S_{\boldsymbol{X}}^{c}}^{-1+}(0), F_{S_{\boldsymbol{X}}^{c}}^{-1}(1) \right] \right\}. \tag{3.8}$$

Notice that our definition of supports differs from the usual one, where the support of $\boldsymbol{X}$ is the smallest closed set C such that $\mathrm{P}[\boldsymbol{X} \in C] = 1$. See Appendix A for a discussion on the supports of $S_{\boldsymbol{X}}$ and $S_{\boldsymbol{X}}^{c}$ defined respectively by (3.7)-(3.8).

3.2 Definition of the quantile risk-sharing rule

We can now define the quantile risk-sharing rule, which was described informally in the introduction to this paper, in a rigorous way.

Definition 3.3. *Under the quantile risk-sharing rule $\mathbb{H}^{\text{quant}} : \chi^n \to \chi^n$, the contribution vector $\boldsymbol{H}^{\text{quant}}$ for a pool $\boldsymbol{X} \in \chi^n$ is given by*

$$\boldsymbol{H}^{\text{quant}} = \boldsymbol{h}^{\text{quant}}\left(S_{\boldsymbol{X}}; F_{\boldsymbol{X}}\right), \tag{3.9}$$

where $\boldsymbol{h}^{\text{quant}} : \mathbb{R} \times \mathcal{F}\left(\chi^n\right) \to \mathbb{R}^n$ is defined by

$$\boldsymbol{h}^{\text{quant}}\left(s; F_{\boldsymbol{X}}\right) = \left(F_{X_1}^{-1(\alpha_s)}\left(F_{S_{\boldsymbol{X}}^{c}}(s)\right), \ldots, F_{X_n}^{-1(\alpha_s)}\left(F_{S_{\boldsymbol{X}}^{c}}(s)\right)\right), \tag{3.10}$$

with α_s following from

$$F_{S_{\boldsymbol{X}}^{c}}^{-1(\alpha_s)}\left(F_{S_{\boldsymbol{X}}^{c}}(s)\right) = s. \tag{3.11}$$

One can prove that for any possible outcome s of $S_{\boldsymbol{X}}$, the quantile risk-sharing contributions are uniquely determined. Consider $s \in (F_{S_{\boldsymbol{X}}}^{-1+}(0), F_{S_{\boldsymbol{X}}}^{-1}(1))$. By (A.3), we have that

$$F_{S_{\boldsymbol{X}}}^{-1+}(0) < s < F_{S_{\boldsymbol{X}}}^{-1}(1) \Rightarrow F_{S_{\boldsymbol{X}}^c}^{-1+}(0) < s < F_{S_{\boldsymbol{X}}^c}^{-1}(1)$$

so that $0 < F_{S_{\boldsymbol{X}}^c}(s) < 1$. This implies that $F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ and $F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ are finite and

$$F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \leq s \leq F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big).$$

Now, define

$$\alpha_s = \begin{cases} \dfrac{F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) - s}{F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) - F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)} \text{ if } F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \neq F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \\ 1 \text{ if } F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) = F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \end{cases}. \tag{3.12}$$

Clearly, (3.11) immediately follows from (3.12). Therefore, any realization s of $S_{\boldsymbol{X}}$ can be expressed as a linear combination of $F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ and $F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ with weights obtained from (3.12), thus as an α_s-quantile. Notice that if $F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \neq F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ then α_s defined in (3.12) is the unique solution of (3.11) while if $F_{S_{\boldsymbol{X}}^c}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) = F_{S_{\boldsymbol{X}}^c}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ then every choice for α_s is a solution to (3.11). Considering (3.5), we have that

$$0 = \sum_{i=1}^{n} \Big(F_{X_i}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) - F_{X_i}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \Big) \Rightarrow F_{X_i}^{-1+}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) = F_{X_i}^{-1}\big(F_{S_{\boldsymbol{X}}^c}(s)\big) \text{ for } i = 1, 2, \ldots, n.$$

Thus every choice of α_s leaves (3.10) unchanged and we set it to 1. This shows that the allocation (3.10) is unique and follows from (3.12).

For any given $s \in \mathbb{R}$ and $\boldsymbol{X} \in \chi^n$, the additivity property (3.5) combined with (3.11) guarantees that

$$\sum_{i=1}^{n} h_i^{\text{quant}}(s; F_{\boldsymbol{X}}) = s \quad \text{whenever } s \in \Big[F_{S_{\boldsymbol{X}}^c}^{-1+}(0), F_{S_{\boldsymbol{X}}^c}^{-1}(1)\Big], \tag{3.13}$$

and hence $\mathbb{H}^{\text{quant}}$ satisfies the self-financing condition (2.1). Furthermore, $\mathbb{H}^{\text{quant}}$ is an aggregate risk-sharing rule by definition. An important observation is that the quantile risk-sharing rule does not require the knowledge of the dependency structure of the joint distribution function $F_{\boldsymbol{X}}$ of loss vector $\boldsymbol{X}$. It suffices to know the marginal distribution functions F_{X_i} of the individual losses X_i. This property will be formalized in the next section. Furthermore, it can be proven that all $F_{X_i}^{-1(\alpha_s)}\big(F_{S_{\boldsymbol{X}}^c}(s)\big)$ are non-decreasing and Lipschitz continuous functions in s, see Denuit, Dhaene & Robert (2022). This observation immediately implies that the contribution vector $\boldsymbol{H}^{\text{quant}}$ is a comonotonic random vector, which means that the quantile risk-sharing rule transforms pools into comonotonic contribution vectors.

For any given comonotonic random vector $\boldsymbol{X}^c$, one particular choice of its support is

$$\text{Support}[\boldsymbol{X}^c] = \big\{\big(F_{X_1}^{-1}(u), \ldots, F_{X_n}^{-1}(u)\big)\big|0 \leq u \leq 1\big\}.$$

This support is not necessarily a connected curve in $\mathbb{R}^n$ but rather a series of ordered connected curves in general; any horizontal segment of one of the marginal distribution functions F_{X_i} would lead to a discontinuity in $\text{Support}[\boldsymbol{X}^c]$. If the endpoints of consecutive curves in $\text{Support}[\boldsymbol{X}^c]$ are connected by straight lines, we obtain a comonotonic connected curve in $\mathbb{R}^n$. We will call this set the connected support of $\boldsymbol{X}^c$ and denote it by $\text{CSupport}[\boldsymbol{X}^c]$. It can be parameterized as follows:

$$\text{CSupport}[\boldsymbol{X}^c] = \big\{\big(F_{X_1}^{-1(\alpha)}(u), \ldots, F_{X_n}^{-1(\alpha)}(u)\big)\big|0 \leq u \leq 1, 0 \leq \alpha \leq 1\big\}.$$

We make the convention here that both $\text{Support}[\boldsymbol{X}^c]$ and $\text{CSupport}[\boldsymbol{X}^c]$ have to be seen as subsets of $\mathbb{R}^n$. We refer to Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a) for more discussion on the notion of connected support. Of course, one may choose to enlarge $\text{Support}[\boldsymbol{X}^c]$ to form a comonotonic connected curve by connecting the endpoints in any other way as long as the curve after connection is comonotonic. However, connecting the endpoints by straight lines is probably the most natural and easiest way to do so, and there seems to be no theoretical reasons to justify other ways.

3.3 The quantile risk-sharing rule as the solution of an optimization problem

The conditional mean risk-sharing rule is based on the conditional expectations of individual losses X_i given their sum $S_{\boldsymbol{X}}$ and is obtained by minimizing the expectation of the squared error (between X_i and a measurable function of $S_{\boldsymbol{X}}$). This property is known as elicitability and has been introduced in a decision-theoretic framework for evaluating the performance of different forecasting procedures (Gneiting (2011)). In the context of risk management, elicitable risk measures allow financial institutions to evaluate and report risks in a way that can be objectively validated. These measures are characterized by their ability to be "elicited" through scoring rules, meaning they can be reliably estimated and verified based on observed outcomes (see e.g. Bellini & Bignozzi (2015)). This quality makes elicitable risk measures particularly valuable in practice, as they allow for objective comparisons and evaluations of risk estimates based on empirical data. In financial regulation, elicitable risk measures are favored because they are easy to implement and verify, fitting within frameworks that aim to enforce consistency across different financial entities.

This section shows that the quantile risk-sharing rule can also be obtained by minimizing the expectation of a specific loss function. To ease exposition, let us assume as in the introduction that each F_{X_i} is 1-to-1, except for a possible positive probability mass at the origin. As for the definition of Kendall's tau, one of the most famous association measures between two random variables, let us introduce an independent copy of the pool $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, which we denote by $\boldsymbol{X'} = (X'_1, X'_2, \ldots, X'_n)$ and let $S_{\boldsymbol{X'}}$ be the sum of its components. We are interested in looking for an aggregate internal function $\boldsymbol{h}$ that minimizes the following loss function

$$\mathrm{E}\left[\sum_{i=1}^{n}\Big(\big|X_i - h_i(S_{\boldsymbol{X'}}; F_{\boldsymbol{X}})\big| + \big|X'_i - h_i(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})\big|\Big)\right],$$

i.e. the sum of the expected absolute values of the differences between the loss X_i of participant

i in pool $\boldsymbol{X}$ and the contribution $h_i(S_{\boldsymbol{X}'}; F_{\boldsymbol{X}})$ of its counterpart in pool $\boldsymbol{X}'$, and of the reciprocal absolute values when the pools are swapped. The introduction of a second pool and the comparison with the contributions of the other pool is linked to the idea of the robustness of the sharing rule with respect to protection from the other pool.

The following lines show that the quantile risk-sharing rule can be obtained as a solution of the following minimization problem:

$$\min_{\boldsymbol{h} \text{ s.t. } \sum_{i=1}^{n} h_i = Id} \mathrm{E}\left[\sum_{i=1}^{n}\left(\left|X_i - h_i(S_{\boldsymbol{X}'}; F_{\boldsymbol{X}})\right| + \left|X_i' - h_i(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})\right|\right)\right].$$

First, note that, for $i = 1, \ldots, n$,

$$\mathrm{E}\left[\left|X_i - h_i(S_{\boldsymbol{X}'}; F_{\boldsymbol{X}})\right|\right] = \mathrm{E}\left[\left|X_i' - h_i(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})\right|\right]$$

and that

$$\mathrm{E}\left[\sum_{i=1}^{n} \left|X_i - h_i(S_{\boldsymbol{X}'})\right|\right] = \mathrm{E}\left[\sum_{i=1}^{n} \mathrm{E}\left[\left|X_i - h_i(S_{\boldsymbol{X}'})\right| \,\middle|\, S_{\boldsymbol{X}'}\right]\right],$$

with

$$\mathrm{E}\left[\left|X_i - h_i(S_{\boldsymbol{X}'})\right| \,\middle|\, S_{\boldsymbol{X}'} = s\right] = \mathrm{E}\left[\left|X_i - h_i(s)\right|\right].$$

It is therefore enough to consider the following problem for each $s \in [F_{S_{\boldsymbol{X}}}^{-1+}(0), F_{S_{\boldsymbol{X}}}^{-1}(1)]$

$$\min_{h_{i,s} \text{ s.t. } \sum_{i=1}^{n} h_{i,s} = s} \sum_{i=1}^{n} \mathrm{E}\left[\left|X_i - h_{i,s}\right|\right]$$

and to prove that its solution corresponds to the quantile risk-sharing rule, i.e.

$$h_{i,s} = F_{X_i}^{-1}\left(F_{S_{\boldsymbol{X}}^{c}}(s)\right), \quad i = 1, \ldots, n. \tag{3.14}$$

The intuition is as follows: for any realization s of $S_{\boldsymbol{X}}$, we minimize the sum over all participants of the expected differences in absolute value of the random losses and the realized contributions. See also Theorem 2 in Dhaene, Tsanakas, Valdez & Vanduffel (2012) where an optimization problem leads to (3.14) in relation to optimal capital allocation. Notice that there is a difference between the first optimization problem where the minimum is with respect to functions summing to identity and the second optimization problem where the minimum is with respect to scalars summing to s. This is why $h_i(s)$ is replaced with $h_{i,s}$.

The corresponding Lagrangian is given by

$$L\left(h_{1,s},\ldots,h_{n,s},\lambda\right)=\sum_{i=1}^{n}\mathrm{E}\left[\left|X_i-h_{i,s}\right|\right]-\lambda\left(\sum_{i=1}^{n}h_{i,s}-s\right),$$

where λ is the Lagrange multiplier related to the constraint $\sum_{i=1}^{n}h_{i,s}=s$. The first-order conditions given by

$$\frac{\partial}{\partial h_{i,s}}L\left(h_{1,s},\ldots,h_{n,s},\lambda\right)=2F_{X_i}\left(h_{i,s}\right)-1-\lambda=0$$

lead to

$$h_{i,s}=F_{X_i}^{-1}\left(\frac{1+\lambda}{2}\right),\quad i=1,\ldots,n.$$

Based on the self-financing condition (2.1), that is,

$$\sum_{i=1}^{n}h_{i,s}=s=\sum_{i=1}^{n}F_{X_i}^{-1}\left(\frac{1+\lambda}{2}\right)=F_{S_{\boldsymbol{X}}^{c}}^{-1}\left(\frac{1+\lambda}{2}\right),$$

we then get

$$\lambda=2F_{S_{\boldsymbol{X}}^{c}}\left(s\right)-1,\quad i=1,\ldots,n,$$

so that we end up with (3.14), that is, with the allocation under the quantile risk-sharing rule.

4 Properties of risk-sharing rules

Properties that risk-sharing rules may (or may not) satisfy have been studied in detail in Denuit, Dhaene & Robert (2022), as well as in Jiao, Kou, Liu & Wang (2022), who also provide axiomatic characterizations of the conditional mean risk-sharing rule. Hereafter, we repeat the definitions of the "comonotonicity" property and the "stand-alone for comonotonic pools" property of risk-sharing rules. We also introduce a new property referred to as "dependence-freedom".

Definition 4.1 (Comonotonicity property)**.** *A risk-sharing rule* $\mathbb{H} : \chi^n \to \chi^n$ *is comonotonic if there exists a function* $\boldsymbol{h}^{\text{com}} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ *such that the contribution vector* $\boldsymbol{H}$ *of any* $\boldsymbol{X} \in \chi^n$ *can be expressed as*

$$\boldsymbol{H} = \boldsymbol{h}^{\text{com}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}}) = \big(h_1^{\text{com}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}}), \ldots, h_n^{\text{com}}(S_{\boldsymbol{X}}; F_{\boldsymbol{X}})\big), \tag{4.1}$$

where each h_i^{com}*,* $i = 1, 2, \ldots, n$*, is non-decreasing in its first argument.*

If $\mathbb{H}$ is comonotonic, then it is an aggregate risk-sharing rule, and for any pool $\boldsymbol{X}$, one has that the contribution vector $\boldsymbol{H}$ is a comonotonic random vector. The quantile risk-sharing rule $\mathbb{H}^{\text{quant}}$ is clearly comonotonic. Notice that the definition here is more restrictive than the one in Denuit, Dhaene & Robert (2022), as here we also require the risk-sharing rule $\mathbb{H}$ to be internal.

Next, we introduce the stand-alone property for comonotonic pools.

Definition 4.2. *A risk-sharing rule* $\mathbb{H} : \chi^n \to \chi^n$ *with internal function* $\boldsymbol{h} : \mathbb{R} \times \mathcal{F}(\chi^n) \to \mathbb{R}^n$ *is stand-alone for comonotonic pools if for any comonotonic pool* $\boldsymbol{X}^c \in \chi^n$*, one has that*

$$\boldsymbol{h}(\boldsymbol{x}^c; F_{\boldsymbol{X}^c}) = \boldsymbol{x}^c \quad \textit{for any } \boldsymbol{x}^c \in \textit{CSupport}[\boldsymbol{X}^c]. \tag{4.2}$$

For any risk-sharing rule which is stand-alone for comonotonic pools, one has that the contribution vector of any comonotonic pool satisfies

$$\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X}^c; F_{\boldsymbol{X}^c}) = \boldsymbol{X}^c.$$

In a comonotonic pool $\boldsymbol{X}$, for any realized loss amounts $x_1, \ldots, x_n$, the corresponding probability levels $p_1, \ldots, p_n$ that solve the equations $x_i = F_{X_i}^{-1}(p_i)$ would always be identical. A more intuitive motivation for the stand-alone for comonotonic pools property is that in a comonotonic pool, no diversification benefit arises from risk-sharing. Therefore, it may be reasonable to require that in such a pool each participant remains with his or her own risk, as stated in the property of stand-alone for comonotonic pools.

In case the risk-sharing rule is stand-alone for comonotonic risks and also aggregate, (4.2) transforms into

$$\boldsymbol{h}\left(s_{\boldsymbol{x}^c}; F_{\boldsymbol{X}^c}\right) = \boldsymbol{x}^c \quad \text{for any } \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c].$$

The next result demonstrates that the quantile risk-sharing rule satisfies the "stand-alone for comonotonic pools" property.

Proposition 4.3. *The quantile risk-sharing rule satisfies the "stand-alone for comonotonic pools" property.*

Proof. Consider a comonotonic pool $\boldsymbol{X}^c \in \chi^n$ and let $\boldsymbol{x}^c$ be a point in CSupport[$\boldsymbol{X}^c$] with $s_{\boldsymbol{x}^c} = s$. By construction, both $\boldsymbol{x}^c$ and $\boldsymbol{h}^{\text{quant}}\left(s_{\boldsymbol{x}^c}, F_{\boldsymbol{X}^c}\right)$ lie in the intersection of the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n \mid s_{\boldsymbol{x}} = s\}$ and CSupport[$\boldsymbol{X}^c$]. As CSupport[$\boldsymbol{X}^c$] is non-decreasing in any of its coordinates, there can be no more than one point in the intersection of this connected support and the hyperplane. We can conclude that $\boldsymbol{x}^c = \boldsymbol{h}^{\text{quant}}\left(s_{\boldsymbol{x}^c}; F_{\boldsymbol{X}^c}\right)$, and hence the stated result holds. □

Let us now define the dependence-free property of a risk-sharing rule that has been discussed in the introduction.

Definition 4.4 (Dependence-free property)**.** *The risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is dependence-free if there exists a function $\boldsymbol{h}^{\text{dep-free}} : \mathbb{R}^n \times (\mathcal{F}(\chi))^n \to \mathbb{R}^n$ such that for any pool $\boldsymbol{X} \in \chi^n$, one has that the contribution vector $\boldsymbol{H}$ is given by*

$$\boldsymbol{H} = \boldsymbol{h}^{\text{dep-free}}\left(\boldsymbol{X}; F_{X_1}, \ldots, F_{X_n}\right).$$

Notice that this requirement is not always desirable. In general, there is no reason why the contribution vectors would be the same in two pools with the same marginal distributions but opposite dependence structures. However, if the dependence comes from the exposition to a common environment, so that all individual losses are positively related and only differ in their marginal distributions, then participants can be willing to adopt a risk-sharing rule fulfilling the dependence-free property, as explained in the introduction. Another reason why participants could be willing to retain this property is when the dependency structure is unknown or hard to estimate.

From Definition 4.4, it follows that in order to determine the contribution vector $\boldsymbol{H}$ under a dependence-free risk-sharing rule, we only need to know the outcome of $\boldsymbol{X}$ and the marginal distribution functions of the individual losses X_i, but not the dependency structure of $\boldsymbol{X}$. Given the outcome $\boldsymbol{x}$ of $\boldsymbol{X}$, the contribution vector remains the same, regardless of what the dependence structure of $\boldsymbol{X}$ is. It is clear from the definition that a dependence-free risk-sharing rule is internal, with internal function $\boldsymbol{h}$ satisfying $\boldsymbol{h}\left(\boldsymbol{X}; F_{\boldsymbol{X}}\right) = \boldsymbol{h}^{\text{dep-free}}\left(\boldsymbol{X}; F_{X_1}, \ldots, F_{X_n}\right)$ for any $\boldsymbol{X} \in \chi^n$.

Example 4.5. *The quantile risk-sharing rule* $\mathbb{H}^{quant}$ *is dependence-free because for any pool* $\boldsymbol{X} \in \chi^n$*, the function* $\boldsymbol{h}^{quant}\left(s_{\boldsymbol{x}}; F_{\boldsymbol{X}}\right)$ *is completely determined by* $s_{\boldsymbol{x}}$ *and the marginal distributions* $F_{X_1}, \ldots, F_{X_n}$*, implying that the knowledge of the dependence structure of* $\boldsymbol{X}$ *is not required.*

In view of the fact that the quantile risk-sharing rule is dependence-free, we can also write $\boldsymbol{h}^{\text{quant}}(s; F_{\boldsymbol{X}})$ as $\boldsymbol{h}^{\text{quant}}(s; F_{X_1}, \ldots, F_{X_n})$.

Remark 4.6. Dependence-free risk-sharing rules do not use the joint distribution function $F_{\boldsymbol{X}}$ but only its marginals $F_{X_1}, \ldots, F_{X_n}$. Let us mention that some rules do not use the joint distribution function at all so that they could be called distribution-free. Formally, a risk-sharing rule is distribution-free if there exists a function $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that the contribution vector $\boldsymbol{H}$ for any pool $\boldsymbol{X}$ is given by $\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{X})$. A first example of distribution-free risk-sharing rules is the stand-alone risk-sharing rule where for any pool $\boldsymbol{X}$ one has that its contribution vector is given by $\boldsymbol{H} = \boldsymbol{X}$. A second example is the uniform risk-sharing rule where $\boldsymbol{H} = \left(\bar{X}_n, \ldots, \bar{X}_n\right)$, with $\bar{X}_n = \frac{X_1 + \ldots + X_n}{n}$.

5 Axiomatic characterization of the quantile risk-sharing rule

In this section, we give an axiomatic characterization of the quantile risk-sharing rule.

Theorem 5.1. *A risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ is the quantile risk-sharing rule if, and only if, it satisfies the following axioms:*

Axiom 1 $\mathbb{H}$ *is aggregate.*

Axiom 2 $\mathbb{H}$ *is dependence-free.*

Axiom 3 $\mathbb{H}$ *is stand-alone for comonotonic pools.*

Proof. As $\mathbb{H}$ satisfies Axioms 1 and 2, there exists a function $\boldsymbol{h}^{\text{aggr}} : \mathbb{R} \times (\mathcal{F}(\chi))^n \to \mathbb{R}^n$ such that the contribution vector of any pool $\boldsymbol{X} \in \chi^n$ is given by $\boldsymbol{h}^{\text{aggr}}\left(S_{\boldsymbol{X}}; F_{X_1}, \ldots, F_{X_n}\right)$. Let $\boldsymbol{X}^c = \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$ be the comonotonic counterpart of $\boldsymbol{X}$. From Axiom 3, we have that

$$\boldsymbol{h}^{\text{aggr}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{x}^c, \qquad \text{for any } \boldsymbol{x}^c \in \text{CSupport}\left[\boldsymbol{X}^c\right].$$

Since the quantile allocation rule is also dependence-free and generalized stand-alone for comonotonic risks by Proposition 4.3, that is, $\boldsymbol{h}^{\text{quant}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{x}^c$, we find that

$$\boldsymbol{h}^{\text{aggr}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) = \boldsymbol{h}^{\text{quant}}\left(s_{\boldsymbol{x}^c}; F_{X_1}, \ldots, F_{X_n}\right) \text{ for any } \boldsymbol{x}^c \in \text{CSupport}\left[\boldsymbol{X}^c\right].$$

From Proposition A.3, we find that

$$\boldsymbol{h}^{\text{aggr}}\left(s;F_{X_1},\ldots,F_{X_n}\right)=\boldsymbol{h}^{\text{quant}}\left(s;F_{X_1},\ldots,F_{X_n}\right),\qquad \text{for any } s\in\left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0),F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right].$$

As Support $[S_{\boldsymbol{X}}]\subseteq\left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0),F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right]$ by (A.1), the above equation implies that

$$\boldsymbol{h}^{\text{aggr}}\left(S_{\boldsymbol{X}};F_{X_1},\ldots,F_{X_n}\right)=\boldsymbol{h}^{\text{quant}}\left(S_{\boldsymbol{X}};F_{X_1},\ldots,F_{X_n}\right), \tag{5.1}$$

which proves the "$\Longleftarrow$" part of the theorem. $\square$

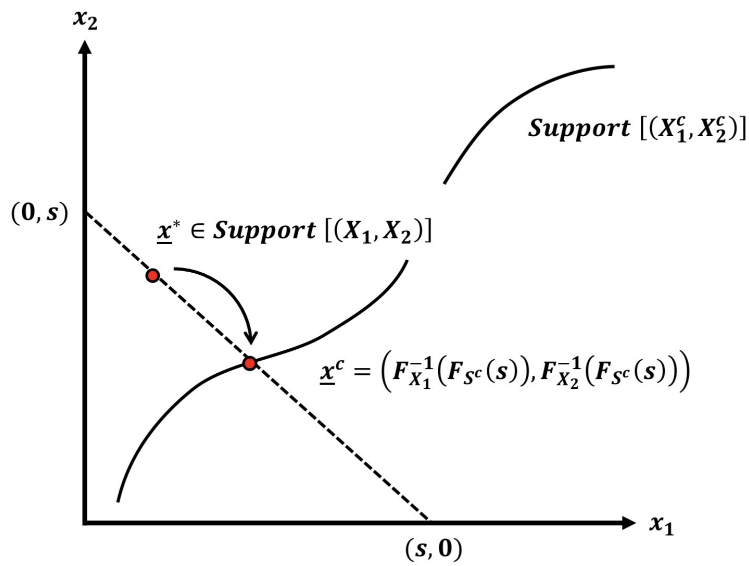

Figure 1: Graphical interpretation of the quantile risk-sharing rule, bivariate case (part I).

In Figure 1, we give a graphical interpretation of the proof of the characterization theorem in the bivariate case. Consider the risk-sharing rule $\mathbb{H}:\chi^2\to\chi^2$ which satisfies the 3 axioms of the theorem and a pool $\boldsymbol{X}=(X_1,X_2)\in\chi^2$. Let (X_1^c,X_2^c) be its comonotonic counterpart. Suppose that the time-1 observable outcome of (X_1,X_2) is given by (x_1^*,x_2^*), with $x_1^*+x_2^*=s$.

First, suppose that the marginal distribution functions F_{X_i} are strictly increasing. Taking into account Axiom 1 ($\mathbb{H}$ is aggregate) we have that the contribution vector of the pool (X_1,X_2) is given by $\boldsymbol{h}\left(x_1^*+x_2^*;F_{\boldsymbol{X}}\right)$ for some function $\boldsymbol{h}:\mathbb{R}\times\mathcal{F}\left(\chi^2\right)\to\mathbb{R}^2$. Let (x_1^c,x_2^c) be the unique point on the intersection of the line $x_1+x_2=s$ and Support$[(X_1^c,X_2^c)]$. We know that (x_1^c,x_2^c) is given by $(x_1^c,x_2^c)=\left(F_{X_1}^{-1}(F_{S_{\boldsymbol{X}}^c}\left(s\right),F_{X2}^{-1}(F_{S_{\boldsymbol{X}}^c}\left(s\right)\right)$. From Axiom 1 ($\mathbb{H}$ is aggregate), we find that

$$\boldsymbol{h}\left(x_1^*+x_2^*;F_{\boldsymbol{X}}\right)=\boldsymbol{h}\left(x_1^c+x_2^c;F_{\boldsymbol{X}}\right).$$

From Axiom 2 ($\mathbb{H}$ is dependence-free), it follows that

$$\boldsymbol{h}\left(x_1^c+x_2^c;F_{\boldsymbol{X}}\right)=\boldsymbol{h}\left(x_1^c+x_2^c;F_{\boldsymbol{X}^c}\right).$$

Axiom 3 ($\mathbb{H}$ is stand-alone for comonotonic pools) leads to

$$\boldsymbol{h}\left(x_1^c+x_2^c;F_{\boldsymbol{X}^c}\right)=\left(x_1^c,x_2^c\right).$$

Summarizing, when the realization of (X_1,X_2) equals (x_1^*,x_2^*), then we have that the realization of the contribution vector is given by

$$\boldsymbol{h}\left(x_1^*+x_2^*;F_{\boldsymbol{X}}\right)=\left(F_{X_1}^{-1}(F_{S_{\boldsymbol{X}}^c}\left(s\right)),F_{X_2}^{-1}(F_{S_{\boldsymbol{X}}^c}\left(s\right))\right).$$

Next, suppose that the marginal distribution functions F_{X_i} are not both strictly increasing in x_i^*, $i=1,2$. In this case, the line $x_1+x_2=s$ has no intersection with Support$[(X_1^c,X_2^c)]$, and we introduce the connected support of this comonotonic random vector. The graphical interpretation of the characterization theorem in this case follows then in a similar way as before, see Figure 2.

Let us now show that the three axioms in Theorem 5.1 are non-redundant (also called independent), which means that no pair of two axioms implies the remaining one. Taking into account that the quantile risk-sharing rule satisfies these three axioms, non-redundancy implies that none of the three axioms can be removed to characterize this risk-sharing rule.

Proposition 5.2. *Axioms 1-3 in Theorem 5.1 are non-redundant.*

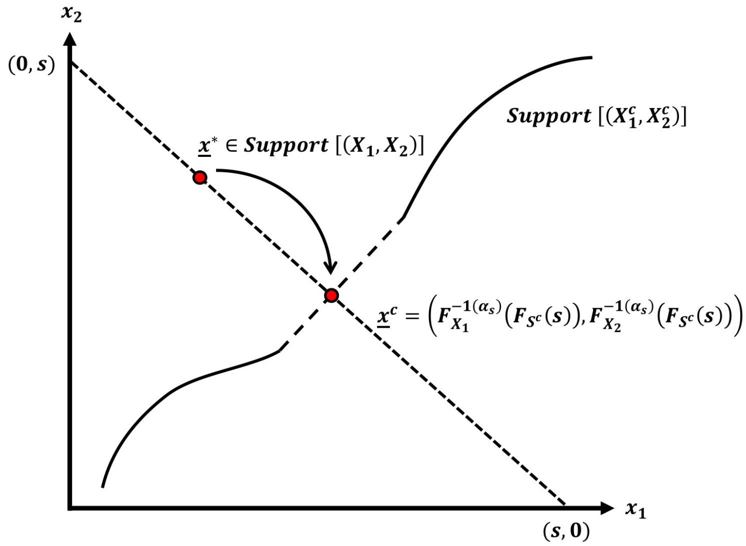

Figure 2: Graphical interpretation of the quantile risk-sharing rule, bivariate case (part II).

Proof. For each of the three axioms, we have to provide an example of a risk-sharing rule which is different from the quantile risk-sharing rule, and which does not satisfy this axiom, while it satisfies the two other axioms.

Axioms 2 and 3, but not Axiom 1: Consider the stand-alone risk-sharing rule $\mathbb{H}^{\text{sa}} : \chi^n \to \chi^n$ with contribution vector $\boldsymbol{H} = \boldsymbol{X}$ for any pool $\boldsymbol{X} \in \chi^n$ and with internal function $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^n$. It is straightforward to prove that $\mathbb{H}^{\text{sa}}$ is "dependence-free" and "generalized stand-alone for comonotonic pools", but does not satisfy the "aggregate" axiom.

Axioms 1 and 3, but not Axiom 2: Define the risk-sharing rule $\mathbb{H} : \chi^n \to \chi^n$ with contribution vector

$$\boldsymbol{H} = \begin{cases} \boldsymbol{X}, & \text{if } \boldsymbol{X} \text{ is comonotonic,} \\ \left(\bar{X}_n, \bar{X}_n, \ldots, \bar{X}_n\right), & \text{otherwise,} \end{cases}$$

and internal function satisfying $\boldsymbol{h}\left(\boldsymbol{x}; F_{\boldsymbol{X}^c}\right) = \boldsymbol{x}$ for any $\boldsymbol{x}$ and any $\boldsymbol{X}^{\boldsymbol{c}}$.

Under this rule, participants are left with their own risk in any comonotonic pool (since there is no diversification in that case) while total losses are distributed uniformly among participants in all other cases. It is straightforward to prove that $\mathbb{H}$ is an "aggregate" risk sharing rule and is "stand-alone for comonotonic pools", but does not satisfy the "dependence-free" axiom.

Axioms 1 and 2, but not Axiom 3: Consider the uniform risk-sharing rule $\mathbb{H}^{\text{uni}} : \chi^n \to \chi^n$ defined for any pool $\boldsymbol{X} \in \chi^n$ by contribution vector

$$\boldsymbol{H}^{\text{uni}} = \left(\bar{X}_n, \bar{X}_n, \ldots, \bar{X}_n\right).$$

It is straightforward to prove that $\mathbb{H}^{\text{uni}}$ satisfies the "aggregate" and "dependence-free" axioms, but not the "stand-alone for comonotonic pools" axiom.

□

6 Numerical illustrations

6.1 Continuous losses

Consider $n = 3$ participants exposed to respective losses X_1, X_2 and X_3. Assume that X_1 is Gamma distributed, with mean 1 and variance 0.5, while X_2 and X_3 are distributed according to LogNormal distributions with mean 1 and variances 1 and 2, respectively. All random variables are assumed to be mutually independent. The probability density functions of these losses are displayed in Figure 3. We observe that the skewness is higher and that the tails are thicker for X_2 and X_3 compared to X_1.

Figure 4 shows the distribution functions $F_{S_{\boldsymbol{X}}}$ and $F_{S^c_{\boldsymbol{X}}}$. We see that these distribution functions only cross once, with $F_{S_{\boldsymbol{X}}}$ dominating $F_{S^c_{\boldsymbol{X}}}$ after this unique crossing point. Clearly, $S_{\boldsymbol{X}^c}$ exhibits

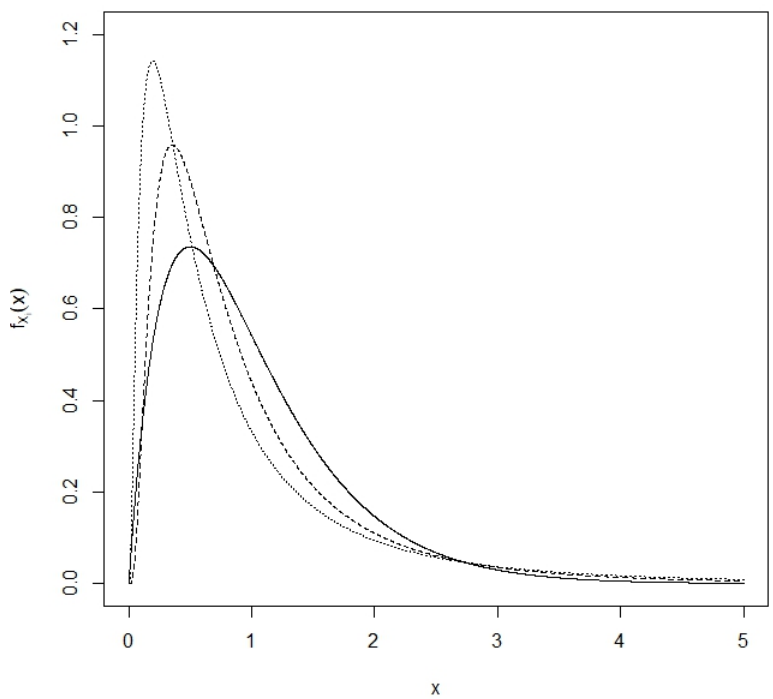

Figure 3: Probability density functions of X_1 (solid line), X_2 (broken line), and X_3 (dotted line) in the pool of Section 6.1.

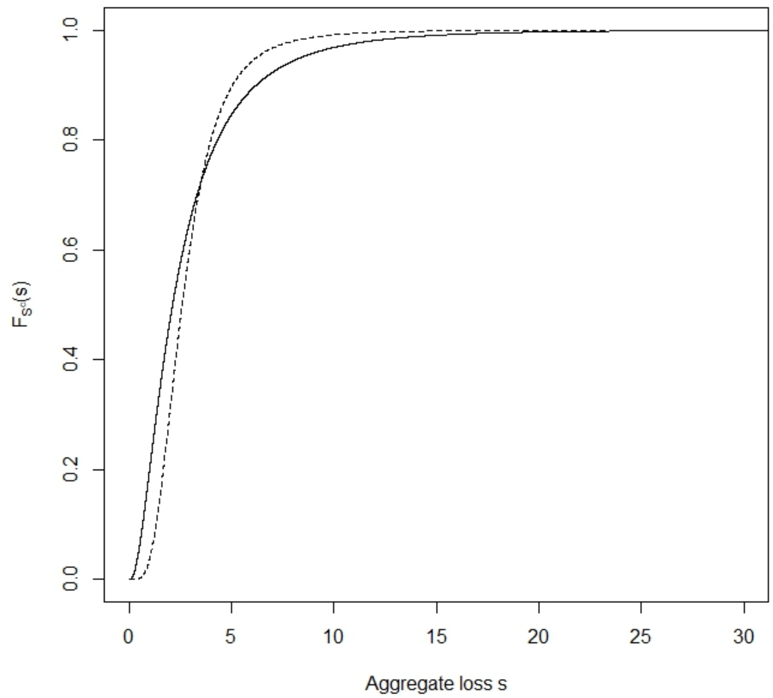

Figure 4: Distribution functions $F_{S^c_{\boldsymbol{X}}}$ (solid line) and $F_{S_{\boldsymbol{X}}}$ (broken line) in the pool of Section 6.1.

more dispersion compared to the actual aggregate loss $S_{\boldsymbol{X}}$.

Individual contributions under the quantile risk-sharing rule are displayed in Figure 5 for each of the three participants. One can verify that the contributions are ordered differently depending on whether the observed value s of $S_{\boldsymbol{X}}$ falls in the central part or the tail part of the distribution. For central values, the contribution paid by participant 1 is higher than those paid by participants 2-3, while it becomes smaller than those paid by participants 2 and 3 when $S_{\boldsymbol{X}}$ assumes large values. A similar comment applies to the contribution paid by participant 2 compared to the one paid by participant 3.

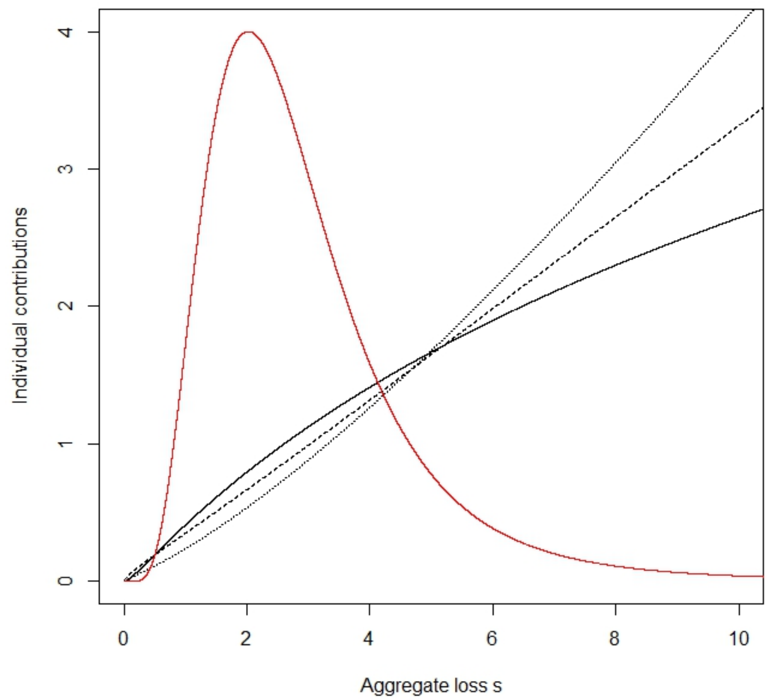

Figure 5: Contributions $F_{X_i}^{-1}(F_{S_{\boldsymbol{X}}^c}(s))$ as a function of the aggregate loss s, for participant 1 (solid line), for participant 2 (broken line), and for participant 3 (dotted line) in the pool of Section 6.1. The red curve is the probability density function of $S_{\boldsymbol{X}}$, properly re-scaled on the y-axis.

6.2 Compound Poisson losses

Following the numerical illustrations in Denuit (2020), consider an insurance pool gathering $n = 35$ participants with 3 risk profiles. Each participant brings a compound Poisson loss X_i to the pool. This means that the loss X_i for participant i to the insurance pool is of the form

$$X_i = \sum_{k=1}^{N_i} C_{i,k} \text{ with } N_i \sim \text{Poisson}(\lambda_i), \quad i = 1, 2, \ldots, \tag{6.1}$$

where the claim severities $C_{i,k}$ are positive, all these random variables being independent. The severities $C_{i,k}$ are assumed to be identically distributed for fixed i (as C_i, say). They are modeled with the help of Beta distributions with group-specific parameters a and b, where the mean is equal to $a/(a+b)$ and the variance is equal to $ab/((a+b)^2(a+b+1))$. The 3 groups have the following characteristics:

Group 1 (low risks): $n_1 = 20$ individuals, $\lambda_i = 5\%$ for $i = 1, \ldots, 20$, and claim sizes distributed as C_1 following the Beta distribution with parameters $a_1 = 2$ and $b = 5$.

Group 2 (medium risks): $n_2 = 10$ individuals, $\lambda_i = 10\%$ for $i = 21, \ldots, 30$, and claim sizes distributed as C_2 following the Beta distribution with parameters $a_2 = 3$ and $b = 5$.

Group 3 (high risks): $n_3 = 5$ individuals, $\lambda_i = 20\%$ for $i = 31, \ldots, 35$, and claim sizes distributed as C_3 following the Beta distribution with parameters $a_3 = 4$ and $b = 5$.

Participants in Group 3 have higher expected severities compared to participants in Group 2 who themselves have higher expected severities compared to participants in Group 1.

All these losses follow zero-augmented distributions, as those encountered in the majority of insurance applications, meaning that $F_{X_i}(0) = \exp(-\lambda_i) > 0$ and F_{X_i} is continuously increasing over $(0,\infty)$. As pointed out by Denuit, Dhaene & Robert (2022, Remark 5.4), we have

$$F_{S^{c}_{\boldsymbol{X}}}(0)=\min\{F_{X_1}(0),\ldots,F_{X_n}(0)\}=\exp(-0.2).$$

Therefore, once the value of $S_{\boldsymbol{X}}$ is known to be equal to s, two situations may occur. Either $F_{S^{c}_{\boldsymbol{X}}}(s)>\max\{F_{X_1}(0),\ldots,F_{X_n}(0)\}=\exp(-0.05)$ and every participant contributes to the total loss, or $F_{S^{c}_{\boldsymbol{X}}}(s)\leq\exp(-0.05)$ and participants with larger no-claim probabilities, i.e. those participants i for which $F_{S^{c}_{\boldsymbol{X}}}(s)\leq F_{X_i}(0)$ do not have to contribute ex post. Stated more precisely, if s is such that $\exp(-0.1)<s\leq\exp(-0.05)$ then participants in Group 1 do not have to contribute whereas if s is such that $\exp(-0.2)<s\leq\exp(-0.1)$, participants in Groups 1 and 2 do not have to contribute. This may lead to undesirable situations since it is reasonable to expect that all participants putting the pool at risk must contribute to $S_{\boldsymbol{X}}$ ex post.

Denuit, Dhaene & Robert (2022) established that this problem disappears when all no-claim probabilities are equal or when the number of participants to the pool becomes sufficiently large (under suitable technical conditions). Given the limited size of the pool, the second argument does not apply here. Since we consider compound Poisson losses, there is an easy way to equalize all no-claim probabilities. It suffices to decompose any loss in Group 2 as the sum of 2 independent compound Poisson losses with the same severity distribution Beta(a_2,b) and Poisson parameter 0.05 and to decompose any loss in Group 3 as the sum of 4 independent compound Poisson losses with the same severity distribution Beta(a_3,b) and Poisson parameter 0.05. We thus work in the augmented pool with $n_1=n_2=n_3=20$. Participants in the initial pool then contribute 2 times the amount calculated in Group 2 and 4 times the amount calculated in Group 3 within the augmented pool.

Figure 6 shows the distribution functions $F_{S_{\boldsymbol{X}}}$ and $F_{S^{c}_{\boldsymbol{X}}}$. Again, we observe that these distri-

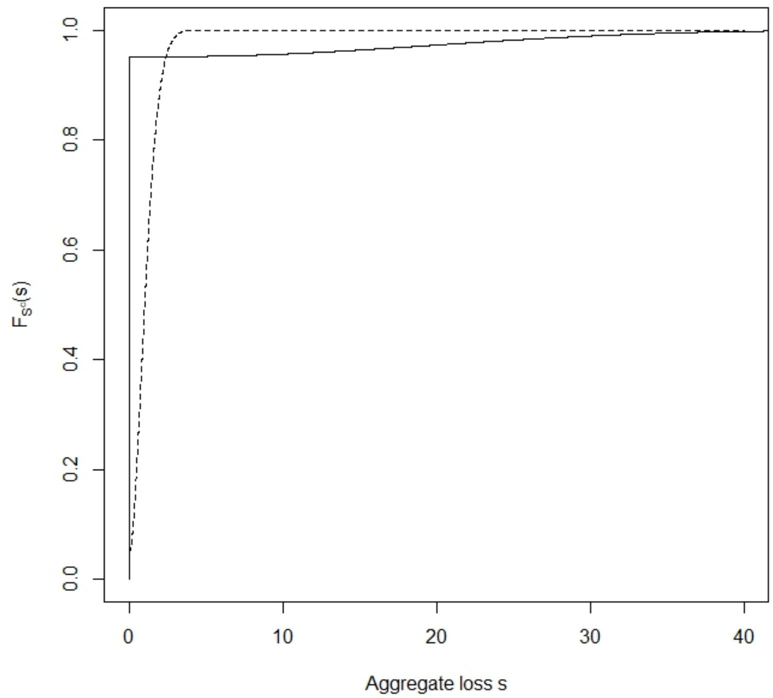

Figure 6: Distribution functions $F_{S^c_{\boldsymbol{X}}}$ (solid line) and $F_{S_{\boldsymbol{X}}}$ (broken line) in the pool of Section 6.2.

bution functions only cross once, with $F_{S_{\boldsymbol{X}}}$ dominating $F_{S^c_{\boldsymbol{X}}}$ after this unique crossing point. As it was the case in Section 6.1, $S^c_{\boldsymbol{X}}$ exhibits more dispersion compared to the actual aggregate loss $S_{\boldsymbol{X}}$. In particular, the probability mass of $S^c_{\boldsymbol{X}}$ at the origin is equal to $\exp(-0.05)$, which is much larger than to the probability mass of $S_{\boldsymbol{X}}$ at the origin, which is equal to $\exp(-3)$, and the tail of $S_{\boldsymbol{X}}$ extends to larger values.

Individual contributions under the quantile risk-sharing rule are displayed in Figure 7 for each of the three groups. We see that the contribution paid by a member of Group 3 increases more rapidly with aggregate loss compared to the contributions paid by members of Groups 1 or 2, due to the larger size of the losses they bring to the pool. The total contribution paid by participants is

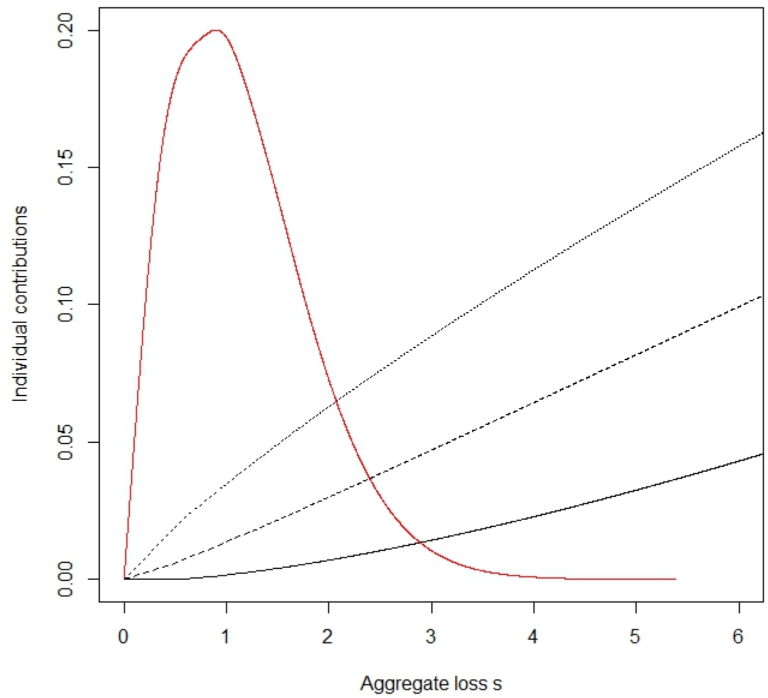

Figure 7: Contributions $F_{X_i}^{-1}(F_{S^c_{\boldsymbol{X}}}(s))$ as a function of s, in Group 1 (solid line appearing at the bottom), in Group 2 (broken line appearing in the middle), and in Group 3 (dotted line appearing at the top) in the pool of Section 6.2. The red curve is the probability density function of $S_{\boldsymbol{X}}$ over $(0, \infty)$, properly re-scaled on the y-axis.

twice the amount displayed in Figure 7 for a participant in Group 2 and 4 times for a participant in Group 3.

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Appendix

A Comonotonicity and supports of distributions

Consider a pool $\boldsymbol{X}$ and its comonotonic counterpart $\boldsymbol{X}^c$, which is defined in the original probability space (see (3.4)). As before, $S^c_{\boldsymbol{X}}$ stands for the sum of the components of $\boldsymbol{X}^c$.

The support of the aggregate claims $S_{\boldsymbol{X}}$ of the pool $\boldsymbol{X}$ is defined by (3.7). Recall that throughout the paper, we make the convention that the interval $\left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]$ has to be replaced by $\left(F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right)$ in case $F^{-1+}_{S_{\boldsymbol{X}}}(0) = -\infty$ and $F^{-1}_{S_{\boldsymbol{X}}}(1) = +\infty$. Similar conventions are made in case only one of the endpoints of the interval $\left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]$ is infinite. One can easily verify that

$$\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]. \tag{A.1}$$

The support of the aggregate claims $S^c_{\boldsymbol{X}}$ of the comonotonic pool $\boldsymbol{X}^c$ is defined by (3.8) where we make a similar convention as before concerning the endpoints of the interval $\left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right]$. In this case, we have that

$$\text{Support}\left[S^c_{\boldsymbol{X}}\right] \subseteq \left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right]. \tag{A.2}$$

Remark that the endpoints of the intervals $\left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right]$ and $\left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right]$ always satisfy the following inequalities:

$$F^{-1+}_{S^c_{\boldsymbol{X}}}(0) \leq F^{-1+}_{S_{\boldsymbol{X}}}(0) \leq F^{-1}_{S_{\boldsymbol{X}}}(1) \leq F^{-1}_{S^c_{\boldsymbol{X}}}(1). \tag{A.3}$$

Example A.1. *It is obvious that* Support $[S^c_{\boldsymbol{X}}]$ *is not always a subset of* Support $[S_{\boldsymbol{X}}]$. *A simple example illustrating this fact is the bivariate random vector* (X_1, X_2), *with* $X_1 = U$ *and* $X_2 = 1 - U$.

In this case, we have that $S_{\boldsymbol{X}} = 1$*, and hence,*

$$\text{Support}\left[S_{\boldsymbol{X}}\right] = \{1\},$$

while taking into account that $S^c_{\boldsymbol{X}} \overset{d}{=} 2U$ *leads to*

$$\text{Support}\left[S^c_{\boldsymbol{X}}\right] = [0, 2].$$

A somewhat less obvious fact is that $\text{Support}\left[S_{\boldsymbol{X}}\right]$ *is not always a subset of* $\text{Support}\left[S^c_{\boldsymbol{X}}\right]$*. In order to illustrate this statement, consider the mutually independent random variables* X_1 *and* X_2*, which are both uniformly distributed over* $[0, 1] \cup [2, 3]$*. Then we have that*

$$\text{Support}\left[S_{\boldsymbol{X}}\right] = [0, 6] \qquad \textit{and} \qquad \text{Support}\left[S^c_{\boldsymbol{X}}\right] = [0, 2] \cup [4, 6].$$

The following result gives conditions under which $\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \text{Support}\left[S^c_{\boldsymbol{X}}\right]$. Remark that we will say that a distribution function F_X is strictly increasing if it is strictly increasing over the interval $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$.

Proposition A.2. *If* F_{X_i} *is strictly increasing,* $i = 1, 2, \ldots, n$*, then*

$$\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \text{Support}\left[S^c_{\boldsymbol{X}}\right] = \left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right].$$

Proof. If all F_{X_i} are strictly increasing, then all $F^{-1}_{X_i}$ are continuous. Taking into account the additivity property (3.5), this implies that $F^{-1}_{S^c_{\boldsymbol{X}}}$ is continuous, and hence, $F_{S^c_{\boldsymbol{X}}}$ is strictly increasing, which implies that $\text{Support}\left[S^c_{\boldsymbol{X}}\right] = \left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right]$. From (A.1), (A.2) and (A.3), it follows then that

$$\text{Support}\left[S_{\boldsymbol{X}}\right] \subseteq \left[F^{-1+}_{S_{\boldsymbol{X}}}(0), F^{-1}_{S_{\boldsymbol{X}}}(1)\right] \subseteq \left[F^{-1+}_{S^c_{\boldsymbol{X}}}(0), F^{-1}_{S^c_{\boldsymbol{X}}}(1)\right] = \text{Support}\left[S^c_{\boldsymbol{X}}\right].$$

This ends the proof. □

Proposition A.3. *For any comonotonic random vector $\boldsymbol{X}^c$, one has that*

$$\{s_{\boldsymbol{x}^c} \mid \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c]\} = \left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0), F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right].$$

Proof. From the definition of CSupport$[\boldsymbol{X}^c]$, one finds that

$$\{s_{\boldsymbol{x}^c} \mid \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c]\} = \left\{\sum_{i=1}^{n} F^{-1(\alpha)}_{X_i}\big(u\big)\Big| 0 \leq u \leq 1, 0 \leq \alpha \leq 1\right\}.$$

Taking into account the additivity property (3.5) leads to

$$\begin{aligned}\{s_{\boldsymbol{x}^c} \mid \boldsymbol{x}^c \in \text{CSupport}[\boldsymbol{X}^c]\} &= \left\{F^{-1(\alpha)}_{S^c_{\boldsymbol{X}}}\big(u\big)\big| 0 \leq u \leq 1, 0 \leq \alpha \leq 1\right\} \\ &= \left[F^{-1+}_{S_{\boldsymbol{X}^c}}(0), F^{-1}_{S_{\boldsymbol{X}^c}}(1)\right],\end{aligned}$$

which proves the stated result. □