A decomposition framework for managing hybrid liabilities

Biwen Ling* Daniël Linders† Jan Dhaene‡ Tim J. Boonen§

Version: September 5, 2025

Abstract

In this paper, we propose a four-step decomposition of hybrid liabilities into a hedgeable part, an idiosyncratic part, a financial systematic part, and an actuarial systematic part. We generalize existing approaches for decomposing hybrid liabilities by incorporating dependence between financial and actuarial markets and allowing heterogeneity in policyholder-specific risks. Our model provides a market- and model-consistent valuation framework which we illustrate using a portfolio of with-profit pure endowment contracts.

Keywords: risk decomposition, systematic risk, market-consistent valuation, mean-variance hedging, incomplete market

1 Introduction

This paper studies the risk management of monetary hybrid liabilities composed of financial and actuarial risks. Financial risks, such as those arising from market price fluctuations, and actuarial risks, such as those related to mortality or longevity, inherently require different pricing and risk management approaches. When a liability represents a non-linear combination of these distinct types of risks, it becomes challenging to disentangle and identify the individual risk components and to effectively integrate appropriate techniques such as financial hedging, diversification, and other actuarial methods for managing the claim. As a result, institutions face significant challenges in accurately valuing and managing such hybrid liabilities, potentially leading to mispricing, inadequate risk mitigation, and regulatory non-compliance.

In this paper we develop a risk management framework for hybrid liabilities that begins with a decomposition of the liability into four different uncorrelated parts. The hedgeable part captures the part of the liability which can be managed by a hedging portfolio. A second part captures

*biwen.ling@kuleuven.be, KU Leuven, Leuven, Belgium.
†daniel.linders@kuleuven.be, KU Leuven, Leuven, Belgium.
‡jan.dhaene@kuleuven.be, KU Leuven, Leuven, Belgium.
§tjboonen@hku.hk, The University of Hong Kong, Hong Kong, China.

non-hedged financial risks. These are the so-called financial derivatives, which can be traded in financial markets, but in an incomplete market, their no-arbitrage value is not unique. A third part captures the part of the claim which can be managed through diversification, which is therefore called the idiosyncratic part. Lastly, the fourth part contains risks that cannot be diversified away or sold in financial markets. These are the actuarial systematic risks. Next, we investigate how to value a hybrid claim employing our decomposition results. We will use replications and risk-neutral pricing to value the two financial parts, where we explicitly account for market incompleteness. The two actuarial parts are priced using an appropriate actuarial risk measure.

We show that our decomposition of a hybrid claim in four parts is optimal in mean-variance sense. From a market-consistent perspective, where the claim must be valued using available market information, it is reasonable to first determine the optimal way to decompose the hybrid claim into a financial part (both hedgeable and non-hedged) and an actuarial part. The financial part is a combination of a hedgeable (linear) part and a non-hedged (non-linear) part. The actuarial part can then again be decomposed in two parts, a systematic actuarial part and an idiosyncratic part. We also show that, under some technical conditions on the expectations, this decomposition is unique if we aim for four uncorrelated parts. The condition on the expectations of the different parts can be relaxed if we shift from uncorrelated parts to orthogonal parts.

Dhaene et al. (2017) decompose a hybrid liability into two parts: a hedgeable part and a non-hedged part, using fair hedgers. They show that using a mean-variance optimization criteria leads to a fair hedger. The hedgeable part can be priced using the replicating portfolio approach whereas an actuarial valuation can be used for the residual, non-hedged part, leading to a market-consistent pricing formula for hybrid claims. This approach was further extended in Delong et al. (2019), Barigou and Dhaene (2019), and Barigou et al. (2019, 2022). Deelstra et al. (2020) decompose a hybrid liability in three parts, taking into account also systematic risks, which require a different pricing approach compared to financial and diversifiable risks. However, their approach still considers an additive valuation in a complete financial market. This was then generalized to non-linear valuation in an incomplete market in Linders (2023). Moreover, Dhaene (2022) introduces a decomposition in four parts, where systematic risks are separated in financial and actuarial systematic risks. In this framework, it was assumed that financial markets are independent from the actuarial market and that the policyholder-specific risks are identically and conditionally independently distributed.

In this paper we will explore in more detail the four-step decomposition of Dhaene (2022). We generalize the assumptions and we show under which conditions such a decomposition is optimal. We also investigate under which conditions the idiosyncratic part is indeed diversifiable. Moreover, we show how to value a hybrid claim by employing the four-step decomposition. This new valuation turns out to be a fair valuation as defined in Dhaene et al. (2017).

Regulatory frameworks, such as Solvency II and IFRS 17, mandate the valuation of insurance products to be market consistent, meaning that the price of a hybrid liability should be ‘in line’ with the prices of traded assets that can be observed in the financial market. Consequently, it is crucial to distinguish between the parts of a liability that can be valued using financial pricing methods (e.g., via replicating portfolios) and the parts that require alternative pricing approaches such as actuarial pricing techniques (e.g., statistical models and diversification principles). Combining actuarial and financial valuation theories for complex liabilities was first proposed in Brennan and Schwartz (1976) for pricing variable annuity contracts, and further explored in Embrechts (2000); see also Bacinello et al. (2021). In Muermann (2008), the authors discuss the pricing of catastrophe insurance using information from CAT bonds; see also Beer and Braun (2022). The idea of pricing financial derivatives in incomplete markets by using the available information in the market was already explored in Cont (2006). In Malamud et al. (2008), the authors consider the market-consistent pricing of insurance claims that combine traded and non-traded risks. In Pelsser and Stadje (2014), it was shown that market-consistent valuations can be characterized by the set of two-step valuations. In Dhaene et al. (2017), the class of hedge-based valuations was introduced and it was shown that in a particular setting the class was equivalent with the two-step valuations. In Linders (2023), the class of 3-step valuations was introduced and it was shown that if one wants to take into account the different nature of systematic risks, the class of hedge-based valuations coincides with the 3-step valuations.

This paper contributes to the literature by generalizing and characterizing the four-step decomposition introduced in the lecture notes of Dhaene (2022). We relax the assumption of market independence and allow for potential correlations between financial and actuarial risks, reflecting the intricate dynamics observed in practice. Additionally, we account for heterogeneity in policyholder-specific risks, acknowledging that individuals may exhibit varying risk profiles and dependencies. By doing so, our framework is applicable to a broader range of financial and insurance products. Moreover, we show how this four-step decomposition can lead to a new market- and model-consistent valuation that values financial derivatives under an appropriate risk-neutral measure, while actuarial risks are valued under the real-world probability measure. The valuation of the financial part relies on the theory of conic finance, which was developed in Madan and Cherny (2010) and Madan and Schoutens (2016). Indeed, our valuation interprets market consistency as the assumption that the financial parts of the hybrid claim (both hedgeable and non-hedged) should be priced using a no-arbitrage argument, i.e. under a risk-neutral measure. In the conic finance framework, we can trade the non-hedged financial part, but the price at which one can buy, is a conservative (i.e. too high) price.

This paper is set out as follows. Section 2 defines the general model setting. Section 3 introduces the 4-step decomposition of claims. Section 4 studies valuation using the 4-step decomposition, and provides three concrete examples. Section 5 provides an application of the 4-step decomposition for product claims. Finally, Section 6 concludes this paper.

2 General setting

Throughout this paper, we work in a one-period framework, assuming today is time 0 and the single period ends at a future deterministic time $T < \infty$. All risks and claims encountered in this paper have to be understood as liabilities which are payable at this future time T. Moreover, we assume that such liabilities are modeled as L^2 random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set of all such random variables is denoted by $\mathcal{C}$.

We assume a liquid, arbitrage-free financial market in which n assets ($n \geq 1$) are traded and a risk-free bank account exists. The risk-free interest rate r is also assumed to be deterministic and constant. Let $\boldsymbol{Y}$ be an $(n+1)$-dimensional random vector representing the time-T payoffs

$$\boldsymbol{Y}=\left(Y^{(0)},Y^{(1)},\ldots,Y^{(n)}\right),$$

where $Y^{(0)}=e^{rT}$ is the time-T payoff of a risk-free bank account, and $Y^{(1)},Y^{(2)},\ldots,Y^{(n)}$ are the payoffs of the n traded assets. These financial assets can be bought and sold at a single price. The time-0 prices observed in the market are denoted by the vector $\boldsymbol{y}$, where

$$\boldsymbol{y}=\left(y^{(0)},y^{(1)},\ldots,y^{(n)}\right).$$

Moreover, market participants can trade any quantity of the financial assets at time $t=0$ at market prices. We denote by $\mathcal{C}^{\boldsymbol{Y}}$ the set of all financial derivatives, defined as follows:

$$\begin{aligned}\mathcal{C}^{\boldsymbol{Y}}&=\left\{S\in\mathcal{C}\,|\,\exists\text{ Borel function } f \text{ s.t. } S=f\left(\boldsymbol{Y}\right)\text{ and }\mathbb{E}\left[f^2\left(\boldsymbol{Y}\right)\right]<\infty\right\}\\&=L^2(\Omega,\sigma(\boldsymbol{Y}),\mathbb{P}).\end{aligned}\tag{2.1}$$

A derivative is thus a linear or non-linear function of the traded assets.

In addition to the traded payoffs $\boldsymbol{Y}$, the set $\mathcal{C}$ also contains non-traded payoffs, which we divide into two parts. The random vector $\boldsymbol{Z}$ represents all non-traded systematic risks. A risk is categorized as a systematic risk if it will impact a large group of policyholders simultaneously. For example, macroeconomic risks and market-wide shocks, as well as systematic insurance-related risks, such as climate risk or longevity risk, are all stored in a d-dimensional vector $\boldsymbol{Z}=(Z_1,Z_2,\ldots,Z_d)$. The set of all risks which depend on financial and systematic risks is denoted by $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ and is defined as follows:

$$\begin{aligned}\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}&=\left\{S\in\mathcal{C}\,|\,\exists\text{ Borel function } f \text{ s.t. } S=f\left(\boldsymbol{Y},\boldsymbol{Z}\right)\text{ and }\mathbb{E}\left[f^2\left(\boldsymbol{Y},\boldsymbol{Z}\right)\right]<\infty\right\}\\&=L^2(\Omega,\sigma(\boldsymbol{Y},\boldsymbol{Z}),\mathbb{P}).\end{aligned}\tag{2.2}$$

We also assume that apart from the traded risks $\mathbf{Y}$ and the non-traded systematic risks $\mathbf{Z}$, there is also a N-dimensional vector $\boldsymbol{X}=(X_1,X_2,\ldots,X_N)$ with risks that can drive a hybrid liability. As a result, we can have hybrid claims which are in $\mathcal{C}$, but which are not in $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$.

3 4-step decomposition of a claim S

Consider a claim S defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$. The claim S can be expressed as a function of several underlying random variables (which are in $\mathcal{C}$), including the financial assets $\boldsymbol{Y}$, but it may also depend on the realizations of $\boldsymbol{Z}$ and $\boldsymbol{X}$. Later, we will specify the claim S as an aggregate liability of an insurance portfolio containing several policyholders. At the moment, however, we do not need this additional assumption.

3.1 The hedgeable and systematic part of a liability

Since S is linked to traded financial risks, we first seek an optimal hedging strategy for the claim S to manage and mitigate the traded financial risks associated with S. Following Dhaene et al.

(2017), we introduce the hedger $\boldsymbol{\theta}$, which is a function assigning a hedge $\boldsymbol{\theta}_S = \left(\theta_S^{(0)}, \theta_S^{(1)}, \ldots, \theta_S^{(n)}\right)$ to any random variable $S \in \mathcal{C}$. The hedge $\boldsymbol{\theta}_S$ for the claim S is a linear combination of the traded assets. We assume here a static hedge and therefore $\boldsymbol{\theta}_S$ is an $(n+1)$- dimensional vector:

$$\boldsymbol{\theta}: \quad \mathcal{C} \rightarrow \mathbb{R}^{n+1}.$$

The i-th component of the vector $\boldsymbol{\theta}_S$ denotes the number of units we hold of the asset $Y^{(i)}$. We also assume that a feasible hedger is normalized, i.e., $\boldsymbol{\theta}_0 = \mathbf{0}$, and is translation invariant, i.e., $\boldsymbol{\theta}_{S+c} = \boldsymbol{\theta}_S + \left(c, e^{-rT}, 0, \ldots, 0\right)$ for any constant $c \in \mathbb{R}$. The payoff at maturity T of the hedge is denoted by Y^h and can therefore be expressed as follows:

$$Y^h = \boldsymbol{\theta}_S \cdot \boldsymbol{Y}, \tag{3.1}$$

where '$\cdot$' denotes the inner product operator between two vectors. A hedging strategy is an n-dimensional real vector. The set of all hedging strategies is denoted by Θ, and we assume non-redundancy of the financial market, meaning that $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$ implies $\boldsymbol{\theta} = \mathbf{0}$. In this paper, we focus on hedgers $\boldsymbol{\theta}$ which are *market consistent*. Market and model consistency of hedgers as well as the notion of fair hedgers was first introduced in Dhaene et al. (2017) and further explored in Barigou et al. (2022), Linders (2023), among others. An overview was provided in Dhaene (2022).

Definition 3.1 (Market-consistent hedgers) *The hedger $\boldsymbol{\theta}$ is said to be a market-consistent hedger if for any real vector $\boldsymbol{v} = \left(v^{(0)}, v^{(1)}, \ldots, v^{(n)}\right)$, we have*

$$\boldsymbol{\theta}_{S+\boldsymbol{v}\cdot\boldsymbol{Y}} = \boldsymbol{\theta}_S + \boldsymbol{v},$$

for all $S \in \mathcal{C}$.

Market consistency of a hedger implies that a claim which consists of a hedgeable part, i.e. a part which can be expressed as a linear combination of the available traded assets, will be hedged using that linear combination and the hedge of the remaining part.

The claim S depends not only on financial assets but also on the non-traded risks. Therefore, we may not be able to perfectly hedge the claim, i.e. the payoff Y^h differs from the liability S. The residual part of the liability S is then defined as what remains of the liability S after subtracting the payoff of the hedging strategy:

$$\text{Residual part} = S - Y^h.$$

This residual part continues to depend on both financial and systematic risks. To better investigate how different realizations of the financial assets $\boldsymbol{Y}$ and the systematic risks $\boldsymbol{Z}$ affect the residual part $S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$, we decompose it into a systematic component Y^s and an idiosyncratic part Y^i using the conditional expectation approach, expressed as follows:

$$S - Y^h = Y^s + Y^i, \tag{3.2}$$

where

$$Y^s = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right], \tag{3.3}$$

$$Y^i = S - \mathbb{E}\left[S \mid \boldsymbol{Y}, \boldsymbol{Z}\right]. \tag{3.4}$$

The systematic part Y^s represents the component of $S - Y^h$ that can be predicted using $\boldsymbol{Y}$ and $\boldsymbol{Z}$, which are factors influencing all observations or policyholders simultaneously. Since $\boldsymbol{Y}$ and $\boldsymbol{Z}$ reflect shared factors, such as market prices or other systematic effects, $\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right]$ models how the expectation of the residual part behaves as a function of the systematic and financial factors. Therefore, we regard Y^s as a *systematic part*. On the contrary, the term 'idiosyncratic' refers to the portion of risk unique to an individual claim or component, unexplained by broader systematic factors. In Section 3.3, we will demonstrate that Y^i can be diversifiable in a more specific context.

Deelstra et al. (2020) decompose the unhedged residual part of a hybrid liability into two different parts to separate the systematic part from the diversifiable part. Note, however, that they assumed that hybrid claims were product claims, and that financial and actuarial risks are independent. Moreover, the financial market was assumed to be complete. The lecture notes of Dhaene (2022, pp. 58–62) generalizes these results to account for incompleteness of the financial market, resulting in the same four-step decomposition as was proposed here. Note, however, that the framework proposed in the context holds for a general hybrid liability, allows for an incomplete financial market and, moreover, incorporates dependence between financial and actuarial risks.

We can interpret Y^s also as the 'between-scenario' part, whereas Y^i can be regarded as the 'within-scenario' part. These interpretations play a crucial role when analyzing the variance of the non-hedged part $S - Y^h$. The following remark elaborates on how the 'between' and 'within' parts are defined in terms of variances.

Remark 3.1 (between vs. within group variance) *Using Expressions (3.3) and (3.4), we can then write:*

$$\mathit{Var}\left[Y^s\right] = \mathit{Var}\left[\mathbb{E}\left[S - Y^h \middle| \boldsymbol{Y}, \boldsymbol{Z}\right]\right], \tag{3.5}$$

$$\mathit{Var}\left[Y^i\right] = \mathbb{E}\left[\mathit{Var}\left[S - Y^h \middle| \boldsymbol{Y}, \boldsymbol{Z}\right]\right]. \tag{3.6}$$

The variance $\mathit{Var}\left[Y^s\right]$ represents the 'between group' variance, whereas $\mathit{Var}\left[Y^i\right]$ represents the 'within group' variance of the non-hedged residual part $S - Y^h$. Indeed, the conditional expectation $\mathbb{E}\left[S - Y^h \middle| \boldsymbol{Y}, \boldsymbol{Z}\right]$ corresponds with the expectation of the non-hedged part in a certain scenario of the systematic risks and the financial assets. Therefore, its variance is a measure for the variability between the different scenarios. The conditional variance $\mathit{Var}\left[S - Y^h \middle| \boldsymbol{Y}, \boldsymbol{Z}\right]$, on the other hand, measures the variance in each scenario for the systematic risks and financial assets. Therefore, its expectation is a measure of the average variability within each scenario.

Using the law of total variance, it follows directly from Expressions (3.5) and (3.6) that we can decompose the variance of the residual part $S - Y^h$ as follows:

$$\mathrm{Var}\left[S - Y^h\right] = \mathrm{Var}\left[Y^s\right] + \mathrm{Var}\left[Y^i\right]. \tag{3.7}$$

This identity follows from the orthogonality property of conditional expectations and is commonly known as the *variance decomposition*. Bühlmann (1995) was the first to apply this decomposition to claim payoffs in life insurance, separating financial investment risk and actuarial mortality risk. Subsequent research has applied similar variance decomposition techniques to analyze investment and insurance risks, most notably Parker (1997), Frees (1998), Marceau and Gaillardetz (1999), Bruno et al. (2000), Christiansen and Helwich (2008). Among these, only Frees (1998) and Christiansen and Helwich (2008) explicitly include systematic mortality risk and both assume that actuarial risk is independent of financial investment risk. The others treat actuarial risk purely as the random variation of individual lifetimes and do not incorporate systematic mortality risk.

The hedger is used to decompose the claim S in two parts, a hedgeable and a non-hedged residual part. Using Expression (3.2), we further decompose the residual part in a systematic part and an idiosyncratic part. Since the hedgeable part only depends on the financial assets $\boldsymbol{Y}$ and the systematic part only depends on the systematic risks $\boldsymbol{Z}$ and the financial assets $\boldsymbol{Y}$, we find that the hedgeable part Y^h and the systematic part Y^s both belong to the set $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$, which was defined in (2.2). In the following theorem, we investigate to what extent this decomposition of the non-hedged residual liability $S-Y^h$ into a systematic and an idiosyncratic part is optimal and unique.

In Theorem 3.2, we show that the systematic part Y^s arises as the solution of a quadratic optimization problem where we approximate the non-hedged residual part with an element in the set $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$. Moreover, the systematic part Y^s is also the unique element in the set $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ that gives a decomposition of the non-hedged part in two uncorrelated parts, provided the expectation of Y^s is equal to the expectation of the residual part $S - Y^h$. Lastly, we show that this condition on the mean can be relaxed if we require the two components to be orthogonal, rather than uncorrelated, where orthogonality is defined via the inner-product condition in (3.9) below.

Theorem 3.2 *Let the residual part $S - Y^h$ be decomposed as*

$$S - Y^h = Y^s + \left(S - Y^h - Y^s\right),$$

where $Y^s \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$. Then the following statements are equivalent.

(1) $Y^s = \mathbb{E}\left[S - Y^h \middle| \, \boldsymbol{Y}, \boldsymbol{Z}\right]$.

(2) *Y^s is the solution to the optimization problem*

$$Y^s = \underset{\xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}}{\arg\min}\, \mathbb{E}\left[\left(S - Y^h - \xi\right)^2\right]. \tag{3.8}$$

(3) *The random variable Y^s satisfies*

$$\mathbb{E}\left[\left(S - Y^h - Y^s\right)\xi\right] = 0, \quad \textit{for any } \xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}. \tag{3.9}$$

(4) *If we have that*

$$\mathbb{E}\left[Y^s\right] = \mathbb{E}\left[S - Y^h\right], \tag{3.10}$$

then Y^s satisfies

$$\mathrm{Cov}\left[S - Y^h - Y^s, \xi\right] = 0, \quad \textit{for any } \xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}. \tag{3.11}$$

Proof. First, we prove the equivalence of statements (1) and (2). Note that if

$$f(\boldsymbol{y},\boldsymbol{z}) = \arg\min_{\pi} \mathbb{E}\left[\left(S - Y^h - \pi(\boldsymbol{Y},\mathbf{Z})\right)^2 | \boldsymbol{Y} = \boldsymbol{y}, \boldsymbol{Z} = \boldsymbol{z}\right],$$

then $f(\boldsymbol{y},\boldsymbol{z}) = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y} = \boldsymbol{y}, \boldsymbol{Z} = \boldsymbol{z}\right]$. Hence, we find that $f(\boldsymbol{Y},\boldsymbol{Z}) = \mathbb{E}\left[S - Y^h | \boldsymbol{Y},\boldsymbol{Z}\right]$, which shows that Y^s is indeed the solution to the maximization problem. Moreover, since the solution is unique, we find the desired equivalence.

Next, we demonstrate the equivalence of statements (2) and (3). We consider the Hilbert space $L^2(\Omega,\mathcal{F},\mathbb{P})$ with the inner product defined as $\langle X, Y\rangle = \mathbb{E}[XY]$. Since $Y^s \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$, $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ is a closed Hilbert subspace of $L^2(\Omega,\mathcal{F},\mathbb{P})$. Let

$$X = S - Y^h,$$

$X \in L^2(\Omega,\mathcal{F},\mathbb{P})$, and $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}} \subset L^2(\Omega,\mathcal{F},\mathbb{P})$, then the projection theorem in Hilbert spaces implies that

(a) there exists a unique element $\hat{X} \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ such that

$$\|X - \hat{X}\| = \inf_{\xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}} \|X - \xi\|,$$

where $\|X\| = \langle X, X\rangle^{\frac{1}{2}} = (\mathbb{E}[X^2])^{\frac{1}{2}}$.

(b) The $\hat{X}$ is also uniquely characterized by:

$$\left\langle X - \hat{X}, \xi \right\rangle = \mathbb{E}\left[(X - \hat{X})\xi\right] = 0, \text{ for any } \xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}.$$

We then find that the statements (2) and (3) are equivalent.

Next, we show that statement (3) implies (4). If (3) holds, we also have (1); that is $Y^s = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y},\boldsymbol{Z}\right]$. It is then straightforward to see that

$$\mathbb{E}[Y^s] = \mathbb{E}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y},\boldsymbol{Z}\right]\right] = \mathbb{E}\left[S - Y^h\right].$$

Therefore, we find that

$$\mathrm{Cov}[S - Y^h - Y^s, \xi] = \mathbb{E}\left[\left(S - Y^h - Y^s\right)\xi\right] - \mathbb{E}\left[S - Y^h - Y^s\right]\mathbb{E}[\xi] = 0,$$

for any $\xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$.

Lastly, we show that statement (4) implies (1). Let Y^s satisfy (3.10). For any L^2 random variable $\xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$, it follows from Expression (3.11) that

$$\mathrm{Var}\left[S - Y^h - Y^s + \xi\right] - \mathrm{Var}\left[S - Y^h - Y^s\right] - \mathrm{Var}[\xi] = 0.$$

Note that $\mathrm{Var}\left[S - Y^h - Y^s + \xi\right]$ can be expressed as

$$\begin{aligned}\mathrm{Var}\left[S - Y^h - Y^s + \xi\right] &= \mathrm{Var}\left[\mathbb{E}\left[S - Y^h - Y^s + \xi \mid \boldsymbol{Y},\boldsymbol{Z}\right]\right] + \mathbb{E}\left[\mathrm{Var}\left[S - Y^h - Y^s + \xi \mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]\\ &= \mathrm{Var}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y},\boldsymbol{Z}\right] - Y^s + \xi\right] + \mathbb{E}\left[\mathrm{Var}\left[S - Y^h \mid \boldsymbol{Y},\boldsymbol{Z}\right]\right].\end{aligned}$$

Additionally, $\text{Var}\left[S - Y^h - Y^s\right]$ can be written as

$$\begin{aligned}\text{Var}\left[S - Y^h - Y^s\right] &= \text{Var}\left[\mathbb{E}\left[S - Y^h - Y^s \mid \boldsymbol{Y}, \boldsymbol{Z}\right]\right] + \mathbb{E}\left[\text{Var}\left[S - Y^h - Y^s \mid \boldsymbol{Y}, \boldsymbol{Z}\right]\right] \\ &= \text{Var}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - Y^s\right] + \mathbb{E}\left[\text{Var}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right]\right].\end{aligned}$$

Then we find that

$$\begin{aligned}&\text{Var}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - Y^s + \xi\right] - \text{Var}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - Y^s\right] - \text{Var}[\xi] \\ =&\text{Var}\left[S - Y^h - Y^s + \xi\right] - \text{Var}\left[S - Y^h - Y^s\right] - \text{Var}\left[\xi\right] \\ =&0.\end{aligned}$$

Hence, we have:

$$\text{Cov}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - Y^s, \xi\right] = 0.$$

Note that the random variable $\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - Y^s$ is an element of $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$. We put $\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - Y^s = M$. Suppose M is a non-degenerate random variable. Then, we have:

$$\text{Cov}\left[M, M\right] = \text{Var}\left[M\right] > 0.$$

However, this contradicts Expression (3.11). Therefore, we conclude that:

$$M = c, \text{ with probability } 1,$$

where $c \in \mathbb{R}$ is a constant. Consequently, Y^s can be expressed as:

$$Y^s = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - M = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - c.$$

From Expression (3.10), we deduce that $c = 0$. Therefore, we have:

$$Y^s = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right].$$

This concludes the proof. ■

Theorem 3.2 provides an 'optimal' way to isolate a systematic part from the non-hedged residual claim, where optimality is defined in mean-variance sense. This residual part depends on the choice of the hedger. Indeed, the hedger determines the hedgeable part, which only consists of financial risks. The systematic parts consists of financial risks which are not yet captured by the hedgeable part and systematic risks. It is then straightforward to show that the random variable $Y^h + Y^s$ is the best approximation of the claim S in the set $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$. Indeed, following the same steps as in the proof of Theorem 3.2, we can show that

$$Y^h + Y^s = \underset{\xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}}{\arg\min}\, \mathbb{E}\left[(S - \xi)^2\right].$$

This result shows that the hedger only distributes the financial risk contained in S between the hedgeable part Y^h and the systematic part Y^s. Therefore, if we want to decompose the claim S in a part that depends only on the financial and the systematic information on the one hand, and a residual part on the other hand, the choice of the hedger $\boldsymbol{\theta}_S$ is not important.

Because the systematic part Y^s contains those financial risks in S which are not captured by Y^h, we try to separate the financial and the actuarial systematic information in Y^s. We apply the conditional expectation approach to decompose Y^s into a financial systematic part Y^s_{fin} and an actuarial part Y^s_{act}:

$$Y^s = Y^s_{fin} + Y^s_{act},$$

where

$$\begin{aligned} Y^s_{fin} &= \mathbb{E}\left[\mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] \mid \boldsymbol{Y}\right] = \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}\right], && (3.12)\\ Y^s_{act} &= \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - \mathbb{E}\left[S - Y^h \mid \boldsymbol{Y}\right] = \mathbb{E}\left[S \mid \boldsymbol{Y}, \boldsymbol{Z}\right] - \mathbb{E}\left[S \mid \boldsymbol{Y}\right]. && (3.13)\end{aligned}$$

It is straightforward to verify that Y^s_{fin} also solves the following optimization problem:

$$Y^s_{fin} = \underset{\xi \in \mathcal{C}^{\boldsymbol{Y}}}{\arg\min}\, \mathbb{E}\left[(Y^s - \xi)^2\right].$$

Hence, Y^s_{fin} can also be interpreted as the best financial derivative to hedge the systematic claim Y^s. However, note that Y^s_{fin} is not necessarily a linear combination of financial assets, in which case a hedging strategy cannot be found to replicate the payoff of Y^s_{fin}. In case one wants to receive the payoff Y^s_{fin}, one needs to buy it over-the-counter at the best market price. Similarly to Theorem 3.2, we can also prove that, under the additional assumption that $\mathbb{E}\left[Y^s_{fin}\right] = \mathbb{E}[Y^s]$, the financial systematic part Y^s_{fin} is characterized as follows:

$$\operatorname{Cov}\left[Y^s - Y^s_{fin}, \xi\right] = 0, \quad \text{for any } \xi \in \mathcal{C}^{\boldsymbol{Y}}. \tag{3.14}$$

The financial systematic part is the unique financial derivative that ensures the residual part is uncorrelated with the set $\mathcal{C}^{\boldsymbol{Y}}$ of financial derivatives, provided their expectations are equal. This leads to the variance decomposition:

$$\operatorname{Var}[Y^s] = \operatorname{Var}\left[Y^s_{fin}\right] + \operatorname{Var}\left[Y^s_{act}\right]. \tag{3.15}$$

More specifically, Y^s_{fin} represents the portion of the systematic part Y^s that can be predicted based on the traded assets $\boldsymbol{Y}$, reflecting the changes in Y^s resulting from shifts in market prices. On the other hand, Y^s_{act} represents the deviations in the systematic part Y^s that are not explained by $\boldsymbol{Y}$. It measures the contribution of $\boldsymbol{Z}$ to the variation in Y^s, conditional on $\boldsymbol{Y}$. In other words, it quantifies the influence of actuarial (non-traded) systematic risks on the systematic part Y^s beyond what is already captured by market prices $\boldsymbol{Y}$.

Following the steps above, we find that a hybrid liability S can be decomposed into

$$S = Y^h + Y^i + Y^s_{fin} + Y^s_{act},$$

with

$$\begin{aligned} Y^h &= \boldsymbol{\theta}_S \cdot \boldsymbol{Y},\\ Y^i &= S - \mathbb{E}[S \mid \boldsymbol{Y}, \boldsymbol{Z}],\\ Y^s_{fin} &= \mathbb{E}[S \mid \boldsymbol{Y}] - \boldsymbol{\theta}_S \cdot \boldsymbol{Y},\\ Y^s_{act} &= \mathbb{E}[S \mid \boldsymbol{Y}, \boldsymbol{Z}] - \mathbb{E}[S \mid \boldsymbol{Y}]. \end{aligned}$$

To illustrate this decomposition, we present Example 3.1 below.

Example 3.1 (Unit-linked insurance liability) *Consider a portfolio with N insurance policyholders, with each policyholder i receives a payoff at maturity T given by*

$$f(\boldsymbol{Y}) \times X_i, \text{ for } i = 1, 2, \ldots, N,$$

where f is a borel-measurable function, and $X_i = 1$ if policyholder i is alive at T, and $X_i = 0$ otherwise. Assume that each X_i is independent of $\boldsymbol{Y}$, and that there exists a systematic mortality r.v. $Z \in \mathcal{C}$ with support A, such that for any $z \in A$, $X_1 \mid Z = z, X_2 \mid Z = z, \ldots, X_N \mid Z = z$ are i.i.d. claims. The aggregate liability S is then given by

$$S = \sum_{i=1}^{N} f(\boldsymbol{Y}) \times X_i.$$

Using the mean-variance hedger $\boldsymbol{\theta}_S^{MV}$ as defined in Definition 3.2, the aggregate liability S can be decomposed into

$$S = Y^h + Y^i + Y^s_{fin} + Y^s_{act},$$

where

$$\begin{aligned} Y^h &= N\,\mathbb{E}[X_1] \times \boldsymbol{\theta}^{MV}_{f(\boldsymbol{Y})} \cdot \boldsymbol{Y}, \\ Y^i &= f(\boldsymbol{Y}) \times \left(\sum_{i=1}^{N} X_i - N\,\mathbb{E}[X_1 \mid Z] \right), \\ Y^s_{fin} &= N\,\mathbb{E}[X_1] \times \left(f(\boldsymbol{Y}) - \boldsymbol{\theta}^{MV}_{f(\boldsymbol{Y})} \cdot \boldsymbol{Y} \right), \\ Y^s_{act} &= N\, f(\boldsymbol{Y}) \times \left(\mathbb{E}[X_1 \mid Z] - \mathbb{E}[X_1] \right). \end{aligned}$$

This example is also considered in Dhaene (2022, p.61), though we note that this setting assumes independence between the policy-specific risk X_i and financial risk $\boldsymbol{Y}$. In Section 5, we will extend this framework to a more general setting where this assumption is relaxed.

Remark 3.3 *For any choice of hedging strategy $\boldsymbol{\theta}$, we have*

$$\mathbb{E}\left[Y^i\right] = \mathbb{E}\left[S - \mathbb{E}\left[S \mid \boldsymbol{Y}, \boldsymbol{Z}\right]\right] = 0, \tag{3.16}$$

$$\mathbb{E}\left[Y^s_{act}\right] = \mathbb{E}\left[\mathbb{E}\left[S \mid \boldsymbol{Y}\right] - \mathbb{E}\left[S \mid \boldsymbol{Y}, \boldsymbol{Z}\right]\right] = 0. \tag{3.17}$$

Moreover, if we choose the hedger $\boldsymbol{\theta}$ such that $\mathbb{E}\left[Y^h\right] = \mathbb{E}\left[S\right]$, then

$$\mathbb{E}\left[Y^s_{fin}\right] = \mathbb{E}\left[\mathbb{E}\left[S \mid \boldsymbol{Y}\right] - Y^h\right] = 0. \tag{3.18}$$

Note that we have established that Y^s and Y^i are uncorrelated, and so are Y^s_{fin} and Y^s_{act}. In the following theorem, we further show that the random variables Y^i, Y^s_{fin}, Y^s_{act} are pairwise uncorrelated, and Y^h is uncorrelated with both Y^i and Y^s_{act}.

Theorem 3.4 *For any choice of hedger $\boldsymbol{\theta}$, we have that*

- *the random variables Y^i, Y^s_{fin}, Y^s_{act} are pairwise uncorrelated,*

- *the hedgeable part Y^h is uncorrelated with each of Y^i and Y^s_{act}.*

Proof. From Theorem 3.2, we find that

$$\operatorname{Cov}\left[Y^i, \xi\right] = 0, \text{ for any } \xi \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}.$$

It follows directly from (3.1), (3.12) and (3.13) that Y^h, Y^s_{fin} and Y^s_{act} are L^2 random variables that are measurable with respect to $\sigma(\boldsymbol{Y}, \boldsymbol{Z})$. Therefore, we have that

$$\operatorname{Cov}\left[Y^i, Y^h\right] = \operatorname{Cov}\left[Y^i, Y^s_{act}\right] = \operatorname{Cov}\left[Y^i, Y^s_{fin}\right] = 0.$$

Similarly, since $Y^h \in \mathcal{C}^{\boldsymbol{Y}}$, it follows from (3.14) that

$$\operatorname{Cov}\left[Y^s_{act}, Y^h\right] = \operatorname{Cov}\left[Y^s - Y^s_{fin}, Y^h\right] = 0.$$

■

We have shown that a claim S, which depends on the financial assets $\boldsymbol{Y}$ and systematic risks $\boldsymbol{Z}$, can always be decomposed into a hedgeable part Y^h and three uncorrelated components, namely Y^s_{fin}, Y^s_{act}, and Y^i, using the conditional expectation approach. Moreover, the idiosyncratic part Y^i and the actuarial systematic part Y^s_{act} are always uncorrelated with Y^h, regardless of the choice of hedger $\boldsymbol{\theta}_S$. However, the correlation between the financial residual part Y^s_{fin} and Y^h is not immediately clear. Indeed, one can end up in a situation where Y^s_{fin} and Y^h are dependent because they are exposed to the same traded risks. We will show in the following section that such a situation is excluded when using the mean-variance hedger, i.e., Y^s_{fin} is also uncorrelated with Y^h, which implies that these four parts will be pairwise uncorrelated.

Subsequently, we consider a more specific framework in Section 3.3 and regard the hybrid liability S as an aggregate claim. We can then discuss the diversifiability of the idiosyncratic part Y^i.

3.2 Mean-variance hedge

In Theorem 3.4, we have shown that Y^i, Y^s_{fin}, and Y^s_{act} are pairwise uncorrelated, and Y^h is uncorrelated with both Y^i and Y^s_{act} regardless of the hedging strategy $\boldsymbol{\theta}_S$. However, we are unable to unravel the dependence between the financial systematic part Y^s_{fin} and the hedgeable part Y^h unless the hedging strategy is specified. Recall that the financial payoffs which are traded in the market are stored in the vector $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}\right)$. We use $\mathcal{H}^{linear}$ to denote the space of all linear combinations of the financial assets:

$$\mathcal{H}^{linear} = \operatorname{span}\{Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}\}.$$

Note there is a one-to-one correspondence between the set $\mathcal{H}^{linear}$ containing random variables which can be expressed as linear combinations of the financial assets and the set Θ with hedgers. Indeed, each hedger in Θ defines a random variable that belongs to the set $\mathcal{H}^{linear}$, and vice versa.

We aim to hedge the claim S such that the non-hedged component, $S - Y^h$, satisfies two conditions: it is uncorrelated with every random variable in the space $\mathcal{H}^{linear}$, and its expectation is zero. In Theorem 3.5, we will demonstrate that these conditions are met if and only if the mean-variance hedging strategy is employed. Furthermore, this approach ensures that the four components of the decomposition are pairwise uncorrelated.

Definition 3.2 (Mean-variance hedge) *For any $S \in \mathcal{C}$, the mean-variance hedger $\boldsymbol{\theta}^{MV}$ is the hedger which assigns to the claim S the hedging strategy $\boldsymbol{\theta}_S^{MV}$ by solving the following minimization problem:*

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\nu} \in \Theta} \mathbb{E}\left[(S - \boldsymbol{\nu} \cdot \boldsymbol{Y})^2\right]. \tag{3.19}$$

Mean-variance hedging is widely applied for hedging contingent claims; see, e.g., Föllmer and Sondermann (1986), Schweizer (1995), Thomson (2005), Dhaene et al. (2017), Barigou and Dhaene (2019), Linders (2023), among others.

In the following theorem, we show a characterization of the mean-variance hedger, which is similar to Theorem 3.2.

Theorem 3.5 *Consider a claim $S \in \mathcal{C}$ and the corresponding hedge $Y^h = \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$. Then, the following statements are equivalent.*

(1) The hedging strategy $\boldsymbol{\theta}_S$ is determined by the mean-variance hedge (3.19), i.e., $\boldsymbol{\theta}_S = \boldsymbol{\theta}_S^{MV}$.

(2) The hedging strategy $\boldsymbol{\theta}_S$ is determined such that non-hedged part $S - Y^h$ is orthogonal to each random variable $Y^{(i)}$:

$$\mathbb{E}[(S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y})Y^{(i)}] = 0, \text{ for } i = 0, 1, \ldots, n. \tag{3.20}$$

(3) The hedging strategy $\boldsymbol{\theta}_S$ is determined such that the non-hedged part $S - Y^h$ is uncorrelated to each random variable $Y^{(i)}$:

$$\operatorname{Cov}\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}, Y^{(i)}\right] = 0, \text{ for } i = 0, 1, \ldots, n, \tag{3.21}$$

and the expectation of the hedgeable part Y^h is equal to the expectation of the claim $\mathbb{E}[S]$:

$$\mathbb{E}\left[Y^h\right] = \mathbb{E}[S]. \tag{3.22}$$

Proof. First, we show that (1) $\iff$ (2):
It directly follows from (3.19) that $\boldsymbol{\theta}_S^{MV}$ is determined from the first-order conditions given in (3.20).

Next, we show that (2) $\implies$ (3):

Given that $Y^{(0)} = e^{rT}$, it follows from (3.20) that

$$\mathbb{E}[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}] = 0.$$

Then we have:

$$\text{Cov}\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}, Y^{(i)}\right] = \mathbb{E}\left[\left(S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right) Y^{(i)}\right] - \mathbb{E}\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right] \times \mathbb{E}\left[Y^{(i)}\right] = 0.$$

Lastly, we show that (3) $\Longrightarrow$ (2):

From (3.21) and (3.22), we find that

$$\mathbb{E}\left[\left(S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right) Y^{(i)}\right] = \text{Cov}\left[S - Y^h, Y^{(i)}\right] + \mathbb{E}\left[S - Y^h\right] \times \mathbb{E}\left[Y^{(i)}\right] = 0,$$

for $i = 0, 1, 2, \ldots, n$.

■

The finite-dimensional result in Theorem 3.5 is the discrete-time analogue of the Föllmer–Schweizer decomposition in continuous time. For an overview of quadratic hedging approaches in the continuous-time setting, including mean-variance hedging via the Föllmer–Schweizer decomposition, see Schweizer (2001).

The financial market with traded assets $\boldsymbol{Y}$ is said to be complete if for any claim $S \in \mathcal{C}^{\boldsymbol{Y}}$, there exists a hedging strategy $\boldsymbol{\nu} \in \Theta$ such that $S = \boldsymbol{\nu} \cdot \boldsymbol{Y}$, almost surely. Therefore, completeness in our one-period model corresponds with all financial derivatives being linear combinations of the traded financial assets, i.e. we have to define that $\mathcal{C}^{\boldsymbol{Y}} = \mathcal{H}^{Linear}$.

Lemma 3.1 *Consider a claim $S \in \mathcal{C}$ and the corresponding hedge $Y^h = \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$. Assume the hedger is market consistent, i.e. $\boldsymbol{\theta}_{S+\boldsymbol{\nu}\cdot\boldsymbol{Y}} = \boldsymbol{\theta}_S + \boldsymbol{\nu}$. If the financial market is complete, there exists a hedger such that $Y^s_{fin} = 0$.*

Proof. If the market is complete, for any $S \in \mathcal{C}$, since $\mathbb{E}[S \mid \boldsymbol{Y}] \in \mathcal{C}^{\boldsymbol{Y}}$, then there exists a hedging strategy $\boldsymbol{\nu}$ such that

$$\mathbb{E}[S \mid \boldsymbol{Y}] = \boldsymbol{\nu} \cdot \boldsymbol{Y}.$$

Define the hedger $\boldsymbol{\theta} : \mathcal{C} \to \mathbb{R}^{n+1}$ such that for any $S \in \mathcal{C}$,

$$Y^h = \boldsymbol{\theta}_S \cdot \boldsymbol{Y} = \mathbb{E}[S \mid \boldsymbol{Y}].$$

Therefore, it directly follows from (3.12) that the financial systematic part $Y^s_{fin} = 0$.

■

Proposition 3.1 *Consider a hybrid claim S and the mean variance hedger $\boldsymbol{\theta}^{MV}$. We have that the claim S can be decomposed as follows*

$$S = \boldsymbol{\theta}^{MV}_S \cdot \boldsymbol{Y} + Y^i + Y^s_{fin} + Y^s_{act},$$

where Y^i, Y^s_{fin}, and Y^s_{act} are given by (3.4), (3.12), and (3.13), respectively. Moreover, the parts $\boldsymbol{\theta}^{MV}_S \cdot \boldsymbol{Y}$, Y^i, Y^s_{act} and Y^s_{fin} are pairwise uncorrelated.

Proof. As demonstrated in Dhaene (2022), the mean-variance hedge of a claim S coincides with the the mean-variance hedge of the conditional expectation $\mathbb{E}[S \mid \boldsymbol{Y}]$:

$$\boldsymbol{\theta}_S^{MV} = \boldsymbol{\theta}_{\mathbb{E}[S|\boldsymbol{Y}]}^{MV}.$$

Using Expressions (3.1) and (3.12), we find that

$$\operatorname{Cov}\left[Y^h, Y_{fin}^s\right] = \operatorname{Cov}\left[\boldsymbol{\theta}_{\mathbb{E}[S|\boldsymbol{Y}]}^{MV} \cdot \boldsymbol{Y}, \mathbb{E}[S \mid \boldsymbol{Y}] - \boldsymbol{\theta}_{\mathbb{E}[S|\boldsymbol{Y}]}^{MV} \cdot \boldsymbol{Y}\right].$$

Then it follows from Theorem 3.5 that

$$\operatorname{Cov}\left[Y^h, Y_{fin}^s\right] = 0.$$

Therefore, from Theorem 3.4, it follows that Y^h, Y^i, Y_{fin}^s, Y_{act}^s are pairwise uncorrelated. ■

3.3 Diversifiability of the idiosyncratic part Y^i

The systematic part Y^s only depends on the financial risks $\boldsymbol{Y}$ and the systematic risks $\boldsymbol{Z}$, whereas the idiosyncratic part Y^i is still a combination of financial, systematic and actuarial risks. In this subsection, we investigate diversification properties of the idiosyncratic part Y^i in case the liability S represents an aggregate liability. Suppose there are N policyholders labeled 1 to N. The random variables $X_1, X_2, \ldots, X_N$ represent the policyholder-specific risks. We assume a heterogeneous portfolio, so the X_i random variables are not necessarily identical. However, we assume that the random variables X_i, $i = 1, 2, \ldots, N$ are conditionally independent, i.e.,

$$\mathbb{P}\left[X_1 \leq x_1, \ldots, X_N \leq x_N \mid \boldsymbol{Y} = \boldsymbol{x}_1, \boldsymbol{Z} = \boldsymbol{x}_2\right] = \prod_{i=1}^{N} \mathbb{P}\left[X_i \leq x_i \mid \boldsymbol{Y} = \boldsymbol{x}_1, \boldsymbol{Z} = \boldsymbol{x}_2\right]. \quad (3.23)$$

Each policyholder will receive a payoff which is a function of its own policyholder-specific risk X_i and the financial market. To be more precise, we assume the liability S is then given by

$$S = \sum_{i=1}^{N} h_i\left(X_i, \boldsymbol{Y}\right). \quad (3.24)$$

We demonstrate in Theorem 3.6 that the idiosyncratic part Y^i, which is given by (3.4), is diversifiable in the sense that the per-policy loss of the idiosyncratic part $\frac{Y^i}{N}$ tends to zero as the number of participants increases. This result is a generalization of Dhaene (2022, p. 60), and Theorem 3.1 in Deelstra et al. (2020), where the authors derive a similar result, but under the assumption the aggregate liability S is a product claim. Moreover, they assume that the systematic risk factors are independent of the financial assets.

Theorem 3.6 *Consider the claim S given by (3.24) and its idiosyncratic part given by (3.4). Assume that the random variables $X_1, X_2, \ldots, X_N$ are conditionally independent; see (3.23). Then Y^i is diversifiable, in the sense that*

$$\frac{Y^i}{N} \xrightarrow{p} 0 \quad \text{as } N \to \infty,$$

where $\xrightarrow{p}$ denotes convergence in probability.

Proof. It directly follows from (3.16) that $\mathbb{E}\left[\frac{Y^i}{N}\right]=0$. And the variance of $\frac{Y^i}{N}$ can be expressed as

$$\begin{aligned}
\operatorname{Var}\left[\frac{Y^i}{N}\right] &= \frac{1}{N^2}\operatorname{Var}\left[S-\mathbb{E}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right] \\
&= \frac{1}{N^2}\left(\operatorname{Var}\left[\mathbb{E}\left[S-\mathbb{E}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]+\mathbb{E}\left[\operatorname{Var}\left[S-\mathbb{E}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]\right) \\
&= \frac{1}{N^2}\left(\operatorname{Var}\left[\mathbb{E}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]-\mathbb{E}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]+\mathbb{E}\left[\operatorname{Var}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]\right) \\
&= \frac{\mathbb{E}\left[\operatorname{Var}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]}{N^2}. \qquad (3.25)
\end{aligned}$$

Using (3.24), and taking into account the conditional independence assumption in (3.23), the variance of $\frac{Y^i}{N}$ can be rewritten as

$$\operatorname{Var}\left[\frac{Y^i}{N}\right]=\frac{\sum_{i=1}^{N}\mathbb{E}\left[\operatorname{Var}\left[h_i\left(X_i,\boldsymbol{Y}\right)\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]}{N^2}. \qquad (3.26)$$

Since $h_i(X_i,\boldsymbol{Y})$ are L^2 random variables on $(\Omega,\mathcal{F},\mathbb{P})$, there exists a constant $M>0$ such that

$$\operatorname{Var}\left[h_i(X_i,\boldsymbol{Y})\right]\leq M \quad \text{for all } i.$$

It follows that

$$\operatorname{Var}\left[\frac{Y^i}{N}\right]\to 0 \text{ as } N\to\infty.$$

Therefore, $\frac{Y^i}{N}$ converges to 0 in probability as $N\to\infty$. ■

Remark 3.7 *Note that combining Theorem 3.6 with Expression (3.4) for the idiosyncratic part Y^i results in*

$$\frac{S}{N}\xrightarrow{p}\mathbb{E}\left[\frac{S}{N}\mid \boldsymbol{Y},\boldsymbol{Z}\right], \quad \text{if } N\to+\infty.$$

In case we hold a large portfolio, the per-policy liability is random and will converge to the conditional expectation. If $h_i\left(X_i,\boldsymbol{Y}\right)=h_i\left(X_i\right)$, we recover the central limit theorem.

Theorem 3.6 indicates that the conditional independence assumption (3.23) is a sufficient condition to show that the claim Y^i is diversifiable. However, conditional independence is not a necessary assumption to demonstrate that Y^i is diversifiable. The following counterexample is presented to illustrate this observation.

Example 3.2 (Conditional independence and diversifiability) *Assume that $X_1|\boldsymbol{Y},\boldsymbol{Z}$ is dependent with $X_i|\boldsymbol{Y},\boldsymbol{Z}$ for $i=2,3,\ldots,N$, while $X_i|\boldsymbol{Y},\boldsymbol{Z}$ is independent of $X_j|\boldsymbol{Y},\boldsymbol{Z}$ for $i,j=2,3,\ldots,N$. In this context, condition (3.23) is violated, however, we can still demonstrate that Y^i is diversifiable. Indeed, it follows from (3.25) that Var $\left[\frac{Y^i}{N}\right]$ is given by*

$$\mathit{Var}\left[\frac{Y^i}{N}\right]=\frac{\sum_{i=1}^{N}\mathbb{E}\left[\operatorname{Var}\left[h_i\left(X_i,\boldsymbol{Y}\right)\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]+\sum_{j=2}^{N}\mathbb{E}\left[\operatorname{Cov}\left[h_1(X_1,\boldsymbol{Y}),h_j(X_j,\boldsymbol{Y})\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right]}{N^2}.$$

Since $\mathrm{Var}\left[h_i\left(X_i, \boldsymbol{Y}\right)\right] < \infty$ *and* $\mathrm{Cov}\left[h_1(X_1, \boldsymbol{Y}), h_j(X_j, \boldsymbol{Y})\right] < \infty$, *we can conclude that* $\mathrm{Var}\left[\frac{Y^i}{N}\right] \to 0$ *as* $N \to \infty$. *We find that* $\frac{Y^i}{N}$ *converges to* 0 *in probability as* $N \to \infty$. *We conclude that that conditional independence is not a necessary condition to show that the claim* $\frac{Y^i}{N}$ *is diversifiable.*

Example 3.2 demonstrates that conditional independence is a sufficient but not necessary condition to prove that Y^i is diversifiable. In convtrast, as shown in Example 3.3, Y^i may fail to be diversifiable under certain conditions. In addition, Appendix A shows that when S is a catastrophe-type liability, the payoff Y^i is no longer diversifiable.

Example 3.3 (Non-diversifiability without conditional independence) *We define liability* S *as the aggregate claim amount where each policyholder has a claim* X_i*:*

$$S = \sum_{i=1}^{N} X_i.$$

We assume $\boldsymbol{Y} = Y^{(0)}$*, i.e. there is only a risk-free bank account and no risky asset. For simplicity, we take* $r = 0$ *and therefore* $Y^{(0)} = 1$. *Furthermore, the random variable* Z *can be represented as the systematic risk factor with* $\mathbb{P}[Z = 1] = 1 - \mathbb{P}[Z = 0] = p$. *Moreover, assume that there are Bernoulli random variables* I_0 *and* I_1 *with success parameters* p_0 *and* p_1*, respectively. The claim amounts* X_i *are given by*

$$X_i = Z I_1 W_i + (1 - Z) I_0 V_i,$$

where $\boldsymbol{W} = (W_1, \ldots, W_N)$ *is multivariate normal with* $\mathit{Corr}\left[W_i, W_j\right] = \rho > 0$ *and* $W_i \sim N\left(\mu_1, \sigma_1^2\right)$ *with* $\mu_1 > 0$. *The random vector* $\boldsymbol{V}$ *is also multivariate normal, but now with independent marginals and* $V_i \sim N\left(0, \sigma_0^2\right)$. *Lastly, we assume that* $Z, I_1, I_0, \boldsymbol{W}, \boldsymbol{V}$ *are all independent of each other.*

If we use that I_1 *is independent of* W_i*,* I_0 *is independent of* V_i*, and* $\mathbb{E}[V_i] = \mathbb{E}[W_i] = 0$*, we find that*

$$\mathbb{E}\left[\mathit{Var}\left[X_i|Z\right]\right] = p \times p_1 \times \left(\sigma_1^2 + \mu_1^2 - p_1\mu_1^2\right) + (1 - p) \times p_0\sigma_0^2.$$

Using the single correlation ρ *between the random variables* W_i *and* W_j *and the independence between* V_i *and* V_j*, we find that the conditional covariance can be calculated as follows:*

$$\mathbb{E}\left[\mathit{Cov}\left[X_i, X_j|Z\right]\right] = p \times p_1 \times \left(\rho\sigma_1^2 + \mu_1^2 - p_1\mu_1^2\right).$$

Therefore, $\mathrm{Var}\left[\frac{Y^i}{N}\right]$ *in this case is given by*

$$\begin{aligned}\mathrm{Var}\left[\frac{Y^i}{N}\right] &= \frac{\mathbb{E}\left[\mathrm{Var}[X_i \mid \boldsymbol{Y}, Z]\right] + (N-1) \times \mathbb{E}\left[\mathrm{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z\right]\right]}{N} \\ &\to p \times p_1 \times \left(\rho\sigma_1^2 + \mu_1^2 - p_1\mu_1^2\right) > 0.\end{aligned}$$

We can conclude that $\mathrm{Var}\left[\frac{Y^i}{N}\right]$ *does not converge to zero, indicating that* Y^i *is not diversifiable in this case.*

3.4 Illustration: pure endowment contract with profit

Consider a financial market consisting of a risk-free bank account and a risky stock market fund. We consider an insurer holding a portfolio consisting of N policyholders. We assume each policyholder has paid a premium P at the start of the contract which is fully invested in the risky stock market fund. The time-T price vector is then given by $\boldsymbol{Y} = \left(e^{rT}, Y^{(1)}\right)$. The time-0 market prices are given by $\left(1, y^{(1)}\right)$. Under the physical measure $\mathbb{P}$, we assume that the log returns of the risky stock market fund can be described by a normal distribution:

$$\log \frac{Y^{(1)}}{y^{(1)}} \sim \mathcal{N}\Big(\big(\mu_f - \tfrac{1}{2}(\sigma_f)^2\big)T, (\sigma_f)^2 T\Big). \tag{3.27}$$

Each policyholder will receive an amount of 1 at maturity T, provided this policyholder is alive. However, since the premium was invested in the risky fund, the investment might allow the insurer to share some of the profits with the policyholder. Hence, the insurer will payout $\alpha\left(Y^{(1)} - K\right)_+$ on top of the defined benefit of 1 at maturity. Therefore, the aggregate liability S is given by

$$S = \left(1 + \alpha\left(Y^{(1)} - K\right)_+\right) \times \sum_{i=1}^{N} X_i. \tag{3.28}$$

Here, $X_1, X_2, \ldots, X_N$ are identically distributed but not necessarily independent, Bernoulli random variables, where $X_i = 1$ if policyholder i survives to maturity T and $X_i = 0$ otherwise.

Conditionally on a common systematic longevity risk $Z \leq 0$, the X_i are i.i.d. with

$$\mathbb{P}\left(X_i = 1 \mid Z = z\right) = e^z. \tag{3.29}$$

The factor Z has a positive effect on the longevity in the portfolio in that if Z increases, the survival probabilities of the policyholders are increasing. If the dynamics of the longevity risk Z is assumed to follow an Ornstein–Uhlenbeck process, then Z follows a normal distribution. To ensure that e^z is a well-defined probability, we censor the normal distribution from above at 0 and take

$$\tilde{Z} \sim \mathcal{N}(\mu_s, \sigma_s^2), \quad Z = \begin{cases} \tilde{Z}, & \text{if } \tilde{Z} < 0, \\ 0, & \text{if } \tilde{Z} \geq 0, \end{cases} \tag{3.30}$$

and assume that $\log \frac{Y^{(1)}}{y^{(1)}}$ and $\tilde{Z}$ are jointly normal with correlation ρ. For more details on this type of product, we refer to Deelstra et al. (2020) and Linders (2023).

In the following proposition we give a closed-form expression for the mean-variance hedge of the claim S given by (3.28).

Proposition 3.2 *Consider the hybrid claim* (3.28). *The financial, systematic and actuarial risks are described by* (3.27), (3.30) *and* (3.29), *respectively. The mean-variance hedge* $\boldsymbol{\theta}_S^{MV} = \left(\theta^{(0)}, \theta^{(1)}\right)$ *in* (3.19) *is then given by*

$$\theta^{(1)} = \frac{\mathrm{Cov}^{\mathbb{P}}\left[Y^{(1)}, S\right]}{\mathrm{Var}^{\mathbb{P}}\left[Y^{(1)}\right]}, \tag{3.31}$$

$$\theta^{(0)} = e^{-rT}\left(\mathbb{E}^{\mathbb{P}}\left[S\right] - \theta^{(1)}\mathbb{E}^{\mathbb{P}}\left[Y^{(1)}\right]\right), \tag{3.32}$$

where

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[Y^{(1)}\right] &= y^{(1)}e^{\mu_f T}, && (3.33)\\
\mathrm{Var}^{\mathbb{P}}\left[Y^{(1)}\right] &= \left(y^{(1)}\right)^2 e^{2\mu_f T}\left(e^{\left(\sigma_f\right)^2 T}-1\right), && (3.34)\\
\mathbb{E}^{\mathbb{P}}\left[S\right] &= Ne^{\mu_s+\frac{1}{2}(\sigma_s)^2}\left[\Phi\left(-c-\sigma_s\right)-\alpha K\Phi_2\left(d_2+\rho\sigma_s,-c-\sigma_s,-\rho\right)\right]\\
&+ N\alpha y^{(1)}e^{\mu_f T+\mu_s+\frac{1}{2}(\sigma_s)^2+\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+\rho\sigma_s,-c-\rho\sigma_f\sqrt{T}-\sigma_s,-\rho\right)\\
&+ N\left(\Phi\left(c\right)-\alpha K\Phi_2\left(d_2,c,\rho\right)\right)\\
&+ N\alpha y^{(1)}e^{\mu_f T}\Phi_2\left(d_1,c+\rho\sigma_f\sqrt{T},\rho\right), && (3.35)\\
\mathbb{E}^{\mathbb{P}}\left[SY^{(1)}\right] &= Ny^{(1)}e^{\mu_f T+\mu_s+\frac{1}{2}(\sigma_s)^2+\rho\sigma_s\sigma_f\sqrt{T}}\Phi\left(-c-\sigma_s-\rho\sigma_f\sqrt{T}\right)\\
&- N\alpha K y^{(1)}e^{\mu_f T+\mu_s+\frac{1}{2}(\sigma_s)^2+\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+\rho\sigma_s,-c-\sigma_s-\rho\sigma_f\sqrt{T},-\rho\right)\\
&+ N\alpha\left(y^{(1)}\right)^2 e^{2\mu_f T+\left(\sigma_f\right)^2 T+\mu_s+\frac{1}{2}(\sigma_s)^2+2\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+\sigma_f\sqrt{T}+\rho\sigma_s,-c-\sigma_s-2\rho\sigma_f\sqrt{T},-\rho\right)\\
&+ Ny^{(1)}e^{\mu_f T}\left[\Phi\left(c+\rho\sigma_f\sqrt{T}\right)-\alpha K\Phi_2\left(d_1,c+\rho\sigma_f\sqrt{T},\rho\right)\right]\\
&+ N\alpha\left(y^{(1)}\right)^2 e^{2\mu_f T+\left(\sigma_f\right)^2 T}\Phi_2\left(d_1+\sigma_f\sqrt{T},c+2\rho\sigma_f\sqrt{T},\rho\right), && (3.36)\\
c &= \frac{\mu_s}{\sigma_s}, && (3.37)\\
d_1 &= \frac{\log y^{(1)}-\log K+\left(\mu_f+\frac{1}{2}\left(\sigma_f\right)^2\right)T}{\sigma_f\sqrt{T}}, && (3.38)\\
d_2 &= \frac{\log y^{(1)}-\log K+\left(\mu_f-\frac{1}{2}\left(\sigma_f\right)^2\right)T}{\sigma_f\sqrt{T}}, && (3.39)
\end{aligned}
$$

$\Phi(x)$ *denotes the CDF for the standard normal distribution, and* $\Phi_2(x,y,\tau)$ *is the CDF of a standard bivariate normal with correlation* τ*.*

The proof of the Proposition 3.2 is given in Appendix B. The following proposition provides a closed form expression for the random variables denoting the systematic parts of the hybrid liability. The proof can be found in Appendix C.

Proposition 3.3 *Consider the hybrid claim (3.28). The financial, systematic and actuarial risks are described by (3.27), (3.30) and (3.29), respectively. The financial systematic part* Y^s_{fin} *can be expressed as*

$$
\begin{aligned}
Y^s_{fin} &= N\left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]-\theta^{(0)}e^{rT}-\theta^{(1)}Y^{(1)} \qquad (3.40)\\
&= N\left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\left[\Phi\left(\frac{m\left(Y^{(1)}\right)}{s}\right)+e^{m\left(Y^{(1)}\right)+\frac{s^2}{2}}\Phi\left(\frac{-m\left(Y^{(1)}\right)}{s}-s\right)\right]\\
&\quad -\theta^{(0)}e^{rT}-\theta^{(1)}Y^{(1)},
\end{aligned}
$$

and the actuarial systematic part Y^s_{act} can be expressed as

$$
\begin{aligned}
Y^s_{act} &= N\left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\times\left(e^Z-\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right) &(3.41)\\
&= N\left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\left[e^Z-\Phi\left(\frac{m\left(Y^{(1)}\right)}{s}\right)-e^{m\left(Y^{(1)}\right)+\frac{s^2}{2}}\Phi\left(\frac{-m\left(Y^{(1)}\right)}{s}-s\right)\right],
\end{aligned}
$$

where

$$
m\left(Y^{(1)}\right) = \mu_s+\rho\sigma_s\frac{\log\frac{Y^{(1)}}{y^{(1)}}-\left(\mu_f-\frac{1}{2}\left(\sigma_f\right)^2\right)T}{\sigma_f\sqrt{T}}, \tag{3.42}
$$

$$
s = \sigma_s\sqrt{1-\rho^2}. \tag{3.43}
$$

The quantity e^Z is the realized survival probability, which is based on the particular longevity risk scenario that eventually unfolds. The conditional expectation $\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]$ corresponds with the expected survival probability, taking into account a particular scenario for the risky fund. Therefore, the difference between the two corresponds with the error that is caused because the survival probability turns out to deviate from our estimate, where we already took into account the financial information. The actuarial systematic part Y^s_{act} reflects the expected payout of the liability that cannot be explained by the information contained in the risky fund. The financial systematic part of the liability corresponds with what is left of the expected payoff of the claim in a given scenario for the risky fund after we use the payout of the hedge to cover the liability.

Proposition 3.4 *Consider the hybrid claim* (3.28)*. The financial, systematic and actuarial risks are described by* (3.27)*,* (3.30) *and* (3.29)*, respectively. Then, the idiosyncratic part Y^i is given by*

$$
Y^i = N\left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\times\left(\frac{\sum_{i=1}^N X_i}{N}-e^Z\right). \tag{3.44}
$$

Proof. It follows directly from (3.4) and Proposition 3.3 that Y^i is given by (3.44). ■

The idiosyncratic part measures the deviations of the experienced survival probability of the portfolio with the realization of the survival probability in a given systematic scenario.

For different portfolio sizes N and correlations ρ, we simulate $50,000$ paths with parameter values given in Table 1. The guaranteed amount at maturity is equal to 1 and we assume here that the single premium paid for this contract was fully invested in the risky fund. The current value of the equity fund is now $y^{(1)} = 0.7$. Note that this could be generalized to a situation where the insurer invests the premium in a portfolio consisting of bonds and the risky equity fund. A bonus will be paid out once the equity fund is above the guaranteed amount. Therefore, we put $K = 1$

We first consider a baseline scenario $\rho = 0$, $N = 500$, which corresponds to the case where the financial asset $Y^{(1)}$ is independent of the non-traded systematic risk Z. Figure 1 presents

μ_f	σ_f	T	μ_s	σ_s	K	α	$y^{(1)}$	r
0.03	0.25	10	-0.28	0.0876	1	0.5	0.70	0.02

Table 1: Simulation parameters

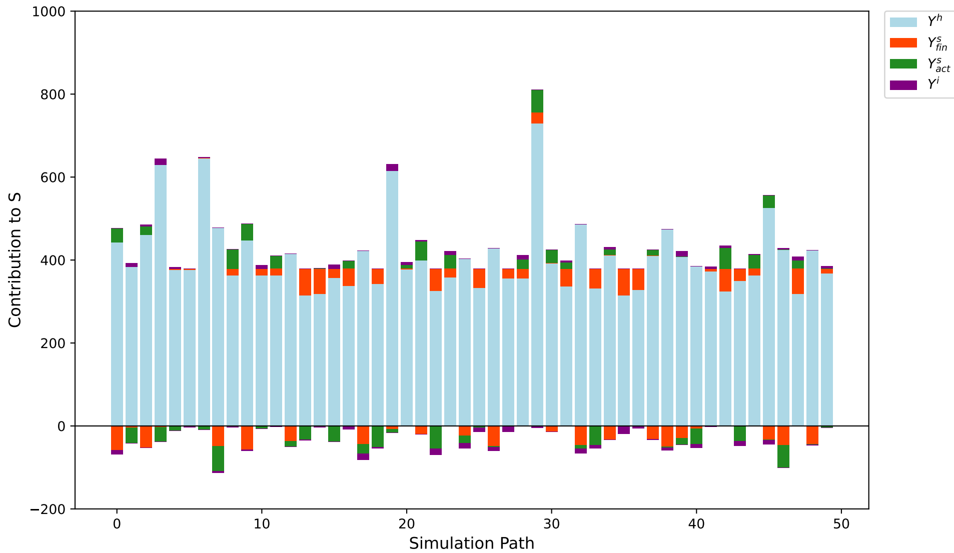


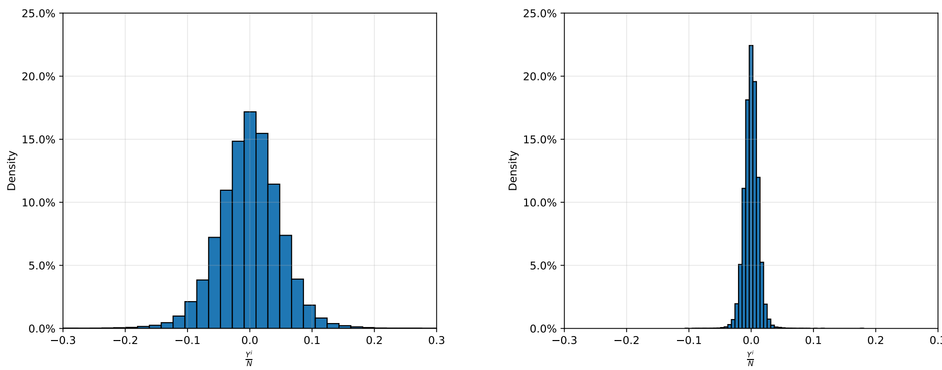
Figure 1: Decomposition of the simulated claim S across the first 50 paths ($N = 500$, $\rho = 0$): each bar is stacked by contributions from Y^h (hedgeable part), Y^s_{fin} (financial systematic part), Y^s_{act} (actuarial systematic part), and Y^i (idiosyncratic part).

a stacked bar plot illustrating how the different decomposition components contribute to the claim S.

Each bar in Figure 1 represents the liability in a randomly generated scenario. The blue segment shows the realization of the hedgeable part Y^h, which dominates almost all paths. By the mean–variance hedge construction we have $\mathbb{E}[Y^h] = \mathbb{E}[S]$, so on average Y^h fully matches the liability. The orange bars depict the financial systematic part Y^s_{fin}, which fluctuates above and below zero with roughly symmetric magnitude. Its mean is approximately zero because it represents the gap between the static hedge Y^h and the conditional expectation $\mathbb{E}\left[S \mid Y^{(1)}\right]$ It captures the 'level of the incompleteness' of the financial market. The green bars show the actuarial systematic part Y^s_{act}, which is even more tightly centered around zero. The expectation of Y^s_{act} is also close to 0, which captures the gap between the conditional expectation $\mathbb{E}^{\mathbb{P}}\left[S \mid Y^{(1)}, Z\right]$ and the conditional expectation $\mathbb{E}^{\mathbb{P}}\left[S \mid Y^{(1)}\right]$. Finally, the purple bars represent the idiosyncratic part, which is vanishingly small at this scale. It arises from individual survival randomness and, by diversification across a large portfolio, is nearly zero in every path.

3.4.1 Impact of N on the idiosyncratic part Y^i

Figure 2 presents the histograms of the idiosyncratic part per policy for a small portfolio size $N = 100$ and a large portfolio size $N = 2000$, under the assumption that financial and actuarial risks are independent. Note, however, that the policyholder-specific risks X_1, X_2, ... are dependent, since they all depend on the systematic risk Z. These two histograms demonstrate that the variance of $\frac{Y^i}{N}$ converges to zero as the portfolio size increases, hence the idiosyncratic part Y^i is clearly diversifiable.



(a) $N = 100$ (b) $N = 2000$

Figure 2: Histograms of the per-policy idiosyncratic component $\frac{Y^i}{N}$ for varying N

3.4.2 Impact of the correlation ρ

We begin by examining how the correlation ρ between the traded asset $Y^{(1)}$ and the non-tradable longevity risk (through the factor $\tilde{Z}$) affects the mean-variance hedge. Figure 3 plots the stock hedge ratio $\theta^{(1)}$ and the bank-account hedge ratio $\theta^{(0)}$ as functions of ρ. From the figure it is clear that $\theta^{(1)}$ is always positive and increases with ρ, which follows directly from (3.31) since a higher correlation raises $\mathrm{Cov}^{\mathbb{P}}\left[Y^{(1)}, S\right]$. Conversely, $\theta^{(0)}$ decreases as ρ increases, as seen from (3.32), $\theta^{(0)}$ decreases as ρ increases, as seen from (3.32): because the total expected payoff $E[S]$ is given in (3.35), any additional hedge allocated to the stock position $\theta^{(1)}E[Y^{(1)}]$ must be financed less through the bank account. In particular, $\mathrm{Cov}^{\mathbb{P}}\left[Y^{(1)}, S\right]$ remains strictly positive for all ρ, ensuring that $\theta^{(1)} > 0$ throughout.

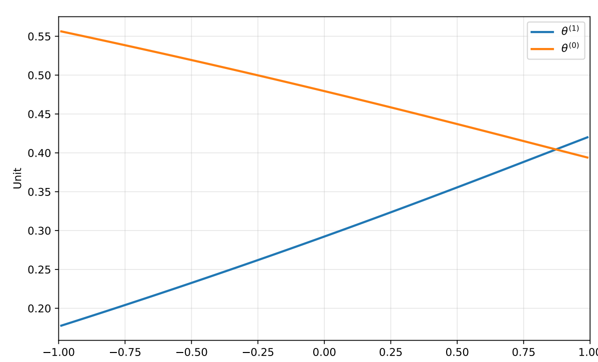


Figure 3: Mean–variance hedge ratios as functions of ρ.

Meanwhile, to assess how the correlation ρ affects the components that consists of the non-hedged part $S - Y^h$, Figure 4 is presented to illustrate the effect of the correlation on the variances of Y^s_{fin}, Y^s_{act}, and Y^i.

Figure 4 shows that as $|\rho|$ increases, the variance of the actuarial systematic part Y^s_{act} decreases, because the financial asset $Y^{(1)}$ contains more information about the systematic factor Z. Moreover, when $\rho = 1$ or $\rho = -1$, we can write

$$Z = a \log \frac{Y^{(1)}}{y^{(1)}} + b,$$

for some $a, b \in \mathbb{R}$. Taking into account that Z has mean μ_s and variance σ_s^2, we find that

$$Z = \begin{cases} \sigma_s \dfrac{\ln Y^{(1)} - \ln y^{(1)} - \left(\mu_f - \frac{1}{2}(\sigma_f)^2\right) T}{\sigma_f \sqrt{T}} + \mu_s, & \text{if } \rho = 1, \\ -\sigma_s \dfrac{\ln Y^{(1)} - \ln y^{(1)} - \left(\mu_f - \frac{1}{2}\sigma_f^2\right) T}{\sigma_f \sqrt{T}} + \mu_s, & \text{if } \rho = -1. \end{cases}$$

It follows from (3.41) that

$$Y^s_{act} = N\Big(1 + \alpha\,(Y^{(1)} - K)_+\Big)\left(e^Z - e^Z\right) = 0.$$

Therefore, the variance of the actuarial systematic part goes to zero as $|\rho| \to 1$. By contrast, the per-policy variance of the idiosyncratic part Y^i remains close to zero since we are holding a large portfolio and Theorem 3.6 showed that in this case, the variance should be close to zero as Y^i is diversified.

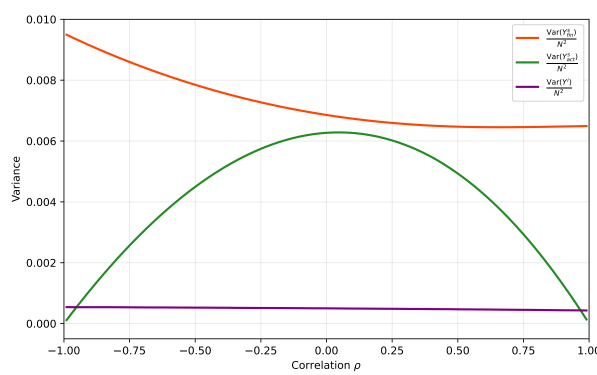


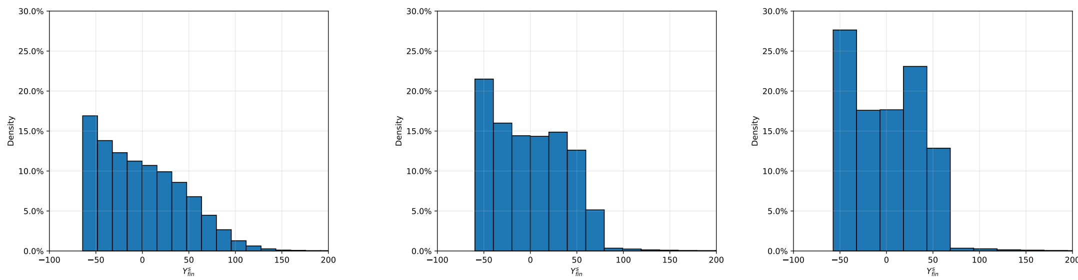
Figure 4: Per-policy ($N = 500$) variances of Y^s_{fin}, Y^s_{act}, and Y^i as functions of ρ.

We now focus on the financial systematic part Y^s_{fin}. From Figure 4 we find that the variance of the financial systematic part decreases as the correlation ρ increases. Intuitively, this means that the mean-variance hedge improves in capturing the financial risks of the hybrid claim as the correlation increases. Figure 5 presents its histograms for various values of the correlation ρ, and Table 2 reports the corresponding variance, skewness and excess kurtosis. In every case the distribution is clearly right-skewed, and the skewness increases with ρ. Likewise, the excess kurtosis is positive and grows with ρ, becoming particularly large at $\rho = 0.8$. Moreover, there is

ρ	Var(Y^s_{fin})	Skew(Y^s_{fin})	Kurt(Y^s_{fin})	Var(Y^s_{act})	Skew(Y^s_{act})	Kurt(Y^s_{act})
-0.8	2185.929	0.986	3.374	532.721	0.095	2.055
0.0	1709.578	1.164	10.133	1594.735	0.151	4.685
0.8	1612.502	1.928	29.350	616.501	-0.026	5.443

Table 2: Sample variance, skewness and excess kurtosis of Y^s_{fin} and Y^s_{act} for various values of ρ.

a finite lower bound for Y^s_{fin}, which occurs when the stock price $Y^{(1)}$ equals the strike K; this is evident in the curve of Y^s_{fin} versus $Y^{(1)}$ in Figure 6.



(a) $\rho = -0.8$ (b) $\rho = 0$ (c) $\rho = 0.8$

Figure 5: Histograms of the financial systematic part Y^s_{fin} for various values of ρ.

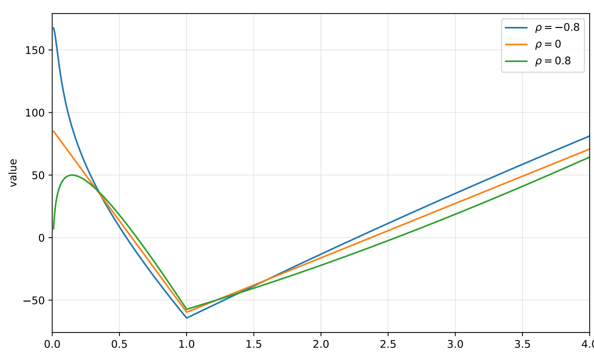
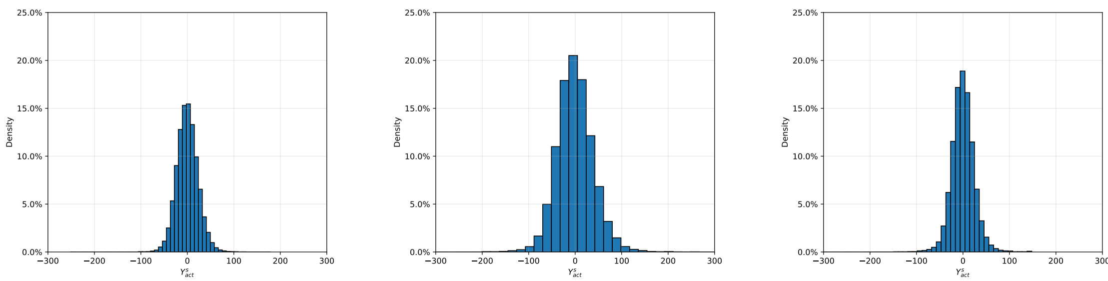


Figure 6: Y^s_{fin} as a function of the stock price $Y^{(1)}$ for various values of ρ

Subsequently, we turn to the actuarial systematic part Y^s_{act} and investigate how the correlation ρ affects Y^s_{act}. Figure 7 shows the histograms of Y^s_{act} for various ρ, and Table 2 lists the corresponding sample variance, skewness, and excess kurtosis. Across all correlation levels, Y^s_{act} remains approximately centered around zero, with skewness close to zero, while its excess kurtosis increases with ρ. This behavior is quite intuitive: as ρ grows, $Y^{(1)}$ and Z are more likely to move together, making Y^s_{res} given in (3.41) prone to more extreme outcomes.



(a) $\rho = -0.8$ (b) $\rho = 0$ (c) $\rho = 0.8$

Figure 7: Histograms of the actuarial systematic part Y^s_{act} for various values of ρ.

4 4-step market-consistent valuation

In the previous section, we considered decomposition formulas for general claims in the set $\mathcal{C}$. In this section, we consider the valuation of such claims. As in Dhaene et al. (2017), a valuation is a mapping $\rho : \mathcal{C} \to \mathbb{R}$, which is normalized

$$\rho[0] = 0,$$

and translation-invariant

$$\rho[S + c] = \rho[S] + e^{-rT}c, \text{ for any constant } c \in \mathbb{R}.$$

Solvency II requires that the valuation of liabilities should be market consistent.[1] The concept of market consistency was first introduced in Cont (2006) when pricing derivatives and further explored in an insurance context in Malamud et al. (2008).

Definition 4.1 (Market-Consistent Valuation) *A valuation $\rho[\cdot]$ is market consistent if for any claim $S \in \mathcal{C}$ and trading strategies $\boldsymbol{v}$, we have:*

$$\rho[S + \boldsymbol{v} \cdot \boldsymbol{Y}] = \rho[S] + \boldsymbol{v} \cdot \boldsymbol{y}. \tag{4.1}$$

Market consistency of a valuation requires that all hedgeable claims, i.e., all claims that can be expressed as linear combinations of the traded financial assets, be priced at their hedging costs. It also implies that in order to price a hedgeable claim, market consistency prescribes that only information from the financial market be used, see Cont (2006).

The actuarial approach to value claims is based on the best estimates and the risk margins. This approach may be preferred for orthogonal claims, which are claims independent of the financial market. We denote the set of orthogonal claims by $\mathcal{C}^{\perp}$. As in Dhaene (2022), we say that a valuation $\rho[\cdot]$ is model consistent if we have that

$$\rho[S] = \pi[S], \quad S \in \mathcal{C}^{\perp}, \tag{4.2}$$

[1]Solvency II requires that the valuation of liabilities be market-consistent, meaning that it should reflect current market information and be aligned with prices observed in deep, liquid, and transparent (DLT) markets.

where $\pi : \mathcal{C} \to \mathbb{R}$ is an actuarial valuation, i.e. a valuation which is law invariant under the probability measure $\mathbb{P}$. Therefore, model consistency of a valuation implies that there exists an actuarial valuation such that the model-consistent valuation coincides with this actuarial valuation on the set of orthogonal claims. This concept was first introduced in Dhaene et al. (2017).

Dhaene et al. (2017) introduce the class of hedge-based valuations. Moreover, it was shown that this class of valuations coincides with the class of valuations which are both market and model consistent, see also Dhaene (2022).

Definition 4.2 *A valuation $\rho[\cdot]$ is a hedge-based valuation if there exists a fair hedger $\boldsymbol{\theta}$ that is both model- and market-consistent, and a model-consistent valuation π such that*

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right]. \tag{4.3}$$

Hedge-based valuations start from a fair hedging strategy $\boldsymbol{\theta}_S$ for the claim S to determine the hedging cost. The remaining residual part is then priced using a model-consistent valuation. Note that the mean-variance hedger is both market- and model-consistent, see Dhaene (2022). However, the hedger $\boldsymbol{\theta}$ used in the hedge-based valuation does not necessarily need to be the mean-variance hedger.

In this section we consider the valuation of hybrid claims S defined in (3.24). We decompose this claim in four parts, where we use the mean-variance hedger to determine the hedgeable part in the decomposition. We then find:

$$S = \boldsymbol{\theta}_S^{MV} \cdot \boldsymbol{Y} + Y^i + Y^s_{fin} + Y^s_{act},$$

where Y^i, Y^s_{fin}, and Y^s_{act} are given by Expressions (3.4), (3.12), and (3.13), respectively. Then using the hedge-based valuation, we find that the valuation of the claim S can be expressed as follows

$$\rho[S] = \boldsymbol{\theta}_S^{MV} \cdot \boldsymbol{y} + \pi\left[Y^i + Y^s_{fin} + Y^s_{act}\right],$$

for some choice of the model-consistent valuation. Since the financial market is assumed to be arbitrage-free, any risk-neutral measure $\mathbb{Q}$ can be used to express the value of the hedgeable part $\boldsymbol{\theta}_S^{MV} \cdot \boldsymbol{y}$ as a discounted risk-neutral expectation. We can then also express the value of the liability as follows:

$$\rho[S] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[Y^h] + \pi\left[Y^i + Y^s_{fin} + Y^s_{act}\right],$$

where $\mathbb{Q}$ is a risk-neutral measure. Hence, market consistency states that we need to value the hedgeable part at its hedging cost. However, in the following examples we show different ways to value the systematic and idiosyncratic parts.

4.1 Example 1: valuation based on the standard-deviation principle

A first idea is to use an actuarial valuation to value the residual part. In this section, we use the standard-deviation principle as the model-consistent valuation π:

$$\pi[S] = e^{-rT}\left(\mathbb{E}^{\mathbb{P}}[S] + \lambda\sigma^{\mathbb{P}}[S]\right), \tag{4.4}$$

where $\lambda \geq 0$ is a risk loading. Note that model consistency only requires a $\mathbb{P}$-law invariance valuation for the orthogonal claims. However, the valuation π given by (4.4) is always $\mathbb{P}$-law invariant. The corresponding valuation is a particular example of a hedge-based valuation, as introduced in Dhaene et al. (2017) in a one-period setting, and further investigated in Barigou et al. (2019), Barigou and Dhaene (2019), and Chen et al. (2021) in a multi-period setting.

Proposition 4.1 *Consider the hedge-based valuation given in* (4.2)*, where the hedger θ is given by* (3.19)*, and the actuarial valuation π is given by* (4.4)*. The value of the insurance liability S given by Expression* (3.24) *can be expressed as follows:*

$$\rho^{MVSD}\left[S\right] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[Y^h] + \lambda e^{-rT}\sqrt{\mathrm{Var}^{\mathbb{P}}[Y^i] + \mathrm{Var}^{\mathbb{P}}[Y^s_{act}] + \mathrm{Var}^{\mathbb{P}}[Y^s_{fin}]}. \tag{4.5}$$

Proof. It follows from (3.16) and (3.10) that

$$\mathbb{E}^{\mathbb{P}}\left[Y^i\right] = \mathbb{E}^{\mathbb{P}}\left[Y^s\right] = 0.$$

From Lemma 3.1, we find that

$$\mathrm{Var}^{\mathbb{P}}\left[S - \boldsymbol{\theta}_S^{MV}\cdot \boldsymbol{Y}\right] = \mathrm{Var}^{\mathbb{P}}\left[Y^s_{act} + Y^s_{fin} + Y^i\right] = \mathrm{Var}^{\mathbb{P}}[Y^s_{act}] + \mathrm{Var}^{\mathbb{P}}[Y^s_{fin}] + \mathrm{Var}^{\mathbb{P}}[Y^i].$$

Therefore, it follows from (4.1) that $\rho^{MVSD}[S]$ is given by (4.5). ∎

A special case of the valuation in (4.5) is presented in Dhaene (2022, pp. 78–80), where a product claim is considered and decomposed into a hedgeable part, a diversifiable part, and a residual part. The risk Y^h is managed using an appropriate hedging strategy. The first term therefore represents the cost to set up this hedging strategy. Note that this cost is indifferent to the choice of the risk neutral measure. The systematic and idiosyncratic parts are managed using a capital buffer. Since we use the variance to build the capital buffer, this buffer is not affected by the dependence between the idiosyncratic and the systematic parts. Otherwise stated, to determine the capital buffer, we only need the distribution of the two systematic parts Y^s_{act} and Y^s_{fin} together with the distribution of the idiosyncratic part Y^i.

4.2 Example 2: valuation based on conditional standard-deviation principle

Generally speaking, actuarial valuation principles, such as the standard-deviation principle in (4.4), are $\mathbb{P}$-law invariant and based on the assumption that actuarial risks are diversifiable. Under the conditional independence assumption in (3.23), Theorem 3.6 demonstrates that the actuarial component Y^i becomes diversifiable as the number of policyholders increases. This justifies the application of an actuarial valuation to Y^i.

The valuation in (4.5) employs the standard-deviation principle to determine the capital buffer for both the systematic part Y^s and the diversifiable part Y^i. The systematic part Y^s consists again of two parts: a financial part, as defined in (3.12), and a actuarial systematic part, as defined in (3.13). Unlike Y^i, the systematic part Y^s is non-diversifiable, which means its per-policy variance for the insurer will not converge to zero if the number of policyholders increases.

Moreover, the financial part Y^s_{fin} reflects pure financial risk. Therefore, one may consider a different approach instead of using the standard-deviation principle to set the risk margin.

Observe that Y^s_{fin}, Y^s_{act}, and Y^i all depend on $\boldsymbol{Y}$. To more effectively manage traded financial risks, one can use a conditional valuation such that the standard-deviation principle is only used using a particular realization of the financial assets. To aggregate over all possible scenarios of the financial assets, we assume that a risk-neutral measure $\mathbb{Q}$ can be specified.

$$\pi[S] = e^{-rT}\left(\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}[S \mid \boldsymbol{Y}] + \lambda\sigma^{\mathbb{P}}[S \mid \boldsymbol{Y}]\right]\right), \tag{4.6}$$

where $\alpha \geq 0$ is a risk loading. The corresponding valuation of the claim S is then a two-step valuation as introduced in Pelsser and Stadje (2014).

Proposition 4.2 *Consider the hedge-based valuation given in* (4.2)*, where the hedger* θ *is given by* (3.19)*, and the model-consistent valuation* π *is given by* (4.6)*. The value of the insurance liability* S *given by Expression* (3.24) *can be expressed as follows:*

$$\rho^{TSSD}[S] = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] + \lambda e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{\mathrm{Var}^{\mathbb{P}}\left[Y^s_{act} + Y^i \mid \boldsymbol{Y}\right]}\right]. \tag{4.7}$$

Proof. It follows from (3.4), (3.12), and (3.13) that

$$\mathbb{E}^{\mathbb{P}}\left[S - Y^h \mid \boldsymbol{Y}\right] = Y^s_{fin}.$$

From (3.4) and (3.13), we get

$$\mathrm{Var}^{\mathbb{P}}\left[S - Y^h \mid \boldsymbol{Y}\right] = \mathrm{Var}^{\mathbb{P}}\left[Y^s_{act} + Y^i \mid \boldsymbol{Y}\right].$$

Therefore, we find that $\rho^{TSSD}[S]$ is given as in (4.7). ∎

Dhaene (2022, pp. 115–117) obtains two-step valuation of a product claim, in which the claim is decomposed into hedgeable, diversifiable, and residual parts. This framework combines conditional expectations and risk adjustments, offering a structured approach to risk management.

4.3 Example 3: Conic market-consistent valuation

Inspired by the examples in the previous two subsections, we will propose a new valuation in this subsection. The mean-variance hedge based valuation (4.5) has the interesting property that the risk margin is composed of marginal variances. Indeed, since we showed in Proposition 3.1 that our claim can be decomposed in four uncorrelated parts, building a risk margin based on variances will not include the covariances. The two-step standard-deviation principle has the advantage that the systematic financial part is valued under a financial valuation rather than a real-world valuation. However, the value of this systematic financial part depends on the choice of the $\mathbb{Q}$ measure. Moreover, the risk margin is not composed of marginal variances. Therefore, in this section, we propose a valuation that takes a prudent approach when employing a financial valuation for the systematic residual part, and that determines the risk margin using only marginal variances.

Assume that we have a claim $S \in \mathcal{C}$. Then the best approximation in mean-variance sense of the claim S in the set $\mathcal{C}^{\boldsymbol{Y}}$ can be determined as follows:

$$Y^f(S) = \underset{\xi \in \mathcal{C}^{\boldsymbol{Y}}}{\arg\min}\, \mathbb{E}\left[(S-\xi)^2\right].$$

Note that Y^f depends solely on the realization of the traded assets $\boldsymbol{Y}$, but it is not necessarily a linear combination of the financial assets. Therefore, this claim Y^f can be interpreted as a financial derivative. Since the market is assumed to be incomplete the price of Y^f may not be unique. A counter party therefore may accept to provide the payoff Y^f in return for a conservative price. To take into account the incompleteness of the market, we therefore determine the price using a supremum over a set $\mathcal{P}$ of possible risk-neutral measures $\mathbb{Q}$. Therefore, the price of the financial part of the hybrid claim S is determined as follows:

$$\text{Price of } Y^f(S) = e^{-rT} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}\left[Y^f\right],$$

for a given set $\mathcal{P}$. This idea is similar to the bid-ask pricing that is applied in conic finance; see, e.g., Madan and Cherny (2010) and Madan and Schoutens (2016). The claim Y^f is a financial claim, but not necessarily hedgeable. However, the 'market' is considered as a counterparty that is willing to accept Y^f, but only in return for a price that makes the cashflow acceptable for the 'market'.

We can now define a new model-consistent valuation:

$$\pi[S] = e^{-rT} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}\left[Y^f\right] + \mathrm{e}^{-rT}\left(\mathbb{E}^{\mathbb{P}}\left[S - Y^f\right] + \lambda \sigma^{\mathbb{P}}\left[S - Y^f\right]\right). \tag{4.8}$$

For an orthogonal claim, the model-consistent valuation π defined in (4.8) coincides with the standard-deviation principle (4.4), and hence π is indeed model consistent. The model-consistent valuation is used in the hedge-based valuation (see Definition 4.2) to determine a price for the part of the claim that cannot be hedged. The model-consistent valuation we propose in (4.8) valuates the financial part by using a non-linear financial valuation to take into account the incompleteness of the market, whereas the remaining part is valuated using a standard-deviation principle.

Theorem 4.1 *Consider the hedge-based valuation given in* (4.2) *where the hedger θ is given by* (3.19)*, and the model-consistent valuation π is given by* (4.8)*. The value of the insurance liability S given by* (3.24) *can be expressed as follows:*

$$\rho^{CMC}[S] = e^{-rT} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] + \lambda e^{-rT}\sqrt{\mathrm{Var}^{\mathbb{P}}\left[Y^i\right] + \mathrm{Var}^{\mathbb{P}}\left[Y^s_{act}\right]}, \tag{4.9}$$

where Y^h is determined using (3.1) *with a fair hedger $\boldsymbol{\theta}$, and Y^i, Y^s_{act}, and Y^s_{fin} are given by* (3.4), (3.13), *and* (3.12)*, respectively. Then, $\rho^{CMC}[\cdot]$ is a fair valuation.*

Proof. If we use the mean-variance hedger, the valuation of the claim S is given by

$$\rho^{CMC}[S] = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[Y^h\right] + \pi\left[S - Y^h\right],$$

with π given by (4.8). One can show that

$$Y^s_{fin} = \underset{\xi \in \mathcal{C}^{\boldsymbol{Y}}}{\arg\min}\, \mathbb{E}\left[\left(S - Y^h - \xi\right)^2\right].$$

Note that $S - Y^h - Y^s_{fin} = Y^i + Y^s_{act}$ and $\mathbb{E}^{\mathbb{P}}\left[Y^i + Y^s_{act}\right] = 0$. Taking into account that Y^i and Y^s_{act} are uncorrelated, we find the desired result.

∎

4.4 Illustration: 4-step market-consistent valuation for the pure endowment contract with profit

Building on the example from Section 3.4, we now employ the 4-step market-consistent valuation framework outlined above to value a pure endowment contract with profit. We still consider the aggregate liability S as defined in (3.28). Closed-form expressions exist for the MSVD valuation in (4.5) of S, as shown in Appendix D.1. It is important to note that this valuation is independent of the specific choice of the risk-neutral measure. Both the two-step valuation and the conic market-consistent valuation involve calculating the risk-neutral expectation of the financial component of the hybrid claim. However, since the market is incomplete, multiple risk-neutral measures may exist. The financial model introduced in (3.27) depends on two parameters. Because the martingale condition must hold for each risk-neutral measure $\mathbb{Q}$, we can characterize each $\mathbb{Q}$ by its risk-neutral volatility parameter $\sigma_{\mathbb{Q}}$.

When $\rho = 0$, it is straightforward to verify that $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ increases as $\sigma_{\mathbb{Q}}$ increases and approaches a constant limit $N\left(e^{\mu_s + \frac{1}{2}(\sigma_s)^2}\Phi(-c-\sigma_s) + \Phi(c)\right)\left(1 + \alpha y^{(1)} e^{rT}\right)$ as $\sigma_{\mathbb{Q}} \longrightarrow \infty$. If $\rho \neq 0$, using Proposition D.1 and the parameters in Table 1, Figure 8 plots $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ as a function of $\sigma_{\mathbb{Q}}$ for different values of ρ. From Figure 8, we find that for non-zero correlation,

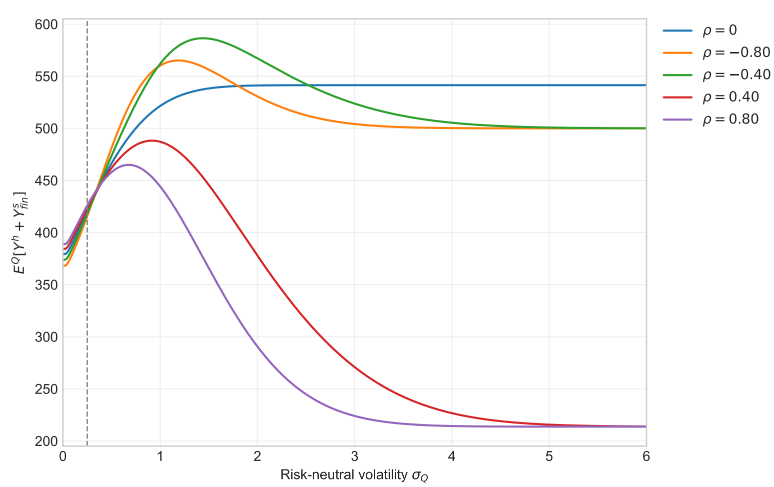


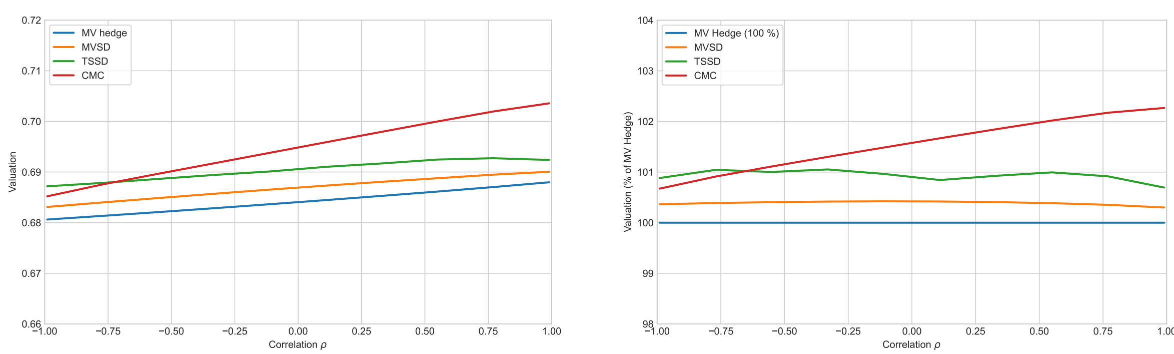
Figure 8: $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ as functions of $\sigma_{\mathbb{Q}}$ for various values of ρ ($N = 500$), the dashed line corresponds with $\sigma_{\mathbb{Q}} = \sigma_f$.

the risk-neutral expectation $\mathbb{E}^{\mathbb{Q}}\big[Y^h + Y^s_{fin}\big]$ initially increases in $\sigma_{\mathbb{Q}}$, attains a unique maximum at an interior point $\sigma_{\mathbb{Q}^*}$, and then decreases as $\sigma_{\mathbb{Q}}$ grows further. In contrast, when $\rho = 0$, $\mathbb{E}^{\mathbb{Q}}\big[Y^h + Y^s_{fin}\big]$ increases monotonically and approaches a constant limit as $\sigma_{\mathbb{Q}} \longrightarrow \infty$, so there is no finite interior maximizer. Consequently, if $\sigma_{\mathbb{Q}}$ is restricted to an interval Σ, the maximum value is achieved at the interior point $\sigma_{\mathbb{Q}^*}$ when $\rho \neq 0$ and $\sigma_{Q^*} \in \Sigma$, or otherwise at the right endpoint of the interval Σ that yields the larger expectation; for $\rho = 0$, the worst-case expectation always occurs at the upper endpoint, or at the asymptote if Σ is unbounded above. Note also that for relatively small values of $\sigma_{\mathbb{Q}}$, the risk-neutral expectation $\mathbb{E}^{\mathbb{Q}}\big[Y^h + Y^s_{fin}\big]$ increases as a function of the correlation, whereas the reverse relationship holds for large volatilities $\sigma_{\mathbb{Q}}$.

For the conditional standard-deviation principle given in (4.6) we choose the risk-neutral measure with $\sigma_{\mathbb{Q}} = \sigma_f$, under which the time-$T$ distribution of the stock price matches that of the classic geometric Brownian motion model. The TSSD valuation in (4.7) of S also admits a closed-form expression, as shown in Appendix D.2. In order to determine the conic market-consistent value (4.9), we employ the closed-form expressions for the variance $\mathrm{Var}\left[Y^i + Y^s_{act}\right]$ derived in Appendix D.3 together with the closed-form expression for $\mathbb{E}^{\mathbb{Q}}\big[Y^h + Y^s_{fin}\big]$, but now for various choices of $\sigma_{\mathbb{Q}}$ in order to determine the supremum.

4.4.1 Comparison of 4-step market-consistent valuation principles

Using the parameters in Table 1, we fix the risk loading at $\lambda = 0.3$ for all three valuation principles: the standard-deviation principle (4.4), the conditional expectation principle (4.6), and the conic market-consistent valuation (4.8). To compare these valuations, we take the time-0 mean-variance hedge value as a benchmark and consider the per-policy value $\frac{\rho[S]}{N}$. Under the conditional standard-deviation principle we take $\mathbb{Q}$ with $\sigma_{\mathbb{Q}} = \sigma_f$, while under the new market-consistent principle we restrict $\sigma_{\mathbb{Q}}$ to the interval $\Sigma = (95\%\sigma_f, 105\%\sigma_f)$.



(a) (b)

Figure 9: Comparison of 4-step market-consistent valuations (MVSD, TSSD, CMC) across correlation ρ: the left panel (a) shows raw valuation levels, and the right panel (b) shows each valuation normalized to the MV hedge (100%).

From Figure 9(a), all three market-consistent valuations (MVSD, TSSD, and CMC) increase as the correlation ρ rises. A larger ρ strengthens the hedge by increasing the covariance between the stock $Y^{(1)}$ and the systematic factor Z, which raises the mean variance hedge value. Equivalently, as ρ grows, asset and liability co-move more closely, making larger liability realizations more likely and, therefore, increasing the expected payoff. Since the mean-variance hedge has the same expectation as the liability, increasing the correlation will drive up each valuation.

The difference between the value of the mean-variance hedge (blue line) and the value under the different valuations represents the capital buffer that is required for holding non-hedged liabilities. We find that MVSD consistently requires the smallest capital buffer, with TSSD only marginally higher. CMC starts below TSSD at negative ρ but overtakes both MVSD and TSSD as ρ increases. Moreover, under MVSD and TSSD the buffer for the unhedgeable part remains nearly constant across ρ, whereas under CMC it increases with ρ.

The conic market-consistent valuation increases as the correlation ρ increases. This behavior can be observed in Figure 8, where the interval Σ is chosen such that we remain on the left side of the plot. In this region, the expectation $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ consistently increases with ρ. However, if the size of the interval Σ were sufficiently enlarged, the conic market-consistent valuation would drastically increase, but would then become a decreasing function of the correlation.

Recall from Figure 5 and Table 2 that the skewness and kurtosis of the financial systematic part Y^s_{fin} increase with ρ, and can become quite large for high ρ. Figure 9(b) shows each valuation as a percentage of the MV hedge (100%). Under MVSD and TSSD, this buffer ratio stays flat or even declines at high ρ. In contrast, the CMC ratio increases steadily with ρ, indicating that this method can allocate a larger capital buffer to the unhedgeable part in response to stronger co-movement between the financial asset and the aggregate liability.

5 Applications of 4-step decomposition for product claims

In this section, we focus on a specific setting where the claims are product claims, i.e. we specify the function h_i in Expression (3.24) to be the product of a financial part and an actuarial part. To simplify the analysis, we assume that the N policyholders select their payoff functions f_i from a finite set of m distinct functions, where $m \leq N$. In this setting, the policyholders are grouped such that all individuals within the same group share the same payoff function. Let $f_j(\boldsymbol{Y})$ denote the payoff function for group $j \in \{1, 2, \ldots, m\}$, and let N_j represent the number of policyholders in group j, satisfying: $\sum_{j=1}^{m} N_j = N$.

Using this grouping, the claim for the k-th policyholder in group j can be expressed as $f_j(\boldsymbol{Y}) \times X_{jk}$, where X_{jk} represents the policyholder-specific variable for the k-th policyholder in group j. The aggregate claim S can now be expressed as follows:

$$S = \sum_{j=1}^{m} \sum_{k=1}^{N_j} f_j(\boldsymbol{Y}) \times X_{jk}. \tag{5.1}$$

For simplicity, we will sometimes use the shorthand notation f_j to denote $f_j(\boldsymbol{Y})$.

We also assume that the systematic risk vector $\boldsymbol{Z}$ is independent of the financial assets $\boldsymbol{Y}$. Conditional on $\boldsymbol{Z} = \boldsymbol{z}$, the actuarial risks X_{jk}, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N_j$, are independent and identically distributed. Moreover, given that $\boldsymbol{Z} = \boldsymbol{z}$, these risks are conditionally independent of the financial assets, i.e., $X_{jk} \perp \boldsymbol{Y} \,\big|\, \boldsymbol{Z} = \boldsymbol{z}$.

The mean-variance hedger will be used to determine the hedgeable part of the claim. We will then use Expressions (3.1), (3.4), (3.13), and (3.12) to decompose the aggregate claim S as follows:

$$S = Y^h + Y^i + Y^s_{act} + Y^s_{fin},$$

where

$$\begin{aligned} Y^h &= \sum_{j=1}^{m} \boldsymbol{\theta}^{MV}_{f_j} \cdot \boldsymbol{Y} \times N_j \mathbb{E}[X_1], \\ Y^i &= \sum_{j=1}^{m} \sum_{k=1}^{N_j} f_j \left(X_{jk} - \mathbb{E}[X_1 | \boldsymbol{Z}] \right), \\ Y^s_{fin} &= \sum_{j=1}^{m} \Big(f_j - \boldsymbol{\theta}^{MV}_{f_j} \cdot \boldsymbol{Y} \Big) N_j \mathbb{E}\left[X_1\right], \\ Y^s_{act} &= \sum_{j=1}^{m} N_j f_j \left(\mathbb{E}\left[X_1 | \boldsymbol{Z}\right] - \mathbb{E}\left[X_1\right] \right). \end{aligned}$$

This decomposition generalizes the special case considered in Example 3.1, where the setting involves a single payoff function ($m = 1$). If we use the mean–variance hedge to determine which part of the hybrid claim will be covered using a hedging strategy, the systematic financial component Y^s_{fin} is driven by the difference between the (non-linear) financial payoff f_j and its hedge $\boldsymbol{\theta}^{MV}_{f_j} \cdot \boldsymbol{Y}$. This difference represents the unhedgeable portion of the financial risk. As the hedging strategy becomes more accurate, the systematic component decreases. However, in an incomplete market, selling non-linear payoff functions that cannot be perfectly replicated comes with a risk, captured by the residual financial part. The actuarial systematic part Y^s_{act} considers the deviations of the expectation of the actuarial risk in a given systematic scenario, from its unconditional counterpart, i.e. aggregated over all possible systematic scenarios.

We can use this decomposition to determine the value of the product claim (5.1). We start by using Expression (4.5) for the mean-variance standard deviation principle. The per-policy MVSD value of the claim S defined in (5.1) can then be expressed as:

$$\frac{\rho^{MVHB}[S]}{N} = \frac{e^{-rT}}{N} \sum_{j=1}^{m} \boldsymbol{\theta}^{MV}_{f_j} \cdot \boldsymbol{Y}_0 \times N_j \mathbb{E}[X_1] + \frac{\lambda e^{-rT}}{N} \sqrt{A + B + C}, \tag{5.2}$$

where

$$A = \sum_{j=1}^{m} N_j \mathbb{E}^{\mathbb{P}}\left[f_j(\boldsymbol{Y})^2\right]\mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}^{\mathbb{P}}\left[X_1 \mid \boldsymbol{Z}\right]\right], \tag{5.3}$$

$$B = \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{j=1}^{m} f_j(\boldsymbol{Y})N_j\right)^2\right]\mathrm{Var}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X_1 \mid \boldsymbol{Z}\right]\right], \tag{5.4}$$

$$C = \mathrm{Var}^{\mathbb{P}}\left[\sum_{j=1}^{m}\left(f_j - \boldsymbol{\theta}_{f_j}^{MV}\cdot\boldsymbol{Y}\right)N_j\mathbb{E}[X_1]\right]. \tag{5.5}$$

The first term in (5.2) represents the cost of the mean-variance hedge. The financial payoff f_j is hedged based on the expected number of financial payouts. The term A represents the aggregate capital buffer required to cover the idiosyncratic part of the hybrid claim. This term grows linearly with the number of policyholders, hence it will vanish in the per-policy case. The term B takes into account the fluctuations of the conditional expectation, and therefore corresponds with the aggregate capital buffer to cover the actuarial systematic part. Finally, term C quantifies the unhedgeable financial risk, representing the systematic financial part.

We also consider the valuation under the two-step standard deviation principle using Expression (4.7). Assume we fix a pricing measure $\mathbb{Q}$, then the per-policy TSSD value of the aggregate claim S can then be expressed as

$$\frac{\rho^{TS}[S]}{N} = \frac{e^{-rT}}{N}\sum_{j=1}^{m} N_j\mathbb{E}^{\mathbb{P}}\left[X_1\right]\cdot\mathbb{E}^{\mathbb{Q}}\left[f_j(\boldsymbol{Y})\right] + \frac{\lambda e^{-rT}}{N}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{D+E}\right], \tag{5.6}$$

where

$$D = \sum_{j=1}^{m} N_j f_j(\boldsymbol{Y})^2\mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}^{\mathbb{P}}\left[X_1 \mid \boldsymbol{Z}\right]\right],$$

$$E = \left(\sum_{j=1}^{m} f_j(\boldsymbol{Y})N_j\right)^2\mathrm{Var}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X_1 \mid \boldsymbol{Z}\right]\right].$$

The term $\mathbb{E}^{\mathbb{Q}}\left[f_j(\boldsymbol{Y})\right]$ represents the price of the financial payoff under the chosen pricing measure $\mathbb{Q}$. Note that in an incomplete market, a trading strategy to replicate this payoff may not exist. The terms D and E are used to determine the capital buffers for the idiosyncratic and actuarial systematic parts, respectively. Note that both D and E are random variables which depend on the realization of the financial risks $\boldsymbol{Y}$.

Lastly, we also determine the value of the hybrid liability using the conic market-consistent valuation by Expression (4.9). The per-policy value of the aggregate claim S can then be expressed as

$$\frac{\rho[S]}{N} = \frac{e^{-rT}}{N}\sup_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{m} N_j\mathbb{E}^{\mathbb{P}}\left[X_1\right]\cdot f_j(\boldsymbol{Y})\right] + \frac{\lambda e^{-rT}}{N}\sqrt{A+B}, \tag{5.7}$$

where A and B are given by (5.3) and (5.4), respectively. Note that the first term of Expression (5.7) represents the hedging of the financial payoff, taking into account the uncertainty about the pricing measure. The capital buffer to account for the non-financial risks is then similar to the valuation derived in (5.2).

6 Conclusion

In this paper, we developed a comprehensive framework for managing hybrid liabilities that intertwine financial and actuarial risks, addressing the challenges of disentangling, pricing, and mitigating these complex risks. We propose a four-step decomposition of liabilities into hedgeable financial risks, diversifiable actuarial risks, non-hedged residual financial risks, and non-diversifiable systematic risk. In this way, we shed light on the interplay between financial and actuarial markets. Our framework incorporates correlations between these markets and accounts for heterogeneity in policyholder-specific risks, making it applicable to a wide range of financial and insurance products.

The key contribution of this paper lies in its ability to bridge the gap between actuarial and financial valuation theories, offering a market- and model-consistent valuation framework that is both practical and robust. Such a valuation framework is called "fair" in Dhaene et al. (2017). This framework enables insurers, pension funds, and financial institutions to manage hybrid liabilities more effectively, ensuring accurate pricing, improved risk mitigation, and regulatory compliance. The provided examples illustrate the applicability of our approach in a real-world contract, highlighting its versatility and utility.

Future research could explore further refinements to the framework, such as incorporating additional layers of risk or adapting it to emerging financial instruments. Additionally, empirical studies could validate the framework's effectiveness in different market conditions and regulatory environments. Ultimately, this paper lays a foundation for the management of hybrid liabilities, contributing to the ongoing evolution of risk management practices in the financial and insurance sectors.

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Appendix

A Non-diversifiable Y^i of a catastrophe liability S

Consider an insurance company with N policyholders, each having an individual loss X_i. The aggregate claim is given by $S = \sum_{i=1}^{N} X_i$, We assume there is a single systematic risk factor, denoted by Z. The random variable Z can be interpreted as a catastrophe risk for a region, defined as

$$Z = \begin{cases} 1, & \text{if a catastrophe occurs;} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbb{P}[Z = 1] = p$, with $p \in (0, 1)$. Additionally, for $i = 1, 2, \ldots, N$, we assume that $X_i|Z$ is independent of $\boldsymbol{Y}|Z$. Conditioned on $Z = 1$, we have

$$X_i|Z = 1 \stackrel{d}{=} \begin{cases} 0, & \text{with probability } 1 - p_1; \\ X_{LN}^1, & \text{with probability } p_1, \end{cases}$$

where $X_{LN}^1 \sim \text{Lognormal}(\mu_1, \sigma_1)$. Moreover, for any $i \neq j$, $\text{Corr}\left[X_i, X_j \mid Z = 1\right] = \rho$, where $\rho > 0$. Alternatively, conditioned on $Z = 0$, we have

$$X_i|Z = 0 \stackrel{d}{=} \begin{cases} 0, & \text{with probability } 1 - p_0 \\ X_{LN}^0, & \text{with probability } p_0, \end{cases}$$

where $X^1_{LN} \sim \text{Lognormal}(\mu_0, \sigma_0)$, with $\mu_0 < \mu_1$ and $p_0 < p_1$. Furthermore, $X_i|Z = 0$ is independent of $X_j|Z = 0$, for any $i \neq j$.

Note that, in this case, although $X_i|\boldsymbol{Y}, Z = 0$ is independent of $X_j|\boldsymbol{Y}, Z = 0$ for $i \neq j$, conditional independence does not hold in general, as $X_i|\boldsymbol{Y}, Z = 1$ is positively correlated with $X_j|\boldsymbol{Y}, Z = 1$. We can find that

$$
\begin{aligned}
\text{Var}\left[X_i \mid \boldsymbol{Y}, Z = 1\right] &= p_1 \times e^{2\mu_1 + 2\sigma_1^2} - p_1^2 \times e^{2\mu_1 + \sigma_1^2}, \\
\text{Var}\left[X_i \mid \boldsymbol{Y}, Z = 0\right] &= p_0 \times e^{2\mu_0 + 2\sigma_0^2} - p_0^2 \times e^{2\mu_0 + \sigma_0^2}.
\end{aligned}
$$

Hence, $\mathbb{E}\left[\text{Var}[X_i \mid \boldsymbol{Y}, Z]\right] < \infty$, and it is given by

$$
\mathbb{E}\left[\text{Var}[X_i \mid \boldsymbol{Y}, Z]\right] = p \times p_1 \times e^{2\mu_1 + \sigma_1^2} \times (e^{\sigma_1^2} - p_1) + (1 - p) \times p_0 \times e^{2\mu_0 + \sigma_0^2} \times (e^{\sigma_0^2} - p_0).
$$

Additionally, we have that

$$
\begin{aligned}
\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z = 1\right] &= \rho \times \left(p_1 e^{2\mu_1 + 2\sigma_1^2} - p_1^2 e^{2\mu_1 + \sigma_1^2}\right), \\
\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z = 0\right] &= 0.
\end{aligned}
$$

Then $\mathbb{E}\left[\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z\right]\right] < \infty$, and it is given by

$$
\mathbb{E}\left[\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z\right]\right] = p \times \rho \times p_1 e^{2\mu_1 + \sigma_1^2} \times (e^{\sigma_1^2} - p_1).
$$

Therefore, $\text{Var}\left[\frac{Y^i}{N}\right]$ in this case is given by

$$
\begin{aligned}
\text{Var}\left[\frac{Y^i}{N}\right] &= \frac{N \times \mathbb{E}\left[\text{Var}[X_i \mid \boldsymbol{Y}, Z]\right] + N \times (N - 1) \times \mathbb{E}\left[\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z\right]\right]}{N^2} \\
&= \frac{\mathbb{E}\left[\text{Var}[X_i \mid \boldsymbol{Y}, Z]\right] + (N - 1) \times \mathbb{E}\left[\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z\right]\right]}{N}.
\end{aligned}
$$

As $N \to \infty$, it is straightforward to see

$$
\lim_{N \to \infty} \text{Var}\left[\frac{Y^i}{N}\right] = \mathbb{E}\left[\text{Cov}\left[X_i, X_j \mid \boldsymbol{Y}, Z\right]\right] = p \times \rho \times p_1 e^{2\mu_1 + \sigma_1^2} \times (e^{\sigma_1^2} - p_1).
$$

Since $\text{Var}\left[\frac{Y^i}{N}\right]$ does not approach 0 as $N \to \infty$, it follows that $\frac{Y^i}{N}$ does not converge to 0 in probability.

B Proof of Proposition 3.2

From (3.27) and (3.28) we immediately obtain $\mathbb{E}^{\mathbb{P}}[Y^{(1)}]$ and $\text{Var}^{\mathbb{P}}[Y^{(1)}]$ as given in (3.33) and (3.34). Set $X = \log Y^{(1)}$ with

$$
X \sim \mathcal{N}\left(\log y^{(1)} + \left(\mu_f - \tfrac{1}{2}(\sigma_f)^2\right)T, (\sigma_f)^2 T\right).
$$

Thus $(X, \tilde{Z})$ is bivariate normal with

$$\boldsymbol{\mu} = \begin{bmatrix} \log y^{(1)} + \left(\mu_f - \frac{1}{2}(\sigma_f)^2\right)T \\ \mu_s \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} (\sigma_f)^2 T & \rho\,\sigma_f\,\sigma_s\sqrt{T} \\ \rho\,\sigma_f\,\sigma_s\sqrt{T} & (\sigma_s)^2 \end{bmatrix}.$$

Using definition (3.28),

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[S] = N\int_{-\infty}^{0}\int_{-\infty}^{\infty} e^z\, f_{X,\tilde{Z}}(x,z)\,\mathrm{d}x\mathrm{d}z + N\alpha\int_{-\infty}^{0}\int_{\log K}^{\infty}(e^x - K)\,e^z\, f_{X,\tilde{Z}}(x,z)\,\mathrm{d}x\mathrm{d}z \\ + N\int_{0}^{\infty}\int_{-\infty}^{\infty} f_{X,\tilde{Z}}(x,z)\,\mathrm{d}x\mathrm{d}z + N\alpha\int_{0}^{\infty}\int_{\log K}^{\infty}(e^x - K)\; f_{X,\tilde{Z}}(x,z)\,\mathrm{d}x\mathrm{d}z.\end{aligned}$$

Consider

$$I = \int_{-\infty}^{0}\int_{\log K}^{\infty} e^{x+z}\, f_{X,\tilde{Z}}(x,z)\,\mathrm{d}x\mathrm{d}z.$$

Let $\boldsymbol{x} = (x, z)^\top$ and $\boldsymbol{L}^\top = (1, 1)$. Then we have that,

$$e^{\boldsymbol{L}^\top\boldsymbol{x}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} = e^{\boldsymbol{L}^\top\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{L}^\top\boldsymbol{\Sigma}\boldsymbol{L}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}^*)^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}^*)},$$

where

$$\boldsymbol{\mu}^* = \boldsymbol{\mu} + \boldsymbol{\Sigma}\boldsymbol{L} = \begin{bmatrix} \log y^{(1)} + \left(\mu_f + \frac{1}{2}(\sigma_f)^2\right)T + \rho\,\sigma_f\,\sigma_s\sqrt{T} \\ \mu_s + (\sigma_s)^2 + \rho\,\sigma_f\,\sigma_s\sqrt{T} \end{bmatrix}.$$

Hence

$$I = e^{\boldsymbol{L}^\top\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{L}^\top\boldsymbol{\Sigma}\boldsymbol{L}}\mathbb{P}\left(X^* \geq \log K, \tilde{Z}^* < 0\right),$$

with $(X^*, \tilde{Z}^*) \sim \mathcal{N}(\boldsymbol{\mu}^*, \boldsymbol{\Sigma})$. Using d_1 and c from (3.38) and (3.37),

$$I = y^{(1)} e^{\mu_f T + \mu_s + \frac{1}{2}(\sigma_s)^2 + \rho\sigma_f\sigma_s\sqrt{T}}\,\Phi_2\left(d_1 + \rho\sigma_s, -c - \rho\sigma_f\sqrt{T} - \sigma_s, -\rho\right),$$

where Φ_2 is the standard bivariate normal CDF.

Applying the same steps to the remaining integrals yields (3.35) for $\mathbb{E}^{\mathbb{P}}[S]$ and (3.36) for $\mathbb{E}^{\mathbb{P}}[S\,Y^{(1)}]$. Therefore, by substituting (3.33), (3.34), (3.35), and (3.36) in (3.31) and (3.32), we can derive the closed-form expressions for $\boldsymbol{\theta}_S^{MV}$.

C Proof of Proposition 3.3

It follows directly from (3.12) that

$$\begin{aligned} Y^s_{fin} &= \mathbb{E}^{\mathbb{P}}\left[\left(1 + \alpha\left(Y^{(1)} - K\right)_+\right)\sum_{i=1}^{N} X_i \,\middle|\, Y^{(1)}\right] - \theta^{(0)}e^{rT} - \theta^{(1)}Y^{(1)} \\ &= \left(1 + \alpha\left(Y^{(1)} - K\right)_+\right)\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N} X_i \,\middle|\, Y^{(1)}, Z\right] \,\middle|\, Y^{(1)}\right] - \theta^{(0)}e^{rT} - \theta^{(1)}Y^{(1)} \\ &= N\left(1 + \alpha\left(Y^{(1)} - K\right)_+\right)\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right] - \theta^{(0)}e^{rT} - \theta^{(1)}Y^{(1)}. \end{aligned}$$

Write $Z = \tilde{Z}\,\mathbf{1}_{\{Z\le 0\}} + 0\,\mathbf{1}_{\{Z>0\}}$. Then

$$\mathbb{E}^{\mathbb{P}}\left[e^{Z} \mid Y^{(1)}\right] = \mathbb{E}^{\mathbb{P}}\left[e^{\tilde{Z}}\mathbf{1}_{\{Z\le 0\}} \mid Y^{(1)}\right] + \mathbb{P}\left[\tilde{Z} > 0 \mid Y^{(1)}\right].$$

Since $\tilde{Z}$ and $\log Y^{(1)}$ are jointly normal with correlation ρ, we have $\tilde{Z} \mid Y^{(1)} \sim \mathcal{N}\left(m\left(Y^{(1)}\right), s^2\right)$, with $m\left(Y^{(1)}\right)$ and s given in (3.42) and (3.43).It is straightforward to find that Y^s_{fin} can be expressed as (3.40).

Meanwhile, from (3.13) we obtain

$$\begin{aligned} Y^s_{act} &= \left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\left(\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N} X_i \,\middle|\, Y^{(1)}, Z\right] - \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N} X_i \,\middle|\, Y^{(1)}\right]\right) \\ &= N\left(1+\alpha\left(Y^{(1)}-K\right)_+\right)\left(e^{Z} - \mathbb{E}^{\mathbb{P}}\left[e^{Z} \mid Y^{(1)}\right]\right) \end{aligned}$$

Using (3.40) for the conditional expectation then yields (3.41).

D Closed-form expressions for 4-step market-consistent valuations

D.1 MVSD valuation

The MVSD valuation in (4.5) of S admits a closed-form expression, since

$$\rho^{MVSD}[S] = \theta^{(0)}y^{(0)} + \theta^{(1)}y^{(1)} + \lambda e^{-rT}\mathrm{Var}^{\mathbb{P}}\left[Y^s_{fin} + Y^s_{act} + Y^i\right],$$

and

$$\mathrm{Var}^{\mathbb{P}}\left[Y^s_{fin} + Y^s_{act} + Y^i\right] = \mathbb{E}^{\mathbb{P}}[S^2] - \left(\mathbb{E}^{\mathbb{P}}[S]\right)^2 - \left(\theta^{(1)}\right)^2\mathrm{Var}^{\mathbb{P}}\left[Y^{(1)}\right], \tag{D.1}$$

where $\theta^{(0)}$, $\theta^{(1)}$ are given by (3.32) and (3.31), $\mathbb{E}^{\mathbb{P}}[S]$ follows from (3.35), $\mathrm{Var}^{\mathbb{P}}\left[Y^{(1)}\right]$ is given by (3.34), and $\mathbb{E}^{\mathbb{P}}[S^2]$ can be expressed as

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}[S^2] \\ = \;& Ne^{\mu_s+\frac{1}{2}(\sigma_s)^2}\Phi(-c-\sigma_s) + N(N-1)e^{2\mu_s+2(\sigma_s)^2}\Phi(-c-2\sigma_s) + N^2\Phi(c) \\ + \;& N\alpha^2\left(y^{(1)}\right)^2 e^{2\mu_f T+(\sigma_f)^2 T}\Big[N\Phi_2\left(d_1+\sigma_f\sqrt{T}, c+2\rho\sigma_f\sqrt{T}, \rho\right) \\ + \;& e^{\mu_s+\frac{1}{2}(\sigma_s)^2+2\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+\sigma_f\sqrt{T}+\rho\sigma_s, -c-\sigma_s-2\rho\sigma_f\sqrt{T}, -\rho\right)\Big] \\ + \;& 2N\alpha(1-\alpha K)y^{(1)}e^{\mu_f T+\mu_s+\frac{1}{2}(\sigma_s)^2+\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+\rho\sigma_s, -c-\sigma_s-\rho\sigma_f\sqrt{T}, -\rho\right) \\ + \;& N\alpha K(\alpha K-2)\left[e^{\mu_s+\frac{1}{2}(\sigma_s)^2}\Phi_2\left(d_2+\rho\sigma_s, -c-\sigma_s, -\rho\right) + N\Phi_2\left(d_2, c, \rho\right)\right] \\ + \;& N(N-1)\alpha^2\left(y^{(1)}\right)^2 e^{2\mu_f T+(\sigma_f)^2 T+\mu_s+2(\sigma_s)^2+4\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+\sigma_f\sqrt{T}+2\rho\sigma_s, -c-2\sigma_s-2\rho\sigma_f\sqrt{T}, -\rho\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2N^2\alpha(1-\alpha K)y^{(1)}e^{\mu_f T}\Phi_2\left(d_1, c+\rho\sigma_f\sqrt{T}, \rho\right) \\
&+ \quad 2N(N-1)\alpha(1-\alpha K)y^{(1)}e^{\mu_f T+2\mu_s+2(\sigma_s)^2+2\rho\sigma_s\sigma_f\sqrt{T}}\Phi_2\left(d_1+2\rho\sigma_s, -c-2\sigma_s-\rho\sigma_f\sqrt{T}, -\rho\right) \\
&+ \quad N(N-1)\alpha K(\alpha K-2)e^{2\mu_s+2(\sigma_s)^2}\Phi_2\left(d_2+2\rho\sigma_s, -c-2\sigma_s, -\rho\right). \qquad \text{(D.2)}
\end{aligned}
$$

Here, c, d_1, d_2 are given by (3.37), (3.38), and (3.39), respectively.

Let $\mathcal{P}$ denote the set of equivalent martingale measures, each uniquely determined by its risk-neutral volatility $\sigma_{\mathbb{Q}} > 0$. Under any $\mathbb{Q} \in \mathcal{P}$, the time-$T$ stock price $Y^{(1)}$ satisfies

$$
\log \frac{Y^{(1)}}{y^{(1)}} \sim \mathcal{N}\Big(\big(r - \tfrac{1}{2}\sigma_{\mathbb{Q}}^2\big)T,\ \sigma_{\mathbb{Q}}^2 T\Big),
$$

so that $e^{-rT}Y^{(1)}$ is a $\mathbb{Q}$-martingale and no-arbitrage requires $\sigma_{\mathbb{Q}} > 0$. Hence the maximal admissible set is

$$
\mathcal{P}_{\max} = \big\{\mathbb{Q}(\sigma) \colon \sigma > 0\big\}.
$$

In applications one typically restricts to a feasible subset by prescribing an interval $\Sigma \subset (0, \infty)$, yielding

$$
\mathcal{P} = \big\{\mathbb{Q}(\sigma) \colon \sigma \in \Sigma\big\}.
$$

The following Proposition demonstrates that $\mathbb{E}^{\mathbb{Q}}\big[Y^h + Y^s_{fin}\big]$ admits a closed-form expression that depends only on the risk-neutral volatility $\sigma_{\mathbb{Q}}$.

Proposition D.1 *Let $\mathbb{Q} \in \mathcal{P}$ be any risk-neutral measure with volatility $\sigma_{\mathbb{Q}} \in \Sigma$.*

- *If $\rho = 0$, then the risk-neutral expectation of $Y^h + Y^s_{fin}$ can be expressed as*

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] \\
&= \quad N\left(e^{\mu_s+\frac{1}{2}(\sigma_s)^2}\Phi(-c-\sigma_s)+\Phi(c)\right)\left[1+\alpha e^{m_{\mathbb{Q}}+\frac{v_{\mathbb{Q}}}{2}}\Phi\left(\frac{-\log K+m_{\mathbb{Q}}+v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}\right)\right. \\
&- \quad \left.\alpha K\Phi\left(\frac{-\log K+m_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}\right)\right]. \qquad \text{(D.3)}
\end{aligned}
$$

- *If $\rho \neq 0$, then the risk-neutral expectation of $Y^h + Y^s_{fin}$ can be expressed as*

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] \\
&= \quad N\left[\Phi\left(\frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}\right)-\alpha K\Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \rho_1\right)\right. \\
&+ \quad \left.\alpha e^{m_{\mathbb{Q}}+\frac{v_{\mathbb{Q}}}{2}}\Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}+bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}+v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \rho_1\right)\right] \\
&+ \quad Ne^{a_1 s+\frac{s^2}{2}+bsm_{\mathbb{Q}}+\frac{b^2s^2}{2}v_{\mathbb{Q}}}\left[\Phi\left(\frac{a_2-b^2sv_{\mathbb{Q}}-bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}\right)\right.
\end{aligned}
$$

$$+ \quad \alpha e^{m_{\mathbb{Q}}+(bs+\frac{1}{2})v_{\mathbb{Q}}}\Phi_2\left(\frac{a_2-(b^2s+b)v_{\mathbb{Q}}-bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}+bsv_{\mathbb{Q}}+v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, -\rho_1\right)$$
$$- \quad \alpha K\Phi_2\left(\frac{a_2-b^2sv_{\mathbb{Q}}-bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}+bsv_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, -\rho_1\right)\Bigg]. \tag{D.4}$$

Here c is given by (3.37), s is given by (3.43), and

$$m_{\mathbb{Q}} = \ln y^{(1)} + \left(r - \tfrac{1}{2}(\sigma_{\mathbb{Q}})^2\right)T, \qquad v_{\mathbb{Q}} = (\sigma_{\mathbb{Q}})^2 T,$$
$$b = \frac{\rho}{\sigma_f\sqrt{(1-\rho^2)T}}, \qquad \rho_1 = \frac{b\sqrt{v_{\mathbb{Q}}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},$$
$$a_1 = \frac{\mu_s}{\sigma_s\sqrt{1-\rho^2}} - b\,\frac{\ln y^{(1)} + (\mu_f - \frac{1}{2}\sigma_f^2)T}{\sigma_f\sqrt{T}}, \qquad a_2 = -a_1 - s.$$

Proof. Let $X = \log Y^{(1)}$. Under any risk-neutral measure $\mathbb{Q}$ with volatility $\sigma_{\mathbb{Q}} \in \Sigma$, $X \sim \mathcal{N}(m_{\mathbb{Q}}, v_{\mathbb{Q}})$ with

$$m_{\mathbb{Q}} = \log y^{(1)} + \left(r - \tfrac{1}{2}(\sigma_{\mathbb{Q}})^2\right)T, \quad v_{\mathbb{Q}} = (\sigma_{\mathbb{Q}})^2 T.$$

If $\rho = 0$, then $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ can be written as

$$\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] = N\left(e^{\mu_s+\frac{1}{2}(\sigma_s)^2}\Phi(-c-\sigma_s) + \Phi(c)\right)\left(1 + \alpha\mathbb{E}^{\mathbb{Q}}\left[(e^X - K)_+\right]\right),$$

where c is given in (3.37). Hence, it is straightforward that $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ can be expressed as (D.3).

If $\rho \neq 0$, then $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right]$ can be written as

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] &= N\mathbb{E}^{\mathbb{Q}}\left[\left(1+\alpha\left(e^X-K\right)_+\right)\Phi(a_1+bX)\right] \\ &+ Ne^{a_1s+\frac{s^2}{2}}\mathbb{E}^{\mathbb{Q}}\left[\left(1+\alpha\left(e^X-K\right)_+\right)e^{bsX}\Phi(a_2-bX)\right].\end{aligned}$$

Since $\mathbb{E}^{\mathbb{Q}}\left[\left(1+\alpha\left(e^X-K\right)_+\right)\Phi(a_1+bX)\right]$ can be expressed as

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\left(1+\alpha\left(e^X-K\right)_+\right)\Phi(a_1+bX)\right] &= \int_{-\infty}^{\infty}\Phi(a_1+bx)f_X^{\mathbb{Q}}(x)\mathrm{d}x \\ &+ \alpha\int_{\log K}^{\infty}e^x\Phi(a_1+bx)f_X^{\mathbb{Q}}(x)\mathrm{d}x \\ &- \alpha K\int_{\log K}^{\infty}\Phi(a_1+bx)f_X^{\mathbb{Q}}(x)\mathrm{d}x.\end{aligned}$$

Let $W \sim \mathcal{N}(0,1)$ be independent of X. Then

$$\int_{-\infty}^{\infty}\Phi(a_1+bx)f_X^{\mathbb{Q}}(x)\mathrm{d}x = \mathbb{P}\left(W - bX \leq a_1\right).$$

Note that $W - bX \sim \mathcal{N}\left(-bm_{\mathbb{Q}}, 1 + b^2 v_{\mathbb{Q}}\right)$, we have that

$$\int_{-\infty}^{\infty} \Phi(a_1 + bx) f_X^{\mathbb{Q}}(x) \mathrm{d}x = \Phi\left(\frac{a_1 + bm_{\mathbb{Q}}}{\sqrt{1 + b^2 v_{\mathbb{Q}}}}\right).$$

Additionally, for $j = 0, 1$, we find that

$$\int_{\log K}^{\infty} e^{jx} \Phi(a_1 + bx) f_X^{\mathbb{Q}}(x) \mathrm{d}x = e^{jm_{\mathbb{Q}} + \frac{j^2 v_{\mathbb{Q}}}{2}} \mathbb{P}\left(W - bX^* \leq a_1, -X^* \leq -\log K\right),$$

where X^* is independent of W, and $X^* \sim \mathcal{N}\left(m_{\mathbb{Q}} + jv_{\mathbb{Q}}, v_{\mathbb{Q}}\right)$. Then

$$W - bX^* \sim \mathcal{N}(-bm_{\mathbb{Q}} - jbv_{\mathbb{Q}}, 1 + b^2 v_{\mathbb{Q}}),$$

and

$$\mathrm{Corr}^{\mathbb{Q}}(W - bX, -X) = \frac{b\left(\mathbb{E}^{\mathbb{Q}}\left[X^2\right] - \left(\mathbb{E}^{\mathbb{Q}}\left[X\right]\right)^2\right)}{\sqrt{1 + b^2 v_{\mathbb{Q}}}\sqrt{v_{\mathbb{Q}}}} = \frac{b\sqrt{v_{\mathbb{Q}}}}{\sqrt{1 + b^2 v_{\mathbb{Q}}}}.$$

Hence,

$$\int_{\log K}^{\infty} e^{jx} \Phi(a_1 + bx) f_X^{\mathbb{Q}}(x) \mathrm{d}x = e^{jm_{\mathbb{Q}} + \frac{j^2 v_{\mathbb{Q}}}{2}} \Phi_2\left(\frac{a_1 + bm_{\mathbb{Q}} + jbv_{\mathbb{Q}}}{\sqrt{1 + b^2 v_{\mathbb{Q}}}}, \frac{-\log K + m_{\mathbb{Q}} + jv_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \frac{b\sqrt{v_{\mathbb{Q}}}}{\sqrt{1 + b^2 v_{\mathbb{Q}}}}\right),$$

where Φ_2 is the bivariate normal CDF with the stated correlation.

Repeating the same steps for the second expectation yields (D.4). ∎

D.2 TSSD valuation

The TSSD valuation in (4.7) of S admits a closed-form expression, since

$$\rho^{TSSD}[S] = e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[Y^h + Y_{fin}^s\right] + \lambda e^{-rT} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{Var}^{\mathbb{P}}\left[Y_{act}^s + Y^i \mid Y^{(1)}\right]\right],$$

where $\mathbb{E}^{\mathbb{Q}}\left[Y^h + Y_{fin}^s\right]$ admits a closed-form expression as shown above, and $\mathbb{E}^{\mathbb{Q}}\left[\mathrm{Var}^{\mathbb{P}}\left[Y_{act}^s + Y^i \mid Y^{(1)}\right]\right]$ can be written as

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{Var}^{\mathbb{P}}\left[Y_{act}^s + Y^i \mid Y^{(1)}\right]\right] &= N\mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2 \mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right] \\ &+ N(N-1)\mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2 \mathbb{E}^{\mathbb{P}}\left[e^{2Z} \mid Y^{(1)}\right]\right] \\ &- N^2\mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2 \left(\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right)^2\right], \end{aligned} \tag{D.5}$$

where

$$f\left(Y^{(1)}\right) = 1 + \alpha\left(Y^{(1)} - K\right)_+.$$

When $\rho = 0$, $\mathbb{E}^{\mathbb{Q}}\left[\mathrm{Var}^{\mathbb{P}}\left[Y_{act}^s + Y^i \mid Y^{(1)}\right]\right]$ can be further simplified to

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\mathrm{Var}^{\mathbb{P}}\left[Y_{act}^s + Y^i \mid Y^{(1)}\right]\right] \\ = \;& N\mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2\right]\left(\mathbb{E}^{\mathbb{P}}[e^Z] + (N-1)\mathbb{E}^{\mathbb{P}}[e^{2Z}] - N\left(\mathbb{E}^{\mathbb{P}}[e^Z]\right)^2\right), \end{aligned} \tag{D.6}$$

where

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2\right] \\
= \;& 1+\alpha^2 e^{2m_{\mathbb{Q}}+2v_{\mathbb{Q}}}\Phi\left(\frac{-\log K+m_{\mathbb{Q}}+2v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}\right)+2\alpha(1-\alpha K)e^{m_{\mathbb{Q}}+\frac{v_{\mathbb{Q}}}{2}}\Phi\left(\frac{-\log K+m_{\mathbb{Q}}+v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}\right) \\
+ \;& \alpha K(\alpha K-2)\Phi\left(\frac{-\log K+m_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}\right),
\end{aligned}
\tag{D.7}
$$

and for $j=1,2$, $\mathbb{E}^{\mathbb{P}}\left[e^{jZ}\right]$ is given by

$$
\mathbb{E}^{\mathbb{P}}\left[e^{jZ}\right]=e^{j\mu_s+\frac{j^2}{2}(\sigma_s)^2}\Phi(-c-j\sigma_s)+\Phi(c). \tag{D.8}
$$

When $\rho\neq 0$, for $j=1,2$, $\mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2\mathbb{E}^{\mathbb{P}}\left[e^{jZ}\mid Y^{(1)}\right]\right]$ can be expressed as

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2\mathbb{E}^{\mathbb{P}}\left[e^{jZ}\mid Y^{(1)}\right]\right] \\
= \;& \Phi\left(\frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}\right)+\alpha^2e^{2m_{\mathbb{Q}}+2v_{\mathbb{Q}}}\Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}+2bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},\frac{-\log K+m_{\mathbb{Q}}+2v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}},\rho_1\right) \\
+ \;& 2\alpha(1-\alpha K)e^{m_{\mathbb{Q}}+\frac{v_{\mathbb{Q}}}{2}}\Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}+bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},\frac{-\log K+m_{\mathbb{Q}}+v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}},\rho_1\right) \\
+ \;& \alpha K(\alpha K-2)\Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},\frac{-\log K+m_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}},\rho_1\right) \\
+ \;& e^{ja_1s+\frac{j^2s^2}{2}+jbsm_{\mathbb{Q}}+\frac{j^2b^2s^2}{2}v_{\mathbb{Q}}}\left[\Phi\left(\frac{-a_1-js-bm_{\mathbb{Q}}-jb^2sv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}\right)\right. \\
+ \;& \alpha^2e^{2m_{\mathbb{Q}}+(2jbs+2)v_{\mathbb{Q}}}\Phi_2\left(\frac{-a_1-js-bm_{\mathbb{Q}}-(jb^2s+2b)v_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},\frac{-\log K+m_{\mathbb{Q}}+(jbs+2)v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}},-\rho_1\right) \\
+ \;& 2\alpha(1-\alpha K)e^{m_{\mathbb{Q}}+(jbs+\frac{1}{2})v_{\mathbb{Q}}}\Phi_2\left(\frac{-a_1-js-bm_{\mathbb{Q}}-(jb^2s+b)v_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},\frac{-\log K+m_{\mathbb{Q}}+(jbs+1)v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}},-\rho_1\right) \\
+ \;& \left.\alpha K(\alpha K-2)\Phi_2\left(\frac{-a_1-js-bm_{\mathbb{Q}}-jb^2sv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}},\frac{-\log K+m_{\mathbb{Q}}+jbsv_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}},-\rho_1\right)\right],
\end{aligned}
\tag{D.9}
$$

and

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2\left(\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right)^2\right] \\
= \ & \Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \rho_1^2\right) \\
+ \ & e^{2a_1s+s^2+2bsm_{\mathbb{Q}}+2b^2s^2v_{\mathbb{Q}}}\Phi_2\left(\frac{a_2-bm_{\mathbb{Q}}-2b^2sv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{a_2-bm_{\mathbb{Q}}-2b^2sv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \rho_1^2\right) \\
+ \ & 2e^{a_1s+\frac{s^2}{2}+bsm_{\mathbb{Q}}+\frac{b^2s^2}{2}v_{\mathbb{Q}}}\Phi_2\left(\frac{a_1+bm_{\mathbb{Q}}+b^2sv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{a_2-bm_{\mathbb{Q}}-b^2sv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, -\rho_1^2\right) \\
+ \ & \alpha^2e^{2m_{\mathbb{Q}}+2v_{\mathbb{Q}}}\Phi_3\left(\frac{a_1+bm_{\mathbb{Q}}+2bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{a_1+bm_{\mathbb{Q}}+2bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}+2v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_1\right) \\
+ \ & 2\alpha(1-\alpha K)e^{m_{\mathbb{Q}}+\frac{v_{\mathbb{Q}}}{2}}\Phi_3\left(\frac{a_1+bm_{\mathbb{Q}}+bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{a_1+bm_{\mathbb{Q}}+bv_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}+v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_1\right) \\
+ \ & \alpha K(\alpha K-2)\Phi_3\left(\frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{a_1+bm_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}}, \frac{-\log K+m_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_1\right) \\
+ \ & e^{2a_1s+s^2+2bsm_{\mathbb{Q}}+2b^2s^2v_{\mathbb{Q}}}\left[\alpha^2e^{2m_{\mathbb{Q}}+(4bs+2)v_{\mathbb{Q}}}\Phi_3\left(d_3, d_3, \frac{-\log K+m_{\mathbb{Q}}+(2bs+2)v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_2\right)\right. \\
+ \ & 2\alpha(1-\alpha K)e^{m_{\mathbb{Q}}+(2bs+\frac{1}{2})v_{\mathbb{Q}}}\Phi_3\left(d_3+\rho_1\sqrt{v_{\mathbb{Q}}}, d_3+\rho_1\sqrt{v_{\mathbb{Q}}}, \frac{-\log K+m_{\mathbb{Q}}+(2bs+1)v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_2\right) \\
+ \ & \left.\alpha K(\alpha K-2)\Phi_3\left(d_3+2\rho_1\sqrt{v_{\mathbb{Q}}}, d_3+2\rho_1\sqrt{v_{\mathbb{Q}}}, \frac{-\log K+m_{\mathbb{Q}}+2bsv_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_2\right)\right] \\
+ \ & 2e^{a_1s+\frac{s^2}{2}+bsm_{\mathbb{Q}}+\frac{b^2s^2}{2}v_{\mathbb{Q}}}\left[\alpha^2e^{2m_{\mathbb{Q}}+(2bs+2)v_{\mathbb{Q}}}\Phi_3\left(d_4, d_5, \frac{-\log K+m_{\mathbb{Q}}+(bs+2)v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_3\right)\right. \\
+ \ & 2\alpha(1-\alpha K)e^{m_{\mathbb{Q}}+(bs+\frac{1}{2})v_{\mathbb{Q}}}\Phi_3\left(d_4-\rho_1\sqrt{v_{\mathbb{Q}}}, d_5+\rho_1\sqrt{v_{\mathbb{Q}}}, \frac{-\log K+m_{\mathbb{Q}}+(bs+1)v_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_3\right) \\
+ \ & \left.\alpha K(\alpha K-2)\Phi_3\left(d_4-2\rho_1\sqrt{v_{\mathbb{Q}}}, d_5+2\rho_1\sqrt{v_{\mathbb{Q}}}, \frac{-\log K+m_{\mathbb{Q}}+bsv_{\mathbb{Q}}}{\sqrt{v_{\mathbb{Q}}}}, \mathbf{C}_3\right)\right]. \qquad \text{(D.10)}
\end{aligned}
$$

$$
\begin{aligned}
d_3 &= \frac{a_2-bm_{\mathbb{Q}}-(2b^2s+2b)v_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}} && \text{(D.11)} \\
d_4 &= \frac{a_1+bm_{\mathbb{Q}}+(b^2s+2b)v_{\mathbb{Q}}}{\sqrt{1+b^2v_{\mathbb{Q}}}} && \text{(D.12)} \\
d_5 &= d_3+bs\rho_1\sqrt{v_{\mathbb{Q}}}, && \text{(D.13)}
\end{aligned}
$$

and

$$
\mathbf{C}_1 = \begin{bmatrix} 1 & \rho_1^2 & \rho_1 \\ \rho_1^2 & 1 & \rho_1 \\ \rho_1 & \rho_1 & 1 \end{bmatrix}, \qquad \mathbf{C}_2 = \begin{bmatrix} 1 & \rho_1^2 & -\rho_1 \\ \rho_1^2 & 1 & -\rho_1 \\ -\rho_1 & -\rho_1 & 1 \end{bmatrix}, \qquad \mathbf{C}_3 = \begin{bmatrix} 1 & -\rho_1^2 & -\rho_1 \\ -\rho_1^2 & 1 & -\rho_1 \\ -\rho_1 & -\rho_1 & 1 \end{bmatrix}.
$$

D.3 CMC valuation

Consider the conic market-consistent valuation principle given in (4.9), that is

$$\rho^{CMC}[S] = e^{-rT} \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}\left[Y^h + Y^s_{fin}\right] + \lambda e^{-rT}\sqrt{\mathrm{Var}^{\mathbb{P}}\left[Y^i + Y^s_{act}\right]},$$

where $\mathrm{Var}^{\mathbb{P}}\left[Y^s_{act} + Y^i\right]$ can be written as:

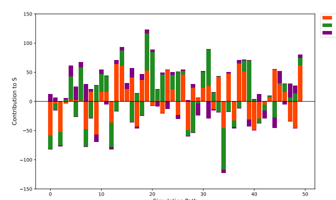
$$\mathrm{Var}^{\mathbb{P}}\left[Y^s_{act} + Y^i\right] = \mathbb{E}^{\mathbb{P}}\left[S^2\right] - N^2\mathbb{E}^{\mathbb{P}}\left[f\left(Y^{(1)}\right)^2\left(\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right)^2\right]. \tag{D.14}$$

Here $\mathbb{E}^{\mathbb{P}}\left[S^2\right]$ is given in (D.2). By the same method used to derive $\mathbb{E}^{\mathbb{Q}}\left[f\left(Y^{(1)}\right)^2\left(\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right)^2\right]$ in (D.10), we can obtain a closed-form for $\mathbb{E}^{\mathbb{P}}\left[f\left(Y^{(1)}\right)^2\left(\mathbb{E}^{\mathbb{P}}\left[e^Z \mid Y^{(1)}\right]\right)^2\right]$ simply by replacing $m_{\mathbb{Q}}$, $v_{\mathbb{Q}}$ with $m_{\mathbb{P}} = \log y^{(1)} + (\mu_f - \frac{1}{2}(\sigma_f)^2)T$, $v_{\mathbb{P}} = \sigma_f^2 T$, respectively.

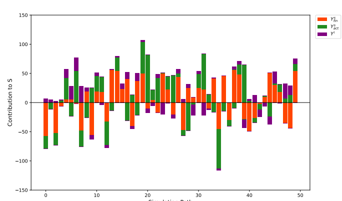
E Decomposition of the unhedgeable part $S - Y^h$ under various values of ρ

(a) $\rho = -0.8$

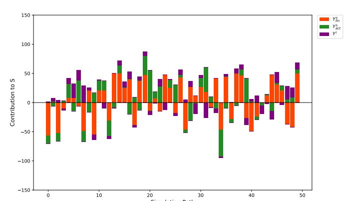
(b) $\rho = -0.4$



(c) $\rho = 0$



(d) $\rho = 0.4$



(e) $\rho = 0.8$

Figure 10: Decomposition of the unhedgeable part $S - Y^h$ across the first 50 paths ($N = 500$): each bar is stacked by contributions from Y^s_{fin} (financial systematic part), Y^s_{act} (actuarial systematic part), and Y^i (idiosyncratic part).